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A means of optimizing an MTI filter for rejecting several types of clutter, which is generated by different mechanisms such as by rain or the ground, is formulated. It is found that the optimal performance of such a filter depends on the spectral density functions, average radar cross sections, and the relative mean doppler frequencies of each type of clutter. It is shown that the optimal improvement factor of such a filter is bounded by the weighted average (weighted in accordance with the radar cross sections of the clutter types) of the improvement factor for the

(Continued)

individual clutter type. It is also shown that the improvement factor of a filter is bounded by two limits. Both of these limits are a function of the relative mean doppler frequency f_0 between the clutter types. As f_0 increases, the performance of the MTI system degrades. The worst improvement factor occurs when f_0 is equal to half of the radar prf.

CONTENTS

MTI OPTIMIZATION IS A MULTIPLE-CLUTTER ENVIRONMENT

INTRODUCTION

The performance of a radar is often limited by echoes from external clutter that are large compared with internal noise. In many practical situations targets have a radial velocity component with respect to the clutter scatterers. In this case, doppler discrimination can be used to enhance the signal-to-clutter ratio. This type of processor has been discussed extensively in the literature [1-6]. Usually the treatment is limited to the case in which the clutter is assumed to be of a single type with a certain spectral density function. However, in reality, the radar echoes may contain more than one type of clutter. For instance, they may contain ground clutter and also weather clutter. These two clutter types may have entirely different spectral density functions, or they have the same spectral density function, but with different mean values and variances. In this report the design of an optimal filter for such a multiple-clutter environment will be presented. The optimization procedure is based on the solution of the minimum eigenvalue of the clutter covariance matrix. The results, therefore, represent a theoretical limit of the performance of such an MTI system. Elements of such a clutter covariance matrix depend not only on the spectral density functions of all the clutter types present, but also depend on the relative radar cross sections and the difference in mean doppler frequencies of the clutter types. In this report, MTI performance as related to these parameters will be discussed, and performance bounds of an MTI system as related to these parameters will be presented.

MTI IMPROVEMENT FACTOR FOR A MULTIPLE-CLUTTER-TYPE ENVIRONMENT

Assume that a sequence of identical radar pulses is transmitted with an equal interval of time, T, between pulses. For simplification, without losing generality, assume that the radar pulses are not modulated and that the received signal is sampled at a rate of twice per range resolution cell.* The correlation function of the radar returns between the ith and jth pulses can be represented [7] by

$$R_{ij} = C \int_{-\infty}^{\infty} G(f) \exp\,[j2\pi fT(i-j)]\; df, \tag{1}$$

where C is the average radar cross section of the clutter and $G(f)$ is the power spectral density function of the clutter. The quantity T is the interpulse spacing.

Assume that the radar return contains two types of clutter. These two types may have two entirely different spectral density functions, or they may have the same spectral density function but different mean dopplers and variances. In any case, they are assumed

*In this case the range resolution cell is equal to the radar pulse length.

Note: Manuscript submitted January 8, 1975.

to be two independent, uncorrelated random processes. The correlation function of such clutter returns can be represented by

$$R_{ij} = C_1 \int_{-\infty}^{\infty} G_1(f) \exp\,[j2\pi fT(i-j)]\; df$$

$$+C_2 \int_{-\infty}^{\infty} G_2(f) \exp\,[j2\pi fT(i-j)]\; df\,. \tag{2}$$

Assume that N radar pulses are transmitted. The received radar signals are sampled and delayed, and then they are weighted by complex weights w_i and summed. The clutter output of this delay-line filter is then

$$P_f = \sum_i \sum_j w_i w_j^* R_{ij}\,, \tag{3}$$

where w_j^* is the complex conjugate of w_j. Since a priori knowledge of the target doppler frequency is not available, it is convenient to assume that the target doppler has a uniformly distributed probability density function. The expected target output power from the delay-line filter is then

$$P_s = E_a \sum_i w_i w_i^*\,, \tag{4}$$

where E_a is the average radar cross section of the target.

The expected input target signal-to-clutter ratio is

$$\mathrm{SCR} = \frac{E_a}{C_1 + C_2}\,. \tag{5}$$

The improvement factor, defined as the ratio of the filter output to the input target SCR, is

$$I = (C_1 + C_2)\,\frac{\sum_i w_i w_i^*}{\sum_i \sum_j w_i w_j^* R_{ij}}\,. \tag{6}$$

The correlation function R_{ij}, besides being a function of the spectral density functions of both clutter types, is also weighted according to the relative amplitudes of the radar cross sections of the clutter returns. One can notice that the form of the improvement factor for a multiple-clutter case is different from that for a single clutter case by a factor $(C_1 + C_2)$. This is necessary in order to take into account the effect of the weighted correlation function.

In the previous formulation the case of two types of clutter is considered. However, this can be extended to cases of more than two types of clutter, provided that these clutter types are independent and uncorrelated random processes. For simplicity in all subsequent discussion, only the case of two clutter types will be treated.

