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Optimal Spectral Estimates

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We consider the general problem of estimating the spectrum of a deterministic signal perturbed by noise. The emphasis of this report is on the reduction of the deterministic part of the error, and we wish to determine when equispaced sampling is optimal, and to what extent aliasing errors can be avoided by the use of nonequispaced sampling. It is shown that equispaced sampling is not always optimal; so the problem reduces to determining whether any practical advantage would be gained by the use of optimal sampling schemes rather than equispaced sampling schemes when the latter are not optimal. The resolution of this question has been further reduced to a calculus problem, the

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20. ABSTRACT (Continued)

minimization of an explicitly given function of N variables. This result has been achieved through the use of techniques borrowed from the theory of Sobolev spaces, which also have the advantage that they permit one to define the efficiency of a spectral estimator in terms of signal energy and signal bandwidth, whereas in the classical theories of numerical quadrature, quadrature errors are expressed in terms of bounds on the kth order derivatives of a function, parameters which have little significance in filter design.

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EXECUTIVE SUMMARY

In engineering and scientific practice, the extraction of information from signals usually involves spectral analysis, a process by which one determines how the energy of a signal is distributed among various frequency bands. In the standard techniques of spectral analysis the data are sampled at equispaced intervals of time. Accuracy and the ability to measure high-frequency components are lost if the data are not sampled at a sufficiently high rate, but the maximum attainable data sample rate is limited by the presence of noise and other technical factors such as limitations on computer storage and processing time. It has been speculated that some of these difficulties might be reduced by the use of data-sampling schemes employing nonequispaced sampling, but the question as to whether this is possible has not yet been settled.

This report is concerned with the spectral analysis of deterministic signals perturbed by noise. The statistics of the noise are assumed to be given (as is often the case), and the emphasis of the report is on the reduction of the deterministic component of the errors in spectral measurement. We develop a new mathematical method which permits one to express such errors as explicitly given functions of the sample points, and using these results we show that equispaced sample points are not always optimal for spectral measurement. Hence the question next arises as to whether or not any *practical* advantage would be gained by the use of optimal sample-point sets rather than equispaced sets when the latter are nonoptimal. Since the errors in spectral measurements have been given as explicit functions of the sample points, the resolution of this question has been reduced to an ordinary problem in the calculus: the minimization of an explicitly given function of N variables. Although this minimization problem is perfectly straightforward, we have not as yet been able to obtain its solution.

OPTIMAL SPECTRAL ESTIMATES

1. DESCRIPTION OF THE PROBLEM AND THE RESULTS

1.1 Introduction

This is a preliminary report describing a recent investigation of the problem of obtaining information about the spectra of deterministic signals perturbed by noise. Given data of the form $f(t) = s(t) + n(t)$ = signal + noise, the problem is to obtain information about the spectrum $\hat{s}(\nu) = \int s(t)e^{-i\nu t}\, dt$ from the sampled data $f(t_1), f(t_2), ..., f(t_N)$ in the time domain $[0, T]$. We should emphasize that the problem is to obtain information about the spectrum of the deterministic component of $f(t)$; the statistics of the noise $n(t)$ will be assumed to be given.

We are interested in obtaining a theory which is applicable to highly oscillatory signals, and in which the errors in spectral estimates are described in terms of physically meaningful parameters.

There are three related but distinct aspects to this general problem: the *detection*, *resolution*, and *measurement* of spectral components. We shall consider the latter aspect first.

1.2 The Problem of Measurement

The errors in spectral estimates consist of two components: a *deterministic* component (or *quadrature error*) which would be present even if noise were completely absent and a *stochastic* component arising from the effects of noise. The emphasis of this report is on the reduction of the deterministic component.

For the moment let us suppose that noise is completely absent. The problem we pose is to construct for each frequency ν a linear filter $f(t) \to f_{\text{est}}(\nu) \equiv \Sigma\beta_j f(t_j)$ which estimates the νth spectral component $\hat{f}(\nu)$. (We write a general linear filter in this form instead of $\Sigma\bar{\beta}_j f(t_j)$ to avoid the inconvenient appearance of the complex conjugates $\bar{\beta}_j$ at a later point in the discussion.) To measure the efficiency of such a filter, we consider the set of all errors $|\hat{f}(\nu) - f_{\text{est}}(\nu)|$ normalized by a Sobolev-space norm which measures the variation of f and *is expressible in terms of signal energy and bandwidth.* We then take the supremum over the class of all functions f which have finite energy and finite bandwidth and thus obtain an expression for the quadrature error explicitly given as a function of the sample points t_j and the "weights" β_j as well as the signal energy and bandwidth.

It is at this point that our analysis differs from the more traditional kind. In the classical theory of numerical quadrature the quadrature error is expressed in terms of *bounds on the kth order derivatives of f*, parameters which have little physical significance and whose use becomes especially dubious for highly oscillatory signals.

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Having expressed the quadrature error as an explicit function of the sample points and weights, we can then, at least in principle, optimize (minimize) the error with respect to these variables. *In particular, we wish to determine if equispaced sample point sets are always optimal.* (Cf. Ref. 1, p. 33.)

It turns out that equispaced sample point sets are *not* always optimal. So we should then consider the question as to whether any *practical* advantage would be gained by using estimates employing optimal sample-point sets when equispaced sets are nonoptimal. We have reduced the resolution of this question to a problem in the calculus — the minimization of an explicit expression involving certain hyperbolic and trigonometric functions evaluated at the sample points. Although this problem is perfectly straightforward, we have not as yet been able to solve it. (See Section 2 for details.)

