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Satellite Spin Axis Control

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Satellite attitude control comprises two parts: determining the orientation in inertial space and positioning a spin axis in this space. Two measured spherical angles of the sun and one computed angle for the earth are used in describing acquisition of the spin axis, avoiding ambiguities in the calculated angles. Maneuvering the spin axis by intermittent thruster pulses can be determined from expressions for rigid-body dynamcis adjusted for the motion of the satellite and its spin axis. Errors and error estimates that enter both parts have been treated; nonimpulsive thrusts and nonconstant spin rates were also considered.

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SATELLITE SPIN AXIS CONTROL

INTRODUCTION

The direction of the satellite spin axis, or *attitude*, is determined by measuring the *solar* aspect angle and the roll angle (azimuth difference) between planes containing the satellite spin axis and the earth and sun vectors, and calculating the earth aspect angle using this data plus the position vectors of the satellite and the sun. The solution of a spherical triangle provides the directions of an orthogonal triad of unit vectors in the satellite reference system (SR); the same triad defined in the local vertical system (LV) allows one to obtain the spin axis in that system from the linear transformations between reference systems.

The spin axis can be brought to any desired direction in LV by means of attitude control (AC) thrusters that provide periodic impulses on a selected azimuth in SR; the torque is applied at right angles to the spin angular momentum on a time schedule that can be obtained from expressions of rigid-body dynamics.

An important purpose of this study is the control of errors, and most of the novel features arise in this connection. First, the earth aspect angle is found unambiguously and, combined with error estimates on the measured angles, pointing error bounds in LV are given. Second, because of judicious choice in the case of one of the SR directions, a matrix inversion involves negligible computational error. Third, the program for precessing the spin axis to a direction specified in SR is developed, with consideration given to accuracy and duration of satellite maneuvers.

GEOMETRY AND NOTATION

The two quantities obtained from the satellite are shown in Fig. 1: the solar aspect (or polar) angle α and the roll angle (or azimuth difference) $\beta = 2\pi\nu(\Delta t)$, where ν is the satellite spin rate and Δt is the time delay between solar and terrestrial radiation detector pulses (note the rotation direction marked on the spin axis, conventionally negative). Earth T has been shown in the (rotating) xz plane to exhibit the roll angle β. The position vectors of the satellite at Q and the sun at S are $\overline{TQ} = \mathbf{r}$ and $\overline{TS} = \mathbf{R}$, respectively. From the satellite, one can locate the earth at $-\mathbf{r}$ and the sun at $\mathbf{R}' = \mathbf{R} - \mathbf{r}$. Unit vectors in these directions will be designated $-\hat{r}$ and $\hat{R}'$. The angle between $\overline{QS}$ and $\overline{QT}$ is γ,

$$\hat{R}' \cdot (-\hat{r}) = \cos\gamma .$$

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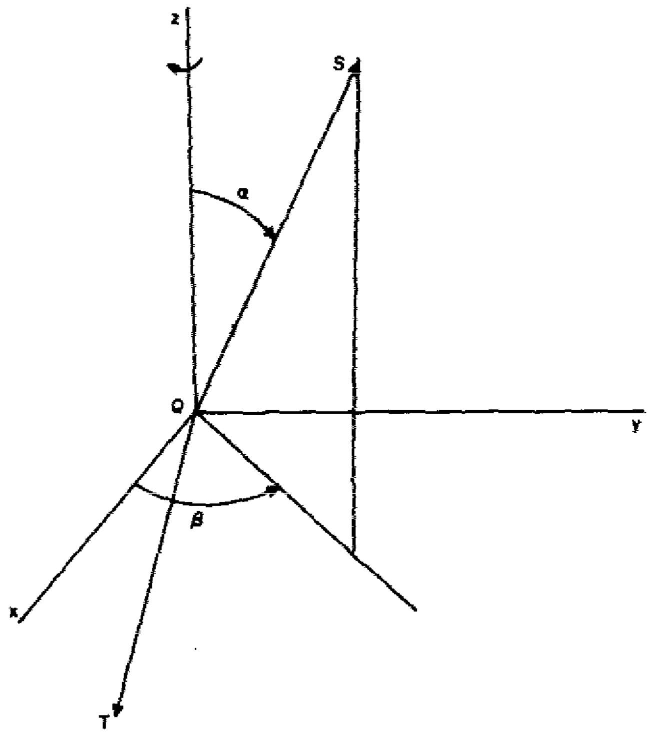

Fig. 1—Solar aspect angle α and the roll angle β in the satellite reference system

The angle between the spin axis QZ and $\overline{QT}$ is γ',

$$\hat{Z} \cdot (-\hat{r}) = \cos\gamma' .$$

These angles are drawn on the unit sphere centered at Q and produce the spherical triangle shown in Fig. 2; γ' is the earth aspect angle that is used in the SR. For use in solving the spherical triangle, dihedral angles opposite α and γ' (denoted α' and β', respectively) are introduced.

