NRL Report 7911

Entire Solutions of the Functional Equation from Percus-Yevick and Gel'fand-Levitan Integral Equations

SAEYOUNG AHN, FRED GROSS, CHARLES OSGOOD, AND CHUNG-CHUN YANG

Mathematical Sciences Staff
Plasma Physics Division

September 10, 1975

NAVAL RESEARCH LABORATORY
Washington, D.C.

SECURITY CLASSIFICATION OF THIS PAGE *(When Data Entered)*

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER NRL Report 7911	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE *(and Subtitle)* ENTIRE SOLUTIONS OF THE FUNCTIONAL EQUATION FROM PERCUS-YEVICK AND GEL'FAND-LEVITAN INTEGRAL EQUATIONS		5. TYPE OF REPORT & PERIOD COVERED Interim report on the NRL problem 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Saeyoung Ahn, Fred Gross, Charles Osgood, and Chung-Chun Yang		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematical Sciences Staff Naval Research Laboratory Washington, D. C. 20375		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NRL Problem B01-11 Project RR 014-02-41-6153
11. CONTROLLING OFFICE NAME AND ADDRESS Department of the Navy Office of Naval Research Arlington, Va 22217		12. REPORT DATE September 10, 1975 13. NUMBER OF PAGES 15
14. MONITORING AGENCY NAME & ADDRESS *(if different from Controlling Office)*		15. SECURITY CLASS. *(of this report)* Unclassified 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT *(of this Report)*

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT *(of the abstract entered in Block 20, if different from Report)*

18. SUPPLEMENTARY NOTES

19. KEY WORDS *(Continue on reverse side if necessary and identify by block number)*

Percus-Yevick integral equation
Gel'fand-Levitan integral equation
Functional equation
Entire solution
Polynomial
Factor
Multiple-valued function
Asymptotic condition

20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

In this report, we present and discuss the most general entire-function solutions to the functional equations of the form $\psi^2(z) + p(z)\,\varphi^2(z) = q(z)$, where z is a complex variable and $p(z)$ and $q(z)$ are given polynomials. These equations are derived from integral equations such as the Percus-Yevick and Gel'fand-Levitan integral equations. The method is based on the theory of functions of one complex variable.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-014-6601 |

SECURITY CLASSIFICATION OF THIS PAGE *(When Data Entered)*

CONTENTS

ENTIRE SOLUTIONS OF THE FUNCTIONAL EQUATION FROM PERCUS-YEVICK AND GEL'FAND-LEVITAN INTEGRAL EQUATIONS

1. INTRODUCTION

Among the many mathematical theories which have developed fully in recent decades is the complex analysis of the functional equation. As F. Dyson pointed out, the progress of both mathematics and physics has in the past been retarded by our unwillingness to listen to one another [1]. Therefore it is rather fortunate that the early development of functional equations met the physical needs arising from the different fields of physics and engineering science. Since physicists and engineers have become involved in quantum mechanics and many-body problems, the real picture has turned out to be complicated and puzzling in many cases. Whereas the system they deal with gets sophisticated and an exact solution becomes difficult to obtain, the necessity is to find physical insights into the problem and some sort of rigorous mathematical solutions, even under a simplification or some approximation of the problem. The functional equation method presents the general solutions, some of which can serve the physical need in a few problems.

We will consider here two integral equations, the Percus-Yevick [2] and Gel'fand-Levitan [3] integral equations, which give rise to functional equations. The former plays an important role in equilibrium statistical mechanics in understanding a certain fluid property, whereas the latter has been used in various fields of science, for example, in the one-dimensional quantum-mechanical potential problem, the plane stratified plasma, the transmission problem of the electromagnetic wave, etc.

Our efforts in this report will be concentrated in rigorous mathematical solutions to the general class of functional equation of the type*

$$\psi^2(z) + g\varphi^2(z) = h, \tag{1}$$

which is extracted from these two fundamental integral equations used in physics and engineering science. The derivation of Eq. (1) depends on physical parameters, and no general rule has yet been found.