1.3 The Problem of Detection

In its deterministic version the problem is to construct for each frequency ν a linear filter $f \to \Sigma\overline{\beta}_j f(t_j)$ with the property that

$$|\Sigma\beta_j f(t_j)| > D_\nu \text{ implies } \hat{f}(\nu) \neq 0, \tag{1.1}$$

where D_ν is a certain threshold value dependent on the weights β_j, the sample points t_j, and the signal parameters (energy and bandwidth). We wish to optimize this scheme in the sense of making the inequality in (1.1) as weak as possible.

Now the reader might object that in the deterministic case one could detect a frequency component $\hat{f}(\nu)$ merely by computing the integral $\int f(t)e^{-i\nu t}\, dt$. However the algorithms for computing such integrals are filters of the type given above, and moreover we have no right to assume that a filter which is optimal for the measurement of a spectral component is also optimal for its detection.

It turns out that, at least for the classes of signals which we have so for considered, a filter which is optimal for measurement is also optimal for detection. But it also turns out that the smallest possible value for the threshold value D_ν is somewhat smaller than the corresponding error bound for measurement.

1.4 The Effects of Noise on Measurement

We shall suppose that we are dealing with data records having some fixed time span T. When noise is absent, the errors in spectral estimates can be made arbitrarily small by making the number of sample points N arbitrarily large. On the other hand, when noise is present, one expects that the stochastic component of the error to remain above a certain level as $N \to \infty$. This is because for data records having fixed length T, N can be increased only by increasing the average sample rate, and when this rate is increased the information contained in the stochastic part of the sampled data becomes more redundant.

More specifically, the deterministic component of the error has the appearance of a bias whose square is a nonhomogeneous polynominal in the weights β_j and whose coefficients are functions of the sample points. The stochastic component is an rms error whose square is a homogeneous polynominal in the weights β_i and whose coefficients are the autocorrelation of the noise evaluated at the sample points. One could effect some kind of a tradeoff between these two kinds of errors, but such an analysis has not yet been attempted.

This apparent limitation that noise imposes on the average sample rate lends greater interest to the problem of determining whether the deterministic component of the error can be significantly reduced below the values obtained by equispaced sampling when N is small and equipspaced sample points are not optimal.

1.5 Resolution

There are two types of resolution phenomena: those which involve the effects of noise and those which do not. We shall also have to consider two classes of signals: periodic and nonperiodic. As in the previous paragraph, we asume that we are dealing with a data record of fixed length T.

Periodic Case

For purely deterministic periodic signals there is no resolution problem. The detection scheme (1.1) never yields a false detection, and the detection (and measurement) of a spectral component with arbitrarily small amplitude can be effected by making N sufficiently large.

Suppose now that noise is present. Then the detection scheme (1.1) will sometimes yield false detections when the noise causes the filter output to rise above the threshhold value D_ν. Similarly the noise will sometimes prevent the detection of a spectral component when one is actually present. The probability of either of these occurrences will depend in part on how strongly the filter responds to frequencies other than the frequency to which it is matched. Now the spectrum of a periodic signal is discrete, and one can redesign the detection filter so that it will have a zero response to any discrete set of frequencies, in particular, to those which are close to the frequency whose detection is desired. Such a "sidelobe suppression" can be made only at the expense of increasing the threshold value D_ν. Also, as mentioned in the last paragraph, one should expect that the existence of noise will impose a limit on the number of sample points N which can be used with a record of fixed time span T.

Nonperiodic Case

In the nonperiodic case there is a resolution problem even when noise is completely absent. This is because what we are measuring is the "truncated" spectrum

$$\hat{f}_T(\nu) = \int_0^T f(t)e^{-i\nu t}\,dt, \tag{1.2}$$

whereas what we really want is the "true" spectrum

$$\hat{f}(\nu) = \int_{-\infty}^{+\infty} f(t)e^{-i\nu t}\, dt. \tag{1.3}$$

From a purely mathematical point of view there is no way out of this difficulty, because for functions f in general the functional values of f on the interval $[0, T]$ do not determine its functional values anywhere else. As is well known, the spectrum $\hat{f}_T$ corresponds (roughly) to the spectrum $\hat{f}$ smoothed (integrated) over a frequency band of width $1/T$.

One can apply the results obtained in the periodic case to the nonperiodic case by extending a function defined on the internal $[0, T]$ to a periodic function. The frequencies of the extended periodic function will be multiples of $1/T$, and the Fourier coefficients (squared) will correspond (roughly) to the amount of signal energy contained in frequency bands of width $1/T$. The accuracy of the process improves as T increases. If noise is now introduced, then the same phenomena will occur as was described for the periodic case.

Remark 1.1. There is one important class of functions f for which the functional values in any internal $[0, T]$, no matter how small, determine the values of f everywhere, namely, the class of "band-limited" signals whose spectra are contained in some bounded interval of ν space. We have not been able to extend our function analytic methods to this class of functions, our difficulties in this regard being conceptual rather than computational. Note that a periodic function is band limited in this sense if it only has a finite number of nonzero Fourier coefficients. Thus the spectrum of a purely deterministic periodic signal can be solved for exactly with only a finite number of sample points. (See Ref. 2.)

