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**Updating the Kalman Filter in Terms of
Correlation Coefficients and Standard
Deviations**

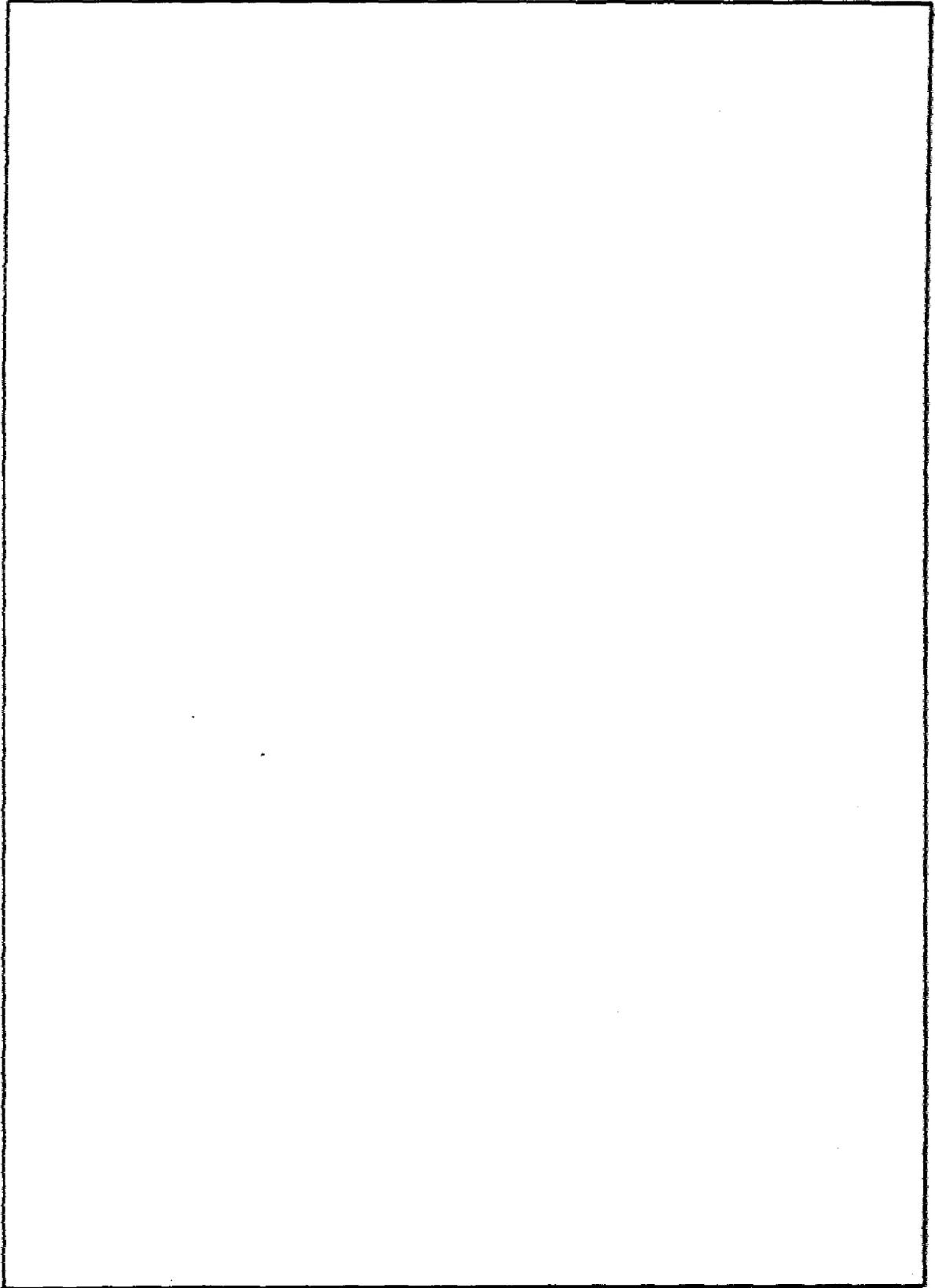
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UPDATING THE KALMAN FILTER IN TERMS OF CORRELATION COEFFICIENTS AND STANDARD DEVIATIONS

INTRODUCTION

The problem of estimating a set of parameters or variables from a set of measurements has long been a problem of interest. Kalman in the early sixties provided a simple recursive estimation procedure by introducing the concept of state and state transition. This procedure in some instances provided simpler implementation than batching techniques. Since Kalman's work a number of numerical procedures have been developed. An excellent account of these procedures as well as historical notes can be found in Bierman's book [1]. Two basic techniques described in Ref. 1 are the SRIF filter (square-root information filter) and the UDU filter. (Actually UDU should be written UDU' , where D is a diagonal matrix, U is an upper triangular matrix, and U' is the transpose.) In both cases the data are pre-whitened using Cholesky factorization. The SRIF filter is then based on the Householder transform. The UDU filter is based on a Cholesky factorization of the smoothed covariance update using one measurement at a time and the modified Gram-Schmidt for updating the predicted covariances. These numerical techniques are claimed to have better numerical stability than use of the Kalman filter equations directly and may be more amenable to hardware implementation.

