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Amplitude Functions and the Analog of the Kelvin Wedge for Three-Dimensional Internal Waves Generated by a Moving Disturbance

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Internal waves are generated by a dipole of arbitrary orientation moving on an inclined rectilinear path in a stratified fluid whose Brunt-Väisälä frequency is constant. Expressions for the asymptotic behavior of the phase and the amplitude are obtained by applying the method of Lighthill. The global geometry of the surfaces of constant phase is considered, and there is shown to be an internal wave cone, analogous to the Kelvin wedge for surface waves. Contours of wave amplitude are calculated for different orientations of the dipole, and various "sidelobe" patterns are found.

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CONTENTS

AMPLITUDE FUNCTIONS AND THE ANALOG OF THE KELVIN WEDGE FOR THREE-DIMENSIONAL INTERNAL WAVES GENERATED BY A MOVING DISTURBANCE

INTRODUCTION

Internal waves in stratified fluids are of interest to geophysicists, planetary scientists, and marine engineers. Moving atmospheric disturbances and the flow of stratified fluids around obstacles can generate internal waves. Attempts have been made to attribute observed atmospheric motions on Mars and Jupiter to such disturbances, at least in part. Internal waves excited by marine vessels propagate energy away from the disturbance, thereby producing wave resistance.

A submerged body moving in a stably stratified fluid can excite internal waves in at least two ways [1]. First, the flow around the body disturbs the isopycnals, and internal waves are generated as equilibrium is restored. This is referred to as the *hull effect*. Second, a turbulent wake of fluid trails the body, and the collapse of this wake excites internal waves. This is the so-called *wake effect*. The relative importance of the two effects is an issue which has not been completely resolved.

The hull-effect problem for a body on a horizontal course is closely related to the *lee-wave* problem of meteorology, for which there is a large literature. Its simplest model consists of a dipole source in a fluid whose velocity is uniform and horizontal and which is endowed with an exponential density profile. An extensive analysis of asymptotic solutions for this model is given by Crapper [2]. Miles [3] considered the equivalent problem in which the body moves in a stationary fluid in his fundamental paper on the subject.

Lighthill [4] has developed an elegant theory to explain properties of anisotropic wave motions. The theory is well suited to the internal-wave problem if the course is rectilinear and the Brunt-Väisälä frequency is constant. Indeed several authors have attacked various portions of the problem using Lighthill's method. Lighthill himself [5] used his method to study the wave pattern associated with a vertically moving disturbance. Rarity [6] used the method in two dimensions to calculate the curve of intersection of the surface of constant phase with a vertical plane containing a velocity vector of arbitrary orientation. More recently Redekopp [7] has treated the three-dimensional wave pattern associated with a disturbance moving horizontally.

Peat and Stevenson [8] have derived expressions for the wave patterns around a body moving in a compressible density-stratified fluid. Their analysis is based on the group-velocity concept. They also compare their theory to wave patterns obtained from experiments for a sphere moving on an inclined course in a stratified fluid.

The purpose of this report is to consider both the phase (wave patterns) and the amplitude of internal waves generated by a disturbance moving on an inclined course. The method of Lighthill is used for a dipole source of arbitrary orientation. Although the wave patterns have

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been calculated by Peat and Stevenson, they are reconsidered here from a different point of view as a natural consequence of the analysis leading to the amplitude functions developed here. Furthermore the surfaces of constant phase have an interesting geometry which has not been discussed elsewhere; namely, there is a conical envelope of the surfaces which is an analog of the Kelvin wedge for surface waves, and this cone could have important consequences as far as observations are concerned.

THE MATHEMATICAL MODEL

A general mathematical model for the dynamics of a stratified fluid with a source is outlined in this section. The model is a standard one (described in Ref. 2, for example) but its description here is appropriate for the sake of completeness and to introduce the notation to be used in what follows.

Let us consider the steady motion of a body in a stratified inviscid fluid. The presence of the body can be accounted for in the equations of motion of the fluid by including a source term in the mass-conservation equation. The model consists of the linearized conservation equations for disturbances of a Boussinesq fluid, written in a frame moving with the body, which are

$$\nabla \cdot \mathbf{v} = Q, \tag{1}$$

$$\rho_o \mathbf{U} \cdot \nabla \mathbf{v} + \rho_o w \mathbf{U}' + \nabla p + g\rho_o\rho \hat{\mathbf{k}} = 0, \tag{2}$$

$$\rho_o \mathbf{U} \cdot \nabla \rho + w\rho_o' = 0, \tag{3}$$

where ρ_0 is the ambient density, which is assumed to be constant except as its derivative ρ_0' enters equation (3) (in accordance with the Boussinesq approximation). The ambient velocity of the fluid (relative to the body), $\mathbf{U} = (U_1(z), U_2(z), U_3(z))$ is allowed to vary with the vertical coordinate z. The Q term in equation (1) represents the body source. Primes denote derivatives with respect to z. The velocity, pressure, and density disturbances due to the body are denoted by $\mathbf{v} = (u, v, w)$, p, and ρ respectively. The analysis is facilitated by eliminating u, v, p, and ρ to obtain a single equation in the vertical velocity disturbance w. After a straightforward reduction we obtain

$$[\mathfrak{D}^2\Delta + \mathfrak{D}U_3'\,\Delta - \mathfrak{D}(U_1''\,\partial_x + U_2''\,\partial_y) + N^2\,\Delta_1]\, w = \mathfrak{D}(\mathfrak{D}Q)', \tag{4}$$

in which $\mathfrak{D} = \mathbf{U} \cdot \nabla$ and $\mathfrak{D}' = \mathbf{U}' \cdot \nabla$, where Δ is the Laplacian operator and N is the Brunt-Väisälä frequency defined by

$$N^2 = -g\rho_0'/\rho_0.$$

Equation (4) is sufficiently general to model waves generated by a disturbance moving on an arbitrary rectilinear course in the presence of either z-dependent shear current or Brunt-Väisälä frequency. In the *main problem* to be treated, $\mathbf{U}$ and N are constant.

LIGHTHILL'S METHOD

Lighthill's theory describes the asymptotic behavior of wave motions far from a localized disturbance. The theory is applicable to wave motions which are anisotropic and/or dispersive.

It is restricted however to homogeneous media or, in mathematical terms, to equations with constant coefficients. A brief description of the method is included in this report because all of the results depend on it and because it is not yet considered a standard tool.