The direction of the spin axis is obtained from the linear transformations between an orthogonal triad constructed from $-\hat{r}$ and $\hat{R}'$ and those of the two reference systems SR and LV. A unit vector $\hat{q}$ is defined by

$$-\hat{r} \times \hat{R}' = \hat{q} \sin\gamma ,$$

and the intermediate triad is completed with

$$\hat{q} \times (-\hat{r}) = \hat{p} .$$

Using

$$-\hat{r} = \hat{x}\cos\left(\gamma' - \frac{\pi}{2}\right) + \hat{z}\cos\gamma'$$

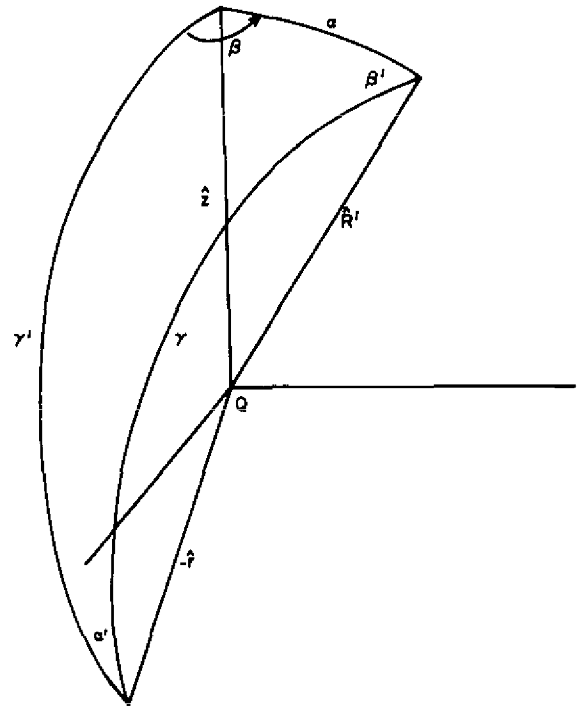

Fig. 2—Spherical triangle extracted from Fig. 1; γ' is the earth aspect angle.

$$\hat{R}' = (\hat{x}\cos\beta + \hat{y}\sin\beta)\sin\alpha + \hat{z}\cos\alpha\,,$$

the triad $(-\hat{r}, \hat{p}, \hat{q})$ can be obtained immediately in SR. The transformation can be written

$$(-\hat{r},\hat{p},\hat{q})^T = \mathsf{K}'(\hat{x},\hat{y},\hat{z})^T$$

where K' is the direction cosine matrix and where column vectors have been represented (for typographical convenience) as the transpose of row vectors. Another direction cosine matrix is available for producing the intermediate triad from LV, the transformation in this case being

$$(-\hat{r},\hat{p},\hat{q})^T = \mathsf{K}(\hat{u},\hat{v},\hat{w})^T;$$

unit vectors in LV are defined with $\hat{u}$ parallel to $\hat{r}$, $\hat{w}$ parallel to the orbital angular momentum

$$\mathbf{L} = \mathbf{r} \times \frac{m\,d\mathbf{r}}{dt}\,,$$

and then

$$\hat{v} = \hat{w} \times \hat{u}\,.$$

Therefore **K** represents a reversal of (not a reflection in) $\hat{r}$ and a rotation about $-\hat{r}$, so that an inverse $\mathbf{K}^{-1}$ can be easily written. The spin axis in LV has direction cosines $(\hat{u}\cdot\hat{z}$, $\hat{v}\cdot\hat{z}$, $\hat{w}\cdot\hat{z})$, which will be obtained as the transpose of $\mathbf{K}^{-1}\mathbf{K}'(0, 0, 1)^T$.

Suppose the spin axis $\hat{z}$ must be processed to a direction $\hat{z}'$ by rotating the angular momentum vector about the direction of

$$\hat{z} \times \hat{z}' = \hat{y}' \sin\theta' .$$

The angular impulse

$$\Delta r(-\hat{y}' \times -\hat{z})F\,dt = \hat{x}'\,dJ ,$$

due to the AC thruster delivering an impulse $-\hat{z}F\,dt$ at a point $-\hat{y}'\,\Delta r$ from Q, can be represented as

$$\hat{x}'J\,d\theta' = d\theta'(\hat{y}' \times \hat{z})2\pi\nu I_z = -d\theta'(\hat{y}' \times \mathbf{J}) .$$

The change of angular momentum due to rotation of axes is the same, except that here the spin and angular momentum vectors are antiparallel. From Fig. 3 it can be seen that if the AC thruster is located on the y axis, it should be pulsed

$$\frac{\pi - \varphi'}{2\pi\nu} = \Delta t'$$

after the earth sensor has initiated a reference pulse. These pulses, of length $\delta t'(<< 1/\nu)$, continue until θ' (the polar angle of $\hat{z}'$, relative to $\hat{z}$) is reduced to zero. Because

$$\delta\theta' = \frac{(F\,\Delta r)\,\delta t'}{2\pi\nu I_z}$$

is used for the increment in θ', for $\Delta\theta'$ the angular displacement of the maneuver, the duration is

$$\frac{\Delta\theta'}{\nu\delta\theta'} = \frac{2\pi I_z \Delta\theta'}{(F\Delta r)\delta t'} = \Delta\tilde{t} .$$

