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The Gravity-Capillary Wave Interaction Applied to Wind-Generated, Short-Gravity Waves

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20. Abstract (Continued)

a $k^{5/2}$ dependence in the no-wind case. Comparisons with more accurate numerical evaluations of the interaction are given.

CONTENTS

THE GRAVITY-CAPILLARY WAVE INTERACTION APPLIED TO WIND-GENERATED, SHORT-GRAVITY WAVES

INTRODUCTION

The second-order, gravity-capillary wave interaction elucidated by Valenzuela and Laing [1] has recently been found to play a major role in the equilibration of short-gravity waves in a wind-wave tank [2]. The exact equations for the interaction require the numerical evaluation of several integrals and thus do not lend themselves easily to the development of a simple, intuitive model of the interaction. Furthermore, the exact theory is developed for irrotational waves and includes no wind-speed dependence except in the form of the directional-wave spectrum.

In this report, the gravity-capillary wave interaction is reexamined for the special case where one component of the resonant triad is in the short-gravity-wave range. Wind-speed effects are included in an *ad hoc* manner by forcing the dispersion relation to conform to that found in microwave scattering experiments [3]. When a k^{-4} capillary wave spectrum is used, the result is a simple, analytical expression for the energy transfer which agrees well with numerical calculations in the no-wind case. The expression shows that the interaction may be envisioned as one between a short-gravity wave and a capillary wave whose group speed equals the phase speed of the gravity wave. The interaction dies out for long-gravity waves, having a $k^{5/2}$ dependence in the no-wind case.

GENERAL THEORY

Valenzuela and Laing developed the theory from the well-known equations for velocity potential and surface deviation in which they included surface tension. These two functions were expanded to third order in wave slope, and each order was considered a Fourier series. From the solutions of differential equations for different orders, they computed the energy transfer up to fourth-order terms. Their results, corrected by a factor of two in the energy transfer, are (symbol definitions follow):

$$\frac{\partial S(k_3,\alpha_3)}{\partial t} = \int_0^\infty \sum_{j=1}^{2} T^{(+)}\left\{\frac{k_3}{k_2} S(k_1,\alpha_1^{(j)})S(k_2,\alpha_2^{(j)}) - \frac{\omega_3}{\omega_2} S(k_1,\alpha_1^{(j)})S(k_3,\alpha_3) - \frac{k_1\omega_3}{k_2\omega_1} S(k_2,\alpha_2^{(j)})S(k_3,\alpha_3)\right\} dk_1 + 2\int_0^\infty \sum_{j=1}^{2} T^{(-)}\left\{\frac{k_3}{k_2} S(k_1,\alpha_1^{(j)})S(k_2,\alpha_2^{(j)}) - \frac{\omega_3}{\omega_2} S(k_1,\alpha_1^{(j)})S(k_3,\alpha_3) + \frac{k_1\omega_3}{k_2\omega_1} S(k_2,\alpha_2^{(j)})S(k_3,\alpha_3)\right\} dk_1 \quad (1)$$

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where

$$T^{(+)} = \begin{cases} \dfrac{2\pi\omega_1^2\omega_2^2|D^{+,+}_{\mathbf{k}_1,\mathbf{k}_2}|^2}{(\partial\omega_2/\partial k_2)k_1^2\omega_3^4|\sin\beta|} & \text{for } |\cos\beta| < 1 \\ 0 & \text{for } |\cos\beta| \geqslant 1 \end{cases}$$

and

$$T^{(-)} = \begin{cases} \dfrac{2\pi\omega_1^2\omega_2^2|D^{-,+}_{\mathbf{k}_1,-\mathbf{k}_2}|^2}{(\partial\omega_2/\partial k_2)k_1^2\omega_3^4|\sin\beta|} & \text{for } |\cos\beta| < 1 \\ 0 & \text{for } |\cos\beta| > 1 \ . \end{cases}$$