Remark 1.2. In many applied problems one is given a band-limited signal with a carrier frequency ν_0 and "bandwidth" B_0. Such functions have the representation

$$f(t) = \int_{\nu_0-B/2}^{\nu_0+B/2} \hat{f}(\nu)e^{i\nu t}\, d\nu.$$

The error functions discussed in the succeeding sections are expressed in terms of another kind of "bandwidth" B, which is the second moment of the spectrum about the axis $\nu = O$. (See Section 2.2.) These two bandwidths are related by

$$B^2 = \nu_0^2 + \delta B_0^2,$$

where δ is a quantity which depends on the shape of the spectrum and is usually on the order of unity. In practice, when dealing with signals having a high carrier frequency, signal processing is preceded by heterodyning which translates the signal spectrum from ν_0 to the origin $\nu = O$. One can therefore apply the results below to the heterodyned signal, the bandwidth B now being identified with B_0. More generally, one would expect that an optimal filter would mimic the process of heterodyning, but we can prove this to be the case only when ν_0/N is small and under the assumption that equispaced sample points are close to optimal.

1.6 Historical Remarks

For deterministic signals the case $\nu = 0$ corresponds to ordinary numerical quadrature, and Sobolev-space techniques have been applied to the problem of numerical quadrature by (not surprizingly) Sobolev and his students. The emphasis of their work has been on the quadrature of functions defined on domains of dimension greater than 1 and on obtaining weights β_j which are optimal with respect to a given (not necessarily optimal) sample-point set and for which the corresponding quadrature formulas are exact on polynomials with a given degree. For a bibliography on the subject the reader is referred to an expository article by Haber [4]. This article also contains an account of some recent number theoretical treatments of the problem of numerical quadrature which have raised doubts concerning the optimality of equispaced sample points.

2. THE ERROR FUNCTIONS FOR DETERMINISTIC SIGNALS

2.1 Notation

We shall adopt a convention whereby the time span over which a function f is defined will be denoted by P is f is periodic and by T if f is nonperiodic. The number of sample points will always be denoted by N, and the set of sample points will always be represented by the sequence $\mathbf{t} = \{t_1, t_2, ..., t_N\}$. The N sample points t_j will always be assumed to be distinct.

All functions under discussion will be assumed to be complex valued (although our results apply equally well to vector-valued functions), and we shall write

$$||f||_0^2 = \int_0^\ell |f(t)|^2 \, dt, \tag{2.1}$$

where ℓ is P or T depending on whether f is periodic or nonperiodic. The operation of differentiation will be denoted by D, so that

$$Df(t) = f'(t). \tag{2.2}$$

Finally a general linear filter will hereafter be written in the form

$$\sum_{j=1}^{N} \overline{\beta}_j f(t_j)$$

instead of

$$\sum_{j=1}^{N} \beta_j f(t_j)$$

to avoid the inconvient appearance of the complex conjugates $\overline{\beta}$ at a later point in the discussion.

2.2 The Error Function $Q_{\nu,c}(\mathbf{t}, \beta)$

For each frequency ν we wish to construct filters of the type

$$\hat{f}_{\text{est}}(\nu) = \sum_{j=1}^{N} \overline{\beta}_j f(t_j), \tag{2.3}$$

which estimate the νth spectral component

$$\hat{f}(\nu) = \int_0^{\ell} f(t) e^{-i\nu t}\, dt, \tag{2.4}$$

where ℓ is P or T as before. For P-periodic functions the only allowed frequencies are of the form $\nu = 2\pi k/P$, k an integer.

To measure the efficiency of the filter (2.3), we define an error function $Q_{\nu,c} = Q_{\nu,c}(\mathbf{t}, \beta)$ by

$$Q_{\nu,c}^2(\mathbf{t}, \beta) = \sup_f \left[\frac{|\hat{f}(\nu) - \hat{f}_{\text{est}}(\nu)|^2}{\|f\|_0^2 + c^2 \|Df\|_0^2} \right], \tag{2.5}$$

where c is a positive number (whose value will be set later) and the supremum is taken over the class of all continuous functions f whose derivatives exist almost everywhere and are square integrable. This class will be denoted by H^1, and it contains sawtoothlike functions whose derivatives are undefined or discontinuous at certain points. We shall presently show that H^1 can also be described as the class of all continuous functions with finite bandwidth.

Since $Q_{\nu,c}^2$ is defined as the supremum of the quantity in the brackets, we have the inequality

$$|\hat{f}(\nu) - \hat{f}_{\text{est}}(\nu)|^2 \leqslant (\|f\|_0^2 + c^2 \|Df\|_0^2) Q_{\nu,c}^2(\mathbf{t}, \beta), \tag{2.6}$$

which holds for all f in the class H^1. It can be shown that this inequality becomes an equality at some functions f, so that (2.6) is in this sense the strongest possible inequality for any given values of ν, c β_j, and t_j.