In this report another factorization is considered for updating the Kalman filter. The factors representing the correlation coefficients and standard deviations are updated in the Kalman filter rather than the factors themselves. Before these results are obtained a method of updating the Kalman filter using only matrix multiplications and additions is first described.

ONE-BY-ONE UPDATING OF THE KALMAN FILTER

The Kalman filter is obtained from modeling the process as state equations, defining a measurement procedure, and best estimating the states of the systems. The state equation and measurement process are defined as

$$X(k) = \Phi(k)X(k-1) + \Gamma(k)W(k)$$

and

$$X_m(k) = H(k)X(k) + V(k),$$

where it is desired to best estimate the n -by-1 state vector $X(k)$. The remaining quantities are an n -by- n state transition matrix $\Phi(k)$, an n -by- p matrix $\Gamma(k)$, an n -by- m measurement matrix $H(k)$, an m -by-1 measurement vector $X_m(k)$, and $W(k)$ and $V(k)$, which are Gaussian noises with the following properties:

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$$E\{W(k)\} = 0,$$

$$E\{W(k)W'(j)\} = S(k)\delta_{jk},$$

$$E\{V(k)\} = 0,$$

$$E\{V(k)V(j)\} = Q(k)\delta_{jk},$$

and

$$E\{W(k)V(j)\} = 0,$$

where δ_{jk} is 1 when $j = k$ and is 0 otherwise. The covariance matrices $S(k)$ and $Q(k)$ are of dimension p by p and m by m respectively.

The best estimate of $X(k)$, denoted by $\tilde{X}(k)$ in the standard Kalman-filter format, is

$$\tilde{X}(k) = \hat{X}(k) + K(k)[X_m(k) - H(k)\hat{X}(k)],$$

where $K(k)$ is the filter gain, given by

$$K(k) = \tilde{P}(k)H'(k)Q^{-1}(k),$$

in which $\tilde{P}(k)$ is the smoothed covariance matrix, given by

$$\tilde{P}^{-1}(k) = \hat{P}^{-1}(k) + H'(k)Q^{-1}(k)H(k).$$

$\hat{P}(k)$ is the predicted covariance matrix, and

$$\hat{P}(k+1) = \Phi(k+1)\tilde{P}(k)\Phi'(k+1) + \Gamma(k+1)S(k+1)\Gamma'(k+1).$$

The prediction is

$$\hat{X}(k+1) = \Phi(k+1)\tilde{X}(k).$$

The filter operates in a predict and correct fashion.

The Kalman filter can be written in a slightly different form by prewhitening the measurement noise. This is a standard practice which can be achieved using Cholesky factorization as described in Ref. 1. The covariance $Q(k)$ of the noise $V(k)$ is factored into the form of

$$Q(k) = LEL' \tag{1a}$$

or

$$Q^{-1}(k) = (L')^{-1}E^{-1}L^{-1}, \tag{1b}$$

where L is a lower triangular matrix with 1's on the diagonal and E is a diagonal matrix with diagonal elements e_j . The algorithm for obtaining L and E is

$$l_{jj} = 1, \quad j = 1, \dots, n-1,$$

$$e_j = q_{jj},$$

$$l_{kj} = q_{kj}/e_j, \quad k = j+1, \dots, n,$$

and

$$q_{ik} = q_{ik} - l_{ij}l_{kj}e_j, \quad k = j+1, \dots, n \text{ and } i = k, \dots, n.$$

With use of equations (1a) and (1b) the Kalman filter can be rewritten as

$$\tilde{X}(k) = \hat{X}(k) + K(k)[Z_m(k) - H(k)\hat{X}(k)], \quad (2)$$

$$K(k) = \tilde{P}(k) H'(k) E^{-1}(k), \quad (3)$$

$$\tilde{P}^{-1}(k) = \hat{P}^{-1}(k) + H'(k) E^{-1}(k) H(k), \quad (4)$$

$$\hat{P}(k+1) = \Phi(k+1)\tilde{P}(k)\Phi'(k+1) + \Gamma(k+1)S(k+1)\Gamma'(k+1), \quad (5)$$

and

$$\hat{X}(k+1) = \Phi(k+1)\tilde{X}(k), \quad (6)$$

where

$$Z_m(k) = L^{-1}X_m(k)$$

and

$$H(k) = L^{-1}H(k).$$

Equations (2), (3), (5), and (6) are easily updated by matrix multiplications and additions. A means of updating equation (4) which does not involve a matrix inverse will now be described.