Here,

$$D^{s_1,s_2}_{\mathbf{k}_1,\mathbf{k}_2} = \frac{i}{2}\Bigg\{(s_1\omega_1 + s_2\omega_2)(k_1k_2 - \mathbf{k}_1\cdot\mathbf{k}_2) + s_1s_2\omega_1\omega_2(s_1\omega_1 + s_2\omega_2) \left(\frac{k_1k_2}{\omega_1^2} + \frac{k_1k_2}{\omega_2^2}\right) - \frac{\omega_3^2}{k_3}\left[\frac{s_1k_1(k_2^2 + \mathbf{k}_1\cdot\mathbf{k}_2)}{\omega_1} + \frac{s_2k_2(k_1^2 + \mathbf{k}_1\cdot\mathbf{k}_2)}{\omega_2}\right]\Bigg\} \tag{2}$$

where s_1 and s_2 give the signs associated with ω_1 and ω_2 in the resonance condition

$$\omega_2 \pm \omega_1 = \omega_3 \ , \ \mathbf{k}_2 \pm \mathbf{k}_1 = \mathbf{k}_3 \ . \tag{3}$$

The angle β between $\mathbf{k}_1$ and $\mathbf{k}_3$ is given by

$$\cos^3\beta \pm \cos^2\beta\left\{\frac{2 + 3(\kappa_1^2 + \kappa_3^2)}{2\kappa_1\kappa_3}\right\} + \cos\beta\left\{\frac{3(\kappa_1^2 + \kappa_3^2)^2 + 4(\kappa_1^2 + \kappa_3^2) + 1}{4\kappa_1^2\kappa_3^2}\right\} \pm \frac{\left\{\kappa_3(1 + \kappa_3^2) + \kappa_1(1 + \kappa_1^2) \pm 2[\kappa_1\kappa_3(1 + \kappa_1^2)(1 + \kappa_3^2)]^{1/2}\right\}^2}{8\kappa_1^3\kappa_3^3} \pm \frac{(\kappa_1^2 + \kappa_3^2)(1 + \kappa_1^2 + k_3^2)^2}{8\kappa_1^3\kappa_3^3} = 0 \tag{4}$$

where upper and lower signs in Eq. (4) go with those in Eq. (3).

The following additional definitions apply in the above equations:

$\mathbf{k}_i$ is the wave number of ith wave,

$\omega_i = \sqrt{gk_i + Tk_i^3}$ is the angular frequency of ith wave,

α_i is the angle of ith wave with symmetry axis,

$S(k_i,\alpha_i)$ is the wave-number spectrum of ith wave in cm^4. Multiplied by $\rho_\omega \omega_i^2/k_i$ this gives the energy spectrum (ρ_ω = density of water),

$k_i = k_i/k_m$,

$g = 980\ \mathrm{cm/s^2}$,

$T = 74\ \mathrm{cm^3/s^2}$,

$k_m = \sqrt{g/T} = 3.64\ \mathrm{cm}^{-1}$.

The sum over j in Eq. (1) accounts for the ambiguity in α_1 and α_2 due to the sign ambiguity in β resulting from Eq. (4).

WIND-DRIVEN, SHORT-GRAVITY WAVE THEORY

We now apply these results to the case in which the subscript 3 refers to a short-gravity wave. In this case, only difference interactions exist; i.e., Eq. (3) can be satisfied only for the lower signs. Furthermore, to include wind-speed effects in the dispersion relation, we write

$$\omega_i = \sqrt{gk_i + Tk_i^3} + \mathbf{k}_i \cdot \mathbf{U}_i \equiv \omega_{io} + \mathbf{k}_i \cdot \mathbf{U}_i \tag{5}$$

where, following Plant and Wright [3],

$$|\mathbf{U}_i| = U_s - \sqrt{\frac{\rho_a}{\rho_\omega}}\,\frac{u_*}{0.41}\, ln\left(1 + \frac{0.277}{z_o k_i}\right) \quad . \tag{6}$$

Here ρ_a and ρ_ω are the densities of air and water, U_s is the surface water velocity at an air friction velocity u_*, and z_0 is the roughness length at u_*. These parameters have all been determined experimentally.