Since the absolute amplitudes of the weighting functions w_i are irrelevant, one can normalize the w_i by the relation

$$w_i' = \frac{w_i}{\sqrt{\sum_i |w_i|^2}}. \tag{7}$$

It is convenient also to normalize the correlation functions R_{ij} by the relation

$$R_{ij}' = \frac{R_{ij}}{C_1 + C_2}. \tag{8}$$

The improvement factor hence becomes

$$I = \frac{1}{\sum_i \sum_j w_i w_j^* R_{ij}}. \tag{9}$$

For simplicity the primed w_i and R_{ij} are all changed to unprimed notations.

The optimization goal then is to find a set of w_i such that the quadratic form

$$Q(\mathbf{w}) = \sum_i \sum_j w_i w_j^* R_{ij} \tag{10}$$

is minimal under the constraint that

$$\sum_i |w_i|^2 = 1. \tag{11}$$

COVARIANCE MATRIX

Before proceeding further, some of the properties of the correlation function R_{ij} will be discussed. For the case of a single clutter type, the correlation function r_{ij} is the Fourier transform of the clutter power spectral density function; that is,

$$r_{ij} = \int_{-\infty}^{\infty} G(f) \exp\,[j2\pi fT(i-j)]\; df, \tag{12}$$

where $G(f)$ is assumed to have a zero mean. If the radar pulses are identical and the clutter scatterers can be represented as a stationary random process, it is well known [8] that this correlation function must be real and even and that its value at the origin is the one, and only one, maximum value in magnitude. That is,

$$r_{ii} > |r_{ij}| \; ; \tag{13a}$$

$$r_{ij} = r_{ji} \, . \tag{13b}$$

The power spectral density function, if it is realizable, must be real, positive, and even; that is,

$$G(f) > 0 \; ; \tag{14a}$$

$$G(f) = G(-f) \, . \tag{14b}$$

Usually the clutter mean doppler will not be zero; however, in this case one can set

$$f' \doteq f - f_0 \tag{15}$$

where f_0 is the mean doppler frequency. Under this condition Eq. (12) becomes

$$r'_{ij} = \exp\left[-j2\pi f_0 T(i-j)\right] \int_{-\infty}^{\infty} G(f') \exp\left[j2\pi f' T(i-j)\right] df' \, . \tag{16}$$

The radar pulse repetition frequency is defined as

$$\text{prf} = \frac{1}{T} \, . \tag{17}$$

One can introduce the following transformation:

$$f'_0 = \frac{f_0}{\text{prf}} \; ; \tag{18a}$$

$$f'' = \frac{f'}{\text{prf}} \, . \tag{18b}$$

Equation (16) then becomes

$$r'_{ij} = \exp\left[-2\pi f'_0 (i-j)\right] \int_{-\infty}^{\infty} G(f'') \exp\left[j2\pi f''(i-j)\right] df'' \, , \tag{19}$$

where $G(f'')$ is normalized such that

$$\int_{-\infty}^{\infty} G(f'')\, df'' = 1 \, .$$

The spectral density function $G(f'')$ is usually assumed to be a bell-shaped curve, with most of the energy concentrated in the neighborhood of $f'' = 0$; it may have a long tail. Furthermore, for an MTI system to be effective, the spectral spread of this $G(f'')$ function must be very small compared with f_r. Under this condition the integral part of Eq. (19) usually is positive, and its value decreases rapidly as the index $|i-j|$ increases. One good example is the case where $G(f'')$ is a Gaussian function.

Assuming that $G(f'')$ is properly normalized, then one has

$$r'_{ii} = 1 \; ; \tag{20a}$$

$$|r'_{ij}| < r'_{ii} \; ; \tag{20b}$$

$$r'_{ij} = r'_{ji} \, . \tag{20c}$$

For the two-clutter case, by use of the same procedure as discussed in the single-clutter case, one finds

$$R_{ij} = \frac{1}{C_1 + C_2} \Big\{ C_1 \exp\left[-j2\pi f_{01}(i-j)\right] r_{ij}^{(1)} + C_2 \exp\left[-j2\pi f_{02}(i-j)\right] r_{ij}^{(2)} \Big\} \, , \tag{21}$$

where

$$r_{ij}^{(1)} = \int_{-\infty}^{\infty} G_1(f) \exp\left[-2\pi f(i-j)\right] \, df \tag{22a}$$

and

$$r_{ij}^{(2)} = \int_{-\infty}^{\infty} G_2(f) \exp\left[-2\pi f(i-j)\right] \, df \, . \tag{22b}$$

The quantity f is the normalized doppler frequency, as shown by Eq. (18b). Quantities f_{01} and f_{02} are respectively the mean doppler of clutter types one and two. Let

$$\rho = \frac{C_2}{C_1}, \qquad C_2 < C_1 \, , \tag{23a}$$

and

$$f_0 = f_{02} - f_{01} \, . \tag{23b}$$

Equation (21) becomes

$$R_{ij} = \frac{1}{1+\rho} \exp\left[-j2\pi f_{01}(i-j)\right] \Big\{ r_{ij}^{(1)} + \rho \exp\left[-j2\pi f_0(i-j) \; r_{ij}^{(2)}\right. \Big\} \, . \tag{24}$$