Let us consider a general wave whose motion may be described by the linear partial differential equation

$$P(\partial_t, \partial_x, \partial_y, \partial_z)\phi = F(x, y, z, t), \tag{5}$$

where P is a polynomial with constant coefficients. Waves generated by a disturbance moving with a constant velocity $\mathbf{U}$ will be steady in a coordinate frame moving with this velocity. Thus, we may use the constraint

$$\mathbf{r} - \mathbf{U}t = \text{const}$$

to eliminate the time derivative in (5). In addition the forcing terms for waves generated by moving disturbances will be steady in the moving frame. The form of the differential equation then becomes

$$P(\mathbf{U} \cdot \nabla, \partial_x, \partial_y, \partial_z)\phi = F(x, y, z). \tag{6}$$

Let us denote the three-dimensional Fourier transform of a function $f(\mathbf{r})$ by

$$\widetilde{f} = \int f(\mathbf{r})\, e^{-\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{r}. \tag{7}$$

If the Fourier transform of F exists, the formal solution of equation (6) is

$$\phi = \frac{1}{(2\pi)^3} \int \frac{\widetilde{F}(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{r}}}{P(\mathbf{U} \cdot \mathbf{k}, \mathbf{k})}\, d\mathbf{k}. \tag{8}$$

The problem is thus transferred to one of evaluating the integral (8). This evaluation is the core of Lighthill's theory.

The asymptotic value of (8) depends critically on the singularities of the integrand. If F is the result of a localized disturbance, $\widetilde{F}$ will have no singularities and the integrand is singular only on the surface

$$P(\mathbf{U} \cdot \mathbf{k}, \mathbf{k}) = 0. \tag{9}$$

This surface is called the *wave-number surface* or the *slowness surface* and is intimately connected with the solution of free waves. In the case of free waves, $F = 0$ and there is a plane-wave solution

$$\phi = e^{i\mathbf{k}\cdot\mathbf{r}}$$

if $\mathbf{k}$ satisfies the dispersion relation

$$P(\mathbf{U} \cdot \mathbf{k}, \mathbf{k}) = 0,$$

which is identical to (9).

Lighthill shows that the surfaces of constant phase consist of the poles of the tangent planes of the slowness surface. This occurs because the surfaces of constant phase satisfy

$$\mathbf{r} \cdot \mathbf{k} = \Phi, \tag{10}$$

and the principle of stationary phase asserts that the dominant contributions to the integral occur where $P = 0$ and Φ is stationary. Applying the method of Lagrange multipliers to this optimization problem with constraints yields the result that $\mathbf{r}$ is parallel to ∇P at the stationary point. From this it follows that

$$\mathbf{r} = \frac{\Phi}{\mathbf{k} \cdot \nabla P} \nabla P, \tag{11}$$

which places $\mathbf{r}$ parallel to the normal to the slowness surface and with magnitude inversely proportional to the distance from the origin to the tangent plane at $\mathbf{k}$. In geometrical language this states that the surfaces of constant phase coincide with the polar reciprocals of the slowness surface. There are two possible choices of direction for the normal. The direction must be chosen so that the radiation condition is not violated.

The amplitude is shown to be inversely proportional to the product of $|\mathbf{r}|$ and the square root of the Gaussian curvature K evaluated at $\mathbf{k}$ if K does not vanish. If $K = 0$, the amplitude can be inversely proportional to $|\mathbf{r}|^{1/2}$ or $|\mathbf{r}|^{5/6}$, depending on the geometry of the surface.

THE PHASE

We now apply Lighthill's method to the "main problem" for internal waves to find the phase of the wave motion. Since the wave patterns consist of surfaces of constant phase, we are concerned here with the wave patterns.

Let $\mathbf{k} = (\xi, \eta, \zeta)$; then the polynomial which defines the slowness surface is

$$(\mathbf{U} \cdot \mathbf{k})^2\, \mathbf{k}^2 - N^2(\xi^2 + \eta^2) = 0. \tag{12}$$

No generality is lost by assuming $U_2 = 0$, because a rotation of the coordinate system in the $\xi\eta$ plane may be applied to remove the second velocity component. Introducing $u = U_1/N$ and $w = U_3/N$ and dividing equation (12) by N^2, we obtain

$$P(\xi, \eta, \zeta) = (u\xi + w\zeta)^2\, (\xi^2 + \eta^2 + \zeta^2) - (\xi^2 + \eta^2) = 0 \tag{13}$$

as the equation for the slowness surface.

An understanding of the surface (13) is fundamental to Lighthill's method, because the solution is derived entirely from its geometry and the source function. The surface is seen to be symmetric with respect to the origin and may be visualized by considering its intersection with various planes.

The curves of intersection with the planes $u\xi + w\zeta = \alpha$ lie on the cones

$$\zeta^2 = \left(\frac{1-\alpha^2}{\alpha^2}\right)(\xi^2 + \eta^2), \tag{14}$$

whose generators pass through the origin and are inclined to the $\xi\eta$ plane at an angle μ satisfying

$$tan^2\mu = \frac{1-\alpha^2}{\alpha^2}. \tag{15}$$

The curves therefore are conics which are hyperbolic, parabolic, or elliptic as $w/u - \tan\mu$ is positive, zero, or negative, respectively. The special cases $w = 0$ and $u = 0$ yield confocal hyperbolas [2, 3] and concentric circles [5] respectively.

Illustrations of the sections of the surface by the coordinate planes $\eta = 0$ and $\xi = 0$ are shown in Figs. 1 and 2 respectively. A family of sections by the planes ζ = const are shown in Fig. 3. The singular point on the self-intersecting curve is a saddle point of the surface. The singular curve divides the family of curves into two subclasses. The first consists of curves with two open branches, each with vertical asymptotes. The second consists of curves with two open branches having a vertical asymptote and a closed ovoid enclosing the origin.

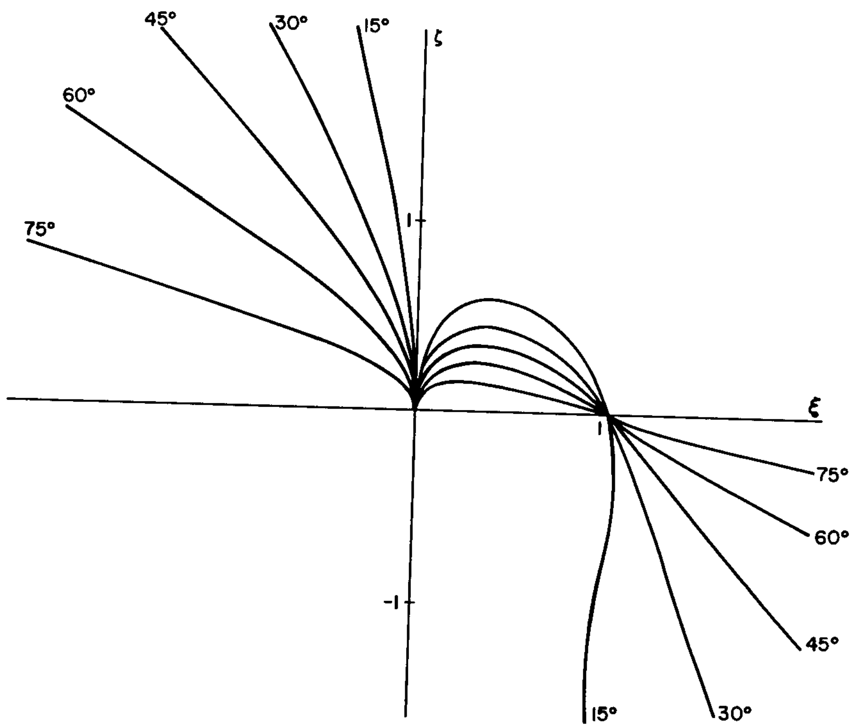


Fig. 1 — Sections of the slowness surface by the plane $\eta = o$ for several values of $\beta' = \arctan (w/u)$