The matrix identity used in converting forms of the Kalman filter [2] is

$$(A^{-1} + B'C^{-1}B)^{-1} = A - AB'(BAB' + C)^{-1}BA, \quad (7)$$

where A and C represent covariances. For the special case in which $BAB' + C$ is a scalar, equation (7) becomes

$$(A^{-1} + B'C^{-1}B)^{-1} = A - \frac{AB'BA}{(C + BAB')}. \quad (8)$$

This result will be used subsequently.

The smoothed covariance update, equation (4), is rewritten as

$$\tilde{P}^{-1} = \hat{P}^{-1} + K'E^{-1}K, \quad (9)$$

where the k th-sample notation has been dropped for notational convenience. The pre-whitened measurement matrix is defined as

$$K = \begin{bmatrix} K_1 \\ K_2 \\ \dots \\ K_i \end{bmatrix}, \quad (10)$$

where K_i represents the i th row. Equation (9) can then be rewritten as

$$\tilde{P}^{-1} = \hat{P}^{-1} + \begin{bmatrix} K'_1 & K'_2 & \dots & K'_m \end{bmatrix} \begin{bmatrix} \frac{1}{e_1} & 0 & \dots & 0 \\ 0 & \frac{1}{e_2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{1}{e_m} \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ \dots \\ K_m \end{bmatrix} \quad (11)$$

Equation (11) can then be rewritten as

$$\tilde{P}^{-1} = \hat{P}^{-1} + K'_1 \frac{1}{e_1} K_1 + K'_2 \frac{1}{e_2} K_2 + \dots + K'_m \frac{1}{e_m} K_m. \quad (12)$$

Equation (12) can be placed in recursive form by

$$\tilde{P}_i^{-1} = \tilde{P}_{i-1}^{-1} + K'_i \frac{1}{e_i} K_i, \quad i = 1, \dots, m, \quad (13)$$

where $\tilde{P}_0^{-1} = \hat{P}^{-1}$ and $\tilde{P}_m^{-1} = \tilde{P}^{-1}$. The matrix identity mentioned previously in equation (8) is applied to equation (13), yielding

$$\tilde{P}_i = \tilde{P}_{i-1} - \frac{\tilde{P}_{i-1} K'_i K_i \tilde{P}_{i-1}}{e_i + K_i \tilde{P}_{i-1} K'_i}, \quad i = 1, \dots, m, \quad (14)$$

where $e_i + K_i \tilde{P}_{i-1} K'_i$ is a scalar, because the K_i as defined in (10) are row vectors. Consequently equation (14) shows that no matrix inverses need be found to update the smoothed covariance. Equation (14) suggests the title of this section: One-by-One Updating of the

Kalman Filter. Each measurement variance, taken one at a time, is prewhitened and used to modify the smoothed covariance. A means of updating the filter in terms of correlation coefficients and standard deviations is considered next.

UPDATING OF THE KALMAN FILTER BY UPDATING THE CORRELATION COEFFICIENTS AND STANDARD DEVIATIONS

A covariance matrix written in terms of correlation coefficients and standard deviations is

$$\tilde{P} = \tilde{D} \tilde{U} \tilde{D}, \tag{15}$$

where the elements of U are $u_{ij} = p_{ij} / \sqrt{p_{ii}} \sqrt{p_{jj}}$ ($i = 1, \dots, n$ and $j = 1, \dots, n$) and the elements of D are $d_j = \sqrt{p_{jj}}$, with the matrix \tilde{D} being a diagonal matrix with diagonal elements d_j which represent the standard deviations. The diagonal elements of U are 1's and the off-diagonal elements are correlation coefficients with values between -1 and 1. The one-by-one updating of the Kalman filter is modified to reflect the factorization (14).

The Kalman-filter equations (2) through (6) will now be reexamined, with the k th-sample notation being eliminated for notational convenience. Equations (2) and (6) remain the same, since they do not involve covariances. These equations are

$$\tilde{X} = \hat{X} + K[Z_m - H\hat{X}]$$

and

$$\hat{X} = \Phi \tilde{X}.$$

The filter gain (equation (3)) is written as

$$K = \tilde{P} H' E^{-1},$$

where each element of the gain matrix K is

$$k_{ij} = \sum_{\ell=1}^n (p_{i\ell} h_{j\ell}) / e_j, \quad i = 1, \dots, n \text{ and } j = 1, \dots, m. \tag{16}$$