The assumption that k_3 refers to a short-gravity wave requires that k_1 and k_2 refer to the capillary-wave range. Thus,

$$gk_3 \gg Tk_3^3 \ , \ gk_1 \ll Tk_1^3 \ , \text{ and } gk_2 \ll Tk_2^3 \ . \tag{7}$$

This implies that $k_3/k_1 = \kappa_3/\kappa_1$ is small and that $\omega_{30}/\omega_{10} \doteq \sqrt{\kappa_3/\kappa_1^3}$. Thus $(\omega_{30}/\omega_{10})^2 < \kappa_3/\kappa_1$. In the following development, we shall drop terms which are higher than first order in κ_3/κ_1.

The resonance condition, Eq. (3), may be written

$$\begin{aligned} \mathbf{k}_2 &= \mathbf{k}_1 + \mathbf{k}_3, \\ \sqrt{gk_3} + \mathbf{k}_3 \cdot \mathbf{U}_3 &= \sqrt{Tk_1^3} + \mathbf{k}_1 \cdot \mathbf{U}_1 - \sqrt{Tk_2^3} - \mathbf{k}_2 \cdot \mathbf{U}_2 \ . \end{aligned} \tag{8}$$

The geometry of this interaction is shown in Fig. 1. Equations (7) and (8) lead immediately to the following relations:

$$k_2 = k_1 + k_3 \cos\beta \tag{9}$$

and

$$\sqrt{gk_3} + k_3 U_3 \cos\alpha_3 = \frac{3Tk_1^2 k_3 \cos\beta}{2\sqrt{Tk_1^3}} + k_3 U_1 \cos\alpha_3 \tag{10}$$

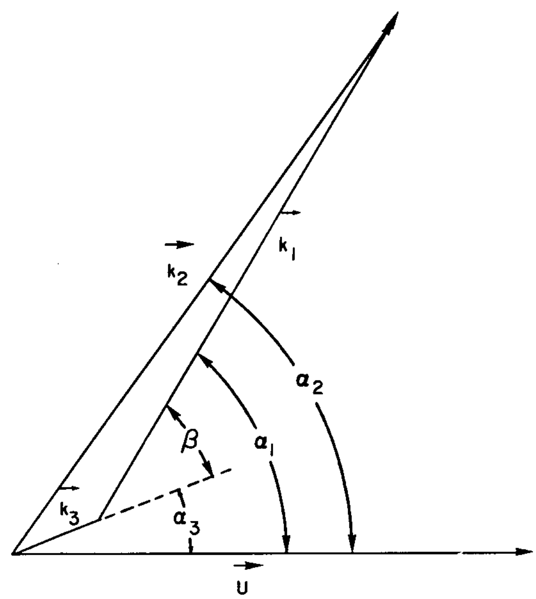

Fig. 1 — Geometry of the gravity-capillary wave interaction applied to a short-gravity wave

where we have used the fact that $U_1 \doteq U_2$. Solving for $\cos\beta$, we obtain

$$\cos\beta = \frac{2k_1[\,c_{30} + (U_3 - U_1)\cos\alpha_3\,]}{3\omega_{10}} \tag{11}$$

where $c_{30} = \sqrt{g/k_3}$ is the gravity-wave phase speed in the absence of wind. Equation (11), then, replaces Eq. (4) when k_3 refers to a short-gravity wave. Note that the sign ambiguity still exists for β.