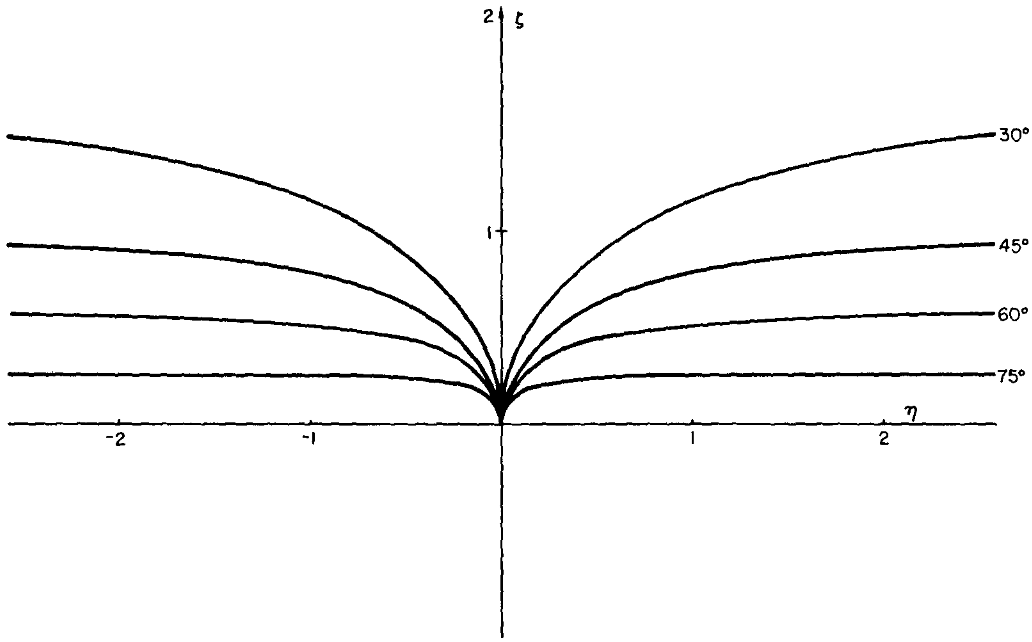


Fig. 2 – Sections of the slowness surface by the plane $\xi = 0$ for several values of $\beta = \arctan (\omega/u)$

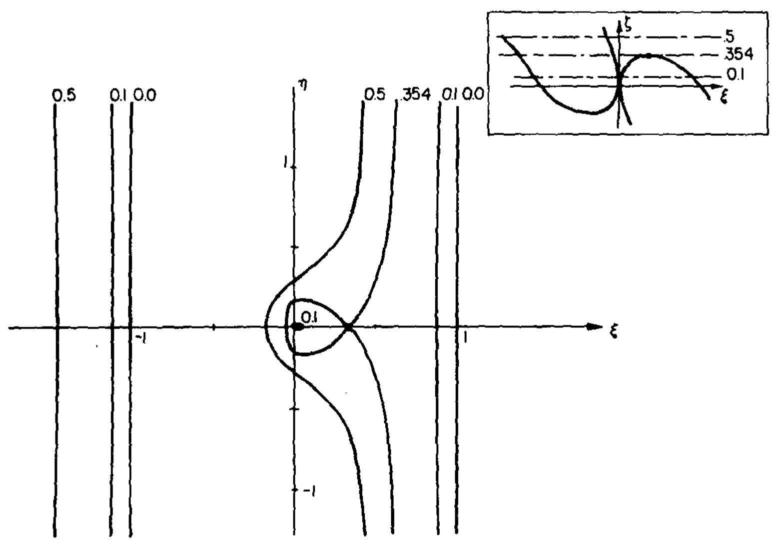


Fig. 3 – Sections of the slowness surface by planes $\zeta = \text{const}$ for $u = \omega = 1$

The proper choice of a normal direction which satisfies the principle of causality is illustrated in Fig. 4. This figure is a representation of the three-dimensional form of the surface. The dashed lines correspond to the level curves of Fig. 3. The broken line is a line of vanishing Gaussian curvature. The surface also contains the straight lines $\zeta = 0$ and $\xi^2 = {}^1/u^2$. The Gaussian curvature also vanishes along these lines.

The equation for a surface of constant phase is obtained by substituting equation (13) into equation (11). The result is

$$-(\operatorname{sgn}\alpha)\mathbf{r} = \frac{1}{\xi^2+\eta^2+\zeta^2}(\xi, \eta, \zeta) + \frac{1}{\alpha}(u, 0, w) - \frac{1}{\xi^2+\eta^2}(\xi, \eta, 0). \tag{16}$$

Since the group velocity is $\mathbf{C} = (NK/k)(\mathbf{k}/k^2 - \mathbf{K}/K^2)$, equation (16) may be written as $-(\operatorname{sgn}\alpha)x = (1/\alpha)(\mathbf{C} + \mathbf{u})$. This is the expression used by Peat and Stevenson [8]. The surface is represented parametrically. Although three parameters appear, only two are independent by virtue of the relation

$$\eta^2 = \frac{\alpha^2\zeta^2}{1-\alpha^2} - \xi^2, \tag{17}$$

which is derived from equation (13).