We now wish to show that $Q_{\nu,c}^2$ is the squared error $|\hat{f}(\nu) - \hat{f}_{\text{est}}(\nu)|^2$ normalized by a quantity which measures the variation of f and is expressible in terms of signal energy and signal bandwidth. In most applications $|f(t)|^2$ has the dimensions of *power* (watts), and $\|f\|_0^2$ is therefore the *signal energy*. For P-periodic functions one easily establishes that

$$\frac{\|Df\|_0^2}{\|f\|_0^2} = \frac{\Sigma \nu^2 |\hat{f}(\nu)|^2}{\Sigma |\hat{f}(\nu)|^2}, \tag{2.7}$$

where ν varies over all the frequencies $2\pi k/P$, k an integer. For nonperiodic functions vanishing at the endpoints the relation is the same with Σ replaced by $\int$. For nonperiodic functions not vanishing at the endpoints the relation is complicated by the addition of terms involving the values of f at the endpoints. For functions of class H^1 (on $(-\infty, +\infty)$) these terms converge to 0 as $T \to \infty$. In any case the left-hand side of (2.7) is related to the spectral spread about the axis $\nu = 0$ and will be called the bandwidth $B(f)$ of f. So, setting

$$B^2(f) = \frac{\|Df\|_0^2}{\|f\|_0^2}\ , \tag{2.8}$$

we write (2.6) in the form

$$|\hat{f}(\nu) - \hat{f}_{\text{est}}(\nu)|^2 \leq \|f\|_0^2(1 + c^2 B^2(f)) Q^2_{\nu,c}(\mathbf{t}, \beta). \tag{2.9}$$

Remark 2.1. There is another type of "bandwidth" more commonly used which measures the spectral moment about a carrier frequency ν_0. See Remark 1.2.

2.3 The Error Function $R_{\nu,c}(\mathbf{t})$

The error function $Q^2_{\nu,c}(\mathbf{t}, \beta)$ will be shown to be a nonhomogeneous quadratic polynomial in the weights β. Hence for any given ν, c, and $\mathbf{t}$ there exists a unique *optimal* value of β which *minimizes* $Q^2_{\nu,c}$. We define

$$R_{\nu,c}(\mathbf{t}) = \inf_{\beta}\ [Q_{\nu,c}(\mathbf{t}, \beta)], \tag{2.10}$$

so that (2.9) with optimal β becomes

$$|\hat{f}(\nu) - \hat{f}_{\text{est}}(\nu)|^2 \leqslant \|f\|_0^2 [1 + c^2 B^2(f)]\, R^2_{\nu,c}(\mathbf{t}), \tag{2.11}$$

this inequality being the strongest possible for given values of ν, c, and $\mathbf{t}$.

Remark 2.2. Since $Q^2_{\nu,c}(\mathbf{t}, \beta)$ is quadratic in β, the calculation of the optimal β requires the inversion of the N-by-N coefficient matrix A of the homogeneous part. The matrix A is a function of $\mathbf{t}$, and we have not been able to obtain closed-form expressions for its inverse except in the case when $\mathbf{t}$ is an equispaced set of sample points. Indeed the major technical accomplishment of our work has been to devise a means of calculating $R_{\nu,c}(t)$ without having to compute A^{-1} or the optimal weights β. The methods used involve the calculus of variations and the theory of Sobolev spaces which has been much used in partial differential equations and differential geometry. (See e.g. Refs. 5 and 6.) We present the results immediately below. The derivations will be given in Ref. 3.

The Result for Periodic Functions

For P-periodic functions there is no loss of generality in assuming that $t_1 = 0$. This is because the P-periodic functions can be identified with the functions defined on a circle with perimeter P, and by rotating the circle one can make t_1 correspond to any point. Alternatively, the P-periodic functions of class H^1 can be identified with functions f on the interval $[0, P]$ with $f(0) = f(P)$, so that the P-periodic functions can be considered as functions on the interval $[0, P]$ with the endpoints 0 and P identified. Since we also assume that the N sample points are distinct, for P-periodic functions there is no loss in generality in assuming that

$$0 = t_1 < t_2 < \dots < t_N < P. \tag{2.12}$$

We also set

$$\begin{aligned} \Delta t_n &= t_{n+1} - t_n, \qquad \text{if } 1 \leqslant n \leqslant N-1, \\ &= P - t_n, \qquad \text{if } n = N. \end{aligned} \tag{2.13}$$

With this understanding our result is

$$R^2_{\nu,c}(\mathbf{t}) = \frac{P}{1 + c^2\nu^2} - \frac{2c}{(1 + c^2\nu^2)^2} \sum_{n=1}^{N} \left[\frac{\cosh\left(\frac{\Delta t_n}{c}\right) - \cos(\nu \Delta t_n)}{\sinh\left(\frac{\Delta t_n}{c}\right)} \right]. \tag{2.14}$$

For $\nu = 0$, this reduces to

$$R^2_{0,c}(\mathbf{t}) = P - 2c \sum_{n=1}^{N} \left[\tanh\left(\frac{\Delta t_n}{2c}\right) \right]. \tag{2.15}$$

The Result for Nonperiodic Functions

For the nonperiodic case we can assume that

$$0 \leqslant t_1 < t_2 < \dots < t_N \leqslant T. \tag{2.16}$$

The sample-point set $\mathbf{t}$ may contain both, one, or none of the endpoints 0 and T. For example, $t_1 = 0$ and $t_N < T$ corresponds to the case when $\mathbf{t}$ contains 0 but not T. Our result in the nonperiodic case is

$$\begin{aligned} R^2_{\nu,c}(t) = {} & \frac{T}{1 + c^2\nu^2} - \frac{2c}{(1 + c^2\nu^2)^2} \sum_{n=1}^{N-1} \left\{ \frac{\cosh\left(\frac{\Delta t_n}{c}\right) - \cos(\nu \Delta t_n)}{\sinh\left(\frac{\Delta t_n}{c}\right)} \right\} \\ & + \frac{c}{(1 + c^2\nu^2)^2} \left[(\nu^2 c^2 - 1) \tanh\left(\frac{t_1}{c}\right) - 2\nu c \frac{\sin \nu t_1}{\cosh\left(\frac{t_1}{c}\right)} \right] \end{aligned} \tag{2.17}$$

