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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The basic equations of classical hydrodynamics of fluids, together with constitutive relations of the medium, have been reviewed. By perturbation expansions the nonlinear equations of acoustic wave propagation currently in use were derived using the acoustic Mach number and Strouhal number as measures of scale (method of Gol'dberg). Several derivations of Burgers'-type equations have been presented, together with a review of solution methods introduced by Hopf and Cole and of the method of characteristics. The classic problem of a piston in a tube was

(Continued)

reviewed, with the theoretical extensions of Blackstock. The complexity of all methods led to a search for new approaches. A promising development is Whitham's theory of nonlinear partial differential equations. Elements of this theory are discussed. It is concluded that theoretical difficulties in nonlinear acoustics remain to be solved.

CONTENTS

NONLINEAR ACOUSTICS: PART I, PHYSICAL PHENOMENA, EXACT BASIC EQUATIONS, AND PROBLEMS IN FORMULATION AND SOLUTION OF APPROXIMATE EQUATIONS

1. PHYSICAL AND EXPERIMENTAL BACKGROUND

Physical and experimental evidence of nonlinear acoustics has been exhibited in the literature [1,2], and several symposia have been devoted entirely to this subject.*

At high acoustic amplitudes or intensities (i.e. the product of acoustic pressure and particle velocity) or high power (the product of acoustic pressure and volume particle velocity) or high energy density (acoustic pressure), significent nonlinear effects occur in gases, liquids, and solids. The physical manifestations of the more important of these effects in liquids and gases are

1. Sound radiation pressure
2. Cavitation
3. Acoustic streaming (quartz wind)
4. Shock waves
5. Generation of Sum and Difference Frequencies
6. Anomalous Absorption
7. Acoustic Self-Demodulation
8. Acoustic Saturation
9. Subharmonic generation.

A brief description of the nature of these effects with simple experiments to illustrate them will emphasize the physical basis of nonlinear acoustics.

2. SOUND RADIATION PRESSURE

In acoustic waves of infinitesimal amplitude which are generated by a piston moving sinusoidally in a medium having amplitude-dependent compressibility the mean acoustic pressure over a period does not differ perceptibly from zero, that is, the mean total pressure is very nearly equal to the static pressure. When the waves are given finite amplitude and are measured at a finite distance from the generating surface, the acoustic pressure experimentally varies nonsinusoidally with spatial coordinate, and its mean value (or radiation pressure) over a period is different from zero. An experiment [2] demonstrating this (Rayleigh) radiation pressure consists in sealing both front and back sides of the diaphragm of a large electrodynamic low-frequency loadspeaker operating in air to

Note: Manuscript submitted May 22, 1974.

*Conference on Nonlinear Acoustics, Navy Underwater Sound Laboratory, New London, Conn., 1968; Conference on Nonlinear Acoustics, Applied Research Laboratory, Texas University, Austin, Texas, 1969 [1a]; Symposium on Nonlinear Acoustics, Naval Research Laboratory, Washington, D.C., 1971 [1b]; and Symposium on Finite Amplitude Wave Effects in Fluids, Copenhagen, Denmark, 1973.

form two pressure-tight chambers and connecting to them fluid manometers, one to each chamber. Upon intense excitation the fluid levels increase in both manometers and by the same amount, indicating an increase of the mean pressure. When the intensity is reduced the fluid levels ultimately reach static (or zero-increment) condition. Rayleigh-type radiation pressure is usually observed in closed spaces.

A second type of sound radiation pressure occurs when a high-intensity, finite-area sound beam (say a sinusoidal plane wave) is launched into a medium at rest. If the beam were effectively free an observer at rest would measure a decrease in mean pressure over a cycle of oscillation of the wave (a result analogous to the Bernoulli effect in incompressible flow). When the boundary medium effectively confines the beam an observer traveling with the beam sees material from the rest medium flowing in and thus measures an increase in the sound beam's mean pressure level over a cycle, the magnitude of increase depending directly on the acoustic intensity. The observer attributes this rise in mean (Langevin) radiation pressure to a force exerted at the surface of the beam by the static medium outside it. This Langevin radiation pressure may be demonstrated [2] by launching a finite-area beam into a tandem (planar-bounded) series of immiscible fluids having almost the same characteristic impedances but different sound speeds. The acoustic intensity in each fluid is very nearly the same, but the energy density (viz, intensity divided by sound speed) differs from fluid to fluid. Inside the beam at fluid interfaces material from high energy-density fluids is forced into low energy-density fluids (interfacial bulges). These bulges disappear when the intensity effectively is reduced to infinitesimal levels. When one of the fluids is a gas and the other a liquid, an interfacial bulge at very high intensity erupts into a fountain (ultrasonic fountain).