Setting

$$w_i' = w_i \exp(-j2\pi f_{01} i) , \tag{25}$$

the quadratic form of Eq. (10) becomes

$$Q(\mathrm{w}) = \sum \sum w_i' w_j'^* R_{ij}' , \tag{26}$$

where

$$R_{ij}' = \frac{1}{1 + \rho} \left\{ r_{ij}^{(1)} + \rho \exp [j2\pi f_0 (i - j)] r_{ij}^{(2)} \right\} . \tag{27}$$

Replacing w_i by w_i', however, still maintains the required constraint that

$$\sum_i |w_i'|^2 = 1 .$$

The problem, therefore, is to find a set of w_i' which will minimize the quadratic form of Eq. (26). For simplification, one can drop all the primed signs in Eq. (26) and write it in a matrix form:

$$Q = |W|^T |R| |W^*| , \tag{28}$$

where $|W|$ is a column matrix:

$$|W| = \begin{vmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{vmatrix} \tag{29a}$$

and $|R|$ is an N-by-N square matrix:

$$|R| = |R_{ij}| . \tag{29b}$$

The matrix R is usually called the covariance matrix. Its elements R_{ij} are defined in Eq. (27). One notices that

$$R_{ii} = 1 \tag{30a}$$

$$R_{ij} = R_{ji}^* . \tag{30b}$$

Hence the matrix R is a Hermitian matrix, and its elements, besides being functions of the clutter spectral density functions $G_1(f)$ and $G_2(f)$, are also functions of ρ, the ratio of the average radar cross section of the two clutter types, and f_0, which is the difference between the mean doppler frequencies of the two clutter types.

MINIMUM EIGENVALUE AND ITS EIGENVECTOR

In previous discussions it was indicated that an effort would be made to find a vector $|W|$ such that the quadratic form of Eq. (28) is minimum under the constraint

$$|W|^T |W^*| = 1 .$$

Matrix R is a Hermitian matrix. It is well known that the eigenvalues and eigenvectors of matrix R possess the following properties [9]:

- The eigenvalues of a Hermitian matrix are real. Furthermore the quadratic form of Eq. (28) represents the filter power output with a constant scale factor. It is, therefore, must be a positive quantity, provided the w_i are not identically zero.* Hence matrix R is positive definite. Accordingly the eigenvalues of matrix R all must be positive [9], and one has

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_{\min} > 0 , \tag{31a}$$

where the λ_i are the eigenvalues which satisfy

$$|R - \lambda I| = 0 , \tag{31b}$$

where I is an identity matrix.

- Eigenvalues and eigenvectors are related by

$$|W^{(i)}|^T |R| |(W^{(i)})^*| = \lambda_i , \tag{32}$$

where $|W^{(i)}|$ is the eigenvector associated with the ith eigenvalue.

- The eigenvectors associated with distinct eigenvalues of a Hermitian matrix are orthogonal. Moreover, these vectors can be normalized and form a complete orthonormal set:

$$\begin{aligned} |W^{(i)}|^T (W^{(i)})^*| &= 1 , \quad i = j ; \\ &= 0 , \quad i \neq j . \end{aligned} \tag{33}$$

Let matrix R be an N-by-N matrix. Since the eigenvectors form a complete set, any vector in this N-dimensional space can be represented as

$$|W| = \sum_i d_i |W^{(i)}| , \tag{34}$$

where the d_i are real constants. For this particular vector, the quadratic form of Eq. (28) becomes

*An exception is when the determinant $|R|$ is zero.

$$Q(|W|) = \sum_i d_i^2 \lambda_i , \tag{35}$$

due to the orthonormal property of $|W^{(i)}|$. It is evident that this quadratic form will assume a minimum value when the vector $|W|$ is so chosen that it is the eigenvector associated with the minimum eigenvalue $\lambda_{\min}$, since d_i^2 is always positive. In this case Eq. (28) becomes

$$Q(|W^N|) = \lambda_{\min} . \tag{36}$$

Therefore, to minimize the clutter output, one has to find the minimum eigenvalue of the covariance matrix R. The eigenvector associated with this minimum eigenvalue comprises the required filter weights. Solving for matrix eigenvalues is a well-known and well-treated problem. It will not be discussed further here. Instead, some relevant properties of this minimum eigenvalue will be pointed out.