(Continued)

$$+ \frac{c}{(1 + c^2\nu^2)^2} \left[(\nu^2 c^2 - 1) \tanh\left(\frac{T - t_N}{c}\right) - 2\nu c \frac{\sin \nu(T - t_N)}{\cosh\left(\frac{T - t_N}{c}\right)} \right], \tag{2.17}$$

where $\Delta t_n = t_{n+1} - t_n$, $(n = 1, ..., N - 1)$ as before. When $\nu = 0$, the result reduces to

$$R^2_{0,c}(t) = T - 2c \sum_{n=1}^{N-1} \left[\tanh\left(\frac{\Delta t_n}{2c}\right) \right] - c \tanh\left(\frac{t_1}{c}\right) - c \tanh\left(\frac{T - t_N}{c}\right). \tag{2.18}$$

2.4 Optimal Sample-Point Sets

The starting point for this discussion is the inequality (2.11), which we wish to make as strong as possible by making the right-hand side as small as possible for given values of signal energy ($= ||f||_0^2$) and bandwidth B.

For each fixed value of ν and c there exists a $\mathbf{t}$ set $\mathbf{t}_\nu(c)$ which minimizes the right-hand side of (2.11) for any given values of $||f||_0^2$ and B. Obviously $\mathbf{t}_\nu(c)$ does not depend on $||f||_0^2$ and B, so that, fixing the value of ν, we have reduced the right-hand side of (2.11) to an expression in which the only variable is c. The final step in this process is to minimize this resulting expression with respect to c, and we shall denote this minimizing value of c by c_ν^*. In general c_ν^* may depend on B: $c_\nu^* = c_\nu^*(B)$. We set $\mathbf{t}_\nu^*(B) = \mathbf{t}_\nu(c_\nu^*(B))$.

Summarizing: For each frequency ν there exists "optimal" values of $c = c_\nu^*(B)$ and $\mathbf{t} = \mathbf{t}_\nu^*(B)$ which minimize the right-hand side of (2.11) for given values of $||f||_0^2$ and B.

These optimal values of c and $\mathbf{t}$ are not necessarily unique. (In the periodic case there exist a continuum of optimal sample-point sets $\mathbf{t}$ obtained by translations of a given point set modulo P, i.e., by rotating the circle on which the P-periodic functions are defined.)

We shall say that any possible value of $\mathbf{t}_\nu^*(B)$ is a strongly optimal **(S - optimal)** *sample-point set for the frequency ν and bandwidth B.*

2.5 The Error Function $L_\nu(\mathbf{t})$

In the discussion above we ought to allow the possibility that the optimal value of c is given at $c = \infty$. Referring to (2.11), this motivates the definition

$$L_\nu(t) = \lim_{c \to \infty} [cR_{\nu,c}(\mathbf{t})]. \tag{2.19}$$

Hence, as $c \to \infty$, the right-hand side of (2.11) converges to $||f||_0^2 B^2(f) L_\nu^2(\mathbf{t})$. What about the left-hand side? We have $\hat{f}_{\text{est}}(\nu) = \Sigma\overline{\beta}_j f(t_j)$, where the weights β are optimal for the given values of c (and the other parameters). It can be shown that β converges to a certain value, say β^*, as $c \to \infty$. Setting $\hat{f}_{\text{est}}(\nu) = \Sigma\overline{\beta}_j^* f(t_j)$ in the limit, the inequality (2.11) at $c = \infty$ becomes

$$|\hat{f}(\nu) - \hat{f}_{est}(\nu)|^2 \leq \|f\|_0^2 B^2(f) L_\nu^2(\mathbf{t}). \tag{2.20}$$

We shall say that a value of **t** *is optimal in the limit (is L optimal) for a given frequency* ν *if it minimizes* $L_\nu(\mathbf{t})$.

An L-optimal set **t** is also S optimal only if the optimal value of c is given by $c = \infty$. If an L-optimal set is not also S optimal, then the right-hand side of (2.20) is greater than the right-hand side of (2.11) with c given the optimal value. Note also that the L-optimal **t** sets depend only on ν and not on B, whereas the S-optimal sets have an (at least apparent) dependence on B which, as discussed in Section 2.4, arose from the (apparent) dependence of the optimal value of c on B.