Langevin radiation pressure is also observed on reflection in a gaseous medium. A plane compression wave, initially at a distance x from a rigid wall, moves $x/2$ units toward the wall. The volume (of isothermal gas) being halved results in the pressure being doubled (Boyle's law). If it moves from x a distance $x/2$ away from the wall, the volume is increased to $3x/2$, and hence the pressure drops to 2/3 of its initial value. Thus a net static pressure is exerted on the wall (Poynting's explanation). The effect evidently increases with amplitude. In both Rayleigh and Langevin types of radiation pressure there is a component acoustic pressure proportional to the square of a field quantity (square of acoustic density in the Rayleigh type and the square of particle velocity in the Langevin type). This accounts for nonzero mean values over a cycle.

The mathematical description of radiation pressure is obtainable from the hydrodynamic equation of motion (in Eulerian variables) (see Eq. (21.6) of this text) and from the equation of state (see Eq. (24.1)). For an ideal fluid (i.e., thermal and viscous terms vanishing), if the gross motion of the fluid is assumed irrotational, one averages Eq. (21.6) over a large interval of time and integrates over space. Since first order terms drop out in the temporal integration process the averaged pressure is of second order. This second order pressure is the acoustic radiation pressure. It is more readily obtained by space time integration of Eq. (24.12), which shows that the second order density (or pressure) is proportional to the algebraic sum of the first order velocity squared and the first order density squared. Rayleigh radiation pressure involves both velocity and density terms. Langevin radiation pressure involves the squared velocity term.

3. CAVITATION

Gases dissolved in liquids under rest conditions come out of solution during the negative phase of infinitesimal-amplitude acoustic signals and redissolve under the positive phase. When the wave is intense the gaseous bubbles formed during the negative phase do not totally disappear during the positive phase. The generation of these voids has three aspects: (a) Quiet degassing, in which the bubbles increase in size with each period of the wave by a process of "rectified diffusion" and eventually rise out of the sound field. Quiet degassing produces no noise in a listening hydrophone. (b) Mechanical resonance of gas bubbles, in which the sound field, at frequency ω, excites the gaseous voids to vibrate and thus generate offspring sounds at integer multiples (harmonics) and at integer fractions (subharmonics) of ω, detectable by a listening hydrophone. (c) Nonlinear collapse of vapor bubbles, in which small explosive ruptures occur, accompanied by instantaneous pressures of great magnitude and hydrophone-detectable noise with a wide-band spectrum. Cavitation generated by sound can be observed by driving a magnetostrictive transducer in air-saturated water. At an appropriate frequency and intensity a swarm of small (cavitation) bubbles is seen to hover near the active surface of the transducer.

Many mathematical models of cavitation have been developed. An extensive review of the best of these is found in Ref. 3.

4. ACOUSTIC STREAMING (OR "QUARTZ WIND")

In the path of a high-intensity or high-amplitude sound beam in fluids there can be observed a time-independent flow (acoustic streaming), particularly noticeable near walls, obstacles, or oscillating bodies. This flow is rotational, i.e., the particle motion is vortical in nature, with a magnitude generally smaller than the particle velocity in the sound wave, and stabilized in time by the viscosity of the medium. Such time-independent streaming is caused by constant (i.e., "direct current") forces in the sound field, directly proportional to sound intensity, and is determined by higher order nonlinear terms in the equation of motion of the flow (Rayleigh's explanation). The scale of this vortical flow is of three sizes: (a) much smaller than a wavelength (Schlichting streaming inside a viscous boundary layer); (b) of the order of a wavelength (Rayleigh streaming); or (c) much larger than a wavelength (Eckart streaming). Echart streaming can be observed when a quartz oscillator radiates high intensity sound whose associated acoustic streaming blows out a nearby candle.