Spherical Triangles

The direction cosine matrix $\mathbf{K}'$ requires the angle γ', shown in Fig. 2 for $\beta < \pi$. One might try the cosine relation

$$\cos\gamma = \cos\alpha\cos\gamma' + \sin\alpha\sin\gamma'\cos\beta ;$$

although $\sin\gamma > 0$ because $\gamma' < \pi$, $\cos\gamma'$ can be of either sign:

$$(\cos\gamma - \cos\alpha\cos\gamma')^2 = \sin^2\alpha(1 - \cos^2\gamma')\cos^2\beta ,$$

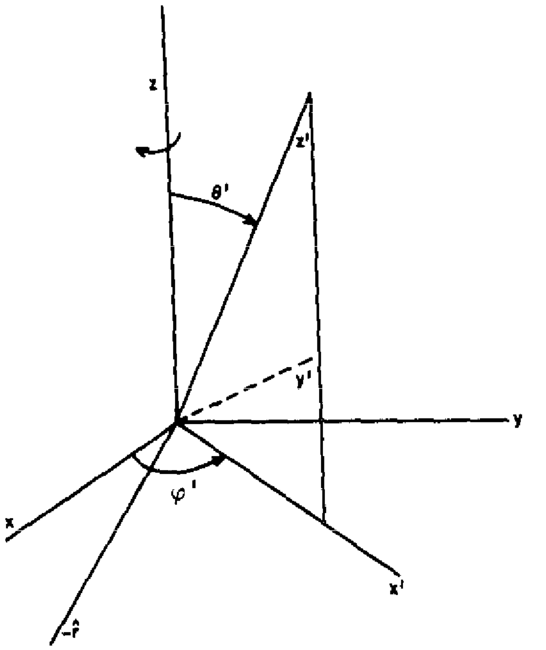

Fig. 3—The spin angular momentum vector $2\pi\nu(-\hat{z})I_z$ is moved to the direction $(-\hat{z}')$ by rotation about $\hat{y}'$ through the angle θ'.

a quadratic in $\cos\gamma'$ with the ubiquitous $\pm$. (Alternative solutions essentially result from the definition of $\beta(\text{mod } 2\pi)$). Further progress with this triangle can be obtained by introducing β' (between α and γ), α' (opposite α), $< \pi/2$ (when $\alpha < \pi/2$), plus the sine relation

$$\sin\alpha' : \sin\alpha = \sin\beta : \sin\gamma = \sin\beta' : \sin\gamma' .$$

The latter gives $\sin\alpha'$ and $\alpha' < \pi/2$, but γ' is still combined with β'. The relation

$$\cos\gamma' = \cos\alpha\cos\gamma + \sin\alpha\sin\gamma\cos\beta'$$

does not alter this situation.

The way out of the ambiguity is to find

$$\alpha' = \arcsin\left(\frac{\sin\alpha\sin\beta}{\sin\gamma}\right) < \pi/2$$

and use it in the two cotangent identities

$$\sin\alpha\cot\gamma = \cos\alpha\cos\beta' + \cot\beta\sin\beta'$$

$$\sin\gamma\cot\alpha = \cos\gamma\cos\beta' + \cot\alpha'\sin\beta'$$

(see, for example, D. S. Meyler and O. G. Sutton, *A Compendium of Mathematics and Physics*, D. van Nostrand, London, 1957, p. 66). Simultaneous solution results in

$$\cos\beta' = \frac{\sin\alpha\cot\gamma\cos\alpha' - \cos\alpha\cos\beta}{\cos\alpha\cos\alpha' - \sin\alpha\cos\beta\cot\gamma}$$

$$\sin\beta' = \frac{\sin\beta\sin\gamma - \sin\alpha\sin\alpha'}{\cos\alpha\sin\gamma\cos\alpha' - \sin\alpha\cos\beta\cos\gamma}$$

giving β' unambiguously. Finally, $\sin\gamma'$ and $\cos\gamma'$ are found from these relations, thus γ' may be determined.

The case $\pi < \beta < 2\pi$ is easily handled: if in Fig. 2 β is replaced by $2\pi - \beta$, signs of β *and* β' are reversed simultaneously in the trigonometric factors. With these modifications, γ' is found from expressions for $\sin\gamma'$ and $\cos\gamma'$.