First we will sketch the derivation of Eq. (1) from the Percus-Yevick integral equation of hard-sphere mixture [5]

$$\sigma_{ij}(r) = A_{ij} - \sum_{\ell=1}^{n} \int \sigma_{i\ell}(y)\sigma_{\ell j}(r-y)\, dy \text{ (in one dimension)}, \tag{2}$$

*Special solutions of this type were obtained before by Penrose and Lebowitz [4].

Manuscript submitted May 22, 1975.

where

$$\sigma_{ij}(r) = \begin{cases} -(\rho_i\rho_j)^{1/2} C_{ij}(r), r < R_{ij} , \\ (\rho_i\rho_j)^{1/2} g_{ij}(r), r \geqslant R_{ij} , \end{cases} \tag{3}$$

n denotes the number of species, and ρ_i is the number density of each species; $C_{ij}(r)$ and $g_{ij}(r)$ are the direct and pair correlation functions respectively. Furthermore, we consider a special case of the Widom-Rowlinson model in two component versions, that is,

$$R_{ij} = \alpha \text{ and } R_{ii} = 0 \quad \text{ in Eq. (3).} \tag{4}$$

Then by defining

$$F_{ij}(z) \equiv \int_0^{R_{ij}} e^{-zr} \sigma_{ij}(r)\, dr , \tag{5a}$$

$$G_{ij}(z) \equiv \int_{R_{ij}}^{\infty} e^{-zr} \sigma_{ij}(r)\, dr , \tag{5b}$$

and

$$U_{ij}(z) \equiv (1 - \delta_{ij}) \int_0^{R_{ij}} e^{-zr} dr \int_{r-R_{ij}}^{0} \sigma_{ii}(y)\sigma_{ij}(r-y)\, dy , \tag{5c}$$

we get the Laplace transform of Eq. (2):

$$\widetilde{F}(z) + \widetilde{G}(z) = \frac{\widetilde{A}}{z} - \widetilde{G}(z)\{\widetilde{F}(z) + \widetilde{F}(-z)\} - \widetilde{U}(z) + \widetilde{U}(-z) , \tag{6}$$

where the tilde denotes the matrix form related with each component and

$$A_{ij} \equiv (\rho_i\rho_j)^{1/2} \left[1 - \sum_{\ell=1}^{n} \rho_\ell \int C_{\ell j}(r)\, dr\right] \tag{7}$$

is a constant.

Our task here is to find $\widetilde{F}(s)$, $\widetilde{G}(s)$, and $\widetilde{U}(s)$ under the condition

$$\int |g_{ij}(r) - 1| dr < \infty . \tag{8}$$

The diagonal elements of Eq. 6 are obtained easily by making use of the Liouville theorem.

After some heavy algebraic manipulation for off-diagonal elements of Eq. (6), we obtain the function equation

$$\psi^2(z) + g(z)\varphi^2(z) = h(z) , \tag{9}$$

where

$$\varphi(z) \equiv \frac{1}{2}\{F_{12}(z) + F_{12}(-z)\}, \tag{10a}$$

$$\begin{aligned}\psi(z) \equiv A_{12} - zU_{12}(z) + zU_{12}(-z) - \frac{1}{2}(z + A_{11})F_{12}(z) \\ + \frac{1}{2}(z - A_{11})F_{12}(z),\end{aligned} \tag{10b}$$

$$g(z) \equiv 4\mu^2 - z^2 , \tag{10c}$$

and

$$h(z) \equiv \mu^2 \equiv \rho_1 A_{11} - A_{11} = \sigma_{12}^2(\alpha) = \text{constant.} \tag{10d}$$

Here $\psi(z)$ and $\varphi(z)$ are even and entire functions of z such that log $|\varphi(z)|$ and log $|\psi(z)|$ have the asymptotic behavior $\alpha|\mathrm{Re}\, z|$ for large z.

In three dimensions a similar analysis shows that

$$g(z) \equiv 4h(z) - z^6 \tag{11a}$$

and

$$h(z) \equiv \mu^2 - \sigma_{12}^2(\alpha)z^2 , \tag{11b}$$

with $\varphi(z)$ and $\psi(z)$ similar to Eqs. (10a) and (10b).

