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The one-dimensional problem of determining the density distribution of a plasma from its reflection of electromagnetic waves can be solved by the Gel'fand-Levitan technique. An extension and simplification of this technique has been proposed, and methods of solving the resulting functional equation were evaluated. The mathematical properties of the reflection coefficient were considered, related to the density distribution.

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FUNCTIONAL EQUATION IN THE PLASMA INVERSE PROBLEM AND SOLUTIONS

INTRODUCTION

Here we discuss some mathematical aspects of the plasma inverse problem, that is, the determination of the density of a plasma from its reflection of electromagnetic waves. In the simplest version of this problem, a cold, collisionless, unmagnetized plasma is stratified so that its density is a function only of x, and transverse electromagnetic waves propagate in the $\pm x$ directions. The density of the plasma is $N(x)$, and it is to be determined from $A(s)$, the complex reflection coefficient. Kay [1] and Balanis [2] have proposed this model as a subject for mathematical study, and also for possible application in remote sensing of the ionosphere.

In the simple one-dimensional problem, the plasma density $N(x)$ can be determined via solution of the Gel'fand-Levitan [3] integral equation. The solution is unique because $N(x) \geqslant 0$. Various techniques for solution of this equation are available, and we propose another one. We extend the Gel'fand-Levitan equation and write it in a more comprehensive form that is amenable to Laplace transformation. The result of Laplace transformation is a linear functional equation, which determines two functions in terms of $A(s)$, the complex reflection coefficient. The asymptotic form of either of the two functions determines $N(x)$, the plasma density. This method and some older methods are then applied to a very simple example. The relation of our equations to the Reimann function and the Friedrichs operator is shown. The analytic properties of $A(s)$ are treated. We conjecture that $A(s)$ can never be an entire function of s, the complex frequency. Simple examples show that $A(s)$ can have one or two branch points at finite values of s. Other examples show that $A(s)$ can be a meromorphic function of s. The conservation of energy gives a relation between $A(s)$ and the transmission coefficient. We combine this relation with the Nevanlinna theory [4]; we obtain a classification of meromorphic reflection coefficients and a new result on the distribution of their poles.

Our new functional equation allows us to generate many analytic solutions to the simple one-dimensional plasma inverse problem. Systematic methods of solution are presented. We suggest a general method for dealing with algebraic functions $A(s)$ that have two branch points at finite s. We shall also consider rational functions $A(s)$. Our two approaches to the rational inverse problem differ somewhat from that given by Kay [5]. As a particular case, we treat the Butterworth functions.

The mathematical discussion presented in this report is not directly applicable to the reflection of electromagnetic waves by the ionosphere. This problem has almost none of the simplifying features listed in the opening paragraph. The indirect application of our results to the ionosphere will be discussed elsewhere.

Manuscript submitted October 22, 1975.

Note: Dr. H.H. Szu is now employed by the Institute for Advanced Study, Princeton, N.J.

FORMULATION AND FUNCTIONAL EQUATION

In this section, we define the simple one-dimensional form of the plasma inverse problem and relate it to functions of a complex variable. We shall obtain a linear functional equation and some growth conditions on the two functions which appear in it. These conditions are perhaps sufficient to ensure the uniqueness of the solution of our functional equation.

The electron density is $N(x)$, and the electromagnetic waves propagate in the $\pm x$ directions. The external magnetic field is zero. With these assumptions, we shall have no changes in the polarization of an incident electromagnetic wave. The electric field is $\mathbf{E}(x,t)$, and we may assume that it is always parallel to the y axis. Since the electron density depends only on x, we shall have no separation of charges in the plasma; this means that

$$\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{j} = 0, \tag{1}$$

where $\mathbf{j}$ is the current density. We neglect reflection of electromagnetic waves by the ions; that means that j is the electron current. For a cold, collisionless plasma we obtain

$$j = \frac{e^2}{m} N(x) \int_{-\infty}^{t} E(x,t')dt' \tag{2}$$

where e and m are the charge and mass of an electron. We assume that $N(x) = 0$ when $x < 0$, and we seek to determine $N(x)$ from the behavior of incident and reflected waves on the negative x axis. The complex reflection coefficient is required at all frequencies; the practical meaning of this statement will not be discussed in this report. Here we adopt the simple physical picture of Kay and Balanis, in which the incident wave is a δ function. This means that

$$E(x,t) = \delta(x-ct) + R(x+ct) \text{ for } x < 0, \tag{3}$$

where c is the speed of light and $R(y)$ is the reflection. The inverse scattering problem is now the determination of $N(x)$ from $R(y)$. The problem has a trivial solution, in which $R(y)$ and $N(x)$ vanish identically.

We have assumed that $N(x) = 0$ for $x < 0$. Causality then requires that $R(y) = 0$ for $y < 0$. The reflection from the plasma is found inside the forward light cone (Fig. 1). These properties are characteristic of the physical or retarded electric field, which satisfies Eq. (3). Solution of the inverse problem depends on the use of a spacelike electric field, which vanishes inside the light cone but not outside. We shall combine these two possibilities to obtain an "entire" electric field which satisfies a simple integral equation.

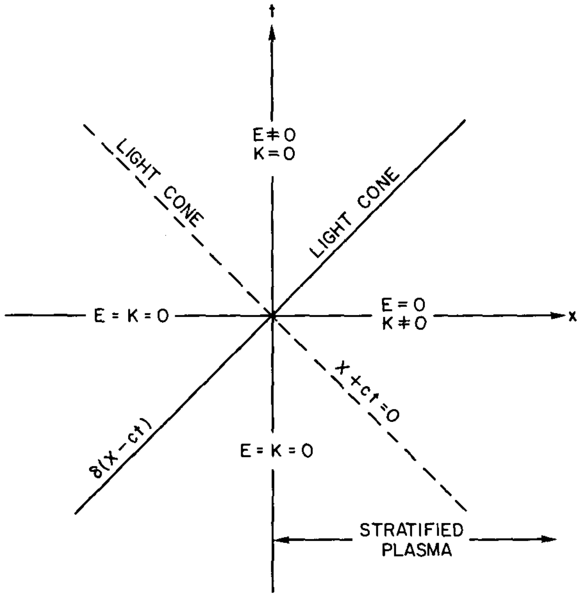

Fig. 1 — The light cone divides the x - t plane into four quadrants. We consider a timelike and a spacelike solution of the wave equation; each solution is the sum of $\delta(x\text{-}t)$ and a bounded part.

In any case, the electric field must satisfy a partial differential equation which is derived from Maxwell's equations. Let

$$q(x) = 4\pi r_e N(x), \tag{4}$$

where $r_e = e^2/mc^2$ is the classical electron radius. Then Eqs. (1) and (2) lead to

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right) E\ (x,t) = q(x)\ E\ (x,t). \tag{5}$$

Here and henceforth we put $c = 1$. Incidentally, we obtain plasma oscillations of long wavelength by considering the case in which $N(x)$ is nearly constant and $\partial^2/\partial x^2$ is negligible. This means that $q(x)$ is the square of the plasma frequency.

The retarded electric field, $E(x,t)$, satisfies Eqs. (3) and (5). It can be written as

$$E(x,t) = \delta(x-t) + R(x+t) + \int_{max(-x,-t)}^{x} K(x,y)[\delta(y-t)+R(y+t)]\,dy. \tag{6}$$

The Marchenko function $K(x,y)$ appears here as a generalized translation operator. This function might well be defined in a different way; other possibilities will be treated. It enters in a simple form when we ask for a reflectionless, spacelike solution of Eq. (5). This solution is $\delta(x-t) + K(x,t)$; it must satisfy

$$\left[\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} - q(x)\right] [\delta(x-t) + K(x,t)] = 0. \tag{7}$$

This solution is reflectionless because we require that

$$K(x,t) = 0 \quad \text{for } x < 0. \tag{8}$$

It is called spacelike because it vanishes inside the light cone (Fig. 1); we require that

$$K(x,t) = 0 \quad \text{for } |t| > x > 0. \tag{9}$$

The singular part of this solution, the δ function, appears explicitly in Eq. (7). We demand that $K(x,t)$ is bounded everywhere. A similar separation into a δ function and a bounded part is possible in the retarded solution, which satisfies Eqs. (3) and (5). We require that $E(x,t) - \delta(x-t)$ is bounded everywhere.

If either the retarded or the spacelike solution of Eq. (5) is known, we can recover Eq. (4) and the plasma density. Equation (7) can be written in the form

$$\left[\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} - q(x)\right] K(x,t) = q(x)\ \delta(x-t),$$

which states that the first partial derivatives of $K(x,t)$ are discontinuous at $x = t$. We find that

$$\frac{\partial}{\partial x} K(x,t) = \frac{\partial}{\partial t} K(x,t) = q(x),$$

where the partial derivatives are evaluated at $t = x^-$, just outside the light cone (Fig. 1). This result can be written as

$$q(x) = 2 \frac{d}{dx} K\ (x,x), \tag{10a}$$

where the limit $t \to x^-$ is implicit. Also, Eq. (7) leads to a claim that $K(x,t)$ and its first partial derivatives vanish at $t = -x$. Similar considerations apply to $E(x,t)$, the retarded electric field. We find that

$$q(x) = -2 \frac{d}{dx} \underset{t \to x^+}{\text{limit}} \left[E(x,t) - \delta(x-t)\right]. \tag{10b}$$

If $t < x$, the retarded electric field must vanish, and Eq. (6) gives

$$R(x+t) + K(x,t) + \int_{-t}^{x} K(x,y) R(y+t)\, dy = 0, \tag{11}$$

which is the Gel'fand-Levitan integral equation. The range of integration is shown in Fig. 2. The conventional approach to the inverse problèmis to solve Eq. (11) for $K(x,t)$ and then obtain $q(x)$ from Eq. (10a). This might be done by iterating the kernel $R(y+t)$. Since $R(y)$ is bounded, the resulting series should converge to the function $K(x,t)$ when x is small; we can show that it converges when

$$0 \leqslant |t| < x < \frac{\pi}{4\ \mathrm{Max}|R(y)|}.$$

This construction suggests that $K(x,t)$ is uniquely determined by $R(y)$.