Explicit expressions for L_ν are easily obtained from (2.14) through (2.18). Using the same expressions for Δt_n as in Section 2.3, we get the following:

In the periodic case

$$L_\nu^2(t) = \frac{P}{\nu^2} - \frac{2}{|\nu|^3} \sum_{n=1}^{N} \left(\frac{1 - \cos \nu \Delta t_n}{|\nu| \Delta t_n} \right). \tag{2.21}$$

At $\nu = 0$ this reduces to (cf. (2.15))

$$L_0^2(\mathbf{t}) = \frac{1}{12} \sum_{n=1}^{N} (\Delta t_n)^3. \tag{2.22}$$

In the nonperiodic case

$$L_\nu^2(\mathbf{t}) = \frac{T}{\nu^2} - \frac{2}{|\nu|^3} \sum_{n=1}^{N-1} \left(\frac{1 - \cos \nu \Delta t_n}{|\nu| \Delta t_n} \right) + \left(\frac{t_1}{\nu^2} - \frac{2}{\nu^3} \sin \nu t_1 \right)$$
$$+ \left(\frac{T - t_N}{\nu^2} - \frac{2}{\nu^3} \sin \nu (T - t_N) \right). \tag{2.23}$$

For $\nu = 0$ this reduces to

$$L_0^2(\mathbf{t}) = \frac{1}{12} \sum_{n=1}^{N-1} (\Delta t_n)^3 + \frac{1}{3} t_1^3 + \frac{1}{3} (T - t_N)^3. \tag{2.24}$$

3. THE PROBLEM OF MEASUREMENT

3.1 Some Consequences of Eqs. (2.14) Through (2.24)

The problem of determining the S-optimal sample-point sets **t** has been reduced to the problem of minimizing the right-hand side of (2.11) with respect to the variables c, $t_1, \ldots, t_N$, where $R_{\nu,c}$ is given by (2.14) or (2.17). Similarly the problem of determining the L-optimal sets **t** has been reduced to the problem of minimizing $L_\nu(\mathbf{t})$ with respect to

t, where L_ν is given by (2.21) or (2.23). Although these problems are perfectly straightforward, we have not as yet been able to obtain their solution. Thus we are not yet able to decide whether any practical advantage would be obtained by the use of optimal sample-point sets over equispaced sets in those cases when the latter are nonoptimal. Another unresolved problem is to determine the optimal weights β. However we do have the following partial results, which we state as propositions. The detailed proofs will appear in Ref. 3.

Proposition 1. Fix N, c, and P (or T). Then there exists a sequence of frequencies $\{\nu_n\}$, with $\nu_n \to \infty$, such that equispaced sample points are not optimal for the given values of N, c, P (or T), and $\nu = \nu_n$.

Proposition 2. Equispaced sample point sets are L-optimal provided that $N \geqslant N_0$, where N_0 depends on ν and P (or T). (The exact dependence of N_0 on ν and P has not yet been determined.)

Proposition 3. For the special case $\nu = 0$, equispaced sample points are *always* optimal, for any value of c. In particular: *For the periodic case* an optimal set **t** is given by $t_n = (n - 1)\, P/N$, $0 \leqslant n < N$. Any other **t** set obtained from this one by translations modulo P is also optimal. *In the nonperiodic case* the *only* optimal **t** set is given by $t_1 = P/2N$, $t_{n+1} - t_n = P/N$, $1 \leqslant n \leqslant N - 1$ (so that $T - t_N = t_1 - 0 = P/2N$).

Proposition 4. Fix the number N of sample points, and let L_ν (equispaced), L_ν (optimal) represent the values of $L_\nu(\mathbf{t})$ evaluated at $\mathbf{t}$ = equispaced set and optimal set respectively. Then

$$L_\nu^2\,(\text{optimal})/L_\nu^2(\text{equispaced}) \geqslant 0.38. \tag{3.1}$$

Proposition 5. Fix the values of ν and c. Then for equispaced sample-point sets the values of the optimal weights β_k converge to their "naive" values

$$\beta_k = \frac{P}{N}\, e^{-i\nu t_k} \tag{3.2}$$

as $N \to \infty$.

3.2 Discussion

The proof of Proposition 1 (in the periodic case) is based on an inspection of (2.14). For equispaced sets one can choose a sequence of frequencies ν such that $\cos(\nu \Delta t_n) = +1$ for all Δt_n, and $R_{\nu,c}^2$ will vary as $(k_1/\nu^2) - (k_2/\nu^4)$, where k_1 and k_2 are certain constants. On the other hand one can choose $N - 1$ of the Δt_n such that $\cos(\nu \Delta t_n) = -1$ for the same frequencies ν, and for such a **t** set $R_{\nu,c}^2$ will vary as $(k_1/\nu^2) - (k_3/\nu^3)$.

It is even more instructive to consider the error function L_ν evaluated at equispaced sets **t**. In the periodic case the only allowed frequencies are of the form $\nu = 2\pi k/P$, k an integer, and setting $\Delta t_n = P/N$ in (2.21) we see that when N divides k, L_ν^2 attains its largest possible value $(= P/\nu^2)$ at equispaced sample-point sets. In other words, if one accepts $L_\nu(t)$ as a measure of efficiency of estimation, as might be suggested by (2.20), then equispaced sample-point sets are sometimes the worst possible. This phenomenon, which occurs when N divides the frequency number k, corresponds to the "aliasing errors" which are much discussed in the literature.

Recall that L_ν gives the best measure of efficiency only if the inequality (2.20) is stronger than (2.11) for all finite values of the parameter c. If this were ever the case, then Proposition 4 would imply that *no* great practical advantage would be gained from the use of optimal over equispaced sample-point sets.