The intersection of the surface of constant phase and the plane $z = \text{const}$ is given by

$$x' = \frac{-\operatorname{sgn}\alpha}{u\,\alpha^3}\left\{(z'\,\alpha - w)[w(2-\alpha^2) - \alpha\xi'] + \alpha^2 u^2\right\}, \tag{18a}$$

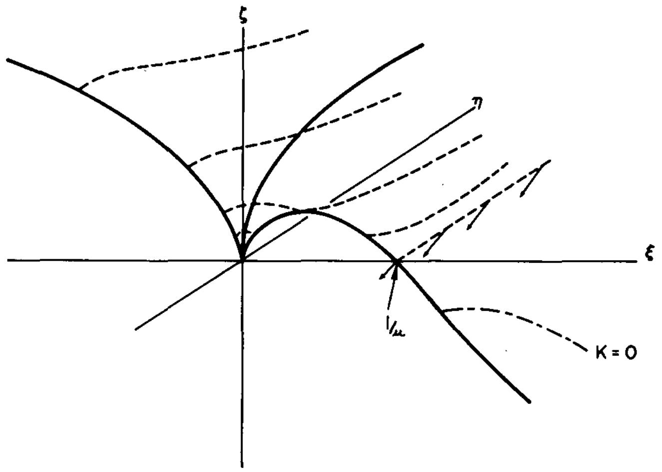


Fig. 4 — An illustration of the slowness surface. The dashed curves show contours of constant ζ. The Gaussian curvature vanishes along the broken curve. The arrows indicate the choice of surface normal which satisfies the radiation condition.

$$y' = \frac{-(\operatorname{sgn}\alpha)(z'\alpha - \omega)}{u\,\alpha^3}\left\{u^2\alpha^2(1-\alpha^2) - [z'\alpha - \omega(2-\alpha^2)]^2\right\}^{1/2}, \tag{18b}$$

where $(x', y', z') = (x, y, z)/\Phi$. It is easily shown that when $\omega = 0$ we obtain the hyperbola

$$\frac{x'^2}{u^2 - z'^2} - \frac{y'^2}{z'^2} = 1 \tag{19}$$

in agreement with the results obtained by Miles and Crapper [2, 3] for this special case.

Examples of the curves representing equation (18) are shown in Figs. 5 and 6. Also shown, Fig. 7, is a family of intersections of the surface of constant phase with the xz plane. This section is the one studied by Rarity [6]; it is shown here for completeness and comparison.

These figures reveal that the surface of constant phase has a cuspsoidal edge, a locus of cusps of the curves (18). This edge is the image of the monoclastic curve $K = 0$ (in Lighthill's terminology) under the mapping (11). It is a closed curve and includes the cusp on the section by the

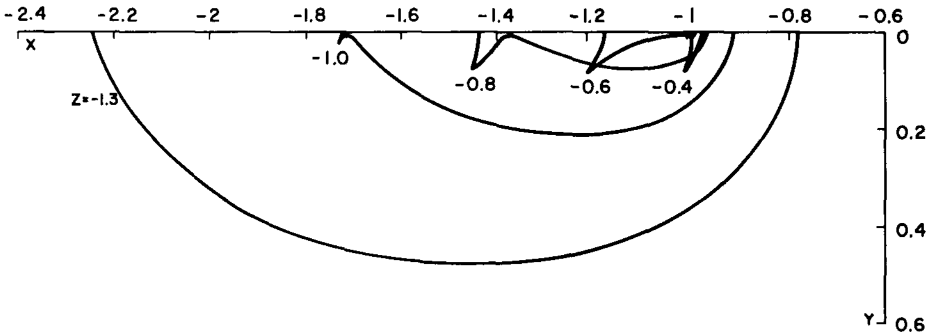


Fig. 5 — Sections of the surface of constant phase by horizontal planes ($u = 1$, $\omega = 1/\sqrt{3}$, $\beta = 30°$)

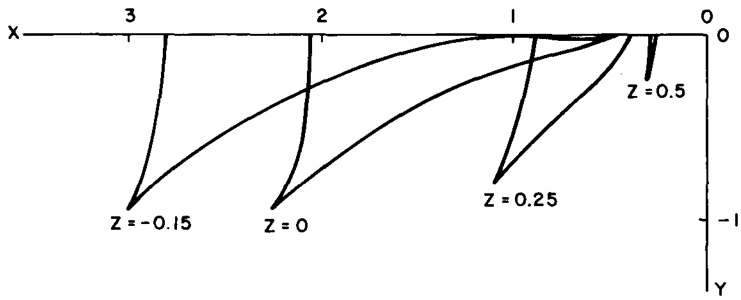


Fig. 6 — Sections of the surface of constant phase by horizontal planes ($u = 1$, $\omega = 0.1$, $\beta = 5.7°$)

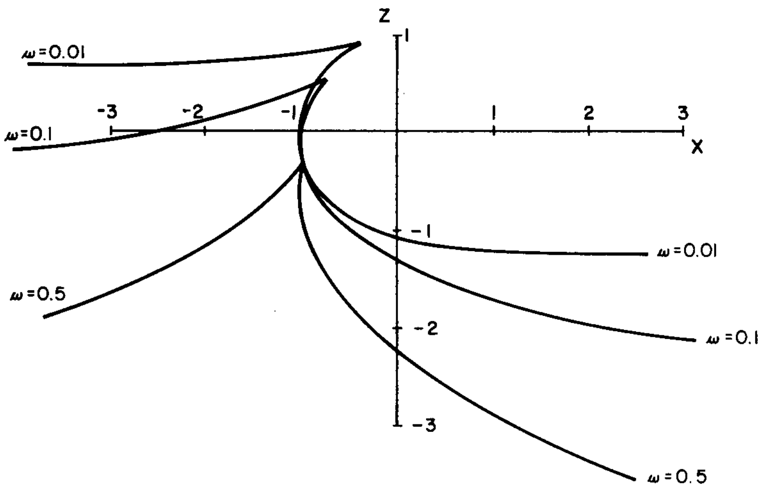


Fig. 7 — Section of the surface of constant phase by a vertical plane containing the velocity vector. The disturbance lies at the origin ($u = 1$)

xz plane (Fig. 8). The envelope of the cuspsoidal edges is the analog of the "Kelvin wedge" of surface wave theory. According to the general theory, and as we shall see in the next section, the amplitude of the waves decays slower with distance along the edge than it does at nonsingular points. To complete the analogy with the surface-wave case, sections of the surface of constant phase which intersect the cuspsoidal edge can be regarded as composed of an "oblique wave" component and a "transverse wave" component. The pattern for downward motion, $w < 0$, may be obtained by replacing z by $-z$ in the preceding.

THE AMPLITUDE

The asymptotic expression for the function ϕ(equation 6) according to Lighthill's theory is

$$\phi \sim \frac{4\pi^2}{r} \Sigma \frac{C\, \widetilde{F}(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{r}}}{|\nabla P|\, \sqrt{|K|}}, \tag{20}$$

where K is the Gaussian curvature of the surface $P = 0$ and the summation is taken over those values of $\mathbf{k}$ for which $\mathbf{r} \times \nabla P = 0$ and where C is $\pm i$ if $K < 0$ and ± 1 if $K > 0$. Equation (20) is valid whenever $K \neq 0$.