Our method of solution uses Eq. (6) directly, without the assumption that $t < x$. We introduce the "entire" electric field

$$\mathcal{E}(x,t) = K(x,t) + \delta(x-t) - E(x,t). \tag{12}$$

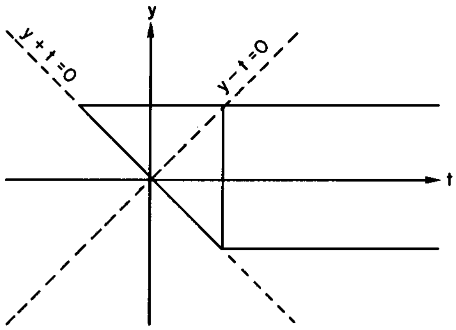

Fig. 2 — Domains of integration for Gel'fand-Levitan equation and extended Gel'fand-Levitan equation are plotted as functions of t. For the Gel'fand-Levitan equation, $t < x$ and the domain appears as a triangle. For the extended Gel'fand-Levitan equation, all values of t are allowed, and the domain appears as a semi-infinite trapezoid.

This is a bounded function which vanishes when $x + t < 0$; see Fig. 1. Indeed, Eq. (10) implies that this function is continuous on the line $x = t$, and that

$$q(x) = 2\frac{d}{dx}\, \mathcal{E}(x,x). \tag{10c}$$

This entire electric field incorporates the solution of the direct scattering problem, for $E(x,t) = -\mathcal{E}(x,t)$ when $t < x$. Also, this function satisfies the integral equation

$$R(x+t) + \mathcal{E}(x,t) + \int_{\max(-x,-t)}^{x} \mathcal{E}(x,y)R(y+t)dy = 0, \tag{13}$$

which is valid for all values of t. The range of integration is plotted in Fig. 2.

The plasma density is independent of time, and we can eliminate t from Eq. (13) by taking Laplace transforms. Also, Laplace transformation will give us a derivation of Eq. (6) from Eq. (7), thus closing the gap in the above reasoning. We can use the Laplace transform rather than the Fourier transform because all electric fields vanish in the backward light cone; see Fig. 1.

The Laplace transform of the retarded electric field is

$$\hat{E}(x,s) = \int_{-\infty}^{\infty} E(x,t)e^{-st}\, dt.$$

Because of the retardation, the integral converges when the real part of s is positive. The condition (3) becomes

$$\hat{E}(x,s) = e^{-sx} + A(s)e^{sx} \quad \text{for } x < 0. \tag{14}$$

Here

$$A(s) = \int_0^\infty R(y)e^{-sy}\,dy$$

is the Laplace transform of the reflection. Clearly, $A(s)$ is analytic when the real part of s is positive. If the real part of s vanishes, then Eq. (14) shows that $A(s)$ is the reflection coefficient for a monochromatic wave; hence, $|A(s)| \leqslant 1$.

The retarded electric field contains a δ function and a bounded part. We define

$$B(x,s) = \int_{-\infty}^{\infty} [E(x,t) - \delta(x-t)]\,e^{-st}dt,$$

and we have

$$|B(x,s)| \leqslant (\text{constant})\frac{e^{-\sigma x}}{\sigma} \quad \text{for } \sigma > 0, \tag{15}$$

where σ is the real part of s. Laplace transformation of Eqs. (5) and (7) gives

$$\left[\frac{\partial^2}{\partial x^2} - s^2 - q(x)\right] \hat{E}(x,s) = 0 \tag{16}$$

and

$$\left[\frac{\partial^2}{\partial x^2} - s^2 - q(x)\right] [e^{-sx} + F(x,s)] = 0, \tag{17}$$

where

$$F(x,s) = \int_{-x}^{x} K(x,t)e^{-st}dt. \tag{18}$$

Equation (8) implies that $F(x,s) = 0$ when $x < 0$. The integration in Eq. (18) extends over the region where $K(x,t)$ is nonvanishing; see (9). This finite range implies that $F(x,s)$ is an entire function of s. Since $K(x,t)$ is bounded, we have

$$|F(x,s)| \leqslant (\text{const.}) \frac{\sinh x\sigma}{\sigma} . \tag{19}$$

Using Eq. (11) and the uniqueness theorem for Laplace transforms, we see that $A(s)$ determines $F(x, s)$ uniquely.

We can now write the one-dimensional plasma inverse problem in terms of relations among these Laplace transforms. A change of sign in Eq. (17) gives

$$\left[\frac{\partial^2}{\partial x^2} - s^2 - q(x)\right][e^{sx} + F(x,-s)] = 0. \tag{20}$$

Equation (18) shows that this change of sign corresponds to time reversal. According to Eqs. (16), (17), and (20), we now have three solutions to an ordinary, second-order differential equation. There must be a linear relation among them. If $s \neq 0$, $e^{-sx} + F(x,s)$ and $e^{sx} + F(x, -s)$ are independent solutions, and it is possible to write $\hat{E}(x,s)$ as a linear combination of them. The coefficients depend on s. Since $F(x,s) = 0$ for $x < 0$, we can use Eq. (14) to determine the coefficients. We find that

$$\hat{E}(x,s) = e^{-sx} + F(x,s) + A(s)[e^{sx} + F(x,-s)].$$

This equation is the Laplace transform of Eq. (6), which has thus been derived from Eq. (7). By continuity, it holds when $s = 0$. Since $\hat{E}(x,s) = e^{-sx} + B(x,s)$, we obtain

$$A(s)[e^{sx} + F(x,-s)] + F(x,s) - B(x,s) = 0, \tag{21}$$

our functional equation for $B(x,s)$ and $F(x,s)$. This equation could have been derived directly from Eq. (13), using

$$F(x,s) - B(x,s) = \int_{-\infty}^{\infty} \mathcal{E}(x,t)e^{-st}dt,$$

but such a derivation would attain brevity at the expense of lucidity. When $A(s)$ is prescribed and the growth conditions of Eqs. (15) and (19) are imposed, the solution of Eq. (21) is probably unique.

After solving for $B(x,s)$ and $F(x,s)$, we can easily recover Eq. (4). If $\partial K/\partial t$ is bounded when x-t is small and positive, partial integration of Eq. (18) gives

$$F(x,s) \sim -K(x,x)\frac{e^{-sx}}{s} \text{ as } s \to -\infty$$

and then Eq. (10a) gives $q(x)$. Alternatively, we can use Eq. (10b). We have

$$\lim_{s\to+\infty} se^{sx}\, B(x,s) = \lim_{s\to-\infty} se^{sx} F(x,s) = -\mathcal{E}(x,x).$$

Finally, Eq. (10c) gives $q(x)$.

We shall show that Eq. (21) can be solved analytically in many cases. If it can be solved, the asymptotic form of either $B(x,s)$ or $F(x,s)$ gives $q(x)$, and, finally, Eq. (4) gives $N(x)$, the unknown plasma density.

A SIMPLE EXAMPLE

The case in which

$$R(y) = -\lambda e^{-\lambda y} \tag{22}$$

can be regarded as the simplest nontrivial example of the plasma inverse problem. This problem was proposed and solved by Kay and Moses in a 1955 research report [6]. This inverse problem will be solved thrice in this section, twice using Eq. (11) and once using Eq. (21).

Iteration of the kernel $R(y + t)$ is the straightforward method of solving Eq. (11). This means that we set

$$K(x, t) = \sum_{n=1}^{\infty} K_n(x, t), \tag{23}$$

where

$$K_1(x,\ t) = -\ R(x+t)$$

and

$$K_{n+1}(x,\ t) = -\int_{-t}^{x} K_n(x,\ y)R(y+t)dy$$

for $n \geqslant 1$. In the present simple example,

$$K_1(x,\ t) = \lambda e^{-\lambda(x+t)},$$

$$K_2(x,\ t) = \lambda^2 e^{-\lambda(x+t)} \int_{-t}^{x} dy\ e^{-2\lambda y},$$

$$K_3(x,\ t) = \lambda^3 e^{-\lambda(x+t)} \int_{-t}^{x} dy\ e^{-2\lambda y} \int_{-y}^{x} dz\ e^{-2\lambda z},$$

and so forth. These functions can be written explicitly in terms of Laguerre polynominals and exponential functions, but it is simpler to write them in terms of the integral operator

$$\hat{I} = \lambda \int_{-t}^{x} dy\ e^{-2\lambda y}.$$

After integration, we may change the variable from t to y; let $\hat{C}$ denote this change. Then we have

$$K_3(x,\ t) = \lambda e^{-\lambda(x+t)}\hat{I}\hat{C}\hat{I}$$

and, in general,

$$K_n(x,\ t) = \lambda e^{-\lambda(x+t)}\hat{I}\hat{C}\hat{I} \ldots \hat{C}\hat{I},$$

where $\hat{I}$ appears $n - 1$ times. The sum of the series (23) is

$$K(x,\ t) = \lambda e^{-\lambda(x+t)}(1 + \hat{I} + \hat{I}\hat{C}\hat{I} + \hat{I}\hat{C}\hat{I}\hat{C}\hat{I} + \dots).$$

Another change from t to y gives

$$K(x,\ y) = \lambda e^{-\lambda(x+y)}(1 + \hat{C}\hat{I} + \hat{C}\hat{I}\hat{C}\hat{I} + \dots).$$

We now have to find the sum

$$\sum(x,\ y) = 1 + \hat{C}\hat{I} + \hat{C}\hat{I}\hat{C}\hat{I} + \dots .$$

This function satisfies

$$\sum(x,\ y) - 1 = \hat{C}\hat{I}\sum(x,\ y).$$

A simple calculation shows that

$$\sum(x,\ y) = e^{\lambda(x+y)}$$

is the solution of this integral equation. Therefore,

$$K(x,\ t) = \lambda \quad \text{for } 0 < |t| < x . \tag{24}$$

Finally,

$$q(x) = 2\lambda\delta(x)$$

is obtained from Eq. (10a). This solution depended on the fact that $R(y+t)$ is a function of y times a function of t; it serves as a separable kernel in Eq. (11).