The mathematical description of acoustic streaming can be approached via use of the basic hydrodynamic equation of motion (in Eulerian variables) (see Eq. (21.6) of this text) and the equation of continuity (Eq. (21.7)). To better display the vortical nature of the fluid motion, the second term on the left-hand side of Eq. (21.6) is rewirtten as

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = \frac{1}{2}\,\nabla v^2 - \mathbf{v}\mathrm{x}(\nabla\mathrm{x}\mathbf{v}).$$

This relation is substituted into the hydrodynamic equations of second order, Eqs. (23.8) and (23.9). The latter equations are then averaged over time. These time-averaged, second-order equations of motion and continuity then serve as basic models for the acoustic streaming process.

5. SHOCK WAVES

A traveling plane wave of infinitesimal amplitude which is sinusoidal at the generating surface remains nearly sinusoidal during propagation to great distances. When its amplitude is greatly increased the associated particle velocity becomes no longer negligible relative to the propagation speed of the wave. During the positive phase of the wave these two speeds add, and during the negative phase they subtract. The wave becomes asymmetrical in sonic velocity. In addition, at high amplitudes the medium is more stiff during the positive phase than the equilibrium stiffness, and less stiff during the negative phase. The propagation speed therefore exceeds infinitesimal wave speed during one half-cycle and is reduced below this value during the next half-cycle. As a result the positions of the maxima in space shift ahead while the positions of the minima shift behind. At any fixed point in space the maxima occur earlier than the comparable time of the infinitesimal wave, and the minima occur later. When the propagating wave has high enough amplitude, a region of space is eventually reached in which the spatial distribution of pressure shows the maxima coinciding with the minima (viz "shock") at intervals corresponding to the frequency of the original sinusoidal wave. At a fixed point in space the time history of the waveform is sawtooth-like. Part of the wave energy has thus been transferred from the fundamental (drive frequency) into higher harmonics. This transfer of energy to the higher harmonics is counterbalanced by increased attenuation loss due to viscosity (and possible heat conduction) which is proportional to the square of the frequency in the first approximation. The energy in the initial high amplitudes is thus eventually lost to the surrounding medium and the wave ultimately becomes infinitesimal once again.

Shock waves can be formed experimentally [2] by attaching a very long hard-walled hollow tube of small bore flared at one end to a source of very highly pressurized compressed air regulated by a valve. When the valve is suddenly opened, the pressure at the valve reaches a maximum in a finite rise time and the waveform is shaped accordingly. After propagating down the tube the waveform between the resting medium ahead and the high-pressure medium behind steepens and the rise time shortens, until eventually a shock wave is formed. This is heard as a sharp clap when the wave exits the flared end of the tube.

The mathematical description of acoustic shock waves is found in Sects. 29, 30, and 36 of this report. Although several mathematical models are discussed in these sections the validity of each model must be ascertained for particular cases. This caution is necessary since a shock wave can be modeled either as a zero-width or as a finite-width discontinuity. The development of harmonics and their subsequent propagation is particularly discussed in Sec. 30 where a number of mathematical models of harmonic formation and absorption are treated.

6. GENERATION OF SUM AND DIFFERENCE FREQUENCIES

When a beam of high-intensity sound at frequency ω_1 is launched into a gaseous or liquid medium, and a second acoustic wave at frequency ω_2 is made to propagate along the axis of the first, in the same direction, a receiver of sound placed in the beam detects not only ω_1 and ω_2 and their harmonics, but also the sum frequency $\omega_1 + \omega_2$ and the difference frequency $\omega_1 - \omega_2$. The pressure level and directionality of the new frequencies depend on the amplitudes or intensities of the original beams, on the width of the high-intensity beam, on the location of the receiver relative to the location of the high-intensity source, and on the nonlinear properties of the medium. An experimental demonstration of this effect can be made by generating two coaxial high-intensity ultrasonic beams in water at slightly different frequencies (say a difference of 5% of the mean frequency of 1 MHz) by means of a single transducer and placing a probe hydrophone at the Rayliegh distance in the beam. Low-level sum and difference frequencies together with harmonics of the main beams will be observed. A mathematical model of the generation of sum and difference frequencies can conveniently be based on the source density equation of Westervelt; see Eqs. (24.18) - (24.21) and Eq. (24.30). These equations have served as starting points for numerous scientific articles on practical applications on nonlinear acoustics [1].