Direction Cosine Matrices

The matrix elements of $\mathbf{K}'$ are found as previously indicated; the first row $(-\hat{r}\cdot\hat{x}, -\hat{r}\cdot\hat{y}, -\hat{r}\cdot\hat{z})$ has been given already. Further,

$$\hat{p}\cdot\hat{x} = (\hat{q}\cdot\hat{y})\cos\gamma'$$

$$\hat{p}\cdot\hat{y} = (\hat{q}\cdot\hat{z})\sin\gamma' - (\hat{q}\cdot\hat{x})\cos\gamma'$$

$$\hat{p}\cdot\hat{z} = -(\hat{q}\cdot\hat{y})\sin\gamma'$$

where

$$\hat{q}\cdot\hat{x} = -\sin\alpha'\cos\gamma' ,$$

$$\hat{q}\cdot\hat{z} = \sin\alpha'\sin\gamma' ,$$

$$\hat{q}\cdot\hat{y} = \frac{\sin\alpha\cos\beta\cos\gamma' - \sin\gamma'\cos\alpha}{\sin\gamma} .$$

The array for $\mathbf{K}$ is written in the same way:

$$\hat{q}\cdot\hat{w} = \cos\psi = -\hat{p}\cdot\hat{v} ,$$

$$\hat{p}\cdot\hat{w} = \sin\psi = \hat{q}\cdot\hat{v} ,$$

$$\mathbf{K} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\psi & \sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix}$$

where ψ is the angle between w and q. The inverse of this matrix is the same as $\mathbf{K}$, as one can show algebraically; it can also be seen using rotation matrices $\mathbf{R}_a(b)$ for rotation through the angle b about the a axis:

$$\begin{aligned} \mathbf{K} &= \mathbf{R}_{w'}(\pm\pi)\mathbf{R}_u(-\psi) \\ &= \mathbf{R}_{q'}(\pm\pi)\mathbf{R}_{-r}(+\psi) \\ &= \mathbf{K}^{-1} \quad \text{where} \quad (w' = q \text{ and } q' = w). \end{aligned}$$

The angle ψ is found from

$$\cos\psi \sin\gamma = \hat{w} \cdot (-\hat{r} \times \hat{R}') = -\hat{v} \cdot \hat{R}'$$

$$(\hat{w} \cdot \hat{R}') \cos\gamma = \hat{w} \cdot (\hat{q} \times \hat{R}') \sin\gamma = (\sin\psi \sin\gamma) \cos\gamma .$$

The direction cosines of the spin axis in LV are found to be

$$\hat{v} \cdot \hat{z} = (\hat{q} \cdot \hat{z}) \sin\psi + (\hat{q} \cdot \hat{y}) \sin\gamma' \cos\psi$$

and

$$\hat{w} \cdot \hat{z} = (\hat{q} \cdot \hat{z}) \cos\psi - (\hat{q} \cdot \hat{y}) \sin\gamma' \sin\psi$$

where

$$\hat{q} \cdot \hat{z} = \sin\alpha' \sin\gamma' ,$$

$$(\hat{q} \cdot \hat{y}) \sin\gamma = \sin\alpha \cos\beta \cos\gamma' - \sin\gamma' \cos\alpha ,$$

$$\sin\psi = \frac{\hat{w} \cdot \hat{R}'}{\sin\gamma}$$

and

$$\cos\psi = \frac{(\hat{v} \cdot \hat{R}')}{-\sin\gamma} .$$

By definition,

$$\hat{u} \cdot \hat{z} = -\cos\gamma' .$$

Pointing Errors

The direction cosines are imprecise because of uncertainties in the basic parameters: $\delta\gamma$, due to $\delta\mathbf{r}$ (satellite position uncertainty—actually, the distance of the sun from the satellite contains the significant error for $-\hat{r}$); $\delta\beta = 2\pi\nu\ \delta t$, due to the inaccuracy in determining the time interval between two radiation detector pulses; and $\delta\alpha$, due to the arrangement of solar detector components and the treatment of pulses originating from the various elements. These three angular uncertainties are considered basic; there are intermediate angles (those with primes) appearing in the direction cosine formulas whose uncertainties can be derived, if needed. There is a dependence of $\delta\beta$ on α:

$$\delta\beta = \frac{0\overset{\circ}{.}5}{\sin\alpha}$$

(from a study of the manufacturer's calibration data), because the pulse shape depends on the radiation entering the detector; a careful arrangement of solar detectors near $\alpha = 0$ or treatment of the pulses produced there can reduce this dependence. Otherwise, the uncertainties can be linerarly superposed in the relations that follow.

There are two methods available to determine the statistics for $\delta\hat{z}$ (an abbreviated notation for the distribution of $\hat{z}$ direction cosines): stochastic, using random variations to the nominal values of the basic parameters; or deterministic, transforming error bounds through the equations. For a first-order estimate, the second method is preferable and recommended; unsatisfactory results can be adjusted by decreasing uncertainties for selected parameters. An analytical procedure for obtaining the pointing error will now be developed, based on the direction cosine relation

$$l_0 l_j + m_0 m_j + n_0 n_j = \cos \epsilon_j$$

where

$$l_j = \hat{u} \cdot \hat{z}, \qquad m_j = \hat{v} \cdot \hat{z}, \qquad n_j = \hat{w} \cdot \hat{z},$$

and the subscript zero designates the nominal case (not without uncertainties).