Another method of solution involves construction of an ordinary differential equation that determines the time dependence of $K(x,\ t)$. The function (22) satisfies the simple equation

The differential operator which appears here can be used to annihilate the first term in Eq. (11), as well as the integral sign. We have

$$\left(\frac{\partial}{\partial t} + \lambda\right) K(x,\ t) + K(x,\ -t)R(0) = 0. \tag{25}$$

Time reversal gives

$$\left(-\frac{\partial}{\partial t} + \lambda\right) K(x,\ -t) + K(x,\ t)R(0) = 0.$$

Both of these differential equations are valid for $-x < t < x$. We now eliminate $K(x,\ -t)$ and use $R(0) = -\lambda$; this gives

$$\frac{\partial^2}{\partial t^2} K(x,\ t) = 0.$$

We have to integrate this differential equation. We set $t = -x$ in Eq. (11) and find

$$K(x,\ -x) = -\ R(0) = \lambda.$$

Therefore,

$$K(x,\ t) = \lambda + (x+t)C(x),$$

where $C(x)$ is a constant of integration. Use of Eq. (25) shows that $C(x) = 0$. Again we obtain Eq. (24).

Finally, we solve this inverse problem by using Eq. (21) and some properties of entire functions. The Laplace transform of Eq. (22) is

$$A(s) = \frac{-\lambda}{s+\lambda}\ .$$

This is a very simple example of a meromorphic reflection coefficient. We substitute this function in Eq. (21) and multiply by a linear factor. The result is

$$-\lambda \left[e^{sx} + F(x,\ -s)\right] + (s+\lambda)F(x,s) = (s+\lambda)B(x,\ s).$$

Since the left-hand side is an entire function of s, so is the right-hand side. Furthermore,

$$(s+\lambda)B(x, s)e^{sx} = -\lambda\left[e^{sx} + F(x, -s)\right]e^{sx} + (s+\lambda)F(x, s)e^{sx} \tag{26}$$

is also an entire function of s. Use of Eq. (15) shows that this function is bounded as $s \to \infty$ in the right half of the complex plane. Use of Eq. (19) shows that the right-hand side of Eq. (26) is bounded by a constant times $|s|$ if $s \to \infty$ while σ (the real part of s) is constant. Also from Eq. (19), we see that Eq. (26) is bounded as $s \to \infty$ in the left half of the complex plane. Since Eq. (26) is an entire function of s, it must be a linear function of s; Liouville's theorem on entire functions is used here. In fact, Eq. (26) must be equal to C, a constant. We obtain

$$B(x, s) = \frac{C}{s+\lambda}\, e^{-sx}$$

and

$$(s+\lambda)F(x, s) - \lambda F(x, -s) = \lambda e^{sx} + Ce^{-sx}. \tag{27}$$

Let us replace s by $-s$ and add the resulting equation to Eq. (27). We obtain

$$s[F(x, s) - F(x, -s)] = 2(\lambda+C)\cosh sx.$$

Since $F(x, s)$ is an entire function of s, this quantity must vanish at $s = 0$. Therefore, $C = -\lambda$ and $F(x, s)$ is an even function of s. Finally, Eq. (27) gives

$$F(x, s) = 2\lambda\,\frac{\sinh sx}{s}\ .$$

The retarded or physical electric field is determined by

$$B(x, s) = -\frac{\lambda}{s+\lambda}\, e^{-sx}\ .$$

We have

$$\mathcal{E}(x, x) = \lambda \quad \text{for } x > 0.$$

Again, we obtain

$$q(x) = 2\lambda\delta(x).$$

REDEFINITION OF K

The Marchenko function $K(x, t)$ is of great importance in the plasma inverse problem, and it can be introduced in various ways. In this section we relate it to the Riemann solution of a hyperbolic equation, and then define it in terms of the transformation operator of Friedrichs. The new definition will turn out to be equivalent to Eqs. (7), (8), and (9).

The hyperbolic equation (5) can be integrated from an earlier time to a later time, assuming that $E(x, 0)$ and $\partial E(x, 0)/\partial t$ are known functions of x. The representation of the solution in terms of these two functions is due to Riemann [7,8]. It is possible because $t = 0$ is never parallel to a characteristic line of Eq. (5). The characteristics of Eq. (5) are diagonal lines in Fig. 1; they show a certain symmetry in x and t, as does the differential operator in Eq. (5). Hence we can integrate Eq. (5) forward in x rather than forward in t. Let us apply Riemann's method to integrate Eq. (5) from x = const. up to a point P (Fig. 3). We shall find E at point P in terms of E and $\partial E/\partial x$ evaluated on the line AB (Fig. 3). The value of E at point P is influenced only by what happens in the triangular domain in Fig. 3. The construction of Riemann's function is simple in the present problem because

$$\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} - q(x)$$

is a self-adjoint differential operator; see Courant and Hilbert [8] for the general procedure. The absence of first derivatives implies that Riemann's function is equal to unity on the characteristics that appear as diagonal lines in Fig. 3. The electric field at point P is given by

$$E_P = \frac{E_A + E_B}{2} + \frac{1}{2}\int_A^B \left(\hat{R}\frac{\partial E}{\partial x} - E\frac{\partial \hat{R}}{\partial x}\right) dt,$$

where E_A and E_B are the values at points A and B, and $\hat{R}$ is the Reimann function. The integration runs along the line AB, where x is a negative constant. We have assumed that $q(x) = 0$ when $x < 0$. Hence, we can decompose the electric field into incident and reflected waves:

$$E(x, t) = I(x-t) + R(x+t) \quad \text{for } x < 0.$$

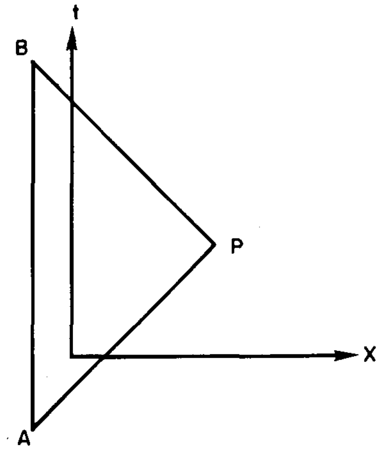

Fig. 3 — Domain used in Riemann's method for calculation of electric field at P in terms of initial values along line AB

This assumption is a small generalization of Eq. (3), where the incident wave was a δ function. Partial integration gives

$$E_P = I_A + R_B - \frac{1}{2}\int_A^B I\left(\frac{\partial \hat{R}}{\partial x} - \frac{\partial \hat{R}}{\partial t}\right)dt - \frac{1}{2}\int_A^B R\left(\frac{\partial \hat{R}}{\partial x} + \frac{\partial \hat{R}}{\partial t}\right)dt.$$

We now assign Cartesian coordinates to the points and lines in Fig. 3. The point P is at $(x,\ t)$ and the line AB has abcissa x'. We have $x' < 0$ and $x' < x$. The explicit formula for the electric field is

$$\begin{aligned} E(x,\ t) &= I(x-t) + R(x+t) \\ &- \frac{1}{2}\int_{t-x+x'}^{t+x-x'} I(x'-t')\left(\frac{\partial \hat{R}}{\partial x'} - \frac{\partial \hat{R}}{\partial t'}\right)dt' \\ &- \frac{1}{2}\int_{t-x+x'}^{t+x-x'} R(x'+t')\left(\frac{\partial \hat{R}}{\partial x'} + \frac{\partial \hat{R}}{\partial t'}\right)dt' . \end{aligned} \tag{28}$$

The Riemann function $\hat{R}$ can depend on x, t, x', and t'; but it does not depend on the choice of $I(x-t)$, the incident pulse.

We now notice that t does not appear explicitly in Eq. (5). This implies that $\hat{R}$ depends on $t - t'$ rather than t and t' separately. Hence we replace $\partial\hat{R}/\partial t'$ by $-\partial\hat{R}/\partial t$ in Eq. (28). Any solution of Eq. (5) must satisfy the resulting relation; we apply it to $\delta(x-t) + K(x, t)$, the reflectionless solution. Hence the Marchenko function is given by

$$K(x,\ t) = -\frac{1}{2}\left(\frac{\partial\hat{R}}{\partial x'} + \frac{\partial\hat{R}}{\partial t}\right)\Bigg|_{x'=t'=0}.$$

We have set $x' = 0$ to simplify this result. We note that $\hat{R}$ is an even function of $t - t'$ because the boundary conditions determining $\hat{R}$ are symmetric about $t - t' = 0$ (Fig. 3). Hence,

$$K(x,\ -t) = -\frac{1}{2}\left(\frac{\partial\hat{R}}{\partial x'} - \frac{\partial\hat{R}}{\partial t}\right)\Bigg|_{x'=t'=\ 0}$$

and

$$E(x,\ t) = I(x-t) + R(x+t) + \int_{t-x}^{t+x} K(x,\ t - t')\ I\ (-t')dt'$$

$$+ \int_{t-x}^{t+x} K(x,\ t' - t)\ R\ (t')dt'.$$

Finally, a change in the integration variables gives

$$E(x,\ \ t) = I(x-t) + R(x+t) + \int_{-x}^{x} K(x,\ y)[I(y-t) + R(y+t)]\,dy.$$

This result is recognized as a generalization of Eq. (6), and its derivation has shown the relation of the Marchenko and Riemann functions.

We now treat Eq. (5) in an entirely different way, related to Laplace transforms and ordinary differential equations. Since $q(x)$ is nonnegative, we deal with a differential operator having a purely continuous spectrum. Suppose that L and L_0 are two such operators; under certain conditions they can be connected by the transformation operator of Friedrichs [9,10]. The connection of this transformation with inverse problems is shown in Faddeyev's review [11]. We extract some ideas from Faddeyev's paper and rearrange them as follows.