7. ANOMALOUS ABSORPTION

Upon initiation of an explosion a shock wavefront initially composed of many frequencies is radiated outward. As the shock progresses the higher frequencies are dissipated at the shock front. By agency of the nonlinear reaction of the medium to high-amplitude sound, new high-frequency energy is resupplied from the low-frequency content of the propagating wave. This process of resupply continues until a distance is reached at which the amplitude of the low-frequency content becomes so low as to effectively shut off nonlinear conversion. An observer at a distance from the source he considers to be acoustic (i.e., low amplitude) finds that the signal contains an anomalously excessive content of high-frequency energy compared to that predicted by low-amplitude theory and an anomalously deficient content of low-frequency energy. He measures nonacoustic behavior on this type at ranges where linear theory should prevail. Unless aware of nonlinear processes, he deduces, as a consequence, attenuation coefficients which are too small at high frequencies and too large at low frequencies [3]. Excessive absorption of low-frequency energy due to finite amplitude is called anomalous absorption. A mathematical model which treats anomalous absorption is briefly noted in Sec. 28 of this report. Greater detail is found in Ref. 3 as noted.

8. ACOUSTIC SELF-DEMODULATION

An acoustic pulse consisting of a high-frequency carrier wave modulated by an envelope is given a large amplitude at the source and then made to propagate through a nonlinear liquid medium. At a distance from the source where the carrier wave has been reduced by attenuation ultil it is negligible, the residual signal is detected by a hydrophone and examined. In the examination a transient signal appears whose time history closely coincides with the onset and cessation of the pulse envelope, and is essentially vanishing in between. A spectral examination of this transient shows strong difference-frequency components arising from the nonlinear interaction of the carrier frequency and the dominant frequencies of the envelope. The transient amplitude of the signal is found to be proportional to the second time derivative of the square of the envelope amplitude. The suppression of the carrier by attenuation in the nonlinear medium and the nonlinear generation of a transient related to (but not identical with) the pulse envelope is called *self-demodulation*, in loose analogy to the demodulation of a pulsed carrier in radio technology by use of a nonlinear electronic detector. The process of acoustic self-demodulation has been mathematically modeled by use of the source density equation of Westervelt (see Eqs. (24.18) - (24.21)). Here the first order pressure is taken to be a (known) high-amplitude, well-collimated, plane-wave pulse with a pulse envelope $f(t - x/c_0)$. By use of an appropriate far-field Green's function, one solves Eq. (24.20) by quadrature. The resultant scattered (second order) pressure is then seen to be proportional to d^2/dt^2 $[f^2(t - x/c)]$ and predicts self-demodulation as noted above.

9. ACOUSTIC SATURATION

When a piston is driven sinusoidally at low amplitude to radiate sound into a medium with amplitude-dependent compressibility, the generated field pressures are sinusoidal. As the amplitude is increased the far-field pressure (i.e., the source level) increases linearly. When the amplitude becomes large enough (determinable in any specific case by frequency, piston size, and the nonlinear properties of the medium) the waveform of sound pressure generated begins to lose its sinusoidal nature and takes on higher harmonic components. The far-field pressure increment is no longer linear with increment of piston amplitude, since the higher harmonic components of the wave are rapidly attenuated with distance from the source. Further increase in piston amplitude beyond this point shows the fundamental component of the acoustic pressure in the far field departing widely from linear dependence on amplitude, eventually levelling off to a steady value independent of source amplitude. Any increase in source power at these levels only transfers energy to harmonics. Since these vanish with distance, increased power does not result in an increase in the far-field source level. The medium is said to be acoustically saturated. A mathematical model of acoustic saturation is described in Sec. 30 in the sequence of equations starting with Eq. (30.23). As with all models of nonlinear acoustic processes, the domain of validity of these equations must be observed in practical applications.