When we deal with different models, we will find different $h(z)$. For example, in the nonadditive mixture of hard spheres [5], we have (in three dimensions)

$$h(z) \equiv a(z)A(z) = \sum_{i=0}^{4} h_i z^{2i} \tag{12a}$$

and

$$g(z) \equiv 4a(z) - z^6 \tag{12b}$$

and the same functions as in Eq. (10) in the case of one dimension.

Next we derive Eq. (1) from the Gelfan'd-Levitan integral equation [6]:

$$B(x+y) + F(x, y) + \int_{\max[-x,\,-y]}^{x} F(x, v)B(v+y)\,dv = 0 \tag{13}$$

where

$$F(x, y) \equiv \begin{cases} K(x, y) & \text{for } x \geqslant y \\ E(x, y) & \text{for } x < y \end{cases} \tag{14}$$

and $K(x, y)$ and $E(x, y)$ are the precursors of inverse scattering and of the propagating wave through the medium when $y = ct$ (c = velocity of light; t = time). Since $B(x)$ is the Fourier transform of the reflection coefficient $b(k)$, we obtain the following Laplace transform

$$A(z)e^{zx} + A(z)F(-z) + F(z) + G(z) = 0 \tag{15}$$

by defining

$$A(z) \equiv b(-ik)\,, \tag{16a}$$

$$F(z) \equiv \int_{-x}^{x} F(x, y)e^{-zy}\,dy\,, \tag{16b}$$

and

$$G(z) \equiv \int_{x}^{\infty} F(x, y)e^{-zy}\,dy\,. \tag{16c}$$

The task in Eq. (15) is to find $F(z)$ and $G(z)$ when $A(z)$ is known. For example, we know $A(z)$ to be

$$A(z) = -\frac{1}{k_0^2}\left(\sqrt{z^2 + k_0^2} - z\right)^2 \tag{17}$$

in the case of the step potential. In this case, we recover the functional equation of the same type with Eq. (10), namely,

$$\psi^2(z) - (z^2 + k_0^2)\varphi^2(z) = 1\,, \tag{18}$$

where

$$\psi(z) \equiv \frac{1}{2}\,\{F(z) + F(-z) + e^{-zx} + e^{zx}\} \tag{19a}$$

and

$$\varphi(z) \equiv \frac{1}{2z}\{F(z) - F(-z) + e^{-zx} - e^{zx}\}. \tag{19b}$$

Given $A(z)$, we find in general a functional equation of type (1).

2. STATEMENT OF THE RESULTS

We investigate in the following sections solutions to the functional equation

$$\psi^2 + g\varphi^2 = h, \tag{20}$$

where g and h are nonzero polynomials and ψ and φ are to be entire functions. We content ourselves in this report with outlining proofs to Theorems 1 through 4 that follow to keep the many details from obscuring the ideas. Full proofs will appear later in a paper.

Theorem 1 is an existence theorem. Theorem 2 shows the existence of a convenient canonical form for those equations of type (20) actually having entire solutions. Theorems 3 and 4 describe the general solution. In the last section, after our condensed proofs, we consider applications.

Theorem 1. There always exist entire solutions ψ and φ of Eq. (20), *unless for some complex number* a *and positive integer α,* $z - a$ *divides* h *to the exact power* $2\alpha - 1$ *while* $(z - a)^{2\alpha}$ *divides* g.

The following definition is needed for the statement of Theorem 2. *Definition.* A functional equation

$$\psi^2 + g_1\varphi^2 = h_1, \tag{21}$$

where g_1 and h_1 are nonzero polynomials with g_1 dividing g and h_1 dividing h, will be said to be constructively equivalent to Eq. 20 if and only if one can construct an algorithm for producing two polynomials $\rho(z)$ and $\xi(z)$ such that whenever ψ_1 and φ_1 are entire solutions of Eq. (21), then $\psi = \rho(z)\psi_1$ and $\varphi = \xi(z)\varphi_1$ are entire solutions of Eq. (20) and conversely, every pair of entire solutions of Eq. (20) is of the form $\rho(z)\psi_1$ and $\xi(z)\varphi_1$ for entire solutions ψ_1 and φ_1 of Eq. (21).