The two linear differential operators L and L_0 are supposed to be second-order, self-adjoint operators; indeed, we assume that

$$L_0 = -\frac{d^2}{dx^2} + s^2. \tag{29}$$

The second term is s^2 times the unit operator, and the assumption that L_0 is self-adjoint does not mean that s^2 is real. Let us assume that

$$V = L - L_0$$

is independent of the parameter s. Friedrichs has studied the unitary transformation which connects L and L_0; it satisfies

$$LU = UL_0. \tag{30}$$

His theory is simplest in the case of L and L_0 having purely continuous spectra, with no discrete spectrum; this case occurs in the plasma inverse problem. Let

$$U = 1 + K,$$

where 1 is the unit operator. Then

$$L(1+K) = (1+K)L_0$$

and, consequently,

$$V(1+K) = KL_0 - L_0K. \tag{31}$$

We want to identify K with the Marchenko function; this requires use of boundary conditions which are not expressed in Eq. (31). But in any case K must commute with the unit operator; this means that the right-hand side of Eq. (31) is independent of s. If the boundary conditions are independent of s and if Eq. (31) can be solved for K, then K is independent of s.

To make this formal manipulation a little more explicit, we consider the functions which are annihilated by L and L_0. Suppose that

$$L\psi = 0 \text{ and } L_0\phi = 0.$$

Then, Eq. (30) implies that $LU\phi = 0$, and $U\phi$ is a function annihilated by L. If $\psi_1 = (1+K)\phi$, then $L\psi_1 = 0$ or $L_0\psi_1 = -V\psi_1$. More explicitly,

$$L_0\psi_1(x) = -\int V(x, y)\psi_1(y)dy,$$

where V has been represented by a kernel. We write this as

$$L_0\psi_1(x) = -\int \delta(x-y)V(y, z)\psi_1(z)dydz$$

and try to divide by L_0. Since L_0 annihilates ϕ, we have

$$\psi_1(x) = \phi(x) + \int G_0(x-y)V(y, z)\psi_1(z)dydz, \tag{32}$$

where G_0 is Green's function for the operator (29); it depends on x-y. The boundary condition

$$G_0(x-y) = 0 \quad \text{for } y > x$$

is a simple choice which makes Eq. (32) into a Volterra integral equation. The integral equation can be solved by an iteration process if $V(x, y)$ is small in some sense. Formal summation of the series gives

$$\psi_1(x) = \int [\delta(x-y) + K(x, y)]\phi(y)dy.$$

The boundary condition on G_0 implies that

$$K(x, y) = 0 \text{ for } y > x. \tag{33}$$

We now turn to the inverse transformation. Let

$$U^{-1} = 1 + K'. \tag{34}$$

Since U is unitary, we must have

$$K + K' + KK' = K + K' + K'K = 0.$$

More explicitly, we have two Volterra integral equations,

$$K(x, y) + K'(x, y) + \int_y^x K(x, z)K'(z, y)dz = 0$$

and

$$K(x, y) + K'(x, y) + \int_y^x K'(x, z)K(z, y)dz = 0.$$

The limits of integration are consistent with the condition

$$K'(x, y) = 0 \quad \text{for } y > x. \tag{35}$$

The condition (33) allows us to write Eq. (31) in a more explicit form

$$\int V(x, z)[\delta(z-y) + K(z, y)]\,dz = \int_{yo}^x K(x, z)L_0(z, y)dz$$

$$- \int_y^\infty L_0(x, z)K(z, y)dz. \tag{36}$$

The limits of integration are chosen so that the first argument of $K(x, y)$ can increase indefinitely, whereas y_0 is the lower bound for the second argument. The kernel corresponding to (29) is

$$L_0(x, y) = -\delta''(x-y) + s^2\delta(x-y).$$

Partial integration shows that, when $x < \infty$, the right-hand side of Eq. (36) is

$$C(x, y) = [K(y, y) - K(x, x)]\delta'(x-y) + \left[\frac{\partial K}{\partial x}(y, y) + \frac{\partial K}{\partial y}(x, x)\right]\delta(x-y)$$

$$+ K(x, y_0)\delta'(y_0-y) - \frac{\partial K}{\partial y}(x, y_0)\delta(y_0-y)$$

$$+ \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)K(x, y).$$

A Taylor series expansion of $K(y, y)$ in powers of $y - x$ shows that the first two terms on the right-hand side are equal. Moreover,

$$C(x, y) = 0 \quad \text{for } y > x. \tag{37}$$

We can now find $V(x, y)$ in terms of $C(x, y)$ by use of the operator (34). Using Eq. (35) and (37), we find that

$$V(x, y) = C(x, y) + \int_y^x C(x, z)K'(z, y)dz.$$

Since $V(x, y)$ vanishes for $y > x$, we are led to write

$$V(x, y) = q(x)\delta(x-y) + Q(x, y)$$

where $Q(x, y)$ is a bounded function that vanishes for $y > x$. We have assumed that V is self-adjoint; this implies that $Q(x, y) = 0$ and

$$V(x, y) = q(x)\delta(x-y).$$

Finally, Eq. (36) becomes

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) K(x, y) + 2\left[\frac{d}{dx} K(x, x)\right] \delta\,(x-y)$$

$$- K(x, y_0)\delta'(y-y_0) - \frac{\partial K}{\partial y}(x, y_0)\delta(y-y_0)$$

$$= q(x)\delta(x-y) + q(x)K(x, y).$$

This shows that $K(x, y)$ and $\partial K/\partial y$ vanish at $y = y_0$ and that

$$q(x) = 2\frac{d}{dx} K(x, x).$$

Also, $K(x, y)$ satisfies

$$\left[\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - q(x)\right] K(x, y) = 0 \quad \text{for } y < x.$$

This differential equation is independent of s, and so are the boundary conditions applied at $y = y_0$ and $y = x$. Therefore, $K(x, y)$ is independent of s. In the plasma inverse problem we assume that $q(x) = 0$ for $x < 0$; this leads to $y_0 = -x$. Finally, $K(x, t)$ satisfies Eqs. (7), (8), and (9).

The condition that $K(x, t)$ and $\partial K/\partial t$ vanish at $t = -x$ has appeared again here; it was mentioned on page. 5. The simple example considered above violates this condition because $q(x)$ has a singularity at $x = 0$; see Eq. (24). The difficulty can be avoided by choosing y_0 slightly less than $-x$.

ANALYTIC PROPERTIES

The Laplace transform $A(s)$ plays an important part in Eq. (21), our linear functional equation. In this section we present examples to show that $A(s)$ can have one or two branch points at finite values of s and then discuss meromorphic functions $A(s)$ in detail. Entire functions $A(s)$ appear not to occur in the plasma inverse problem.

A simple example of a meromorphic function was treated in pages 12 to 14. As another meromorphic example, we take the function

$$A(s) = \frac{-[(b-c)s + (a^2+bc)]\left(\frac{\sinh dr}{r}\right) - (b+c)\cosh dr}{D(s)}, \tag{38}$$

where $r = (s^2+a^2)^{1/2}$, and

$$D(s) = [2s^2 + (b+c)s + (a^2+bc)]\left(\frac{\sinh dr}{r}\right) + (2s+b+c)\cosh dr.$$

The numbers a, b, c, and d are real and positive. Use of Eq. (21) gives

$$B(x,s) = \frac{2s[\cosh r(d-x) + \left(\frac{s+c}{r}\right)\sinh r(d-x)]}{D(s)} - e^{sx} \quad \text{for } 0 \leqslant x < d$$

and

$$B(x, s) = \left[\frac{2se^{ds}}{D(s)} - 1\right] e^{-sx} \quad \text{for } x \geqslant d.$$

The entire function $F(x, s)$ is given by

$$F(x, s) = \cosh rx - (s-b)\left(\frac{\sinh rx}{r}\right) - e^{-sx} \quad \text{for } 0 \leqslant x < d$$

and

$$F(x,\ s) = (\cosh dr)\left\{\frac{(b+c)\sinh\ s(x-d)}{s} + [\exp\ -\ s(x-d)]\right\}$$

$$-\ (s-b)\left(\frac{\sinh\ dr}{r}\right)\left[\frac{c\ \sinh\ s(x-d)}{s} + \cosh\ s(x-d)\right]$$

$$+\ (r\ \sinh\ dr)\ \frac{\sinh\ s(x-d)}{s}\ -\ e^{-sx}$$

for $x \geqslant d$. These functions satisfy the conditions of Eqs. (15) and (19). Their asymptotic forms give

$$\mathcal{E}(x,\ x) = (1/2)(b + a^2x) \quad \text{for } 0 < x < d$$

and

$$\mathcal{E}(x,\ x) = (1/2)(a^2d + b + c) \quad \text{for } x > d.$$

Finally,

$$q(x) = a^2H(x)H(d-x) + b\delta(x) + c\delta(x-d),$$

where $H(x)$ is the Heaviside step function. Note that we can obtain an algebraic function from Eq. (38) by taking the limit as $d \rightarrow +\infty$. The limit depends on the sign of the real part of r; and in the limit branch points appear at $s = \pm\ ia$.