10. GENERATION OF SUBHARMONICS

A piston vibrating sinusoidally at frequency ω is set to radiate into a cavity containing a medium with amplitude-dependent compressibility. Since the radiation is never strictly monochromatic, the piston motion excites weak-amplitude cavity modes. When the piston amplitude increases it has been observed that beyond a threshold the amplitudes of cavity modes suddenly increase to significant amounts. Examination of the cavity-trapped radiation shows the presence of subharmonic components whose origin is traceable to the nonlinear properties of the cavity medium. Subharmonic generation usually occurs in pairs of frequencies such that the sum of a pair is the parent driving frequency.

A mathematical description of the generation of subharmonics is briefly noted in Sec. 48 in connection with nonlinear acoustic waves in tubes. Additional mathematical treatment is to be found in discussions of the theory of resonant interactions in Sec. 49.

CONCLUSION TO SECTIONS 1-10

All the physical manifestations of nonlinear effects discussed above are, together, events of relatively small magnitude, *if observed over short distances or times* (on the scale of distance or time to shock formation). However, over long distances and long times (similarly scaled), the physical manifestations may become prominent and must then be taken into consideration. When they are considered it soon becomes evident that great difficulties arise in attempts to formulate the problem and boundary conditions in mathematical terms and to find methods of solution. We trace the origin of these difficulties, first, to the need for accurately describing the material properties of the nonlinear medium; second, to the need for describing the deformation of the medium as finite deformation; and third, to the nonlinearity of the basic constitutive equations and the hydrodynamic equations of motion with accompanying difficulty of solution. We take up these points sequentially in the following sections.

11. MATERIAL PARAMETERS AND THERMODYNAMIC EQUILIBRIUM

The distortion of the waveform during its propagation in a nonlinear medium (fluids or solids) is attributable to three classes of effects, namely diffusion, convection, and nonlinearity. In diffusion the waveform undergoes reduction in amplitude over range (attenuation or absorption) due to viscosity and heat conduction and/or a transfer of energy between degrees of freedom of the medium (relaxation). Diffusion of wave energy with corresponding distortion also occurs during propagation when the component wavelets of a wave packet travel at different speeds depending on frequency (dispersion). In convection, the waveform is distorted when the particle velocity convects the wave, i.e., adds to, or subtracts from, the wave speed depending on the phase (plus or minus) of the wave. The sequence of speed-up (plus phase) and slow-down (minus phase) results in a wave distortion (namely wave steepening) which corresponds to a transfer of energy from low-frequency to high-frequency components (development of harmonics). In nonlinearity the waveform distorts because a change in acoustic pressure is accompanied by a nonlinear change in acoustic density, which causes harmonics to appear in the signal wave, as in the case of convection.

The rate and magnitude of distortion depend on certain parameters of the medium. The most important of these are (a) dynamic viscosity and bulk viscosity; (b) the dispersion curve (phase velocity vs frequency) and absorption curve (absorption vs frequency) of the medium, caused by relaxation (these curves also define the relaxation frequency and relaxation time); (c) the coefficient of thermal conductivity; (d) the phase velocity of mode propagation of the medium vs frequency (dispersion of mode); (e) the compressibility, thermal expansion, and mass density of the medium; and (f) the sound speed. The magnitudes of these parameters depend, in general, on pressure, temperature, amplitude, frequency, component concentration, molecular constitution, boundaries, etc. In practical cases values of material parameters are determined only by relying on the simplest hypotheses.

These material parameters form the physical basis for defining the relationships among acoustic field quantities (pressure, density, etc.). In formulating these relations the question arises, are these relations in thermodynamic equilibrium? This question is now treated briefly.