We must now simplify Eq. (1). First note that difference interactions require that $s_1 = -$ and $s_2 = +$. Thus $D^{+,+}_{\mathbf{k}_1,\mathbf{k}_2} = 0$ and $T^{(+)} = 0$, so that only the first integral in Eq. (1) contributes. Let us concentrate first on $D^{-,+}_{\mathbf{k}_1,-\mathbf{k}_2}$ which we shall simply call D. Substituting Eqs. (3) and (9) into Eq. (2) and using Eq. (7) yields

$$D = \frac{i}{2}\left\{2\omega_3 k_1(k_1 + k_3\cos\beta) - \omega_3 k_1(k_1 + k_3\cos\beta)\left(\frac{\omega_1+\omega_3}{\omega_1} + \frac{\omega_1}{\omega_1+\omega_3}\right) + \frac{\omega_3^2}{k_3}\left[\frac{k_1^2 k_3\cos\beta}{\omega_1} + \frac{(k_1 + k_3\cos\beta)k_1 k_3\cos\beta}{\omega_1+\omega_3}\right]\right\} \tag{12}$$

or

$$D = +\frac{i\omega_3^2 k_1\cos\beta}{2}\left(\frac{k_1}{\omega_1} + \frac{k_1 + k_3\cos\beta}{\omega_1+\omega_3}\right). \tag{13}$$

Thus, to first order in κ_3/κ_1,

$$D = +\frac{i\omega_3^2 k_1^2\cos\beta}{\omega_1}. \tag{14}$$

The expression for the energy transfer may be written to this same order as

$$\frac{\partial S(k_3,\alpha_3)}{\partial t} = 2\int_0^\infty \sum_{j=1}^{2} T^{(-)}\left\{\frac{k_3}{k_1} S(k_1,\alpha_1^{(j)})\,S(k_1,\alpha_1^{(j)}) - \left[\frac{\omega_3}{\omega_1}\left(1-\frac{\omega_3}{\omega_1}\right)S(k_1,\alpha_1^{(j)}) - \frac{\omega_3}{\omega_1}\left(1-\frac{k_3}{k_1}\cos\beta\right)\left\{S(k_1,\alpha_1^{(j)}) + \left.\frac{\partial S(k,\alpha_1^{(j)})}{\partial k}\right|_{k_1} k_3\cos\beta + \left.\frac{\partial S(k_1,\alpha^{(j)})}{\partial \alpha^{(j)}}\right|_{\alpha_1^{(j)}} \sqrt{2}\,\frac{k_3}{k_1}\cos\beta\right\} S(k_3,\alpha_3)\right]\right\} dk_1 \tag{15}$$

or

$$\frac{\partial S(k_3,\alpha_3)}{\partial t} = 2\int_0^\infty \sum_{j=1}^{2} T^{(-)} \left\{ \frac{k_3}{k_1} S^2(k_1,\alpha_1^{(j)}) + \right.$$
$$\frac{\omega_3}{\omega_1}\left(\frac{\omega_3}{\omega_1} - \frac{k_3}{k_1}\cos\beta\right) S(k_1,\alpha_1^{(j)})S(k_3,\alpha_3) +$$
$$\frac{\omega_3 k_3 \cos\beta}{\omega_1}\left.\frac{\partial S(k,\alpha_1^{(j)})}{\partial k}\right|_{k_1} S(k_3,\alpha_3) + \tag{16}$$
$$\left.\frac{\sqrt{2}\omega_3 k_3\cos\beta}{\omega_1 k_1}\left.\frac{\partial S(k_1,\alpha^{(j)})}{\partial\alpha}\right|_{\alpha_1^{(j)}} S(k_3,\alpha_3)\right\} dk_1 .$$

Since all terms inside the braces are of order κ_3/κ_1 or higher, we need to evaluate $T^{(-)}$ only to lowest order:

$$T^{(-)} = \frac{4\pi\omega_1 k_1^3\cos^2\beta}{3|\sin\beta|}\left[\frac{3\omega_{10} + 3k_1U_1\cos\alpha_1^{(j)}}{3\omega_{10} + 2k_1U_1\cos\alpha_1^{(j)}}\right], \quad |\cos\beta| < 1. \tag{17}$$