The equations leading from (α, β, γ) to (l_j, m_j, n_j) are arranged, the subscript j indicating what uncertainties have been assigned to the basic parameters. Six or eight cases can be considered (for three parameters), depending on whether the variations are singly or triply applied: think of a cube, with center representing the nominal case—single variations correspond to face centers and multiple variations to the corners. Next, the six cases can be examined in pairs ($\gamma \pm \delta\gamma/2$, etc.) to discover which parameter variation has the largest effect. Finally, the pointing error can be determined from

$$\cos \epsilon = \frac{1}{8} \sum_j \cos \epsilon_j = \frac{1}{8} \sum_j (l_0 l_j + m_0 m_j + n_0 n_j)$$

where configurations j involve variations to all three parameters, a total of $2^3 = 8$ cases. The summands have the same signs for reasonably small (i.e., acceptable) uncertainties in the basic parameters.

Some remarks on this choice of method might be useful. The angles ϵ_j are, in fact, differences; two-dimensional representation (on the unit sphere) has been simplified, leaving the average ϵ (or root mean square, for small variations to α, β, and γ) a measure of the pointing accuracy. The effects of separate variations (the $2 \times 3 = 6$ cases) should be handled first, if only to verify the average error as meaningful: weighting the different uncertainties would produce an asymmetric figure to be oriented in LV. (This would be the case for small α; the hypercone would have a larger semivertex angle corresponding to larger $\delta\beta$.) It should be realized that the distributions of errors are probably different in each parameter, which means that combination of root mean squares may be misleading. Because the intention of the error analysis is to determine whether the pointing error exceeds half a degree or so, these minutiae should be of marginal usefulness.

SPIN AXIS DISPLACEMENT

The satellite spin axis is unaffected by the main gravitational field (inverse square term) because it is in free fall. In the absence of other *external* forces, the reference system aligned with the spin axis (assumed to be aligned with the angular momentum), and passing through the centroid Q, is equivalent to an *inertial* reference. Then

$$\mathbf{N} = \frac{d\mathbf{J}}{dt} = \frac{d(\hat{J}|\mathbf{J}|)}{dt} = \frac{|\mathbf{J}|d\hat{J}}{dt} + \frac{\hat{J}d|\mathbf{J}|}{dt}$$

because

$$\mathbf{N} = \Delta r[(-\hat{y}') \times (-\hat{z})]F$$

produces $d\hat{J}/dt$ perpendicular to $\hat{J}$ and leaves

$$|\mathbf{J}| = 2\pi\nu I_z$$

unchanged. The case for misaligned AC thrusters must be treated using Euler's equations of rigid-body dynamics (see Appendix).

The rotation of $\hat{J}$ about $\hat{y}'$, moving $\hat{z}$ to $\hat{z}'$ by means of torques applied in SR, tilts the xz reference plane; in addition, the orbital motion of the satellite will shift $-\hat{r}$ during the course of the maneuver. The rotation of $\hat{x}$ about $\hat{z}$ is

$$(\hat{w} \cdot \hat{z})\frac{d\varphi}{dt} - 2\pi\nu = \omega_3$$

where φ is the true anomaly of the satellite and $\hat{w} \cdot \hat{z} = \cos\theta$. The largest value of $\dot{\varphi}$ is for the synchronous orbit, when

$$\dot{\varphi} = 0.73 \times 10^{-4} \text{ rad/s.}$$

Therefore, the correction to the AC thrusting rate $1/\nu$ is

$$\frac{2\pi}{\omega_3} - \frac{1}{\nu} = \left(\nu - \frac{\dot{\varphi}}{2\pi}\cos\theta\right)^{-1} - \frac{1}{\nu} = \frac{\frac{\dot{\varphi}\cos\theta}{2\pi}}{\nu\left(\nu - \frac{\dot{\varphi}}{2\pi}\cos\theta\right)} ,$$

less than timing errors δt connected with $\delta\beta = 2\pi\nu\ \delta t$. The change of φ' due to $\dot{\varphi}$ during the maneuver may not be negligible, as the following estimate shows: a 40-min maneuver with satellite orbital inclination less than 30° relative to the ecliptic plane produces (ν = 60 RPM = 1 Hz)

$$\Delta\varphi' \gtrsim 40 \times 60 \times 0.73 \times 10^{-4} \times \frac{\sqrt{3}}{2} \doteq \frac{1}{4} \times \frac{3}{4} \times \frac{4}{5} \text{ rad} \approx 10^\circ .$$