In the hydrodynamic equations to be presented later, formulas appear involving p, ρ, and s arguments (i.e., pressure, density, and entropy, respectively) based on equilibrium thermodynamics. However, a continuum having velocity and temperature gradients is not in thermodynamic equilibrium. Thus the nonequilibrium (subscript *ne*) quantities p_{ne}, ρ_{ne}, s_{ne} will contain not only equilibrium quantities (subscript *e*) but also quantities proportional to powers of the gradients. In the case of fluid pressure in the presence of viscosity (say, bulk viscosity ζ) it is known [4] that in the first approximation,

$$p_{ne} = p_e + \zeta v_{i,i}, \qquad \zeta \text{ positive.} \tag{11.1}$$

Thus the difference between the equilibrium p_e and nonequilibrium p_{ne} values of pressure is of first order in the velocity gradient. A similar argument applied to entropy s shows this difference to be of second and higher orders in the temperature gradient, but not of the first. Since viscosity coefficient ζ and thermal conductivity coefficient χ are parameters of the fluid describing first order effects of velocity and temperature gradients, their appearance in the hydrodynamic equations indicates the limits of validity of such equations, namely, to the cases where higher order space derivatives of velocity, temperature, etc., are negligible. When these derivatives are large one must remodel the equations so that

$$p_{ne} = p_e + \zeta v_{i,i} + \zeta'(v_{i,i}{}^2 + \zeta''(v_{i,i})^3 + \ldots \tag{11.2}$$

$$s_{ne} = s_e + \lambda_s'(T_{,i})^2 + \lambda_s''(T_{,i})^3 + \ldots . \tag{11.3}$$

The hydrodynamic equations will then contain the additional material parameters ζ', ζ'', λ_s'', etc. However, systematic inclusion of these parameters greatly increases the difficulty of solving the equations of motion that use them.

12. CONSTITUTIVE RELATIONS

The addition of terms containing these higher order material properties to the hydrodynamic equations can be formulated as a set of more generalized constitutive relations, i.e., relations between stress and strain in the field of viscoelastic deformation, and pressure, density, and entropy (or temperature) in the field of thermodynamics. These enlarged constitutive relations are considered next.

We consider first the viscoelastic stress-strain relations and refer all deformations to rectangular coordinates χ_α. Two tensors of importance in generalizing the stress-strain relations are the (a) generalized deformation-rate (or Rivlin-Ericksen) tensor $\mathsf{A}^{(M)}(\tau)$ which is read "Mth order derivative with respect to time τ of the generalized deformation-rate tensor"; (b) the stress-rate tensor $\mathbf{t}^{(M)}$ which is read "Mth order time derivative of the generalized stress rate." Their definitions in terms of tensor components $A_{\alpha\beta}$, $t_{\alpha\beta}$ are [4a, 4b]

$$A_{\alpha\beta}^{(M)} = v_{\alpha\beta}^{(M)} + v_{\beta,\alpha}^{(M)} + \sum_{K=1}^{M-1} \binom{M}{K} v_{\mu,\alpha}^{(M-K)} v_{\mu,\beta}^{(K)}, \quad M = 1,2.... \tag{12.1}$$

$$t_{\alpha\beta}^{(M)} = \frac{D^M t_{\alpha\beta}}{D\tau^M} + \sum_{K=1}^{M} \left[\binom{M}{K} \frac{D^{M-K} t_{k\lambda}}{D\tau^{M-K}} \quad \sum_{H=j}^{K} \binom{K}{H} v_{k,\alpha}^{(K-H)} v_{\lambda,\beta}^{(H)} \right] \tag{12.2}$$

in which the vector velocity v_α and its $M + 1$ time derivatives are given by [4c] in the form

$$v_\alpha^{(M+1)}(\tau) = \frac{Dv_\alpha^{(M)}(\tau)}{D\tau} = \left[\frac{\partial}{\partial\tau} + v_\beta^{(1)}(\tau)\frac{\partial}{\partial x_\beta}\right]^M v_\alpha^{(1)}(\tau). \tag{12.3}$$

With these definitions the generalized stress-strain constitutive equation is written in the form of a functional, i.e., functions of functions (bracket notation), $f_{\alpha\beta}[\ \ ] = 0$. This means the value of $f_{\alpha\beta}$ depends continuously on the entire range of the parameter dependence of the functions in the brackets. The functions appearing in the bracket are deformation gradient, velocity gradient, deformation rate, stress rate, etc. An explicit formulation (one of several possible formulations), would be [4d]

$$f_{\alpha\beta}\left[\frac{\partial x_\lambda}{\partial X_L}(\tau);\ A_{\lambda\mu}^{(1)}(\tau), \ldots, A_{\lambda\mu}^{(N)}\mathop{(\tau)}_{\tau=0}^{t};\ t_{\lambda\mu}(\tau), t_{\lambda\mu}^{(1)}(\tau), \ldots,\right.$$