The application of this expression to the main problem of internal waves is expedited by expressing the surface $P(\mathbf{k}) = 0$ in the parametric form

$$\mathbf{k} = \lambda\, \mathbf{k}, \tag{21}$$

where

$$\mathbf{k} = (\sin\theta \cos\phi,\ \sin\theta \sin\phi,\ \cos\theta)$$

and

$$\lambda = 1/(u \cos\phi + w \cot\theta).$$

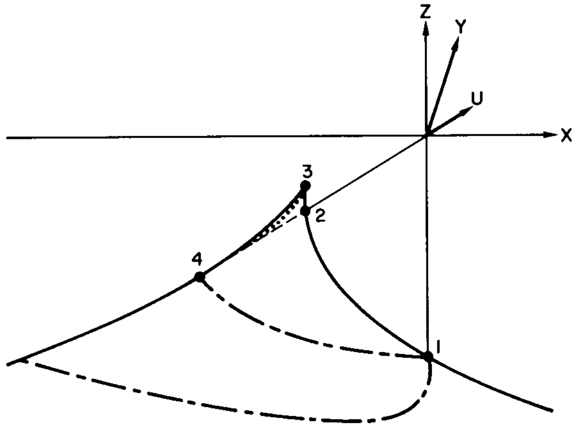


Fig. 8 – The surface of constant phase. Point 1 is the image of the saddle point on the slowness surface, point 2 is the image of the straight line, point 3 is the image of the inflection point on the section by the plane $\eta = 0$, and point 4 is the image of the "point at infinity." The broken curve is the image of the separatrix. The dotted curve is the image of the monoclastic curve $K = 0$ and is the cuspsoidal edge.

The Gaussian curvature may then be expressed as

$$K = -\left[\frac{\lambda\cos\theta}{\lambda_2^2 + (\lambda^2 + \lambda_1^2)\sin^2\theta}\right]^2 (\sin^2\theta + \lambda^2 u^2 \sin^2\phi + 2\lambda w \cot\theta), \tag{22}$$

where $\lambda_1 = \partial\lambda/\partial\theta$ and $\lambda_2 = \partial\lambda/\partial\phi$. The magnitude of ∇P may be obtained from

$$|\nabla P|^2 = 4\lambda^2 \sin^2\theta[(1 + \sin^2\theta) - 2\lambda u \cos\phi + \lambda^2(u^2 + w^2)]. \tag{23}$$

The factor $\widetilde{F}(\mathbf{k})$ depends in general on the physical mechanism producing the waves. The hull effect can be modeled by a dipole source, and the collapsing wave can be modeled by a quadropole source [3]. We shall concentrate on the former. Since the three-dimensional structure of the waves has been emphasized, allowance should be made for an arbitrary orientation of the dipole. Let the direction cosines for the axis of the dipole be $\mathbf{e} = (\ell, m, n)$. Then the source function Q becomes

$$Q = M(\mathbf{e}\cdot\nabla)\,\delta(\mathbf{r})$$

$$= M[\ell\delta'(x)\,\delta(y)\,\delta(z) + m\,\delta(x)\,\delta'(y)\,\delta(z) + n\,\delta(x)\,\delta(y)\,\delta'(z)], \tag{24}$$

where M specifies the dipole strength. Substituting (24) into (4) and applying the Fourier transform, we obtain

$$\widetilde{F}(\mathbf{k}) = M\zeta(u\xi + w\zeta)^2(\ell\xi + m\eta + u\zeta). \tag{25}$$

An alternate, compact form of this expression is

$$\widetilde{F} = M(u^2 + w^2)\lambda^4 \cos\theta\cos\tau\cos^2\sigma,$$

where σ is the angle between **k** and the velocity vector and τ is the angle between **k** and the dipole axis.

Before discussing the amplitude in more detail, we should examine the curves of vanishing Gaussian curvature on the slowness surface. This curve limits the applicability of equation (20) and determines the internal wave cone analogous to the Kelvin wedge which has been alluded to before. The equation $K=0$ is satisfied if one of the two factors on the right-hand side of equation (22) vanishes. The first factor vanishes only for $\theta=\pi/2$. This corresponds to the straight line which lies on the slowness surface and produces point 2 on the surface of constant phase illustrated in Fig. 8. The internal wave cone is determined by the vanishing of the second factor:

$$\sin^2\theta + \lambda^2 u^2 \sin^2\phi + 2\lambda w \cot\theta = 0. \tag{26}$$

Equation (26) is seen to depend on u and w only in their ratio u/w. Consequently the internal wave cone depends on the direction of motion but not on the speed. Examples of the boundary of the cone are illustrated in Fig. 9, where **r** is expressed in spherical coordinates as

$$\mathbf{r} = r(\cos\vartheta \cos\varphi, \cos\vartheta \sin\varphi, \sin\vartheta). \tag{27}$$

The cone contains the line collinear with the direction of motion as a generator.

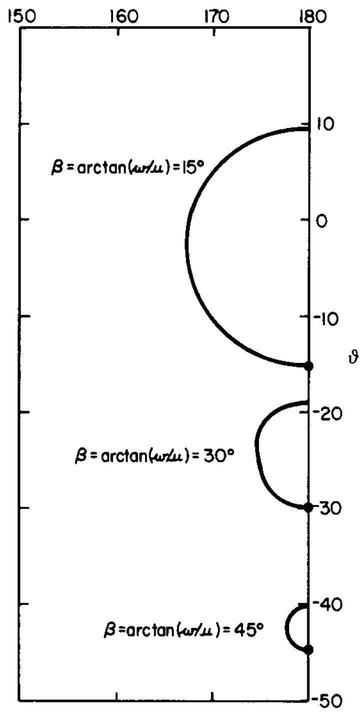


Fig. 9 — Boundaries of the internal wave cone analogous to the Kelvin wedge for ship waves. The filled circle is directly aft of the disturbance.