The eigenvalues of a matrix are the roots of the polynomial $|R - \lambda I| = 0$. Therefore

$$\sum_{i=1}^{N} \lambda_i = \sum_{i=1}^{N} R_{ii} , \tag{37a}$$

and

$$\prod_{i=1}^{N} \lambda_i = \det R \leqslant \prod_{i=1}^{N} R_{ii} . \tag{37b}$$

Since the R_{ii} of the covariance matrix are all unity, the minimum eigenvalue of matrix R is bounded by

$$0 \leqslant \lambda_{\min} \leqslant 1 . \tag{38}$$

This minimum eigenvalue depends on the values of the elements R_{ij} of the covariance matrix R. The elements R_{ij} in turn are a function of the clutter spectral density function and the correlation time. It can be seen intuitively that the spectral spread of the spectral density function affects the correlation function. If the spectral spread is very great, the correlation function between pulses will be reduced; hence R_{ij} becomes smaller. For example, white Gaussian noise has a uniform spectrum across the band; the correlation function of such noise has the form

$$R_{ij} = \delta_{ij} . \tag{39}$$

The covariance matrix in this case becomes a unitary matrix. The eigenvalues of this matrix are all equal to unity. Under this condition an MTI filter has a 0-dB improvement factor, no matter how the filter weights are chosen. On the other hand, if the spectral density is concentrated at the mean doppler frequency and becomes an impulse function, the correlation between any two pulses would be perfect, and the elements in the covariance matrix are all equal to unity.* If binomial weights are chosen, the minimum eigenvalue becomes zero, and the improvement factor becomes infinite. A spectral distribution which lies between these two extreme cases must have a minimum eigenvalue which lies within these two limits. Hsiao [11] showed that for a Gaussian spectral density function, the improvement factor is a monotonic function of the variance of the clutter doppler.

For a multiple-clutter case, this becomes much more complicated. One may notice from Eq. (24) that the elements R_{ij} are functions of the clutter cross-section ratio ρ, the relative mean doppler frequency f_0, and the spectral density functions of the two clutter types. However, as a parallel analogy, one can conjecture that no matter how these quantities (ρ, f_0, and the spectral spread of the two clutter types) vary, if the R_{ij} are reduced, the minimum eigenvalues will increase, and vice versa. In the following sections some of these effects will be examined.

EFFECT OF CLUTTER CROSS-SECTION RATIO

For simplification, assume that the difference of the mean doppler frequencies f_0 is zero. Then the covariance matrix elements R_{ij} become (see Eq. (27))

$$R_{ij} = \frac{1}{1+\rho}\left[r_{ij}^{(1)} + \rho r_{ij}^{(2)}\right], \tag{40}$$

where matrices $|R|$, $|r_1|$, and $|r_2|$ have matrix elements R_{ij}, $r_{ij}^{(1)}$, and $r_{ij}^{(2)}$ respectively. The minimum eigenvalue of matrix R can be represented as

$$\lambda_m(R) = \frac{1}{1+\rho}\left[|W_0|^T|r_1||W_0^*| + \rho|W_0|^T|r_2||W_0^*|\right], \tag{41}$$

where $|W_0|$ is the eigenvector associated with the minimum eigenvalue of matrix R. Matrices $|r_1|$ and $|r_2|$ are respectively the covariance matrices for clutter types 1 and 2 alone. They are positive definite. Hence they must have a positive, minimum eigenvalue of themselves. There is only one vector in N-dimension space which is associated with this minimum eigenvalue. Vector $|W_0|$ is not necessarily this minimum eigenvector. One has†

$$\lambda_{\min}(r_1) \leqslant |W_0|^T|r_1||W_0^*| \tag{42a}$$

and

$$\lambda_{\min}(r_2) \leqslant |W_0|^T|r_2||W_0^*|. \tag{42b}$$

Therefore

$$\lambda_{\min}(R) \geqslant \frac{1}{1+\rho}\left[\lambda_{\min}(r_1) + \rho\lambda_{\min}(r_2)\right]. \tag{43}$$

Equality occurs only when the minimum eigenvectors of matrices R, r_1, and r_2 are identical. Equation (43) can be written as

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*This effect on the correlation function has been discussed by DiFranco and Rubin [10].

†This is really the Courant-Fisher min-max theorem. For details see Ref. 9.

$$\lambda_{\min}(R) \geqslant \frac{C_1}{C_1 + C_2} \lambda_{\min}(r_1) + \frac{C_2}{C_1 + C_2} \lambda_{\min}(r_2) . \tag{44}$$

Equation (44) can be easily extended to a general case of more than two clutter types:

$$\lambda_{\min}(R) \geqslant \frac{1}{C_1 + C_2 + \ldots C_K} [C_1 \lambda_{\min}(r_1) + C_2 \lambda_{\min}(r_2) + \ldots C_K \lambda_{\min}(r_K)] . \tag{45}$$