A method of determining the azimuth φ' will now be described. For the motion of $\hat{z}$, one needs the final spin axis $\hat{z}'$, the rotation axis $\hat{y}'$ (fixed if $\hat{z} \to \hat{z}'$ along a circumference of the unit sphere), and the angle θ'. These quantities are combined to give

$$\hat{z} = R_{y'}(-\theta')\hat{z}' .$$

Starting at $\hat{z}'$, a rotation about $\hat{y}'$ through $-\theta'$ produces $\hat{z}$. In the same manner,

$$-\hat{r} = R_w(\varphi)\,(-\hat{\pi})$$

where $-\hat{\pi}$ is a unit vector in the direction of satellite apogee, $\hat{w}$ is parallel to the orbital angular momentum, and φ is the true anomaly of the current position. The unit vectors $-\hat{r}$ and $\hat{z}$ are then combined to give

$$-\hat{r} \cdot \hat{z} = \cos\gamma' ,$$

$$\hat{y}' \cdot [\hat{z} \times (-\hat{r})] = (\hat{y}' \cdot \hat{y}) \sin\gamma' = \cos\varphi' \sin\gamma' ,$$

$$\hat{y}' \times [\hat{z} \times (-\hat{r})] = \hat{y}' \times (\hat{y} \sin\gamma') = -\hat{z} \sin\varphi' \sin\gamma'$$

$$= \hat{z}[\hat{y}' \cdot (-\hat{r})] + (\hat{y}' \cdot \hat{z})\hat{r} = -\hat{z}(\hat{r} \cdot \hat{y}') .$$

From $\cos\gamma'$ one obtains $\gamma' < \pi$ (unambiguously), whence $\sin\gamma'$ is available for the remaining equations, thus giving φ'.

RELAXATION OF $\delta t' << 1/\nu$

In this section the adjustments for $\delta t'$ when it is not small *vs* $1/\nu$ are developed, and optimum values of the AC thruster pulse are explored. The time delay

$$\Delta t' = \frac{\pi - \varphi'}{2\pi\nu}$$

has been determined for the impulsive case, while in fact the AC thruster has rolled through

$$2\pi\nu\, \delta t' = \delta\varphi' .$$

If

$$-\varphi' - \frac{\pi}{2} = \tilde{\varphi}$$

represents the AC thruster azimuth in SR for the impulsive case ($\delta t' << 1/\nu$),

$$\delta J = \int dt'(F\Delta r)\cos(\widetilde{\varphi} - \varphi')$$

$$= F\Delta r \int \frac{d\varphi'}{2\pi\nu}\cos(\widetilde{\varphi} - \varphi')$$

$$= \frac{F\Delta r}{2\pi\nu} 2 \sin\frac{\delta\varphi'}{2}$$

is the result for arbitrary $\delta t'$. The net rotation of **J** is $\delta\theta' = \delta J/|\mathbf{J}|$, but reduced somewhat from the previous value. Expanding

$$2\sin\frac{\delta\varphi'}{2} = \delta\varphi' - \frac{1}{3}\left(\frac{\delta\varphi'}{2}\right)^3 + \ldots$$

gives

$$\delta J = \frac{F\Delta r}{2\pi\nu}\left[\delta\varphi' - \frac{(\delta\varphi')^3}{24} + \ldots\right] = F\Delta r\left[1 - \frac{(\delta\varphi')^2}{24} + \ldots\right]\delta t',$$

the second term in square brackets being the correction factor. For

$$\nu = 1 \text{ Hz and } \delta t' = 0.^s06, \qquad \frac{(\delta\varphi')^2}{24} = 0.0006\pi^2 = 0.006;$$

this indicates that $\Delta\widetilde{t}$ is about 0.6% longer than in the impulsive approximation.

The optimum value of $\delta t'$ must lie between zero (minimum fuel waste) and $1/(2\nu)$ (minimum duration for maneuver). Let

$$\dot{J} \equiv \frac{\delta J}{\delta t'} = 2(F\Delta r)\sin\frac{\delta\varphi'/2}{\delta\varphi'}$$

represent the efficiency condition and

$$\dot{\theta}' = \nu\delta\theta' = F\Delta r\frac{\sin\dfrac{\delta\varphi'}{2}}{2\pi^2\nu I_z}$$

the efficacy condition. Optimum $\delta t'$ would maximize the product

$$\dot{J}\cdot\dot{\theta}' = \frac{(F\Delta r)^2}{2\pi^2\nu I_z}\,\frac{\sin^2(\delta\varphi'/2)}{\delta\varphi'/2}$$

Substituting

$$S = \frac{\delta\varphi'}{2} = \pi\nu\delta t'$$

in radians, $(\sin S)^2/S$ is extremal for S_0 such that

$$0 = \frac{d}{dS}\,\frac{\sin^2 S}{S} = \frac{-1}{S^2}\sin^2 S + \frac{2}{5}\sin S\cos S = \frac{\sin S}{S}\left(2\cos S - \frac{\sin S}{S}\right),$$

with solutions $S_0 = k\pi$ (k an integer different from zero) and the zeros of

$$2S_0 - \tan S_0 = 0 \neq S_0 .$$

The transcendental equation has its first zero somewhat greater than $\pi/4$ (at about 1.165 rad or 66.75°), a much larger value than the $0.06\pi = 0.19$ used. A plot of

$$\frac{\dot{J}}{F\Delta r} = \frac{2}{\delta\varphi'}\sin\frac{\delta\varphi'}{2}$$

is shown in Fig. 4, which illustrates the decrease of the correction factor with increasing $\pi\nu\delta t' = \delta\varphi'/2$; it can be seen that fuel conservation is an important consideration.