Now let us return to the calculation of the amplitude. The task remaining is to find those $\mathbf{k}$ such that for a given $\mathbf{r}$

$$\mathbf{r} \times \nabla P(\mathbf{k}) = 0.$$

For the case of horizontal motion ($w = 0$) we may quote Miles' [3] result

$$\mathbf{k} = \frac{1}{u(\sin^2\vartheta + \cos^2\vartheta \ \sin^2\varphi)^{3/2}} [\cos\vartheta \ \sin\vartheta \ \cos\varphi(\sin^2\vartheta + \cos^2\vartheta \ \sin^2\varphi),$$

$$-\cos^3\vartheta \sin\vartheta \cos^2\zeta \ \sin\zeta, \cos^2\vartheta \ \sin^2\varphi + (\sin^2\vartheta + \cos^2\vartheta \ \sin^2\varphi) \ \sin^2\vartheta].$$

For the general case

$$\nabla P || \ \mathbf{U} - \frac{\cos\theta}{\lambda} \ \hat{\theta},$$

where $\hat{\theta} = \partial\hat{\mathbf{k}}/\partial\theta$, and the condition to be satisfied becomes

$$\left(\frac{1}{\lambda} \cos\theta\right) \mathbf{r} \times \hat{\theta} = \mathbf{r} \times \mathbf{U}. \tag{28}$$

No analytic solution to equation (28) has yet been found, so we must now resort to numerical methods. Furthermore the solutions are no longer simple. Solutions of multiplicity three occur inside the internal wave cone, and directly aft of the disturbance the solution is of multiplicity two.

Figure 10 illustrates numerical solutions in the domain of simple solutions for the case $u = w = 1$. The solutions are presented as a family of curves of θ and ϕ versus ϑ and φ. Details of the structure in the domain of triple solutions, bounded by the internal wave cone, are shown in Fig. 11 for the same case $u = w = 1$. The occurrence of the multiple solutions is explained by the form of the surface of constant phase, as illustrated in Fig. 8. The mulitplicity is simply the number of times a ray from the origin in the direction of $\mathbf{r}$ intersects the surface of constant phase.

Finally we illustrate the calculation of the amplitude by plotting contours of the angular dependent portion. From equation (20) we may write

$$|\phi|^2 = \left(\frac{4\pi^2}{r}\right)^2 A(\vartheta, \varphi), \tag{29}$$

where

$$A = \widetilde{F}^2 / |\nabla P|^2 \ |K| \tag{30}$$

in the domain of simple solutions. Figure 12 shows contours of $\log A$ for three different orientation of the dipole: (a) parallel to the direction of motion, (b) perpendicular to the direction of motion but in the plane of the velocity vector, and (c) horizontal and perpendicular to the direction of motion. The dashed curve in these figures outlines the boundary of the internal wave cone.

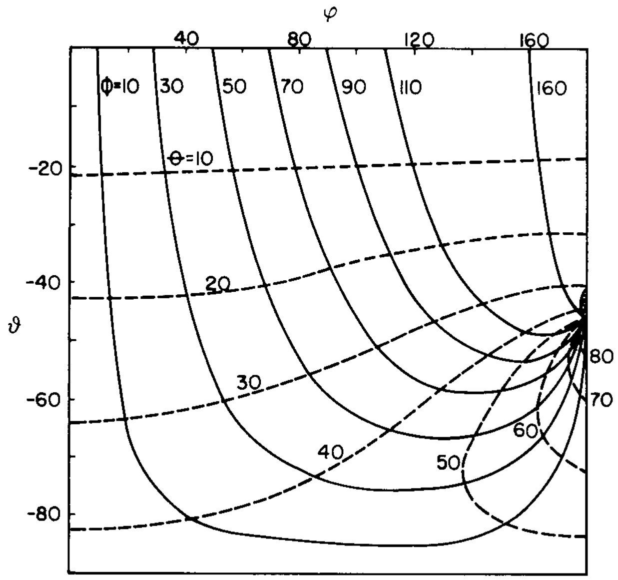


Fig. 10 — Curves representing solutions of the equation $\nabla P \times \mathbf{r} = 0$ for the case $u = w = 1$. On the solid curves ϕ is constant, and on the dashed curves θ is constant. The hatched region is the domain of multiple solutions enclosed by the cuspsoidal edge.

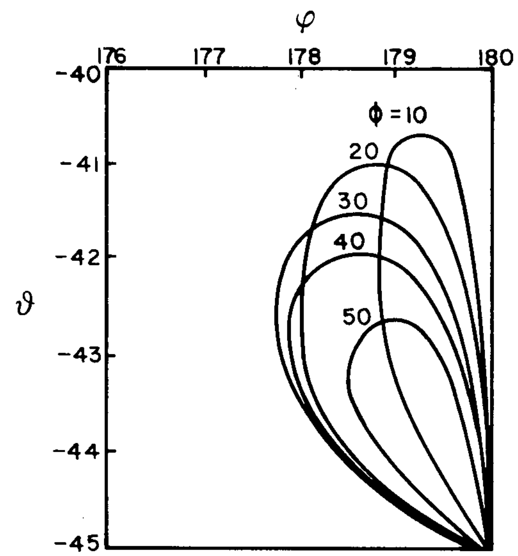
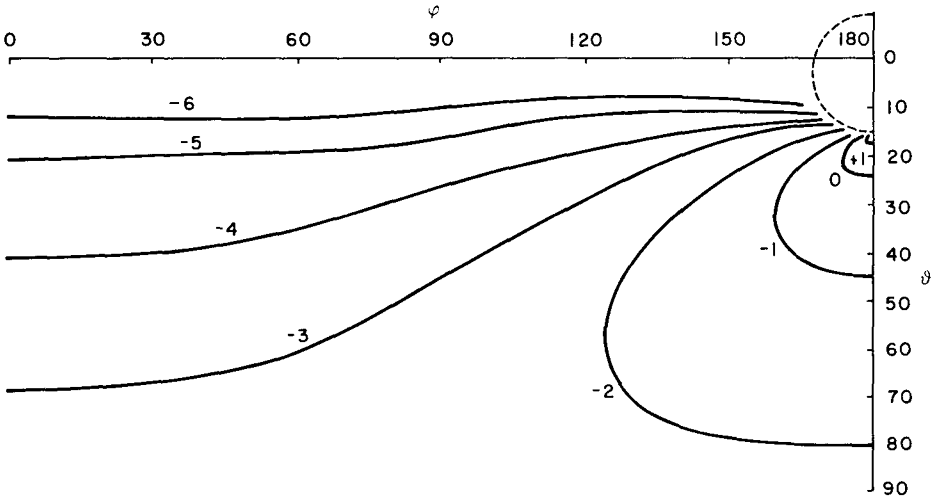
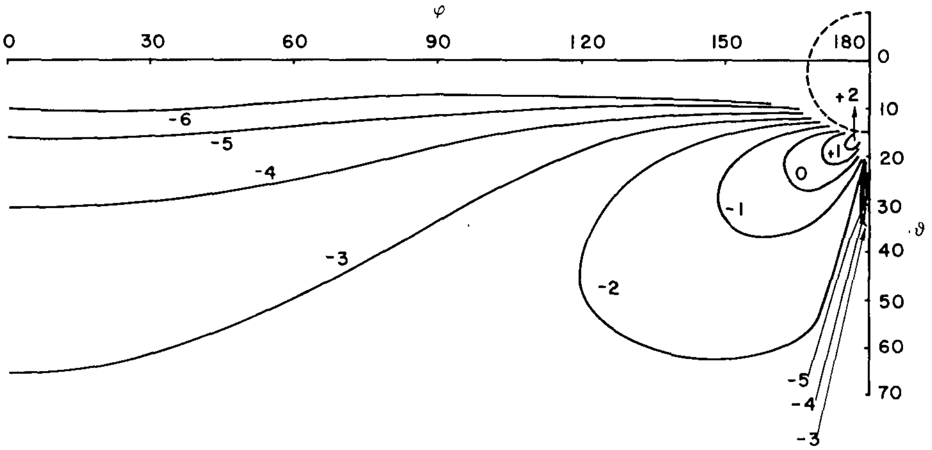