Another reflection coefficient with the same branch points is

$$A(s) = \frac{s(r-s+b)\ -\ (b+c)r\ -\ (a^2+b^2+bc)}{s(r+s+b)\ +\ (b+c)r\ +\ (a^2+b^2+bc)}\ . \tag{39}$$

Here a, b, and c are positive constants. We draw the branch line along the imaginary axis from $-ia$ to ia, so that r is an odd function of s. If x is positive, the solution of Eq. (21) is

$$B(x,\ s) = \frac{2s\left(r + \dfrac{b}{1+bx}\right)e^{-rx}}{s(r+s+b) + (b+c)r + (a^2+b^2+bc)} - e^{-sx}$$

and

$$F(x,s) = \cosh rx + (b+c-s)\frac{\sinh rx}{r} - \frac{b \sinh rx}{r(1+bx)}$$

$$+ b^2(b+c-s)\frac{rx \cosh rx - \sinh rx}{r^3(1+bx)} - e^{-sx} .$$

From these functions we find that

$$\mathcal{E}(x,\ x) = \frac{a^2 x}{2} + \frac{b^2 x}{1+bx} + \frac{c}{2} ,$$

and hence

$$q(x) = \left[a^2 + \frac{2b^2}{(1+bx)^2}\right] H(x) + c\delta(x).$$

This example shows that $A(s)$ can have a pair of branch points on the imaginary axis, whose location gives the limit of $q(x)$ as $x \to +\infty$. In this example, the limit exists and does not vanish. If $q(x)$ decreases slowly as $x \to +\infty$ we may expect that $A(s)$ has branch points at 0 and ∞. For example, suppose that

$$A(s) = \frac{(2a+1)H^{(1)}_{a-1}(ibs) - 4iaH^{(1)}_{a}(ibs) - (2a-1)H^{(1)}_{a+1}(ibs)}{(2a-1)H^{(1)}_{a+1}(ibs) - 4iaH^{(1)}_{a}(ibs) - (2a+1)H^{(1)}_{a-1}(ibs)} . \qquad (40)$$

Here a and b are positive constants and $H^{(1)}$ is the Hankel function of the first kind. Using the circuit relation for this function [12], we can show that $A(se^{2\pi i}) = A(s)$ only when $2a$ is an odd integer. Furthermore, $A(s)$ has a pole on the positive real axis if $a < 1/2$; hence we assume $a \geqslant 1/2$. If x is positive, the solution of Eq. (21) is

$$B(x,\ s) = \frac{-8ia\left(\frac{x+b}{b}\right)^{1/2} H_a^{(1)}(\text{is}(x+b))}{(2a-1)H_{a+1}^{(1)}(ibs) - 4iaH_a^{(1)}(ibs) - (2a+1)H_{a-1}^{(1)}(ibs)} - e^{-sx}$$

and

$$F(x,s) = \frac{\pi b^{1/2}s}{16a}(x+b)^{1/2}[(2a+1)H_{a-1}^{(1)}(ibs)-4iaH_a^{(1)}(ibs)-(2a-1)H_{a+1}^{(1)}(ibs)]H_a^{(2)}[\text{is}(x+b)]$$

$$- \frac{\pi b^{1/2}s}{16a}(x+b)^{1/2}[(2a+1)H_{a-1}^{(2)}(ibs)-4iaH_a^{(2)}(ibs)-(2a-1)H_{a+1}^{(2)}(ibs)]H_a^{(1)}(\text{is}\ (x+b)) - e^{-sx}.$$

$F(x,\ s)$ is an entire function of s. To show this, we begin by noting that $H_a^{(1)}(ibs)$ $H_a^{(2)}(\text{is}(x+b))$ - $H_a^{(2)}(ibs)H_a^{(1)}(\text{is}(x+b))$ is an entire function, and then we use the recursion relations [12] to make entire functions out of Hankel functions of orders a-1, a, and a+1. From the asymptotic form of $B(x,\ s)$ or $F(x,\ s)$, we obtain

$$\mathcal{E}(x,\ x) = \frac{(a^2 - 1/4)x}{2b(x+b)} \quad .$$

Finally,

$$q(x) = (a^2 - 1/4)/(x + b)^2.$$

The reflection coefficient $A(s)$ can profitably be considered together with the transmission coefficient $T(s)$. If $xq(x) \to 0$ as $x \to +\infty$, we can demand that

$$\hat{E}\,(x,\ s) \sim T(s)e^{-sx} \quad \text{as } x \to +\infty. \tag{41}$$

This condition and Eq. (14) serve as boundary conditions for the differential equation (16). The derivative satisfies $d\hat{E}/dx \sim -sT(s)e^{-sx}$ as $x \to +\infty$, whereas differentiation of Eq. (14) gives $d\hat{E}/dx$ at negative values of x. If $s \neq 0$, we notice that $\hat{E}(x, -s)$ is another solution of Eq. (16), with different limiting forms. The Wronskian of these two solutions gives

$$1 - A(s)A(-s) = T(s)T(-s) \quad \text{for } s \neq 0. \tag{42}$$

If s^2 is real and negative, this relation can be derived from the conservation of energy for electromagnetic waves of frequency is.

If $s = 0$, our boundary conditions give $d\hat{E}/dx = 0$ for $x < 0$ and $d\hat{E}/dx \to 0$ as $x \to +\infty$. Because of (4), $q(x)$ is real and nonnegative. These three conditions require that $\hat{E}(x, 0) = 0$ for all x, unless $q(x)$ vanishes identically. Therefore, we have

$$1 + A(0) = T(0) = 0, \tag{43}$$

and Eq. (42) is valid at $s = 0$. If $q(x)$ does vanish identically, $A(s)$ and $T(s) - 1$ also vanish identically, and Eq. (42) is satisfied, but not Eq. (43).

The present assumption that $xq(x) \to 0$ as $x \to +\infty$ is probably not strong enough to establish any analytic properties of $A(s)$. If there is some positive number N such that

$$q(x)e^{Nx} \to 0 \quad \text{as } x \to +\infty, \tag{44}$$

then it is perhaps possible to show that $A(s)$ is analytic except for isolated points by using Regge's method [13]. We shall not try to prove that Regge's method succeeds. Regge does show that branch points can occur, even if we assume (44). To show that $A(s)$ and $T(s)$ are meromorphic functions, we must assume that (44) holds for all positive numbers N. To prove that they are meromorphic, we introduce $f(x, s)$, a solution of (16) which satisfies

$$f(x, s) \sim e^{-sx} \quad \text{as } x \to +\infty. \tag{45}$$

This condition was suggested by Jost [14], and it is somewhat simpler than (41). We can write Eq. (16) in the form of an integral equation, using Eq. (45):

$$f(x, s) = e^{-sx} - \frac{1}{s}\int_x^\infty [\sinh s(x-y)]\, q(y) f(y, s)\, dy.$$

This equation can be solved by iterating the kernel. We set

$$f(x, s) = \sum_{n=0}^{\infty} f_n(x, s) \tag{46}$$

where

$$f_0(x,\ s) = e^{-sx}$$

and

$$f_n(x,\ s) = -\frac{1}{s}\int_x^\infty [\sinh s(x-y)]\,q(y)f_{n-1}(y,\ s)dy \quad \text{for } n \geqslant 1.$$

The series (46) converges for all values of s because

$$\int_x^\infty |\sinh s(x-y)|\,q(y)dy$$

always converges. Therefore $f(x,\ s)$ is an entire function of s. We differentiate (46) term by term to show that df/dx is an entire function of s. Since (45) differs from (41) only by a factor of $T(s)$, we have

$$f(x,\ s) = \frac{1}{T(s)}\ e^{-sx} + \frac{A(s)}{T(s)}\ e^{sx}$$

when x is negative; this implies that

$$sf(x,\ s) + \frac{df}{dx} = 2s\,\frac{A(s)}{T(s)}\ e^{sx}$$

and

$$sf(x,\ s) - \frac{df}{dx} = \frac{2s}{T(s)}\ e^{-sx}\ .$$

Hence, $s/T(s)$ and $sA(s)T(s)$ are entire functions of s. We conclude that $A(s)$ and $T(s)$ are meromorphic functions. Incidentally, (43) says that $s = 0$ is a zero of $T(s)$, and we now see that it must be a simple zero.

We introduced $A(s)$ and $B(x, s) = \hat{E}(x,s) - e^{-sx}$ as Laplace transforms of bounded functions. If σ, the real part of s, is positive, then $A(s)$ and $B(x, s)$ are analytic functions of s. The bound in Eq. (15) shows how $B(x, s)e^{sx}$ vanishes as $s \to \infty$ and $\sigma \to +\infty$; the same argument can be applied to $A(s)$, which is of order $1/\sigma$. The case in which σ is bounded while $s \to \infty$ must now be considered. We can claim that $\int_0^\infty |R(y)|dy$ converges and that $|R'(y)|$ is integrable over any finite interval. Let $s = \sigma + i\tau$; we assume that σ is bounded and nonnegative. We can now show that, for any positive ϵ,

$$\left| \int_0^\infty R(y)e^{-sy}\,dy \right| < \epsilon$$

when $|\tau|$ is sufficiently large. To prove this, we choose a positive number M such that

$$\int_M^\infty |R(y)|dy < (1/2)\epsilon.$$

We have

$$\left| \int_0^\infty R(y)e^{-sy}\,dy \right| \leqslant \left| \int_0^M R(y)e^{-\sigma y}e^{-i\tau y}\,dy \right| + \int_M^\infty |R(y)|e^{-\sigma y}\,dy,$$

and we use partial integration to show that the right-hand side is

$$< \frac{|R(M)|}{|\tau|} + \frac{|R(0)|}{|\tau|} + \frac{\sigma}{|\tau|}\int_0^M |R(y)|dy + \frac{1}{|\tau|}\int_0^M |R'(y)|dy + (1/2)\epsilon.$$

We now choose $|\tau|$ so large that this sum is $< \epsilon$. We conclude that $A(s) \to 0$ uniformly as $s \to \infty$, provided that $\sigma \geqslant 0$. A similar argument applies to $B(x, s)e^{sx}$, for every positive x. Using (41), we see that $T(s) \to 1$ uniformly as $s \to \infty$, provided that $\sigma \geqslant 0$.