For the canonical representation we have:

Theorem 2. Each equation of type (20) *which actually has at least one pair of entire solutions is constructively equivalent to some equation of the form*

$$\psi^2 + qp\varphi^2 = qq_1, \tag{22}$$

where p, q, *and* q_1 *are relatively prime nonzero polynomials and* q *has no multiple zeros.*

We next proceed to describe the structure of the general solution of Eq. (22).

Theorem 3. Let us choose branches of $\sqrt{qp}$ *and* $\sqrt{qq_1}$. *A pair of entire functions* ψ *and* φ *is a solution of Eq.* (22) *if and only if it is of the form*

$$\psi = \sqrt{qq_1}\,\sin(\sqrt{qp}\,\gamma(z))$$

$$\varphi = \sqrt{p^{-1}q_1}\,\cos(\sqrt{qp}\,\gamma(z)),$$

where $\gamma(z)$ *may be any multiple valued function which is analytic and single valued on the complex plane with suitable cuts from the zeros of* pqq_1 *to* $z = \infty$ *removed such that: (a) at each zero* z_j *of* q, $\gamma(z) - n_j\pi(\sqrt{qp})^{-1}$ *is analytic for some integer* n_j; *(b) at each zero* z_k *of* p, $\gamma - [\ell_k + (1/2)]\,\pi(\sqrt{qp})^{-1}$ *is analytic at* $z = z_k$ *for some integer* ℓ_k; *(c) at each zero* z_r *of* q_1 *of multiplicity* m_r, $\gamma(z) + (1/2)t_r i \log(z - z_r)\,(\sqrt{qp})^{-1}$ *is analytic for some integer* $t_r, -m_r \leqslant t_r \leqslant m_r$, *such that* $(m_r \pm t_r)/2$ *is an integer; (d) given a solution pair* ψ_1 *and* φ_1 *and any associated* γ, *say* γ_1, *then* γ_2 *is another associated* γ *if and only if* $\sqrt{qp}\,(\gamma_1 - \gamma_2) = 2n_1\pi$ *for some integer* n_1.

(Note if γ corresponds to the solution ψ_1, φ_1, then $\gamma + \pi(\sqrt{qp})^{-1}$ corresponds to the pair $-\psi_1$, $-\varphi_1$ and $-\gamma$ corresponds to the pair $-\psi_1$, φ.)

Theorem 4. Let $s(z)$ *equal the product of the distinct zeros of* $q_1(t)$. *For each* p, q, q_1 *and sets of integers* n_j, ℓ_k, *and* t_r *as in Theorem 3, there exists a function* $\gamma_1(z)$ *satisfying the conditions of Theorem 3 concerning* $\gamma(z)$. *Further,* $\gamma_1(z)$ *may be written in the form*

$$\gamma_1(z) = (\sqrt{p(z)q(z)})^{-1}\int_a^z \sqrt{p(t)q(t)}\,(s(t))^{-1}f(t)\,dt$$

for some polynomial $f(t)$ *and some complex number* a *which is not a zero of* pqq_1.

It is clear that any $\gamma(z)$ as in Theorem 3, corresponding to a particular set of n_j, ℓ_k, and t_r, and the $\gamma_1(z)$ in Theorem 4, corresponding to the same set of n_j, ℓ_k, and t_r, differ by an entire function.

3. METHOD AND PROCEDURE

We first prove the nonexistence of solutions under the conditions asserted in Theorem 1. Clearly $(z - a)^{2\alpha-1}$ divides ψ^2, so $(z - a)^{\alpha}$ divides ψ and $(z - a)^{2\alpha}$ divides both ψ^2 and g. But then $(z - a)^{2\alpha}$ divides h, contrary to the hypothesis, so there are no solutions. The remainder of Theorem 1 is a consequence of Theorems 2, 3, and 4.