$$\left. t_{\lambda\mu}^{(M)}(\tau)\right] = 0, \quad (N > M). \tag{12.4}$$

Here τ takes on all values between 0 and t (where t is the present), in agreement with the concept of a functional. It is noted that $A_{\alpha\beta}^{(M)}$, $t_{\alpha\beta}^{(M)}$, and $v_\alpha^{(M+1)}$ are all nonlinear when $M > 1$. In this equation one also understands that the functional form implies the possibility of multiterm products of the functionals with each other and with themselves.

Among the models of viscoelastic behavior the most commonly used is the *simple viscoelastic material.* To specify this material we begin with a broad class of materials for which $M = 0$ for stress flux; i.e. the behavior of these materials is independent of time derivative of stress. Recalling the definition of $A_{pq}^{(M)}$, we see that the constitutive equation reduces to [4e]

$$t_{pq} = F_{pq}\left[\chi_{\lambda,L}(\tau);\ v_{\lambda,\mu}^{(1)}(\tau) \ldots, v_{\lambda,\mu}^{(N)}(\tau)\right], \tag{12.5}$$

where F_{pq} is again a continuous functional over the range $0 < \tau < t$, and is an ordinary function at $\tau = t$. For regular functions one can make a power series expansion of t_{pq} in a polynomial in $x_{\ell,L}$, $v_{\ell,L}^{(1)}$, etc., the coefficients of which are continuous functionals of $\chi_{\lambda,L}(\tau)$, $v_{\lambda,L}^{(1)}(\tau)$, etc. The result is [4f]

$$t_{pq} = F_{pq}\left[\mathop{\chi_{\lambda,L}(\tau)}_{\tau=0}^{t}; x_{k,K}, A_{\ell m}^{(1)}, \ldots, A_{\ell m}^{(N)}\right]. \tag{12.6}$$

Of this class of materials we select a stress-strain model independent of the history of displacement gradients (that is, we do not include $\chi_{\lambda,\ell}(\tau)$). Then [4g]

$$t_{pq} = F_{pq}\left[x_{k,K}; A_{\ell m}^{(1)}, \ldots A_{\ell m}^{(N)}\right] \tag{12.7}$$

Next we take the model to represent an isotropic material. Then $x_{k,K}$ and $A_{\ell m}^{(N)}$ can be replaced by deformation tensors **c** and **d** so that the functional equation is reducible to the ordinary function [4h],

$$\mathbf{t} = \mathbf{F}(\mathbf{c}, \mathbf{d}); \qquad \mathbf{c} = \mathbf{c}[C_{k\ell}]; \qquad \mathbf{d} = \mathbf{d}[d_{k\ell}]$$

$$c_{k\ell} = C_{KL} X_{K,k} X_{L,\ell}; \qquad d_{k\ell} = \frac{1}{2}\,[v_{k,\ell} + v_{\ell,k}] \tag{12.8}$$

in which $\mathbf{c}$ is Green's deformation tensor and $\mathbf{d}$ is the Euler deformation rate tensor. Since $\mathbf{F}$ is an isotropic matrix polynomial of two symmetric matrixes, it can be expressed in a canonical form (i.e. in finite powers of $\mathbf{c}$, $\mathbf{d}$) with coefficients $a_{k\ell}$. The result is the constitutive equation for a simple viscoelastic isotropic material, [4i], where I is the identity matrix;

$$\begin{aligned} \mathbf{t} = a_{00}\mathbf{I} &+ a_{10}\mathbf{c} + a_{20}\mathbf{c}^2 + \\ &+ a_{01}\mathbf{d} + a_{02}\mathbf{d}^2 \\ &+ a_{11}(\mathbf{cd} + \mathbf{dc}) + a_{12}(\mathbf{cd}^2 + \mathbf{d}^2\mathbf{c}) \\ &+ a_{21}(\mathbf{c}^2\mathbf{d} + \mathbf{dc}^2) + a_{22}(\mathbf{c}^2\mathbf{d}^2 + \mathbf{d}^2\mathbf{c}^2). \end{aligned} \tag{12.9}$$

The linear part of this constitutive relation is

$$\mathbf{t} = a_{00}\mathbf{I} + a_{01}\mathbf{d} \tag{12.10}$$

in which, for compressible viscous fluids, one has [4j]

$$a_{00} = -\pi + \lambda_v I_d; \qquad a_{01} = 2\mu_v$$

where $I_d (= d_{i,i})$ is the first invariant of deformation rate tensor $\mathbf{d}$, and λ_v, μ_v are viscosity coefficient (Sec. 20). This is the conventional model of a single viscoelastic material, and the one commonly used in the theory of nonlinear acoustics.