Using the factorization in equations (15) and (16) yields the gains

$$k_{ij} = \sum_{\ell=1}^n (u_{i\ell}) \frac{h_{j\ell} d_i d_\ell}{e_j}$$

The prediction covariance update in equation (5) is next examined for the case of zero process noise. The equation is

$$\hat{P} = \Phi \tilde{P} \Phi'.$$

The factorization described in equation (15) is used in yielding

$$\hat{D}\hat{U}\hat{D} = \Phi\tilde{D}\tilde{U}\tilde{D}'\Phi'. \quad (17)$$

Since $\tilde{D} = \tilde{D}'$, equation (17) can be rewritten as

$$\hat{D}\hat{U}\hat{D} = \tilde{D}\{(\tilde{D}^{-1}\Phi\tilde{D})\tilde{U}(\tilde{D}^{-1}\Phi\tilde{D})'\}\tilde{D}. \quad (18)$$

The bracketed term of equation (18) is factored in the form of (15), yielding

$$(\tilde{D}^{-1}\Phi\tilde{D})\tilde{U}(\tilde{D}^{-1}\Phi\tilde{D})' = R\hat{U}R. \quad (19)$$

The forms (18) and (19) then yield

$$\hat{D} = \tilde{D}R,$$

which is the simple product of two diagonal matrices.

The last equation to consider is the smoothed covariance, which relies on the one-by-one update previously described. The equation is

$$\tilde{P}_i = \tilde{P}_{i-1} - \frac{\tilde{P}_{i-1}\kappa_i'\kappa_i\tilde{P}_{i-1}}{e_i + \kappa_i\tilde{P}_{i-1}\kappa_i'} \quad (20)$$

for $i = 1, \dots, m$ and $\tilde{P}_0 = \hat{P}$. By use of the factorization (15), equation (20) becomes

$$\tilde{D}_i\tilde{U}_i\tilde{D}_i = \tilde{D}_{i-1}\left(\tilde{U}_{i-1} - \frac{\tilde{U}_{i-1}\tilde{D}_{i-1}\kappa_i'\kappa_i\tilde{D}_{i-1}\tilde{U}_{i-1}}{e_i + \kappa_i\tilde{D}_{i-1}\tilde{U}_{i-1}\tilde{D}_{i-1}\kappa_i'}\right)\tilde{D}_{i-1}, \quad (21)$$

where $\tilde{U}_0 = \hat{U}$, $\tilde{D}_0 = \hat{D}$, $\tilde{U}_m = \tilde{U}$, and $\tilde{D}_m = \tilde{D}$. The vector V_{i-1} is defined as

$$V_{i-1} = \frac{\kappa_i\tilde{D}_{i-1}\tilde{U}_{i-1}}{\sqrt{e_i + \kappa_i\tilde{D}_{i-1}\tilde{U}_{i-1}\tilde{D}_{i-1}\kappa_i'}}$$

Equation (21) can then be written as

$$\tilde{D}_i\tilde{U}_i\tilde{D}_i = \tilde{D}_{i-1}(\tilde{U}_{i-1} - V_i'V_i)\tilde{D}_{i-1}. \quad (22)$$

Equation (22) is factored into the correlation coefficient form, where

$$\tilde{U}_{i-1} - V_i'V_i = R\tilde{U}_iR.$$

Therefore

$$\tilde{D}_i = \tilde{D}_{i-1}R,$$

which is the product of two diagonal matrices. This completes the development.

The role the DUD factorization, in terms of standard deviations and correlation coefficients, plays in the Kalman filter is that of the scale factors, and this factorization is numerically different than a straightforward implementation. This may be an advantage in some problems in which the range of standard deviations are known and fixed-point hardware could probably be used to an advantage. It should be noted that in the computations the values of the elements of U are usually near 1 and the values of the elements of D usually change only slightly from the values of the elements of the previous D . This indicates that for some problems good numerical accuracy may be achieved.

The updating of the Kalman filter in terms of correlation coefficients and standard deviations is similar in many respects to Carlson's UDU -filter update found in Ref. 1. Both rely on the same form of the Kalman filter in the one-by-one update form as previously presented. Both filters update factors of the covariances, and both require a refactorization of a portion of covariance equations to obtain the desired form. They are different in that the factorization is different.

SUMMARY

The Kalman filter was briefly reviewed, and a one-by-one covariance update technique was shown to operate such that the entire Kalman filter can be updated with only matrix multiplications and additions, given that the noise is prewhitened. The one-by-one update operates on one measurement at a time in a simple fashion, and after all measurements are used, the Kalman filter is updated.

The covariances are factored in terms of correlation coefficients and standard deviations, and the Kalman filter is updated in terms of these factors. The technique is similar in principle to Carlson's technique but with a different factorization.

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1. G.J. Bierman, *Factorization Methods for Discrete Sequential Estimation*, Academic Press, New York, 1976.
2. M. Aoki, *Optimization of Stochastic Systems: Topics in Discrete Time Systems*, Academic Press, New York, 1967.