Fig. 11 — Detail of the domain of multiple solutions of $\nabla P \times \mathbf{r} = 0 (u = w = 1)$

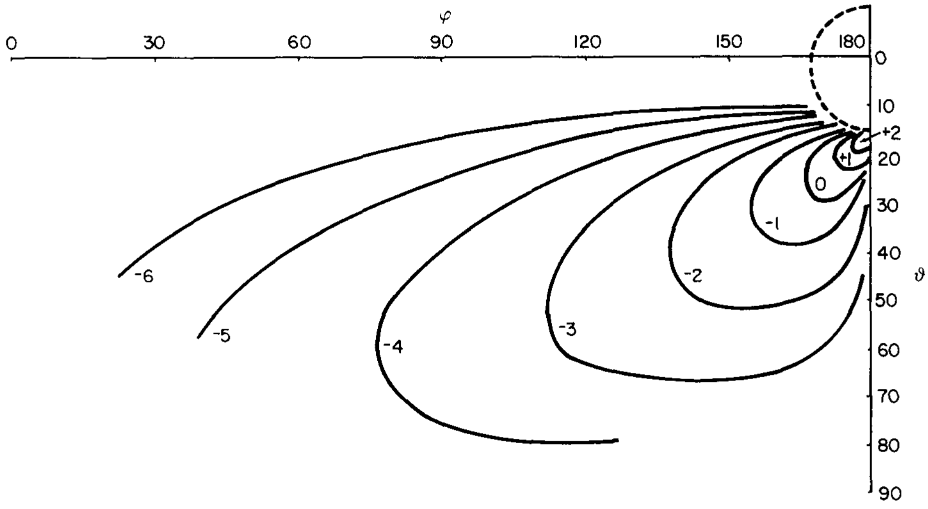


(a) For **e** || **U**, $\beta = 15°$



(b) For **e** ⊥ **U**, **e** ⊥ $\hat{\mathbf{j}}$, $\beta = 15°$

Fig. 12 – Contours of log A



(c) For $\mathbf{e} \perp \mathbf{U}$, $\mathbf{e} \perp \hat{\mathbf{k}}$, $\beta = 15°$

Fig. 12 (Continued) – Contours of log A

The spherical propagation (r^{-1} decay of the amplitude) is not valid along the boundary of the internal wave cone. In these directions Lighthill's results for a general monoclastic curve apply. It is shown that in this case the amplitude decays like $r^{-5/6}$, intermediate between cylindrical and spherical propagation.

CONCLUSION

A salient feature of the wave patterns is the internal wave cone. The most intense waves should occur on the boundary of this cone. The cone may or may not intersect the surface, depending on the inclination of the course of the disturbance. The extent of the cone is critically dependent on the inclination, being most pronounced for nearly horizontal motion, but is independent of the speed. It is conceivable however that the amplitude at the waves on the cone could depend on the speed.

Although the geometrical explanation of the internal wave cone is entirely satisfactory, it would be illuminating to determine a physical reason for its existence. In the case of ship waves Whitham [9] shows by a simple construction that the Kelvin wedge arises because the group velocity for water waves is parallel to the phase velocity but the group speed is 1/2 the phase speed. This physical property alone confines ship waves to the well-known wedge of half-angle 19°28′. In the case of internal waves the group velocity lies in the vertical plane containing the phase velocity but is perpendicular to it. If we denote the phase velocity by $\mathbf{c}$, the group velocity by $\mathbf{C}$, and let

$$\mathbf{c} = c(\cos\chi \cos\psi, \sin\chi \cos\psi, \sin\psi),$$

then

$$\mathbf{C} = (c \tan \psi)(\cos\chi \; \sin\psi, \sin\chi \sin\psi, -\cos\psi). \tag{31}$$

Following Whitham's argument, plane internal wave fronts can be stationary relative to the moving disturbance only if the projection of **U** onto **c** has magnitude c, as illustrated in Fig. 13. Thus, the locus of admissable phase velocities is a sphere with diameter **U**. Equation (31) yields the attendant group velocities which are admissable. The geometrical construction is shown in Fig. 14 when the plane containing **U** and **c** is vertical. A similar construction valid in any vertical plane is obtained replacing the circle of diameter U by circle of intersection of the sphere of admissable phase velocities and the given plane. The locus of group velocities shown in Fig. 14 is illustrated in Fig. 15 when $\beta = 15°$. This locus may be interpreted as defining the domain of influence for waves originating at 0 (as in Fig. 14) and propagating until the disturbance moves from 0 to P. A series of such loci is shown in Fig. 16. These figures are the internal-wave analogs of the water-wave loci shown in Figs. 12.2 and 12.3 of Whitham [9]. The loop is formed from plane wave components for which the sense of the vertical component is opposite to that of **U**. The portion of the curve outside the loop is formed from plane wave components whose vertical component of group velocity has the same sign as that of **U**. The waves inside the internal wave cone correspond to group velocities which form the loop. Figure 14c shows that the loop enlarges as β is decreased and that the lower extremity is directly aft of the disturbance. These features have already been noted in the discussion of Fig. 9. The inclination of the ray from the origin to the upper extremity of the internal wave cone can now be seen to be the same as the stationary value of the inclination of the vector $\rho = \mathbf{C} - \mathbf{P}$, that is, the value of θ which renders the following expression stationary:

$$f = \frac{\cos\beta\ \sin\theta\ \cos^2\theta + \sin\beta(1 + \sin^2\theta)\ \cos\theta}{-\cos\beta\ \cos^3\theta + \sin\beta\ \sin^3\theta}. \tag{32}$$

To summarize the physical argument, the wave pattern is the result of a superposition of successive anisotropic wave groups of the form shown in Fig. 15. The internal wave cone is the envelope of the family of looped portions of these waves, in contrast to the analogous families of circles which lead to Mach cones for acoustic waves and Kelvin wedges for surface waves.