We now return to the entire function $s/T(s)$ and estimate its behavior as $s \to \infty$, without restrictions on σ. If $f(s)$ is an entire function, and $M(r)$ is the maximum value of $|f(s)|$ on the circle $|s| = r$, then the order of f is defined as [15]

$$\limsup_{r \to \infty} \frac{\ln \ln M(r)}{\ln r} .$$

We can now show that the order of $s/T(s)$ is $\geqslant 1$, unless

$$T(s) = \frac{s}{s+\lambda} \tag{47}$$

where λ is a constant. To prove this, we assume that $s/T(s)$ has order < 1 and show that we are led to Eq. (47). The assumption of order < 1 means that we have

$$\left|\frac{s}{T(s)} \frac{1}{s-1}\right| < a \exp(br^c) \tag{48}$$

when $\sigma < 0$ and $r = |s|$ is sufficiently large. The constants a, b, and c are positive, and $c < 1$. By the Phragmén-Lindelöf theorem [15], the left-hand side of (48) is bounded by a constant when $\sigma < 0$. Then we have

$$\left|\frac{s}{T(s)}\right| < (\text{const.})\, |s-1|$$

when $\sigma < 0$. An inequality of this form also holds when $\sigma \geqslant 0$ and r is sufficiently large because $T(s) \to 1$ when $s \to \infty$ in the right half-plane. We now use Liouville's theorem on entire functions and obtain $s/T(s) = \kappa s+\lambda$, where κ and λ are two constants. Since $T(s) \to 1$ in the right half-plane, we must have $\kappa = 1$. Therefore $T(s)$ is given by Eq. (47). This completes the proof, but we could also prove that Eq. (47) leads to

$$A(s) = \frac{-\lambda}{s+\lambda} e^{-2\alpha s} ,$$

which differs only by a translation (along the x axis) from the simple case already studied.

To obtain more detailed results about the meromorphic functions $A(s)$ and $T(s)$, we need to assume that $q(x)$ decreases rapidly as $x \to +\infty$. In the remainder of this section, we assume that there are positive numbers δ and ϵ such that

$$q(x) < (\text{const.}) \exp(-\delta x^{1+\epsilon}) \tag{49}$$

when x is positive. This hypothesis is somewhat stronger than (44), and it leads to several interesting results on the functions $A(s)$ and $T(s)$. The bound (49) need not exclude δ functions in $q(x)$; a finite number of them can easily be accommodated by slight modifications in the following argument, which is taken from Sartori [16]. We wish to estimate the sum (46). Since $q(x)$ satisfies (49), the integral $I_q = \int_0^\infty q(x)dx$ must exist. If $n > 0$ and $x \geqslant 0$, we have the following bound for each term in the series (46):

$$|f_n(x,s)| < \frac{I_q^{n-1}}{|s|^n} \exp(-|s|x) \int_0^\infty \exp(2|s|y)q(y)dy. \tag{50}$$

This can be proved by induction. We have

$$|\sinh s(y-x)| < \exp|s|(y-x)$$

when $y > x$; we use this and $|f_0(x, s)| \leqslant \exp(|s|x)$ to show that (50) is valid for $n = 1$. To complete the proof by induction, we use the same bound for the hyperbolic sine again. A similar bound for $|\cosh s(y-x)|$ is available; it can be combined with (50) to show that

$$\left|\frac{d}{dx} f_n(x, s)\right| < \frac{I_q^{n-1}}{|s|^{n-1}} \exp(-|s|x) \int_0^\infty \exp(2|s|y)q(y)dy \tag{51}$$

when $n > 0$ and $x \geqslant 0$.

We know that $f(x, s)$ and $df(x,s)/dx$ are entire functions of s. From the bounds (50) and (51), we have

$$|f(x,s)| < \exp(|s|x) + \frac{\exp(-|s|x)}{|s|-I_q} \int_0^\infty \exp(2|s|y)q(y)dy$$

and

$$\left|\frac{d}{dx} f(x, s)\right| < |s|\exp(|s|x) + \frac{|s|\exp(-|s|x)}{|s| - I_q} \int_0^\infty \exp(2|s|y)q(y)dy$$

when $|s| > I_q$. To estimate the integral which occurs here, we consider the entire function

$$\int_0^\infty \exp(xz-\delta x^{1+\epsilon})dx = \frac{1}{1+\epsilon} \sum_{n=1}^{\infty} \Gamma\left(\frac{n}{1+\epsilon}\right) \left(\frac{1}{\delta}\right)^{\frac{n}{1+\epsilon}} \frac{z^{n-1}}{\Gamma(n)} . \tag{52}$$

Application of a standard test to the coefficients of the power series [15] shows that this integral is a function of order $(1+\epsilon)/\epsilon$ and the type is finite. Hence, when $|s|$ is sufficiently large, the logarithm of $\int_0^\infty \exp(2|s|y)q(y)dy$ is bounded by a constant times $|s|^{(1+\epsilon)/\epsilon}$. This means that $f(x, s)$ and $df(x,s)/dx$ are entire functions of order $(1+\epsilon)/\epsilon$, at most. This holds at $x = 0$, and therefore $s/T(s)$ and $sA(s)/T(s)$ are entire functions of order $(1+\epsilon)/\epsilon$, at most.

In order to show that $sA(s)/T(s)$ can actually be an entire function of any order > 1, we need a calculation for a specific function $q(x)$. Suppose that $q(x) = \exp(-x^{1+\epsilon})$ when $x > 0$. Then

$$f_1(0, s) = \frac{I_q}{2s} - \frac{1}{2s} \int_0^\infty \exp(-2ys - y^{1+\epsilon})dy$$

and

$$\frac{d}{dx} f_1(0,s) = -\frac{I_q}{2} - (1/2) \int_0^\infty \exp(-2ys - y^{1+\epsilon})dy.$$

The integral (52) can now be used to show that

$$sf_1(0, s) + \frac{d}{dx} f_1(0, s) \tag{53}$$

is an entire function of order $(1+\epsilon)/\epsilon$. The difference $sf_1(0, s) - df_1(0,s)/dx$ and the functions $sf_0(0, s)$ and $df_0(0,s)/dx$ are of relatively slow growth as $s \to \infty$. The estimates (50) and (51) show that $\sum_{n=2}^{\infty} sf_n(x, s)$ and $\sum_{n=2}^{\infty} df_0(0,s)/dx$ are smaller than (53) by a factor of magnitude $|s|^{-1}$ as $s \to \infty$. Therefore, $sA(s)/T(s)$ is an entire function of order $(1+\epsilon)/\epsilon$, and it grows faster than $s/T(s)$ as $s \to \infty$.

The functions $A(s)$ and $T(s)$ themselves are meromorphic rather than entire, and Nevanlinna [17] has generalized the concept of order (defined above for entire functions) to handle meromorphic functions. The entire function $f(0, s)$ has infinitely many zeros, and their distribution was briefly discussed by Sartori. The functions $A(s)$ and $T(s)$ have, in general, infinitely many poles, and we shall obtain results on their distribution by use of the modern theory of meromorphic functions.

The entire function $s/T(s)$ has finite order within the Nevanlinna scheme; indeed, it has the same order as was calculated above with the simple definition applicable to entire functions. Let $\rho(f)$ denote the Nevanlinna order of a function $f(s)$. Then the elementary properties of the Nevanlinna order [4] give

$$\rho(s/T) = \rho(1/T) = \rho(T). \tag{54}$$

We have seen that

$$\rho(T) \geqslant 1 \tag{55}$$

except in the case (47). We shall show that

$$\rho(A) = \rho(T), \tag{56}$$

except in the case (47). We remark that equality does not always hold in (55) because $\rho(sA/T) = \rho(A/T) > 1$ in the specific example quoted above, and $\rho(A/T) > 1$ implies that $\rho(A) > 1$ or $\rho(T) > 1$.

Any meromorphic function of finite order can be written as the quotient of two Weierstrass products times an exponential function [4]. Thus,

$$A(s) = \frac{N(s)}{D(s)} \exp\left[P(s)\right], \tag{57}$$

where $N(s)$ and $D(s)$ are entire functions of the Weierstrass product form and $P(s)$ is a polynomial. For $T(s)$, we can write a similar formula. Since $T(s)$ has only one zero, which is known to be simple, we have

$$T(s) = \frac{s}{d(s)} \exp\left[p(s)\right],$$

where $d(s)$ is a Weierstrass product and $p(s)$ is a polynomial. Since $A(s)$ and $T(s)$ are analytic in the right half oif the s plane and are bounded on the imaginary axis, all the zeros of $d(s)$ and $D(s)$ are in the left half-plane. Then Eq. (42) implies that $d(s) = 0$ iff $D(s) = 0$. Since $d(s)$ and $D(s)$ are Weierstrass products, they are the same. Therefore

$$T(s) = \frac{s}{D(s)} \exp\left[p(s)\right] . \tag{58}$$

The denominator $D(s)$ is an entire function having order $\rho(D)$ and zeros $s_1, s_2, s_3, \ldots$. Except in the case (47), $\rho(D) \geqslant 1$ and the sum

$$\sum_{n=1}^{\infty} \frac{1}{|s_n|^{\alpha}} \tag{59}$$

converges or diverges depending on the value of α, which is real. One can show that $\rho(D)$ is the greatest lower bound of those α's for which (59) converges [18]. This result indicates that $\rho(D)$ is an index of the density of poles of $A(s)$ and $T(s)$ as $s \to \infty$. For this reason, we want to show that

$$\rho(D) = \rho(T). \tag{60}$$

This equation and Eqs. (54), (55), and (56) are our new results for meromorphic reflection coefficients.

We begin the proof by showing that the polynomial $p(s)$ which appears in Eq. (58) has degree $\leqslant \rho(D)$. In the simple case (47), $p(s)$ is a constant and this inequality becomes $0 \leqslant 0$, which is valid. Let us assume that the degree of $p(s)$ is $> \rho(D)$. Then the behavior of $T(s)$ as $s \to \infty$ in the right half-plane is the same as that of exp $p(s)$; but $T(s) \to 1$ in this limit. Hence, $p(s)$ is a constant and its degree is zero. Then our assumption gives $\rho(D) < 0$, which is impossible.

We now know that $p(s)$ has degree $\leqslant \rho(D)$. Use of Eq. (58) and the elementary properties of the Nevanlinna order gives us $\rho(T) \leqslant \rho(D)$. Furthermore, the connection between the poles of $T(s)$ and the convergence of (59) gives $\rho(T) \geqslant \rho(D)$. These two inequalities imply Eq. (60).