The quantity $\lambda_{\min}(R)$ represents the minimum clutter output for a multiple-clutter environment, whereas $\lambda_{\min}(r_1)$, $\lambda_{\min}(r_2)$, ... are respectively the minimum clutter outputs for single-clutter types. The sum on the right side of Eq. (45) is the weighted average of the minimum clutter output when the MTI is optimized against a single-clutter type. The weighting factors are the clutter cross sections C_1, C_2, ..., and C_K. As an example, assume that one can design an optimal MTI filter for ground clutter and achieve a minimum clutter output of 10^{-6} (equivalent to a 60-dB improvement factor) and design another optimal MTI filter for weather clutter with which one can only achieve a minimum clutter output of 10^{-2} (20-dB improvement factor). Assume that the radar cross section of the ground clutter is 100 times larger than that of the weather clutter (equivalent to 20 dB). If a single MTI filter is designed to reject both clutter types simultaneously, the clutter output of this filter is then

$$\lambda_{\min}(R) \geqslant \frac{1}{1 + 100} (100 \times 10^{-6} + 10^{-2})$$

$$\geqslant 10^{-4} .$$

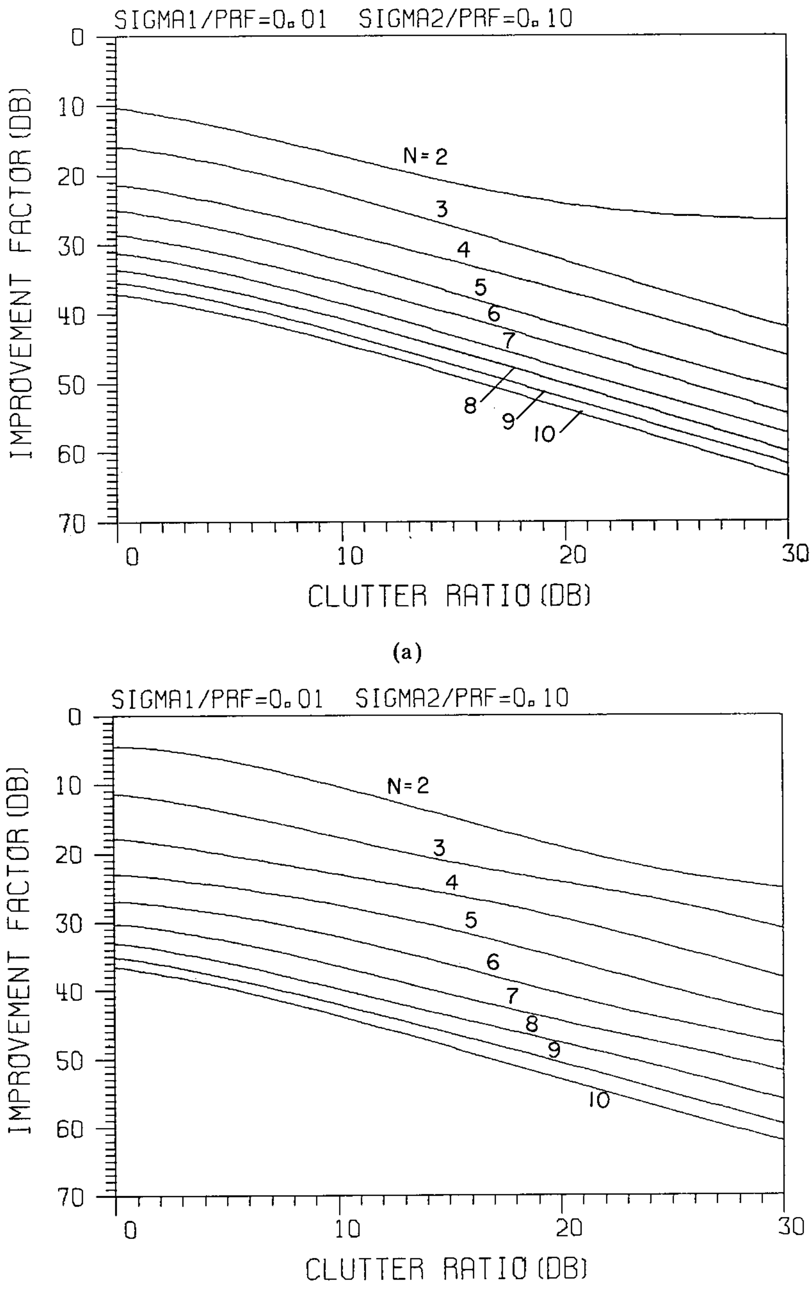
The improvement factor of this MTI filter is no better than 40 dB.

In the derivation of Eq. (45) it is only assumed that signal returns from the clutter types can be represented as independent random processes. No specific spectral density function is assumed. Therefore this result should be quite general.