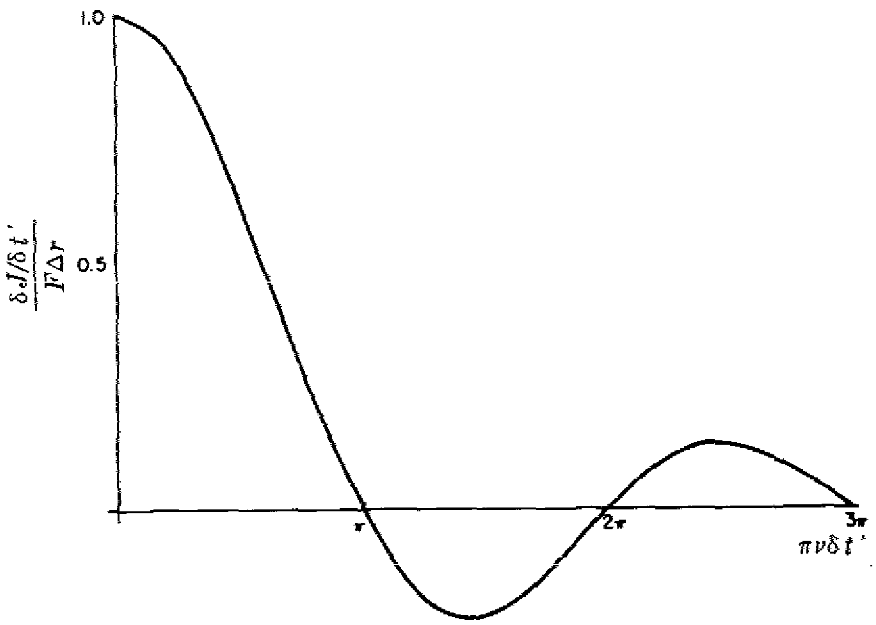

Fig. 4—Normalized thrusting rate, equal to the correction factor $(\delta J/\delta t')/F\Delta r$ for nonimpulsive thrusting; the duration of one pulse $\delta t' = \delta\varphi'/(2\pi\nu)$

The term

$$\frac{\partial \mathbf{J}}{\partial t} = \hat{x}' \frac{d}{dt}(I_x w_1) + \hat{y}' \frac{d}{dt}(I_y w_2) + \hat{z} \frac{d}{dt}(I_z w_3)$$

involves partial differentiation neglecting the motion of the principal axes. The Euler equations are

$$I_x \dot{w}_1 + w_2 I_z w_3 - w_3 I_y w_2 = N_1 = y' F_z - z F_{y'} ,$$

$$I_y \dot{w}_2 + w_3 I_x w_1 - w_1 I_z w_3 = N_2 = z F_{x'} - x' F_z ,$$

$$I_z \dot{w}_3 + w_1 I_y w_2 - w_2 I_x w_1 = N_3 = x' F_{y'} - y' F_{x'} ,$$

with an obvious notation for the torques described in the body reference system.

THRUST MISALIGNMENT

The torques are now perturbed by thrust misalignment, which can enter through $\delta(\Delta\mathbf{r})$, variation in the point of application $\Delta\mathbf{r}$, and also the direction $\delta\hat{F}$ of the nozzle. These quantities can be used with the structural tolerances estimated by the fabricator, but they must be put into Cartesian form for use here.* The thrust $-\hat{z}F$ is nominally applied at $\Delta\mathbf{r} = -\hat{y}'\Delta r + \hat{z}l$ (relative to the centroid Q), with $\delta(\Delta r)$ distributed on a disk in the $x'y'$ plane and δF distributed within a cone whose axis is along $-\hat{z}$. Then

$$\delta N_1 < F\delta\,(\Delta r) + lF\,\delta\hat{F} = F[(l\,\delta\hat{F} + \delta\,(\Delta r)] > \delta N_2$$

(to first order)

$$\delta N_3 < 2\delta\,(\Delta r)\,F\,\delta\hat{F}$$

(of second order). The effect of the misalignment is to introduce an uncertainty in Δr less than $l \times$ (semivertex angle or nozzle angle error) + (lateral error disk radius), a torque of magnitude less than F times this same quantity in the second equation (involving $\dot{w}_2$), and a somewhat smaller torque δN_3 perturbing w_3. Because the body is nearly axisymmetric ($I_x \doteq I_y$), the last Euler equation shows that $w_3 = \dot{\psi} + \dot{\varphi}\cos\theta$ is least perturbed by thrust misalignment; under the same symmetry conditions, the "wobble" arising from misalignment or the situation that $\vec{w} \neq \hat{z}\dot{\psi}$ initially can be treated directly using the Euler equations.