We shall sketch a proof of Theorem 2 after the following example. Consider

$$\psi^2 + z^2(z-1)(z-2)(z-3)^2\varphi^2 = z^2(z-1)^2(z-2)(z-4)\,. \qquad (23)$$

We may write this as

$$\psi^2 + [z^2(z-1)(z-2)]\,[(z-3)^2]\,\varphi^2 = [z^2(z-1)^2(z-2)]\,[z-4]\ ,$$

where the polynomials in square brackets are each relatively prime except for the first and third, which indeed have all their distinct linear factors in common. If the first and third factors above had no multiple zeros, we would have the desired form already. Instead we notice that $z(z-1)$ must divide ψ and, using this, that $z-1$ divides φ. Then setting $\psi = z(z-1)\psi_1$ and $\varphi = (z-1)\varphi_1$, we have

$$\psi_1^2 + [(z-1)(z-2)]\,[(z-3)^2]\,\varphi_1^2 = [z-2]\,[z-4]\ , \tag{24}$$

which is in the desired form. As we see from this example, the only difficulty in the preceding procedure is in dealing with multiple zeros which occur in either the first or third square brackets. It is possible to prove that ψ or φ is always divisible by factors which allow us to carry out cancellation as in the example. (If we cancel out a factor completely on one side but not on the other side, then it is no longer a common factor, and it should be moved to either the second or fourth square brackets respectively.) This indicates the proof of Theorem 2.

4. CONSTRUCTION OF THE SOLUTIONS

Next we shall sketch the proof of Theorem 3. Suppose that ψ and φ are two entire functions which are solutions of an equation of Eq. (22) type. We may factor $\psi^2 + qp\varphi^2$ into

$$(\psi + i\sqrt{qp}\,\varphi)(\psi - i\sqrt{qp}\,\varphi).$$

If we set

$$\psi + i\sqrt{qp}\,\varphi = i\sqrt{qq_1}\,e^{i\sqrt{qp}\,\gamma(z)} \tag{25}$$

and

$$\psi - i\sqrt{qp}\,\varphi = -i\sqrt{qq_1}\,e^{-i\sqrt{qp}\,\gamma(z)}\ , \tag{26}$$

this will define a multiple-valued function $\gamma(z)$ which is then certainly analytic on the complex plane with suitable cuts, from the zeros of pqq_1 to $z = \infty$, removed, as required by the statement of Theorem 3.

We next analyze the behavior of $\gamma(z)$ at z_j, a zero of q. Analytically continuing Eq. (25), once around z_j, we necessarily obtain Eq. (26). Thus we are eventually led to the conclusion that $\gamma(z) - n_j\pi(\sqrt{qp})^{-1}$ is analytic at $z = z_j$ for some integer n_j.

We can write, using Eqs. (25) and (26),

$$\psi(z) = \sqrt{qq_1}\,\sin(\sqrt{qp}\,\gamma(z)) \tag{27a}$$

and

$$\varphi(z) = \sqrt{p^{-1}q_1}\cos(\sqrt{qp}\,\gamma(z)). \tag{27b}$$

By use of several trigometric identities, we can show that if the functions $\psi(z)$ and $\varphi(z)$ are to be analytic at $z = z_k$, a zero of p, then

$$\gamma(z) - \left(\ell_k + \frac{1}{2}\right)\pi(\sqrt{qp})^{-1}$$

is analytic at $z = z_k$ for some integer ℓ_k. (The cases of z_k having even and odd multiplicity are considered separately.) One of the major problems is showing that

$$\gamma(z) - \left(\ell_k + \frac{1}{2}\right)\pi(\sqrt{qp})^{-1}$$

does not have a pole at $z = z_k$ after we know that it is single-valued there.