Figures 1a and 1b show some computed results for a two-clutter case. The spectral density functions of both clutter types are assumed to be Gaussian functions which have the form

$$G(f) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{f^2}{2\sigma^2}\right) . \tag{46}$$

The standard deviation σ is normalized with respect to the radar prf. The first clutter type has a σ of 0.01, and the second clutter type has a σ of 0.1. The optimal MTI improvement factor is then computed by varying the clutter cross-section ratio. It is assumed that the first clutter type has a larger radar cross section. The clutter cross-section ratio is plotted in dB on the abscissa. The number of MTI canceling pulses varies from $N = 2$ to 10, and a family of curves is plotted. Figure 1a shows the case when the relative mean doppler frequency $f_0 = 0$; Fig. 1b shows the case for $f_0 = 0.25$. One can see that the improvement factor becomes better when the clutter ratio ρ increases, because the first clutter type is weighted more heavily. Since its spectral spread is narrower (it has a smaller standard deviation), the overall improvement factor becomes better, as



(a)

(b)

Fig. 1—Optimal MTI improvement factor vs clutter cross-section ratio for the two-clutter case (N = number of pulses)

indicated in Eq. (45). As a comparison, Fig. 2 shows the improvement factor for a single-clutter case. In this plot the improvement factor is plotted against the normalized standard deviation of the spectral density function of Eq. (46). One notices that for $\sigma = 0.01$, a two-pulse canceller can achieve an improvement factor of 28.2 dB, whereas for $\sigma = 0.1$, the improvement factor of this same two-pulse canceller is only 8 dB. If the two-clutter types have the same radar cross section (0-dB clutter ratio), one has

$$\lambda_{\min}(R) \geqslant \frac{1}{1+1}\left(10^{-2.82} + 10^{-0.8}\right)$$

$$\geqslant 0.08 .$$

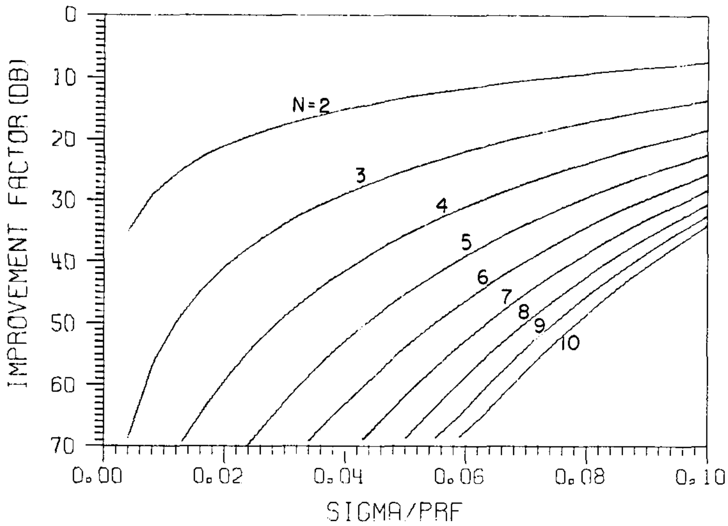


Fig. 2—Optimal MTI improvement factor vs clutter doppler standard deviation for the single-clutter case

The improvement factor of the two-clutter case would be less than 11 dB. One sees from Fig. 1a that the actual improvement factor is about 10.3 dB. When the relative mean doppler frequencies of the clutter types are assumed to be zero, the MTI filter response needs only a single notch to reject both clutter types. However, when the mean doppler frequencies of these two clutter types are different, two notches are required. One can see intuitively that this, in general, would cause an increase in the clutter output.

Comparing Figs. 1a and 1b, one sees that, in general, introducing a difference between the mean doppler frequencies of the two clutter types tends to degrade the improvement factor. This confirms the intuitive conjecture. In the next section this effect of the mean doppler frequency will be discussed in more detail.

EFFECT OF MEAN DOPPLER FREQUENCY

The covariance matrix element R_{ij} is a function of the relative mean doppler frequency between the clutter types, f_0, as shown in Eq. (27) which is now repeated:

$$R_{ij}(f_0) = \frac{1}{1 + \rho} \left\{ r_{ij}^{(1)} + \rho \exp\left[-j2\pi f_0 (i-j)\right] r_{ij}^{(2)} \right\} . \tag{47}$$

Since $i - j$ is an integer, $R_{ij}(f_0)$ is a periodic function, that is,

$$R_{ij}(f_0) = R_{ij}(n + f_0) ,$$

where n is an integer. Therefore we shall be concerned only with the case in which

$$0 \leqslant f_0 \leqslant 1 .$$

Furthermore, since

$$R_{ij}(f_0) = R_{ij}^*(1 - f_0)$$

and the eigenvalue can be expressed as

$$\lambda = |W|^T |R| |W^*| ,$$

where $|W|$ is the eigenvector associated with the eigenvalue λ, one has

$$\lambda^* = |W^*|^T |R^*| |W| ,$$

where $|R^*|$ is the conjugate of $|R|$. Because λ is real, changing the matrix R to its conjugate does not affect the eigenvalue. Therefore $\lambda_{\min}(R)$ must be a function which is symmetrical with respect to $f_0 = 0.5$. In the subsequent discussion consideration will be limited to the case in which

$$0 \leqslant f_0 \leqslant 0.5 .$$

Construct a series of matrices as follows:

$$A_2 = \begin{vmatrix} 1 & R_{12} \\ R_{21} & 1 \end{vmatrix} ; \tag{48a}$$

$$A_3 = \begin{vmatrix} 1 & R_{12} & R_{13} \\ R_{21} & 1 & R_{23} \\ R_{31} & R_{32} & 1 \end{vmatrix} ; \tag{48b}$$

$$A_N = \begin{vmatrix} 1 & R_{12} & \cdots & R_{1N} \\ R_{21} & 1 & \cdots & R_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ R_{N1} & R_{N2} & \cdots & 1 \end{vmatrix} ; \tag{48c}$$

where the R_{ij} are elements of the covariance matrix as defined in Eq. (47). According to the Sturmian Separating Theory [9],

$$\lambda_{\min}(A_2) \geqslant \lambda_{\min}(A_3) \geqslant \ldots \geqslant \lambda_{\min}(A_N), \tag{49}$$

where $\lambda_{\min}(A_n)$ represents the minimum eigenvalue of matrix A_n which is the covariance matrix of an MTI filter having n canceling pulses.

The minimum eigenvalue of A_2 can be computed directly as

$$\lambda_{\min}(A_2) = 1 - |R_{12}|. \tag{50}$$

According to Eq. (49), one finds

$$\lambda_{\min}(A_n) \leqslant 1 - |R_{12}|. \tag{51}$$

According to Gerschgorim's theorem [12],

$$|\lambda - R_{ii}| \leqslant \sum_{j \neq i} |R_{ij}| \tag{52}$$

for a certain index i. Since $R_{ii} = 1$ and $\lambda_{\min}$ is positive and less than unity, one has

$$\lambda_{\min}(A_n) \geqslant 1 - \sum_{j \neq i} |R_{ij}|. \tag{53}$$

Accordingly the $\lambda_{\min}(A_n)$ is bounded such that

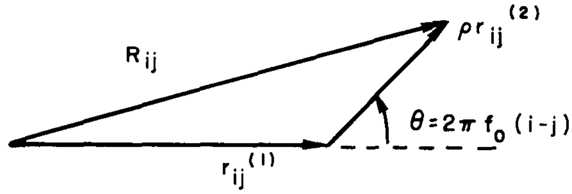
$$1 - |R_{12}| \geqslant \lambda_{\min}(A_n) \geqslant 1 - \sum_{j \neq i} |R_{ij}|. \tag{54}$$

The summation of $|R_{ij}|$ can be either greater than or less than unity. If this summation is greater than unity, the right side of the inequality of Eq. (54) becomes negative. However, since $\lambda_{\min}$ is always positive, the bound of $\lambda_{\min}$ then becomes

$$\lambda_{\min}(A_n) > 0, \quad \text{if} \quad \sum_{j \neq i} |R_{ij}| \geqslant 1. \tag{55}$$

Referring to Eq. (47), the amplitude of R_{ij} can be viewed as the amplitude of the sum of two vectors (Fig. 3). As was discussed earlier, in most cases both $r_{ij}^{(1)}$ and $r_{ij}^{(2)}$ are positive. Furthermore, without losing generality, one can assume that

$$r_{ij}^{(1)} > \rho r_{ij}^{(2)}.$$



$\theta = 2\pi f_0 (i-j)$

Fig. 3—Amplitude of the covariance matrix element R_{ij}

As f_0 increases from zero to 0.5, the vector $\rho r_{ij}^{(2)}$ sweeps around the tip of the vector $r_{ij}^{(1)}$. It is evident that $|R_{ij}|$ has a maximum value when $f_0 = 0$ and decreases to a minimum value when f_0 increases to 0.5. Therefore both the upper and lower bounds shown in Eq. (54) have a minimum value at $f_0 = 0$ and increase as f_0 increases from 0 to 0.5.

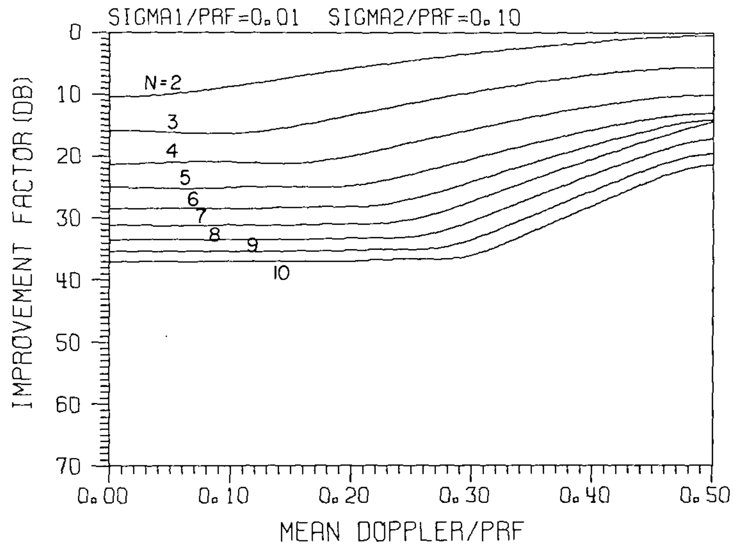
It has been pointed out earlier that when the correlation function R_{ij} decreases, the clutter output tends to increase. Therefore one can conjecture that not only the upper and lower bounds of $\lambda_{\min}(A_n)$ increase, but that the minimum clutter output $\lambda_{\min}(A_n)$ itself also increases as f_0 increases from 0 to 0.5.