The remaining question is the order of $A(s)$; we want to prove Eq. (56). From Eq. (58) we have

$$T(s)T(-s) = \frac{-s^2}{D(s)D(-s)} \exp[p(s) + p(-s)]. \tag{61}$$

Using the fact that $D(s)$ has all its zeros in the left half-plane, we can easily see that $D(s)D(-s)$ has the same order as $D(s)$. Also, an argument similar to that used above shows that $T(s)T(-s)$ has the same order as $D(s)$. Then Eq. (42) shows that $A(s)A(-s)$ has the same order as $D(s)$. Now $A(s)A(-s)$ could have order lower than that of $A(s)$, but not higher. Therefore we have

$$\rho(A) \geqslant \rho(D) = \rho(T). \tag{62}$$

The next step is to prove that

$$\mathrm{Max}\ [\rho(N),\ \rho(D)] \geqslant \rho(A), \tag{63}$$

except in the simple case (47). To prove this, assume that $\rho(N) < \rho(A)$ and $\rho(D) < \rho(A)$. Then the behavior of $A(s)$ as $s \to \infty$ in the right half-plane is the same as that of exp $[P(s)]$, and $A(s) \to 0$ requires that the degree of $P(s)$ is $\leqslant 1$. Then our assumption gives $\rho(D) < 1$, and Eq. (60) gives $\rho(T) < 1$. This is possible only in the case (47), which proves (63).

We can now complete the proof of Eq. (56). We have seen that $A(s)A(-s)$ has the same order as $D(s)$; and (57) gives

$$A(s)A(-s) = \frac{N(s)N(-s)}{D(s)D(-s)}\ \exp[P(s) + P(-s)]. \tag{64}$$

Assume that Eq. (56) is false, and exclude the simple case (47). Then (62) and (63) give

$$\rho(N) \geqslant \rho(A) > \rho(D) = \rho(T) \geqslant 1. \tag{65}$$

Now the zeros of Eq. (64) are the same as those of $N(s)N(-s)$. Since $N(s)$ is a Weierstrass product, $N(s)N(-s)$ has the same order as $N(s)$; it has order > 1. Therefore, $N(s)N(-s)$ has infinitely many zeros, and a sum similar to (59) can be formed from them; it converges only when $\alpha \geqslant \rho(N)$. However, the Nevanlinna theory gives a bound for the density of the zeros, or an upper bound for the α's at which this series diverges [4]. Since Eq. (64) has the same order as that of $D(s)$, the series must converge if $\alpha > \rho(D)$. Hence $\rho(D) \geqslant \rho(N)$. This contradicts Eq. (65), and the contradiction shows that Eq. (56) is true. This concludes our general discussion of meromorphic functions, but rational functions $A(s)$ will appear in the next section.

METHODS OF SOLUTION

A general method for finding solutions of Eq. (21) has not been found. In this section, we consider first algebraic functions $A(s)$ having branch points only at $s = \pm ia$, and indicate a possible method of solution. Then we consider rational functions $A(s)$; Kay's solution of this problem [5] is rewritten in two different ways. Among rational functions, the Butterworth functions appear as a limiting case.

For algebraic functions such as Eq. (39), we determine $F(x, s)$ indirectly, by introducing two other entire functions:

$$F_e(x, s) = (1/2)[F(x, s) + F(x, -s)] + \cosh sx$$

is an even entire function, and

$$F_0(x, s) = (1/2)[F(x, s) - F(x, -s)] - \sinh sx$$

is an odd entire function. With this notation, Eq. (21) becomes

$$F_e(x, s) + \left[\frac{1-A(s)}{1+A(s)}\right] F_0(x, s) = \frac{B(x,s)+e^{-sx}}{1+A(s)} . \tag{66}$$

We consider the case (39) and multiply Eq. (66) by $r + b$. Then

$$(r+b)F_e(x,s) + [s^2 + (b+c)r + a^2 + b^2 + bc] \frac{F_0(x,s)}{s} = G(x,s),$$

where $G(x,s)$ is a function which decreases exponentially as $\sigma \to \infty$. The bound (19) implies that $G(x, s)$ can increase no faster than a polynomial times $\exp(-xs)$ as $\sigma \to \infty$. We define

$$\psi(s) = bF_e(x,s) + (s^2 + a^2 + b^2 + bc) \frac{F_0(x,s)}{s}$$

and

$$\phi(s) = F_e(x, s) + (b + c) \frac{F_0(x,s)}{s} .$$

Then $\psi(s) + r\phi(s) = G(x, s)$. Since $\psi(s)$ and $\phi(s)$ are even entire functions, $\psi(s) - r\phi(s) = G(x, -s)$. The product of these two equations is

$$\psi^2 - (s^2 + a^2)\,\phi^2 = G(x,s)G(x,-s). \tag{67}$$

The right-hand side is an even entire function, and it cannot increase exponentially as $s \to \infty$; hence it is a polynomial. The recent work of Gross, Osgood, and Yang on such quadratic functional equations [19] suggests that $G(x,s)G(s, -s)$ should be a linear function of s^2, in order to make the solution of Eq. (67) unique, up to a few constants. Hence, we assume that

$$\psi^2 - (s^2 + a^2)\phi^2 = A^2(s^2 + a^2) + B^2, \tag{68}$$

where A and B are constants to be determined; they may depend on x. To solve Eq. (68), we write it as

$$(B\psi + iAr^2\phi)^2 + r^2(A\psi - iB\phi)^2 = (A^2r^2 + B^2)^2.$$

Here $B\psi + iAr^2\phi$ and $A\psi - iB\phi$ are even entire functions, which must be of order 1. The solution is

$$B\psi + iAr^2\phi = \pm\,(A^2r^2 + B^2)\cos ixr \tag{69}$$

and

$$A\psi - iB\phi = \pm\,(A^2r^2 + B^2)\,\frac{\sin ixr}{r}\,. \tag{70}$$

The coefficient ix has been chosen so that (19) may be satisfied. The same condition (19) requires the ambiguous sign in Eq. (70) to be the same as that in Eq. (69). Since the signs of A and B are yet undetermined, the ambiguous signs in Eq. (69) and (70) can now be dropped. These equations yield

$$\psi(s) = iAr\sinh rx + B\cosh rx$$

and

$$\phi(s) = -iA \cosh rx - B \frac{\sinh rx}{r} .$$

Now

$$\frac{F_0(x,s)}{s} = \frac{\psi(s)-b\phi(s)}{r^2}$$

must be an even entire function; this requires that $B = -iAb/(1+bx)$. To satisfy condition (19), we choose $A = i$. Then

$$F_e(x, s) = \cosh rx + (b+c) \frac{\sinh rx}{r} - \frac{b \sinh rx}{r(1+bx)}$$

$$+ b^2(b+c) \frac{rx \cosh rx - \sinh rx}{r^3(1+bx)}$$

and

$$F_0(x, s) = - \frac{s \sinh rx}{r} + b^2 s \frac{\sinh rx - rx \cosh rx}{r^3(1+bx)} .$$

These functions yield the $F(x, s)$ given in page 24. The method can perhaps be generalized to treat all other algebraic functions having the same branch points.

We proceed to consider rational functions $A(s)$. If $A(s)$ is rational and $q(x)$ satisfies condition (44) for all positive values of N, then $T(s)$ is given by Eq. (47) and $A(s)$ must have the simple form considered in the example. In general, $A(s)$ does not have this form, and hence $q(x)$ cannot satisfy (44) for all N. This shows that $q(x)$ decreases no faster than exponentially as $x \to +\infty$.

Our method, that of solving Eq. (21) and obtaining $q(x)$ from the asymptotic forms of $F(x, s)$ and $B(x, s)$, implies that we need only the asymptotic forms. We may expect that

$$B(x,\ s) \sim \left[\frac{b_1(x)}{s} + \frac{b_2(x)}{s^2} + \ldots\right] e^{-sx} \tag{71}$$

as $\sigma \to +\infty$. Since $A(s)$ is a rational function, it must be analytic except at a finite number of poles; and Eq. (21) requires $B(x, s)$ to have the same singularities as $A(s)$. Using (15) and (19), we can show that $B(x, s)e^{sx}$ is a rational function of s. Then Eq. (21) leads to

$$F(x,\ s) = \frac{\sum_j g_j(x)s^j e^{sx} + \sum_j h_j(x)s^j e^{-sx}}{\sum_j d_j(x)s^j}\ , \tag{72}$$

where the degrees and coefficients of the polynomials are to be determined. We can obtain a relation among the degrees by substituting Eqs. (71) and (72) into Eq. (21) and separating the terms in e^{-sx}. The terms in e^{sx} give a number of equations in $d_j(x)$, $g_j(x)$, and $h_j(x)$. Further equations in $d_j(x)$, $g_j(x)$, and $h_j(x)$ come from the requirement that $F(x, s)$ is an entire function of s. The solution for any rational function $A(s)$ can be obtained by this method. A more explicit solution can be obtained only if some restrictions are imposed on the rational function $A(s)$.