The relations (2.11) and (2.20) give bounds on the absolute errors $|\hat{f}(\nu) - \hat{f}_{est}(\nu)|$, but they cannot be used to obtain bounds on the relative errors $|\hat{f}(\nu) - \hat{f}_{est}(\nu)|/|\hat{f}(\nu)|$, since, e.g., $\hat{f}(\nu)$ may be 0 when $\hat{f}_{est}(\nu)$ is not 0. Another and perhaps better way of looking at these results is to interpret them as imposing bounds on the *energy* lost (or gained) in going from a signal $f(t)$ to the "reconstituted" signal $f_{est}(t)$ defined by

$$f_{est}(t) = \sum_{\nu} \hat{f}_{est}(\nu) e^{i\nu t}$$

(periodic case) or

$$f_{est}(t) = \int_0^T \hat{f}_{est}(\nu) e^{i\nu t}\, dt$$

(nonperiodic case). To see how this works, let us evaluate (2.20) at a set of N equispaced sample points and take the periodic case. Then using (2.21) we get $L_\nu^2 \sim (1/12)P^3/N^2$ as $N \to \infty$. So 2.20 becomes

$$|\hat{f}(\nu) - \hat{f}_{est}(\nu)|^2 \leqslant ||f||_0^2 B^2(f)\ \frac{1}{12}\ \frac{P^3}{N^2}, \qquad N \text{ large.} \tag{3.3}$$

Recall that $||f||^2$ has the dimensions of *energy* (joules), so that $|\hat{f}(\nu) - \hat{f}_{est}(\nu)|^2$ has the dimensions of joules per hertz and can be regarded as the amount of energy at the frequency ν which is lost or gained in forming $f_{est}(t)$. The total amount of energy lost or gained over all frequencies is obtained by summation (or integration). Since the right-hand side of (3.3) does not contain ν, an upper bound to the total energy difference can be approximated by multiplying the right-hand side by B = bandwidth. Or to put the matter another way, the fractional energy loss or gain is given by

$$\frac{B|\hat{f}(\nu) - \hat{f}_{est}(\nu)|^2}{||f||_0^2} \leqslant \frac{1}{12}\ \frac{(BP)^3}{N^2}, \qquad N \text{ large.} \tag{3.4}$$

Hence the quantity $(BP)^3/N^2$ measures the efficiency of equispaced sample points over all frequencies. (Note that in going from (2.21) to (3.3) we assumed that $\nu \Delta t_n = \nu P/N$ was small for each frequency ν under consideration. Hence (3.4) is only valid when BP/N is small.)

4. THE PROBLEM OF DETECTION

4.1 First Detection Scheme

The problem here, in its deterministic setting, is to construct filters $f \to \Sigma \overline{\beta}_j f(t_j)$ with the property that $\hat{f}(\nu) \neq 0$ whenever the filter output exceeds a certain threshold value. Our first detection scheme is based on the filters $\hat{f}_{\text{est}}(\nu)$ and the use of (2.9) to obtain

$$\frac{|\hat{f}_{\text{est}}(\nu)|^2}{\|f\|_0^2 [1 + c^2 B^2(f)]} > Q^2_{\nu,c}(\mathbf{t}, \beta) \text{ implies } \hat{f}(\nu) \neq 0. \tag{4.1}$$

All this relation says is that if $|\hat{f}_{\text{est}}(\nu)|$ exceeds the largest possible *error*, then $\hat{f}(\nu)$ must be nonzero.

4.2 Second Detection Scheme

We shall new develop a detection scheme in a more systematic fashion and show that a set of weights β which are optimal for measurement are also optimal for detection. We shall also show that the right-hand side of (4.1) is somewhat larger than necessary and that spectral components $\hat{f}(\nu)$ of arbitrarily small energy can be detected by making the number N of sample points sufficiently large.

Our starting point is the construction of a threshold values $D_{\nu,c}(\mathbf{t}, \beta)$ defined by

$$D^2_{\nu,c}(\mathbf{t}, \beta) = \sup_{\hat{f}(\nu) = 0} \left[\frac{|\Sigma \overline{\beta}_j f(t_j)|^2}{\|f\|_0^2 (1 + c^2 B^2(f))} \right]. \tag{4.2}$$

Since $D^2_{\nu,c}$ is the supremum of the quantity in the brackets taken over all signals whose νth component vanishes, we have

$$\frac{|\Sigma \overline{\beta}_j f(t_j)|^2}{\|f\|_0^2 [1 + c^2 B^2(f)]} > D^2_{\nu,c}(\mathbf{t}, \beta) \text{ implies } \hat{f}(\nu) \neq 0. \tag{4.3}$$

We now wish to optimize the detection scheme (4.3) with respect to $\mathbf{t}$ and β.

We first optimize with respect to β as follows: For each fixed values of ν, c, and $\mathbf{t}$ we choose the value of β which minimizes $D^2_{\nu,c}(\mathbf{t}, \beta)$ subject to a constraint which normalizes the weights β by matching the filter to the frequency ν. More specifically we shall require that

$$\Sigma \overline{\beta}_j e^{i\nu t_j} = u,$$

where u is a nonzero parameter. It turns out that the precise value of u selected has no effect on the detection scheme, because u will occur as a factor on both sides of (4.3) with β optimal. Hence we define

$$D^2_{\nu,c}(\mathbf{t}) = \inf_{\beta} [D^2_{\nu,c}(\mathbf{t}, \beta) | \Sigma\beta_j e^{i\nu t_j} = u], \tag{4.5}$$

and we let β^* denote the optimal value of β, so that

$$D_{\nu,c}(\mathbf{t}) = D_{\nu,c}(\mathbf{t}, \beta^*). \tag{4.6}$$

Let γ denote the weight vector which is optimal for measurement for given ν, c, and $\mathbf{t}$, so that from Section 2.3 we have