When $|\cos\beta| \geqslant 1$, $T^{(-)} = 0$. Thus, the lower limit of the integral in Eq. (16) may be replaced by k_l, the value of k_1 when $\cos\beta = 1$. From Eq. (11),

$$3\sqrt{Tk_l}/2 + U_1\cos\alpha_3 = \sqrt{g/k_3} + U_3\cos\alpha_3. \tag{18}$$

The wind-speed dependence of the term in brackets in Eq. (17) is very weak. For conditions of practical interest, it varies from about 1 to 1.3. Thus we shall evaluate it at k_l for $\alpha_1 = 0$ and denote it as A. Then,

$$T^{(-)} = \frac{4A\pi\omega_1 k_1^3\cos^2\beta}{3|\sin\beta|} \text{ where } A \doteq \frac{3\sqrt{Tk_l} + 3U_1}{3\sqrt{Tk_l} + 2U_1}. \tag{19}$$

EVALUATION FOR k^{-4} CAPILLARY-WAVE SPECTRA

Capillary-wave spectra measured in wind-wave tanks fall off approximately as k^{-4} and have an angular dependence close to $\cos^2\alpha$ [4]. Thus we now assume that

$$S(k,\alpha) = C(u_*)k^{-4}\cos^2\alpha \tag{20}$$

for the capillary waves. Then, if Eq. (11) is used to evaluate ω_3/ω_1, Eq. (16) yields

$$\frac{\partial S(k_3,\alpha_3)}{\partial t} = 2\int_{k_l}^{\infty}\sum_{j=1}^{2}\frac{4A\pi\omega_1 k_1^3\cos^2\beta}{3|\sin\beta|}\left\{\frac{C^2k_3\cos^4\alpha_1^{(j)}}{k_1^9} - S(k_3,\alpha_3)\right.$$

$$\left[\frac{7C\omega_3k_3\cos\beta\cos^2\alpha_1^{(j)}}{2\omega_1k_1^5} + \frac{k_3\omega_3CU_1\cos^2\alpha_1^{(j)}(\cos\alpha_3 - 3/2\cos\beta\cos\alpha_1^{(j)})}{\omega_1^2k_1^4}\right. \tag{21}$$

$$\left.\left. + \frac{2\sqrt{2}C\omega_3k_3\cos\beta\cos\alpha_1^{(j)}\sin\alpha_1^{(j)}}{\omega_1k_1^5}\right]\right\} dk_1$$

or

$$\frac{\partial S(k_3,\alpha_3)}{\partial t} = \int_{k_l}^{\infty}\sum_{j=1}^{2}\left\{\frac{A\pi C^2\omega_1k_3\cos^2\beta\cos^4\alpha_1^{(j)}}{3k_1^6|\sin\beta|} - \right.$$

$$\frac{8A\pi C\omega_3k_3\cos^3\beta}{3k_1^2|\sin\beta|}\left[\frac{7\cos^2\alpha_1^{(j)}}{2} + 2\sqrt{2}\cos\alpha_1^{(j)}\sin\alpha_1^{(j)}\right]S(k_3,\alpha_3) - \tag{22}$$

$$\left.\frac{8A\pi C\omega_3k_3U_1\cos^2\beta\cos^2\alpha_1^{(j)}(\cos\alpha_3 - 3/2\cos\beta\cos\alpha_1^{(j)})S(k_3,\alpha_3)}{3\omega_1k_1|\sin\beta|}\right\} dk_1.$$

Fig. 1 shows that $\alpha_1 = \alpha_3 \pm \beta$ which accounts for the sum over j in the above equations.

Two cases are of particular importance, namely, when $\alpha_3 = 0$, and when an integral over α_3 is performed. For $\alpha_3 = 0$, we note that $\alpha_1 = \pm\beta$ so that the sum over j causes the term involving sin $\alpha_1^{(j)}$ to drop out. Then we have

$$\frac{\partial S(k_3,0)}{\partial t} = \int_{k_l}^{\infty}\frac{16A\pi C^2\omega_1k_3\cos^6\beta\, dk_1}{3k_1^6|\sin\beta|} - \tag{23}$$

$$S(k_3,0)\left\{\int_{k_l}^{\infty}\frac{56A\pi C\omega_3k_3\cos^5\beta\, dk_1}{3k_1^2|\sin\beta|} + \int_{k_l}^{\infty}\frac{8A\pi C\omega_3k_3U_1\cos^4\beta(2-3\cos^2\beta)dk_1}{3\omega_1k_1|\sin\beta|}\right\}$$