A classification of rational functions $A(s)$ is furnished by $T(s)$, the transmission coefficient. If $q(x)$ satisfies (44) for all values of N, then $T(s)$ can have only one zero. But in fact $q(x)$ does not satisfy (44) for all values of N. Moreover, Eq. (42) suggests that $T(s)$ must be a rational function. Then $T(s)T(-s)$ is an even function of s with finitely many zeros; when $A(s)$ is given, they can be found from Eq. (42). The multiplicity of these zeros is different in the different rational cases. Because of Eq. (43), $s = 0$ must be a double zero of $T(s)T(-s)$. Let us assume that all other zeros are simple; this simplifying assumption is copied from the work of Kay [5]. Also, we use his assumption that $A(s)$ has only simple poles; thus, we write

$$A(s) = \sum_{j=1}^{n} \frac{\alpha_j}{s+\lambda_j}\ ,$$

where $\alpha_j \neq 0$ and the real part of λ_j is positive. Since $B(x, s)$ has the same singularities as $A(s)$, we write

$$B(x,\ s) = \sum_{j=1}^{n} \frac{\beta_j(x)}{s+\lambda_j}\, e^{-sx} \ , \tag{73}$$

where the functions $\beta_j(x)$ are to be determined. Now, $F(x,\ s)$ is a linear combination of e^{sx} and e^{-sx}, with rational coefficients. Both coefficients must have poles. If $F(x,\ s)$ contains a term such as

$$\frac{\sinh\ x(s+\lambda_j)}{s+\lambda_j} \ ,$$

then Eq. (21) implies that $B(x,\ s)$ has a pole at $s = \lambda_j$; this is impossible. If we change the sign of s in Eq. (21) and try to solve the two resulting equations for $F(x,\ s)$ and $F(x,\ -s)$, we must divide by the determinant of this system of equations, which is (42). The two rational coefficients in $F(x,\ s)$ are therefore expected to have poles at the zeros of (42). Suppose the zeros of (42) are $\kappa_1 = -\kappa_1 = 0$, κ_2, $-\kappa_2$, κ_3, $-\kappa_3$, . . . κ_n, and $-\kappa_n$. Then the κ's and λ's must be disjoint sets of complex numbers. We can expect to write $F(x,\ s)$ as a linear combination of the functions

$$\frac{\sinh\ x(s \pm \kappa_j)}{s \pm \kappa_j} \ ,$$

with coefficients depending only on x. In fact, we want to write

$$F(x,\ s) = 2 \sum_{j=1}^{n} f_j(x) \left[A(\kappa_j)\ \frac{\sinh\ x(s-\kappa_j)}{(s-\kappa_j)} - \frac{\sinh\ x(s+\kappa_j)}{(s+\kappa_j)} \right], \tag{74}$$

where the functions $f_j(x)$ are to be determined. Substitute this into Eq. (21) and separate the terms in e^{sx}; they are

$$A(s) + A(s) \sum_{j=1}^{n} f_j(x) \left[A(\kappa_j)\ \frac{\exp(\kappa_j x)}{(s+\kappa_j)} - \frac{\exp(-\kappa_j x)}{(s-k_j)} \right]$$

$$+ \sum_{j=1}^{n} f_j(x) \left[A(\kappa_j) \frac{\exp(-\kappa_j x)}{(s-\kappa_j)} - \frac{\exp(\kappa_j x)}{(s+\kappa_j)} \right] . \tag{75}$$

This is a rational function of s, and it vanishes as $s \to \infty$. It must vanish identically; to achieve this, we set the residue equal to zero at each possible pole. Equation (74) was contrived so that (75) has zero residue at $s = \kappa_j$ and at $s = -\kappa_j$. We must have zero residue at $s = -\lambda_j$; this gives

$$\sum_{k=1}^{n} \left[A(\kappa_k) \frac{\exp(\kappa_k x)}{\lambda_j - \kappa_k} - \frac{\exp(-\kappa_k x)}{\lambda_j + \kappa_k} \right] f_k(x) = 1 \tag{76}$$

for $j = 1, 2, \ldots, n$. We now have n linear equations to determine the n unknown functions $f_j(x)$. We could determine the functions $\beta_j(x)$ by examining the terms in e^{-sx} which appear in Eq. (21); but in fact Eq. (73) is no longer needed. To determine $q(x)$ we need only $F(x, s)$; this requires that we solve Eq. (76). Let us rewrite Eq. (76) as

$$\sum_{k=1}^{n} M_{jk}(x) f_k(x) = \exp(-\lambda_j x),$$

where $M(x)$ is an $n \times n$ matrix with elements

$$M_{jk}(x) = A(\kappa_k) \frac{\exp(-\lambda_j x + \kappa_k x)}{\lambda_j - \kappa_k} - \frac{\exp(-\lambda_j x - \kappa_k x)}{\lambda_j + \kappa_k} .$$

To complete this calculation, we assume that the inverse matrix exists. Then

$$f_j(x) = \sum_{k=1}^{n} [M(x)^{-1}]_{jk} \exp(-\lambda_k x)$$

and

$$\mathcal{E}(x,\ x) = -\lim_{s\to-\infty} se^{sx}\ F(x,\ s)$$

$$= \sum_{j=1}^{n} [A(\kappa_j)\exp(\kappa_j x) - \exp(-\kappa_j x)] f_j(x)$$

$$= -\sum_{j=1}^{n} \sum_{k=1}^{n} \left[\frac{d}{dx}\ M_{kj}(x)\right] [M(x)^{-1}]_{jk}$$

$$= -\ \text{Trace}\ M(x)^{-1} \frac{d}{dx}\ M(x).$$

Using a convenient identity, we write this as

$$\mathcal{E}(x,\ x) = -\ \frac{d}{dx}\ \ell n\ \frac{\det M(x)}{\det M(0)}\ .$$

The denominator is inserted to ensure that $\mathcal{E}(x,\ x)$ is continuous at $x = 0$; $q(x)$ and $\mathcal{E}(x,\ x)$ must vanish for $x < 0$. Finally, we have

$$q(x) = -\ 2\ \frac{d^2}{dx^2}\ \ell n\ \frac{\det M(x)}{\det M(0)}\ ,$$

which agrees with Kay's solution of the problem [5]. Each element of $M(x)$ increases or decreases exponentially as $x \to +\infty$; and, in general, $q(x)$ will decrease exponentially as $x \to +\infty$.

This solution of the rational inverse problem certainly does not apply when all the κ's vanish, although this could perhaps be treated as a limiting case. We now assume that all the κ's vanish and also make the simplifying assumption that $A(s)$ has no zeros. These two assumptions imply that $A(s)$ is one of the Butterworth functions [20]. The first assumption gives

$$A(s)A(-s) = \frac{D(s)D(-s) - (is)^{2n}}{D(s)D(-s)} ,$$

where $D(s)$ is a polynomial; and then the second assumption gives $D(s)D(-s) =$ (const.) $+ (is)^{2n}$. By changing the scale of s, we can arrange to have $D(s)D(-s) = 1 + (is)^{2n}$ and $A(s)A(-s) = [1 + (is)^{2n}]^{-1}$. Since $A(0) = -1$, we finally have

$$A(s) = -\left[\sum_{j=0}^{n} a_j s^j\right]^{-1} .$$

The constants $a_0, a_1, \ldots, a_n$ are real and positive because $A(s)$ has all its poles in the left half-plane. With this notation, $a_0 = a_n = 1$. Since there are $2n$ terms in (74), we expect that

$$F(x, s) = \frac{1}{s^{2n}}\left[\sum_j g_j(x)s^j e^{sx} + \sum_j h_j(x)s^j e^{-sx}\right] \tag{77}$$

in the limit as all κ's vanish. The condition (19) implies that $j \leqslant 2n - 1$ in both of these sums. The last of the h's gives the function we want:

$$h_{2n-1}(x) = - \mathcal{E}(x, x).$$

Furthermore, $B(x, s)$ is a rational function times e^{-sx}, and its poles must be the poles of $A(s)$. Hence

$$B(x, s) = A(s)b(x, s)e^{-sx} \tag{78}$$

where $b(x, s)$ is a polynomial in s of degree n-1. We now substitute Eqs. (77) and (78) into Eq. (21). The terms in e^{sx} give

$$s^{2n} + \sum_j \dot{h}_j(x)(-s)^j + \frac{1}{A(s)}\left[\sum_j g_j(x)s^j\right] = 0.$$

This requires that $g_n(x) = 1$ and that $g_j(x) = 0$ when $j > n$. Moreover, $F(x, s)\, e^{sx}$ can have no pole at $s = 0$; this means that

$$\left[\sum_{j=0}^{n} g_j(x)\, s^j\right] e^{2sx} + \sum_{j=0}^{2n-1} h_j(x) s^j$$

vanishes at $s = 0$, together with its first $2n-1$ derivatives. Let us define $\xi_k = (2x)^k/k!$, and write these two conditions on the g's and h's as

$$\sum_{k=j}^{n-1} a_{n+j-k} g_k(x) - (-1)^{n+j} h_{n+j}(x) = -a_j$$

and

$$\sum_{k=0}^{n-1} \xi_{n+j-k} g_k(x) + h_{n+j}(x) = -\xi_j,$$

for $j = 0, 1, 2, \ldots, n-1$. We have eliminated s and obtained $2n$ linear inhomogeneous equations. Solution by Cramer's rule gives

$$\mathcal{E}(x, x) = \frac{\begin{vmatrix} 1 & a_{n-1} & \cdots & a_1 & (-1)^{n+1} & 0 & \cdots & a_0 \\ 0 & 1 & \cdots & a_2 & 0 & (-1)^{n+2} & \cdots & a_1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & a_{n-1} \\ \xi_n & \xi_{n-1} & \cdots & \xi_1 & 1 & 0 & \cdots & \xi_0 \\ \xi_{n+1} & \xi_n & \cdots & \xi_2 & 0 & 1 & \cdots & \xi_1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \xi_{2n-1} & \xi_{2n-2} & \cdots & \xi_n & 0 & 0 & \cdots & \xi_{n-1} \end{vmatrix}}{\begin{vmatrix} 1 & a_{n-1} & \cdots & a_1 & (-1)^{n+1} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & a_2 & 0 & (-1)^{n+2} & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & (-1)^{2n} \\ \xi_n & \xi_{n-1} & \cdots & \xi_1 & 1 & 0 & \cdots & 0 \\ \xi_{n+1} & \xi_n & \cdots & \xi_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \xi_{2n-1} & \xi_{2n-1} & \cdots & \xi_n & 0 & 0 & \cdots & 1 \end{vmatrix}}$$

The $2n \times 2n$ determinants which appear here can easily be reduced to $n \times n$ determinants. Finally, we compute

$$q(x) = 2 \frac{d}{dx} \mathcal{E}(x, x),$$

which gives the plasma density.

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