$$R_{\nu,c}(\mathbf{t}) = Q_{\nu,c}(\mathbf{t}, \gamma). \tag{4.7}$$

Using function analytic techniques from the theory of Sobolev spaces, one can write down explicit expressions for $D^2_{\nu,c}(\mathbf{t}, \beta)$ and $D^2_{\nu,c}(\mathbf{t})$. It turns out that β^* is a scalar multiple of γ, and our second detection scheme becomes

$$\frac{|\Sigma\overline{\gamma}_j f(t_j)|^2}{\|f\|_0^2[1 + c^2B^2(f)]} > R^2_{\nu,c}(\mathbf{t}) \left[1 - \frac{(1 + c^2\nu^2)}{P} R^2_{\nu,c}(\mathbf{t})\right] \text{ implies } \hat{f}(\nu) \neq 0. \tag{4.8}$$

Note that the right-hand side of (4.8) is smaller than $R^2_{\nu,c}(\mathbf{t})$, which in turn is the smallest possible value of $Q^2_{\nu,c}(\mathbf{t}, \beta)$, which is the right-hand side of the detection scheme (4.1). Recall also that the optimal weights γ depend on ν, c, and $\mathbf{t}$. In particular, multiplying (4.8) by c^2 and taking the limit as $c \to \infty$, we get

$$\frac{|\Sigma\overline{\gamma}_j f(t_j)|^2}{\|f\|_0^2 B^2(f)} > L^2_\nu(\mathbf{t}) \left[1 - \frac{\nu^2}{P} L^2_\nu(\mathbf{t})\right] \text{ implies } \hat{f}(\nu) \neq 0, \tag{4.9}$$

where γ_j are now the limiting values of the weights γ as $c \to \infty$.

Having optimized the detection scheme with respect to the weights, we should now consider the problem of optimizing with respect to the variables $\mathbf{t}$ (and c) for given values of signal energy $\|f\|_0^2$ and bandwidth B. Rewriting (4.8) in the form

$$|\Sigma\overline{\gamma}_j f(t_j)|^2 > \|f\|_0^2[1 + c^2B^2(f)] R^2_{\nu,c}(\mathbf{t}) \left[1 - \frac{(1 + c^2\nu^2)}{P} R^2_{\nu,c}(\mathbf{t})\right] \text{ implies } \hat{f}(\nu) \neq 0, \tag{4.10}$$

one is tempted to choose those values of $\mathbf{t}$ and $\mathbf{c}$ which are optimal for *measurement*, because these are the values of $\mathbf{t}$ and c which make $[1 + c^2B^2(f)] R^2_{\nu,c}(\mathbf{t})$ minimum. However, as $\mathbf{t}$ and c vary, so do the weights γ_j, and we have the conceptual difficulty of defining how the sensitivity of the detection scheme changes as both sides of (4.10) are varied in this way.

On the other hand one can justify choosing the values of $\mathbf{t}$ and c to be those which are optimal for measurement on the following grounds: Any scheme which is optimal for measurement will estimate $\hat{f}(\nu)$ at least as well as schemes based on equispaced sampling, and for equispaced $\mathbf{t}$ we know (Proposition 5, Section 3.1 that the left-hand side of (4.10) is equal to square of the Riemann sum $(P/N)\Sigma e^{i\nu t_j} f(t_j)$ plus a small quantity that goes to 0 as $N \to \infty$. It can also be shown that this Riemann sum converges uniformly to $\hat{f}(\nu)$ for all functions f with given signal energy and bandwidth. Moreover for equispaced $\mathbf{t}$ we know that the right-hand side of (4.10) goes to 0 as $1/N^2$ as $N \to \infty$ (cf. Section 3.2). Hence we have the following proposition.

Proposition 4.1. For equispaced or optimal $\mathbf{t}$, the left-hand side of (4.10) converges uniformly to $|\hat{f}(\nu)|^2$ as $N \to \infty$, and the right-hand side goes to 0 at least as fast as $1/N^2$. Hence in the deterministic setting one can detect arbitrarily small values of $|\hat{f}(\nu)|$ by making N sufficiently large.

Remark 4.1. There does not appear to be any *practical* difference between the two detection schemes, because the right-hand sides of (4.1) (with $Q_{\nu,c}(\mathbf{t}, \beta) = Q_{\nu,c}(t, \gamma) = R_{\nu,c}(\mathbf{t})$) and (4.8) differ only by a term which varies as $R^4_{\nu,c}(\mathbf{t})$, and $R_{\nu,c}(\mathbf{t})$ is small for good values of $\mathbf{t}$. However the theoretical development of the second detection scheme provides more information about the optimal weights γ, and it has the additional advantage that it can be easily modified to serve the purpose of sidelobe cancellation, as will be shown in the next paragraph.

4.3 Resolution

We shall now confine our discussion to the periodic case and consider the problem of resolution. As was mentioned in Section 1, when noise is present one might wish to reduce or eliminate the response of a detection filter to frequencies ν' which are close to the frequency ν to which the filter is matched. Then, referring to (4.5), to achieve this result all one need do is to require that the "optimal" weights β_j satisfy the additional constraints $\Sigma\overline{\beta}e^{i\nu' t_j} = 0$ as well as the constraint $\Sigma\overline{\beta}_j e^{i\nu t_j} = u$. The calculations involved in obtaining such a set of weights appears to be perfectly straightforward, though rather laborious, and will be undertaken later.

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