Let z_r be a zero of q_1. In Eq. (25) let $\psi + i\sqrt{qp}\varphi$ vanish to the order $(m_r + t_r)/2$ at $z = z_r$. Then $\psi - i\sqrt{qp}\varphi$ vanishes to the order $(m_r - t_r)/2$. Also, from Eq. (25),

$$\sqrt{q}\exp\left[\sqrt{qp}\gamma(z) - \frac{1}{2}t_r\log(z - z_r)\right]$$

is analytic and nonvanishing at $z = z_r$; hence

$$\sqrt{qp}\gamma(z) - \frac{1}{2}t_r\log(z - z_r)$$

is analytic at $z = z_r$. Obviously t_r is an integer and so is $(m_r \pm t_r)/2$.

One must then go through carefully to show that the necessary conditions are indeed sufficient.

We next sketch the proof of Theorem 4. We shall indicate how we can show the existence of a polynomial $f(t)$ such that for some a in C, which is not a zero of pqq_1, and for each j, k, and r as in Theorem 3,

$$\int_a^{z_j}\sqrt{p(t)q(t)}\,(s(t))^{-1}f(t)\,dt = n_j\pi, \tag{28}$$

$$\int_a^{z_k}\sqrt{p(t)q(t)}\,(s(t))^{-1}f(t)\,dt = \left(\ell_k + \frac{1}{2}\right)\pi, \tag{29}$$

and

$$\sqrt{pq}\,(s)^{-1}f \text{ has a residue of } t_r i/2 \text{ at } z = z_r. \tag{30}$$

It is possible to verify that Eqs. (28) through (30) together imply the statement of Theorem 4. Suppose that we have indexed the set of z_j and z_k by z_u. Then we can show that it suffices to find a collection of polynomials f_u such that

$$\int_a^{z_u} \sqrt{p(t)q(t)} f_{u_1}(t)\, dt = \delta_u^{u_1}, \tag{28}$$

where $\delta_u^{u_1}$ is the Kronecker delta. Indeed, we can set

$$f = \sum_u c_u f_u s(z) + \sum_r \frac{1}{2} it_r (\sqrt{p(z_r)q(z_r)})^{-1} s(z)(z - z_r)^{-1}$$

for appropriately chosen constants c_u. The existence proof for the f_u's is fairly complicated. We assume that no such polynomials exist; this is shown to be equivalent to the matrix

$$\left(\int_a^{z_u} \sqrt{p(t)q(t)} t^k\, dt \right)$$

with an infinite number of rows, corresponding to $k = 0, 1, \ldots$, and a finite number of columns, corresponding to the different z_u's, having rank less than the number of its columns. Changing variables so that the paths of integration are all from 0 to 1, we have that some linear combination of square roots of differing polynomials (the $p(t)q(t)$ after the differing changes of variables), with not identically zero coefficients, has every moment on [0, 1] equal to zero. We can prove that this means that the linear combination of square roots must be identically zero. If $\sqrt{pq}$ is not a polynomial and the point a was chosen "appropriately" previously, we can show that the differing square roots are linearly independent functions. (We analyze the locations of the singularities of the square roots.) This gives the desired result if $\sqrt{pq}$ is not a polynomial. The case where $\sqrt{pq}$ is a polynomial is an easier but separate one. This completes our outline of the proof of theorem 4.

5. APPLICATION TO SPECIFIC CASES

We next apply the results stated in the second section to the cases of greatest physical interest.

To aid us, we shall need the following results:

Corollary (to Theorems 1 through 4). Set $\delta = (1/2)$ *deg* (pq). *Let us choose arbitrarily a polynomial* F *in* z. (a) *If* $\delta > 1$ *is an integer, there exists some* $\sqrt{\mathrm{pq}}\gamma$, *as in Theorem 3, such that, at* $z = \infty$, $\sqrt{\mathrm{pq}}\gamma$ *is asymptotic to* $\mathrm{F} + \mathrm{O}(z^{\delta - 1})$. (b) *If* $\delta > 1/2$ *is not an integer, there exists some* $\sqrt{\mathrm{pq}}\gamma$, *as in Theorem 3, such that, at* $z = \infty$, $\sqrt{\mathrm{pq}}\gamma$ *is asymptotic to* $\sqrt{z}\, F + \mathrm{O}(z^{\delta - 1})$. (c) *If* δ *equals* 0 *or* 1, *then there exists some* $\sqrt{\mathrm{pq}}\gamma$, *as in Theorem 3, such that, at* $z = \infty$, $\sqrt{\mathrm{pq}}\gamma$ *is asymptotic to* $\mathrm{F} + \mathrm{O}(\log z)$. *If* $\delta = 1/2$, *then there exists some* $\sqrt{\mathrm{pq}}\gamma$, *as in Theorem 3, such that, at* $z = \infty$, $\sqrt{\mathrm{pq}}\gamma$ *is asymptotic to* $\sqrt{z}\mathrm{F} + \mathrm{O}(1)$. (d) *If*