Note that Eqs. (11) and (18) indicate that

$$\cos\beta = \sqrt{k_l/k_1}\ , \text{ and } \sin\beta = \sqrt{1-k_l/k_1}\ . \tag{24}$$

If we use this and let $U_1 \doteq U_s$ which is independent of k_1, Eq. (23) becomes

$$\frac{\partial S(k_3,0)}{\partial t} = \frac{16A\pi C^2 k_3 k_l^3 T^{1/2}}{3}\int_{k_l}^{\infty}\frac{dk_1}{k_1^7\sqrt{k_1-k_l}} + \frac{16A\pi C^2 k_3 k_l^{7/2} U_s}{3}\int_{k_l}^{\infty}\frac{dk_1}{k_1^8\sqrt{k_1-k_l}} - S(k_3,0)\left\{\frac{56A\pi C\omega_3 k_3 k_l^{5/2}}{3}\int_{k_l}^{\infty}\frac{dk_1}{k_1^4\sqrt{k_1-k_l}} + \frac{8A\pi C\omega_3 k_3 k_l^2 U_s}{3}\int_{k_l}^{\infty}\left[\frac{2}{\omega_1 k_1^{5/2}\sqrt{k_1-k_l}} - \frac{3k_l}{\omega_1 k_1^{7/2}\sqrt{k_1-k_l}}\right]dk_1\right\} . \tag{25}$$

Now, the only troublesome integral is the last one which contains ω_1 in the denominator. Using Eq. (24), we may write

$$\omega_1^{-1}k^{-5/2} = k^{-4}[T^{1/2} + k_l^{1/2}U_s k_1^{-1}]^{-1} \tag{26}$$

As k_1 goes from k_l to ∞, the term in brackets goes from $[T^{1/2} + U_s k_l^{-1/2}]^{-1}$ to $T^{-1/2}$ while k^{-4} goes rapidly from k_l^{-4} to zero. Thus, the main contribution of the term in brackets will be near k_l, so we approximate ω_1 by

$$\omega_1 \doteq k_1^{3/2}\,[T^{1/2} + U_s k_l^{-1/2}]. \tag{27}$$

Once again note that no approximation is involved if the wind speed is zero.

The integrals in Eq. (25)are now easily evaluated to give

$$\frac{\partial S(k_3,0)}{\partial t} = 11.9T^{1/2}AC^2k_3k_l^{-7/2} + 11.0AC^2k_3k_l^{-4}U_s - S(k_3,0)\left[57.5AC\omega_3k_3k_l^{-1} - \frac{5.14AC\omega_3k_3k_l^{-3/2}U_s}{T^{1/2} + U_s k_l^{-1/2}}\right] \tag{28}$$

If the wind speed is zero, this equation becomes

$$\frac{\partial S(k_3,0)}{\partial t} = 205C^2T^4g^{-7/2}k_3^{9/2} - 129CTg^{-1/2}k^{5/2}S(k_3,0) . \tag{29}$$

This expression has been compared with the original theory by using a computer program developed by Valenzuela and Laing [1]; we used $S(k_3,0) = 0.0064k_3^{-4}$ for $k_3 \geqslant 0.364$ cm^{-1} and zero otherwise. The results are given in Table 1.