In general the correlation function R_{ij} becomes negligibly small when the correlation time is sufficiently long. Therefore, for practical purposes the lower bound $1 - \Sigma_{i \neq j} |R_{ij}|$ of $\lambda_{\min}(A_n)$ reaches a limit when N is sufficiently large. In other words, increasing the number of pulses does reduce the minimum clutter output up to a point, but beyond that point a further increase does not yield a better result.

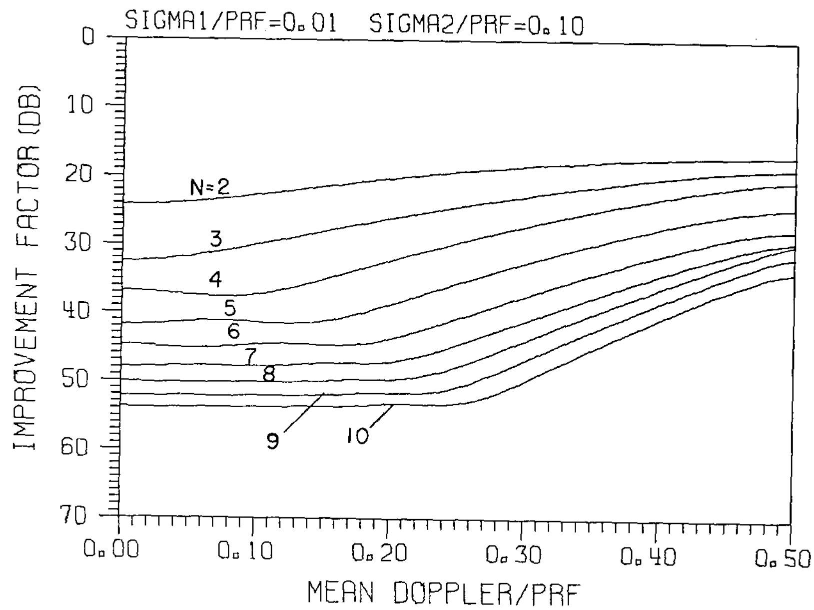
From the previous discussion one can arrive at the following conclusions:

- The minimum clutter output is bounded by two limits, as shown in Eq. (54).
- Both the upper bound and lower bound of Eq. (54) increase as f_0 increases towards 0.5.
- It is also conjectured that in general the minimum clutter output $\lambda_{\min}(R)$ increases as f_0 increases.
- Since the lower bound $1 - \Sigma_{i \neq j} |R_{ij}|$ reaches a limit as the number of radar pulses becomes infinite, the MTI improvement factor as a function of the number of radar pulses is bounded.

To illustrate these points, some computed results are shown in Figs. 4a and 4b. Again the clutter spectral density functions are assumed to be Gaussian. The normalized standard deviations of the two clutter types are respectively 0.01 and 0.1. Figure 4a shows the case when the clutter cross-section ratio is equal to 1. One notices from this plot that



(a) Clutter cross-section ratio = 1 dB



(b) Clutter cross-section ratio = 20 dB

Fig. 4—Optimal MTI improvement factor vs relative mean doppler frequency (N = number of pulses)

- The best improvement factor occurs when the mean doppler frequency is zero.
- Increasing the relative mean doppler frequency degrades the MTI performance. A worst case occurs when $f_0 = 0.5$.
- Increasing the number of radar pulses in general improves the performance; however, as the number of pulses becomes large, the incremental improvement in performance decreases rapidly.

Figure 4b shows the case when the clutter cross-section ratio is 20 dB. Since it is assumed that the clutter type with a lower standard deviation has a larger radar cross section, the improvement factor in general is better in this case. However, other properties are similar to those of Fig. 4a.

CONCLUSION

In this report the design of an optimal MTI filter to reject multiple clutter types is presented. It is found that the optimal performance of such a filter depends on the spectral density functions, average radar cross sections, and the relative mean doppler frequencies of these clutter types.

It is shown that the optimal improvement factor of such a filter is bounded by the weighted average of the improvement factors for the individual clutter types. The average is weighted in accordance with the radar cross section of the clutter types. It is also shown that the minimum clutter output (the reciprocal of the optimal improvement factor) of such a filter is bounded by two limits. Both of these limits are functions of the relative mean doppler frequency of the clutter types, f_0. As this f_0 increases towards 0.5, the performance of the MTI system degrades. The worst performance occurs when f_0 is equal to half of the radar prf.

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