The directional dependence of the amplitude of the waves is quite different for various orientations of the dipole. This directional dependence is best described by the orientation

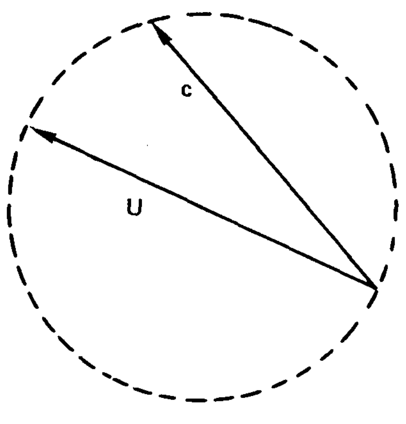
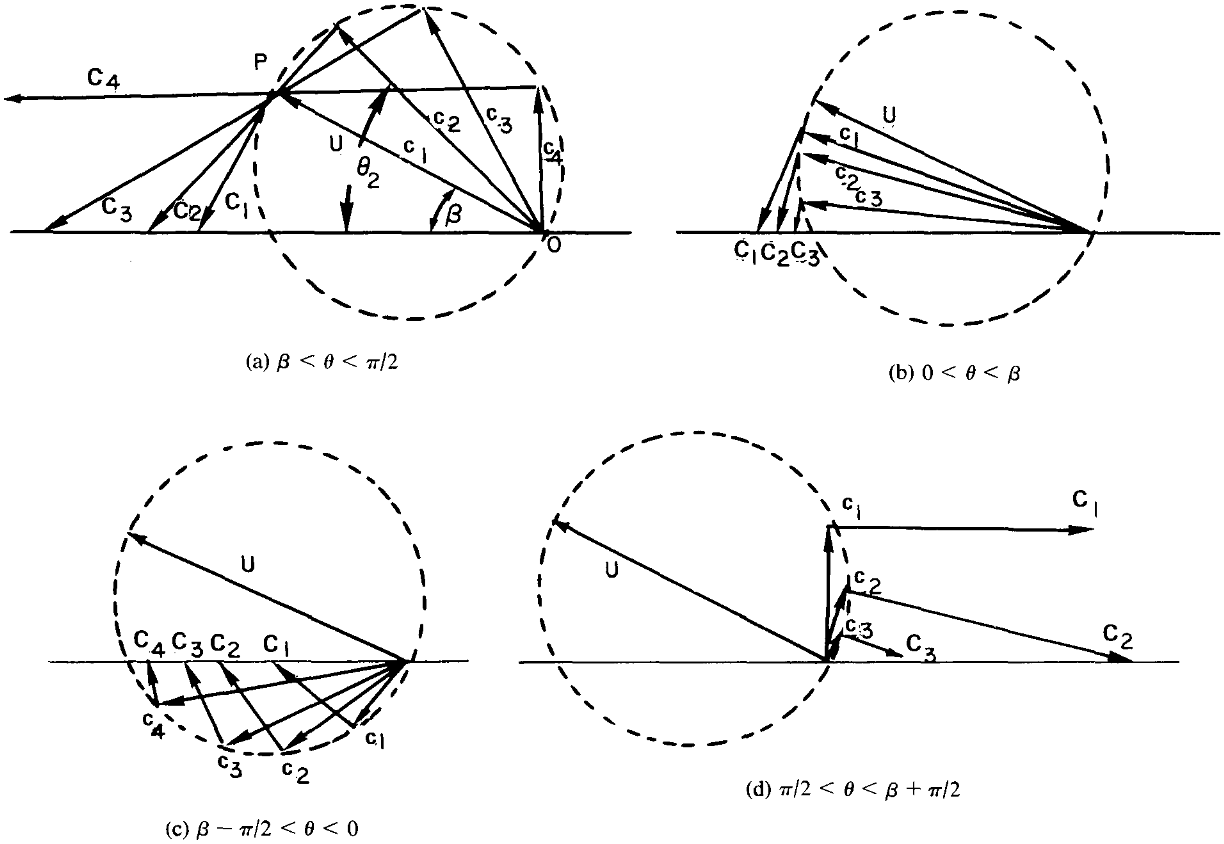


Fig. 13 – Admissable phase velocities for a stationary wave pattern



(a) $\beta < \theta < \pi/2$

(b) $0 < \theta < \beta$

(c) $\beta - \pi/2 < \theta < 0$

(d) $\pi/2 < \theta < \beta + \pi/2$

Fig. 14 — Construction of admissable group velocities

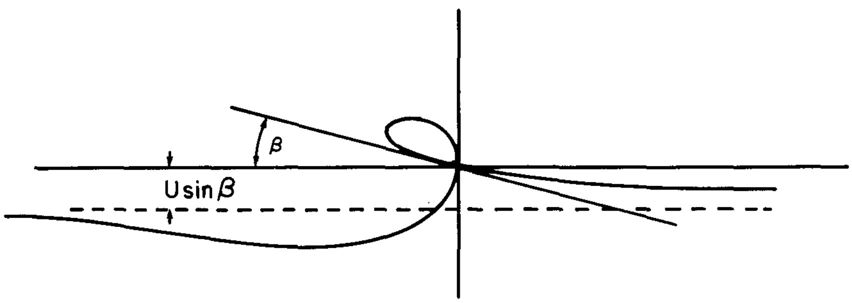


Fig. 15 — Locus of admissable group velocities when $\beta = 15°$

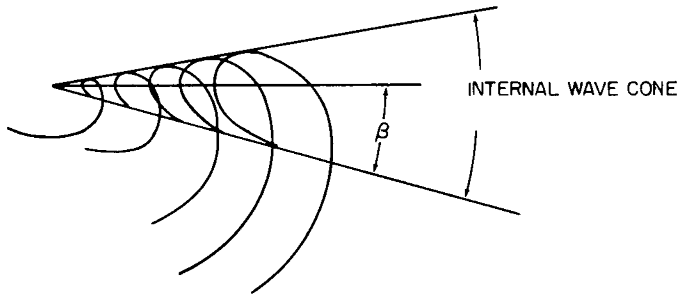


Fig. 16 – The internal wave cone as the envelope of admissable group velocities

of "sidelobes" which lie below the internal wave cone in the illustrative example. The amplitude is also seen to fall off quite rapidly with angular displacement from the internal wave cone.

Finally, we should emphasize the idealized nature of the model. Many important physical effects have been omitted. For example, we have used the Boussinesq approximation, the Brunt-Väisälä frequency has been fixed, nonlinear terms are neglected, the shape of the body is not accounted for, and no account of the surface of the fluid is included. However, it is precisely these idealizations which render the model useful, because the *main* features are preserved in the problem and a detailed picture is derived without recourse to large-scale numerical calculation.

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