Table 1 — Comparison of Energy Fluxes Predicted by Original Theory and by Eq. (29)

k_3 (cm^{-1})	$\partial S(k_3,0)/\partial t$ (Eq. 1) (cm^4)	$\partial S(k_3,0)/\partial t$ (Eq. 29) (cm^4)
0.364	-4.86×10^{-2}	-5.71×10^{-2}
0.728	-1.60×10^{-2}	-2.02×10^{-2}
1.09	-8.09×10^{-3}	-1.10×10^{-2}
1.46	-4.96×10^{-3}	-7.10×10^{-3}
1.82	-3.40×10^{-3}	-5.10×10^{-3}

Now let us consider the second case mentioned above, that of integrating the energy transfer given by Eq. (22) over α_3. We assume the following form for $S(k_3,\alpha_3)$ in conformity with wave-tank experiments:

$$S(k_3,\alpha_3) = \begin{cases} S(k_3)\cos^2\alpha_3 \quad , & |\alpha_3| \leqslant \pi/2 \, , \\ 0 \quad , & |\alpha_3| > \pi/2 \, . \end{cases} \tag{30}$$

Furthermore, in order to simplify the integrations over α_3, we assume that $\alpha_i = 0$ in all dispersion relations so that all dependence on α_3 is in the spectral form. In view of the *ad hoc* manner in which wind speed is being incorporated in the theory, this approximation is probably satisfactory. As usual, the approximation becomes exact for zero wind speed.

The integrals over α_3 are now straightforward if one recalls that $\alpha_1^1 = \alpha_3 + \beta$ and $\alpha_1^2 = \alpha_3 - \beta$. We obtain

$$\frac{\partial S(k_3)}{\partial t} = \frac{2}{\pi}\frac{\partial}{\partial t}\int_{-\pi/2}^{\pi/2} S(k_3,\alpha_3)\, d\alpha_3 = 8\pi C^2 k_3 \int_{k_l}^{\infty} \frac{A\omega_1 \cos^2\beta dk_1}{k_1^6 |\sin\beta|} -$$

$$S(k_3)\left[\frac{28\pi C\omega_3 k_3}{3}\right]\int_{k_l}^{\infty} \frac{A(1 + 2\cos^2\beta)\cos^3\beta dk_1}{k_1^2 |\sin\beta|} - \tag{31}$$

$$S(k_3)\left[\frac{4\pi C\omega_3 k_3}{3}\right]\int_{k_l}^{\infty} \frac{AU_1(1 + 2\cos^2\beta)\cos^2\beta(2 - 3\cos\beta)dk_1}{\omega_1 k_1 |\sin\beta|} .$$

Then, letting $U_1 \doteq U_s$ and using Eq. (26) for ω_1 in the last integral and Eq. (24) for $\sin\beta$ and $\cos\beta$, we obtain

$$\frac{\partial S(k_3)}{\partial t} = 8A\pi C^2 k_3 T^{1/2} k_l \int_{k_l}^{\infty} \frac{dk_1}{k_1^5\sqrt{k_1 - k_l}} +$$
$$8A\pi C^2 k_3 U_s k_l^{3/2} \int_{k_l}^{\infty} \frac{dk_1}{k_1^6\sqrt{k_1 - k_l}} -$$
$$S(k_3)\left\{ \frac{28A\pi C\omega_3 k_3}{3}\left[\int_{k_l}^{\infty} \frac{k_l^{3/2}\,dk_1}{k_1^3\sqrt{k_1 - k_l}} + 2\int_{k_l}^{\infty} \frac{k_l^{5/2}\,dk_1}{k_1^4\sqrt{k_1 - k_l}}\right] + \right. \tag{32}$$
$$\frac{8A\pi C\omega_3 k_3 U_s}{3[T^{1/2} + U_s k_l^{-1/2}]}\left[\int_{k_l}^{\infty} \frac{k_l\,dk_1}{k_1^3\sqrt{k_1 - k_l}} - \frac{3}{2}\int_{k_l}^{\infty} \frac{k_l^{1/2}\,dk_1}{k_1^{5/2}\sqrt{k_1 - k_l}} + \right.$$
$$\left.\left. 2\int_{k_l}^{\infty} \frac{k_l^2\,dk_1}{k_1^4\sqrt{k_1 - k_l}} - 3\int_{k_l}^{\infty} \frac{k_l^{5/2}\,dk_1}{k_1^{9/2}\sqrt{k_1 - k_l}}\right]\right\} .$$