$\delta > 0$ *and both* $\sqrt{\mathrm{pq}}\gamma_1$ *and* $\sqrt{\mathrm{pq}}\gamma_2$ *are asymptotic to either* F *or* $\sqrt{\mathrm{z}}\mathrm{F}$ *for some polynomial* F, *up to the error term given above for the appropriate value of* δ, *and if further* γ_1 *and* γ_2 *both give rise to the same constants* $\mathrm{n_j}$, ℓ_k, *and* $\mathrm{t_r}$, *then* $\gamma_1 \equiv \gamma_2$. (e) ***In parts*** *(a) through (c), given any allowable set of values of the constants* $\mathrm{n_j}$, ℓ_k, *and* $\mathrm{t_r}$ *(see Theorem 3), we may require also that the* γ *to be constructed there corresponds to these values of* $\mathrm{n_j}$, ℓ_k, *and* $\mathrm{t_r}$.

Proof. We use the formula

$$\sqrt{p(z)q(z)}\gamma(z) = \int_a^z \sqrt{p(t)q(t)}(s(t))^{-1}f(t)\,dt + \sqrt{pq}g,$$

where g is an entire function. Now

$$\sqrt{p(t)q(t)}(s(t))^{-1}f(t)$$

has an expansion about $z = \infty$ in descending integral powers of t if δ is an integer, or descending odd integral powers of $t^{1/2}$ if δ is not an integer. Thus the integral is of the same form as the integrand, except that if δ is an integer, there may now be a $\log z$ term, and if δ is not an integer, there may now be a constant term. If δ is an integer, set the terms where the power of z is nonnegative equal to G. If δ is not an integer, set the terms where the power of z is positive equal to G. We can find a series h, in descending integral powers of z, such that

$$h = (\sqrt{pq})^{-1}(F - G)$$

if δ is an integer or

$$h = (\sqrt{pq})^{-1}(\sqrt{z}F - G)$$

if δ is not an integer. Let the polynomial g be defined to be such that $h - g$ vanishes at $z = \infty$. Then

$$\sqrt{pq}g = F - G + O(z^{\delta - 1})$$

if δ is an integer, and

$$\sqrt{pq}g = \sqrt{z}F - G + O(z^{\delta - 1})$$

if δ is not an integer. With this choice of g we obtain the desired asymptotic form for $\sqrt{pq}\gamma$, up to an error which depends on the nature of δ. If δ is an integer, the error is the larger of $O(z^{\delta - 1})$ and $O(\log z)$. If δ is not an integer, the error is the larger of $O(z^{\delta - 1})$ and $O(1)$. This proves parts (a) through (c) of the Corollary. We next prove part (d).

By the remark after Theorem 4, we see that we must have

$$\sqrt{pq}\gamma_1 - \sqrt{pq}\gamma_2 = \sqrt{pq}g,$$

where g is entire. Also $\sqrt{pq}g$ is asymptotic to $O(z^{\delta-1})$, $O(\log z)$, or $O(z^{\delta-1/2})$. In the first and third cases,

$$\lim_{z\to\infty} |g(z)| = 0 ,$$

so

$$g \equiv 0 .$$

Also, if $\delta = 1$, the same conclusion holds. This proves the Corollary. *Lemma. If, for any two solutions* ψ *and* φ *of Eq. (20),* $\sqrt{pq}\gamma$ *is asymptotic to* $\alpha\sqrt{z} + O(1)$, *where* $\alpha \neq 0$ *is real, then on any angular sector about* $z = 0$ *not including the negative real axis, we see that* $|\psi|$ *and* $|\varphi|$ *are asymptotic to*

$$\exp\,(\alpha|\mathrm{Re}\sqrt{z}| + O(|\log z|)), \text{ at } z = \infty .$$

Proof. Trivial.