"WOBBLE"

In the absence of torques, the angular momentum is invariant in an inertial description. It is possible for spin angular momentum ($\mathbf{S} = \hat{z}I_z w_3$) to contribute to the total angular

*Compare the treatment of R. S. Armstrong, "Errors Associated With Spinning-Up and Thrusting Symmetric Rigid Bodies," Jet Propulsion Laboratory Technical Report 32644, Pasadena, Calif., Feb. 15, 1965, Sect. IV-A.

momentum $\mathbf{J}$, and when $\hat{z} \neq \hat{J}$, there is usually a relative motion of these two vectors. This bears a resemblance to the precession of a body spinning about a constrained axis due to gravitational torques, illustrating the *gyroscopic* effect of applied forces. In free-fall, motion of the spin axis about the invariant angular momentum direction is due to nonzero $\dot{\varphi}_0$ and/or $\dot{\theta}_0$, initial values of precession and nutation that are damped out if the spinning body is supported.

For the case $I_x = I_y$, w_1 and w_2 are essentially the same except for a phase factor represented by the functions of ψ. If

$$\exp(\pm i\psi) = \cos\psi \pm i \sin\psi ,$$

is used,

$$w_1 \pm iw_2 = (\dot{\theta} \pm i\dot{\varphi} \sin\theta) \exp(\pm i\psi) \equiv w_\pm$$

is a convenient combination, using complex number notation ($i^2 = -1$). If

$$I_x = I' = I_y ,$$

the equations

$$I'\dot{w}_\pm \mp iw_3(I_z - I')w_\pm = N_\pm \equiv N_1 \pm iN_2$$

and

$$I_z\dot{w}_3 = I_z \frac{d}{dt}(\dot{\varphi}\cos\theta + \dot{\psi}) = N_3 = x'F_{y'} - y'F_{x'}$$

are produced from the Euler equations to facilitate the discussion of "wobble" and its relation to the spin rate.

Notice first that in the absence of torques, $\dot{w}_3 = 0 = N_\pm$, and

$$\frac{dw_\pm}{w_\pm} = \pm \frac{iw_3}{I'}(I_z - I')dt$$

integrates immediately to

$$ln\left(\frac{w_\pm}{w_0}\right) = \pm iw_3 t\left(\frac{I_z}{I'} - 1\right) ,$$

$$w_\pm = w_0 \exp\left[\pm iw_3 t\left(\frac{I_z}{I'} - 1\right)\right] ,$$

$$w_0^2 = (w_1 + iw_2)(w_1 - iw_2) = w_1^2 + w_2^2 = \dot{\theta}_0 + \dot{\varphi}_0^2 \sin^2\varphi_0 .$$

In the constants w_0 and $\dot{\varphi}_0 \cos\theta_0 + \dot{\psi}_0$, it is possible to take $\dot{\varphi}_0 \neq 0 = \varphi_0$ and $\dot{\theta}_0 = 0 \neq \theta_0$ because of the axial symmetry of the body. (Alternatively, it is possible to take $\dot{\theta}_0 \neq 0 = \theta_0$ and $\dot{\varphi}_0 = 0 \neq \varphi_0$. In this case spin about the body $\hat{z}$ axis is combined with tumbling about a space axis.) The angular velocity in addition to the spin is "wobble."

Finally, it is of interest to examine how wobble and spin are influenced by thrust misalignments. The spin rate is affected least because δN_3 is of second order; however,

$$N_{\pm} = F[l \cdot \delta \hat{F} + \delta(\Delta r)](1 + i)$$

produces an impulsive change to the harmonic solution

$$\delta w_{\pm} = \frac{N_{\pm}\delta t'}{I'}$$

This result is obtained from the nonhomogeneous solution

$$w'_{\pm} = \frac{N_{\pm}}{\mp i w_3 (I_z - I')}\left[1 - \exp\left(\pm\, i w_3 t \frac{I_z - I'}{I'}\right)\right]$$

by setting $t = \delta t'$, the duration of a pulse, and expanding the exponential function to obtain a first-order factor. Because the AC thrusters are fired periodically, $\delta w_{\pm}$ can produce a resonance change to w_0, and this has been used in a wobble damper that decreases $|w_0|$ by modifying the firing schedule. The spin angular momentum is thereby brought to a new direction at the same time the wobble angle (between $\hat{J}$ and $\hat{z}$) is decreased, without changing the spin rate appreciably.