Finally, evaluating the integrals and combining numerical values where possible, we have

$$\frac{\partial S(k_3)}{\partial t} = 21.6T^{1/2}AC^2 k_3 k_l^{-7/2} + 19.4AC^2 k_3 k_l^{-4} U_s -$$
$$S(k_3)[92.2AC\omega_3 k_3 k_l^{-1} - \frac{13.4AC\omega_3 k_3 k_l^{-3/2} U_s}{T^{1/2} + U_s k_l^{-1/2}} . \tag{33}$$

This expression has been checked against a direct numerical integration of Eq. (31) for $C = 0.01$ and the exact value of A. The results are given in Table 2. For zero wind speed, Eq. (33) becomes

$$\frac{\partial S(k_3)}{\partial t} = 369C^2 T^4 g^{-7/2} k^{9/2} - 207CTg^{-1/2} k_3^{5/2} . \tag{34}$$

Table 2 — Comparison of Energy Fluxes Predicted by Eqs. (31) and (33)

k_3 (cm^{-1})	u_* (cm/s)	$\partial S(k_3)/\partial t$ (Eq. 31) (cm^4)	$\partial S(k_3)/\partial t$ (Eq. 33) (cm^4)
0.11	15	$2.96 \times 10^{-9} - (1.93 \times 10^{-2})S(k_3)$	$3.81 \times 10^{-9} - (2.35 \times 10^{-2})S(k_3)$
0.11	60	$1.84 \times 10^{-8} - (3.28 \times 10^{-2})S(k_3)$	$2.71 \times 10^{-8} - (4.11 \times 10^{-2})S(k_3)$
1.15	15	$6.87 \times 10^{-5} - (7.06)S(k_3)$	$9.53 \times 10^{-5} - (7.75)S(k_3)$
1.15	60	$2.73 \times 10^{-4} - (11.2)S(k_3)$	$4.51 \times 10^{-4} - (13.9)S(k_3)$

SUMMARY AND DISCUSSION

We have assumed that $k_3 \ll 3.64\ \mathrm{cm}^{-1}$ and derived a simplified form for the gravity-capillary wave interaction. For the zero wind-speed case, this assumption is sufficient to reduce the form of the interaction to the simple analytical expressions given in Eqs. (29) and (34). The first term in each expression is several orders of magnitude smaller than the second so that energy is always drained from the short-gravity wave.

To include wind-speed effects in the problem, the dispersion relation was modified as given in Eq. (5). This procedure is by no means rigorously justified but is an attempt to include the predominant effect of air flow in the problem, considering that a rigorous analysis of wave-wave interactions in a shear flow is extremely complex and currently evades elucidation.

Three additional approximations were necessary to simplify solutions for a wind-dependent dispersion relation. Two of these are given in Eqs. (19) and (27); the third consisted of setting $\cos \alpha_i$ equal to one in all dispersion relations. Such dispersion relations are good approximations for small angles where wave heights and energy transfers are large. The results of these manipulations are given in Eqs. (28) and (33). Table 2 shows that the first two terms in these equations are very small. Furthermore, the last terms in the equations are less than about 10% of the third terms. Thus, a rather good approximation to the gravity-capillary interaction for an arbitrary wind speed may be written

$$\frac{\partial S(k_3,0)}{\partial t} = -57.5\omega_3 k_3 k_l^3 S(k_3,0) S(k_l.0), \tag{35}$$

$$\frac{\partial S(k_3)}{\partial t} = -92.2\omega_3 k_3 k_l^3 S(k_3) S(k_l)\ , \tag{36}$$

where

$$3\sqrt{Tk_l}/2 + U_s = \sqrt{g/k_3} + U_3. \tag{37}$$

To this extent, then, the gravity-capillary wave interaction for short-gravity waves may be considered to be an interaction between a short-gravity wave and a capillary wave whose group speed equals the gravity-wave phase speed.

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