As an example, we now consider a more general Penrose-Liebowitz equation:

$$\psi^2 + (az^2 + b)\varphi^2 = r(z^2) , \tag{31}$$

where a and b are complex numbers, $a \neq 0$, $r(z^2) \neq 0$ is a polynomial in z^2 with complex coefficients, and $(az + b)$ does not divide $r(z)$. We are required to find even entire solutions of Eq. (31), ψ and φ such that, for some real $\alpha \neq 0$, $\log \psi$ and $\log \varphi$ are each asymptotic at $z = \infty$ to

$$\alpha|\mathrm{Re}\, z| + O(|\log z|)$$

on any angular sector about $z = 0$ not containing either the positive or negative imaginary axis.

Set $w = z^2$. By Theorem 1 we always have a solution. By the Corollary, with $\delta = 1/2$, and the Lemma, we have that for each integral choice of n_1 and the allowable choices of t_r, there are entire solutions $\psi_1(w)$ and $\varphi_1(w)$ with $\log|\psi_1(w)|$ and $\log|\varphi_1(w)|$ each asymptotic to

$$\alpha|\mathrm{Re}\sqrt{w}| + O(|\log w|)$$

on any angular sector about $z = 0$ which does not contain the negative real axis. Then $\psi_1(z^2)$ and $\varphi_1(z^2)$ satisfy the desired conditions.

From Theorem 3(d) and the comment after the statement of Theorem 3, we see that if our only free parameter is n_1, then the only solutions are $\pm\psi_1$ and $\pm\varphi_1$. This will be the case if q_1 is a nonzero constant. If q_1 is linear in z^2, then we have two choices of t_1, that is $t_1 = \pm 1$. By interchanging γ and $-\gamma$, if necessary, we may take $t_1 = 1$. Then by the comment after the statement of Theorem 3, the uniqueness of

ψ_1 and φ_1, up to sign, follows. If $q_1(w)$ has degree larger than one, we cannot have uniqueness, even up to sign.

If we weaken the asymptotic condition slightly, there is also not uniqueness. There exists an entire function $E(z)$ which is bounded on every ray out from zero but which is not a constant and is not even of finite order of growth. (The bound is not uniform, of course.) Adding any such function (we can construct many) to $(\sqrt{qp})^{-1}\gamma$, we would obtain new solutions ψ and φ with $\log|\psi|$ and $\log|\varphi|$ each asymptotic to

$$\alpha|\mathrm{Re}\sqrt{w}| + O(|\log w|)$$

on every ray out from zero, except the negative real axis.

REFERENCES

1. F.J. Dyson, "Missed Opportunities," Josiah Willard Gibbs Lecture, AMS Meeting, Jan. 17, 1972.

2. J.K. Percus and G.J. Yevick, *Phys. Rev.* **110**, 1 (1958); J.L. Lebowitz, *Phys. Rev.* **133**, A895 (1964).

3. I.M. Gel'fand and B.M. Levitan, "On the Determination of a Differential Equation From Its Spectral Function," *Amer. Math Soc. Trans.*, Ser. 2, 1, 253 (1955).

4. O. Penrose and J.L. Lebowitz, *J. Math Phys.* **13**, No. 5, 604 (1972).

5. S. Ahn and J.L. Lebowitz, *J. Chem. Phys.* **60**, No. 2, 523 (1974); J.L. Lebowitz and D. Zomick, *J. Chem. Phys.* 54, 3335 (1971).

6. I. Kay, "The Inverse Scattering Problem When the Reflection Coefficient is a Rational Function, *Commun. Pure Appl. Math.* **13**, 371 (1960).