The stress-strain constitutive equation is one of a set of equations of state needed for solution of the hydrodynamic equations. A second set describes the thermodynamic relations between the thermodynamic pressure π, density ρ, and temperature T or entropy s and their increments $\Delta\pi$, Δs, etc. Near the equilibrium state we can write

$$\rho = \rho(\pi, s)$$

$$\begin{aligned} \rho = \rho_0 &+ \left(\frac{\partial\rho}{\partial\pi}\right)_s \Delta\pi + \left(\frac{\partial\rho}{\partial s}\right)_\pi \Delta s \\ &+ \frac{1}{2!}\left(\frac{\partial^2\rho}{\partial\pi^2}\right)_s (\Delta\pi)^2 + \frac{1}{2!}\left(\frac{\partial^2\rho}{\partial s}\right)_\pi (\Delta s)^2 \end{aligned} \tag{12.11}$$

or

$$\pi = \pi(\rho, s)$$

$$\pi = \pi_0 + \left(\frac{\partial \pi}{\partial \rho}\right)_s \Delta\rho + \frac{1}{2!}\left(\frac{\partial^2 \pi}{\partial \rho^2}\right)_{ss} (\Delta\rho)^2$$

$$+ \left(\frac{\partial \pi}{\partial s}\right)_\rho \Delta s + \frac{1}{2!}\left(\frac{\partial^2 \pi}{\partial s^2}\right)_{\rho\rho} (\Delta s)^2 + \ldots . \tag{12.12}$$

The number of terms to be included in this power series depends on the order (first, second, etc.) of acoustic quantities to be used in a required approximation. We discuss approximations in Sec. 16. The second and third terms (r.h.s.) of Eq. (12.12) are conveniently written in the form

$$\left(\frac{\partial \pi}{\partial \rho}\right)_s \Delta\rho = A\,\frac{\Delta\rho}{\rho_0} \quad ; \qquad \left(\frac{\partial^2 \pi}{\partial \rho^2}\right) = \frac{B}{2}\left(\frac{\Delta\rho}{\rho_0}\right)^2$$

$$A = \rho_0 \left(\frac{\partial \pi}{\partial \rho}\right)_s ; \qquad B = \rho_0^2 \left(\frac{\partial c^2}{\partial \rho}\right)_s \tag{12.13}$$

The ratio B/A which has special significance, i.e.,

$$\frac{B}{A} = \frac{\rho_0}{c_0^2}\left(\frac{\partial^2 c^2}{\partial \rho}\right)_s , \tag{12.14}$$

is the crucial material parameter in the modeling of the nonlinear properties of fluids. For later reference we note here that the ratio B/A in the theory of fluids is related to the ratio of specific heats γ in the theory of gases. When nonlinear processes are under consideration one can interchange the group $(1 + B/A)$ which is applicable to fluids with the symbol γ which is applicable to gases.

13. ATTENUATION

Attenuation (or absorption) is a basic phenomenon in nonlinear acoustics. Its numerical value depends on material parameters, frequency, amplitude, and wave shape. We touch briefly here on plane waves exhibiting moderate absorption over long distances and derive the classic formulas for this type of diffusion.

From the energy equation (Eq. (22.7)) we consider here energy loss in a plane wave per unit volume due to viscous absorption and heat conduction (dimensions: $(N\text{-}m/s)/m^3$). If the intensity I_0 (dimensions: $(N\text{-}m/m^2)/s$) of a plane wave is initially distributed over an area A_0, we can define an attenuation 2α (intensity loss per unit distance) by the ratio [4k],

$$2\alpha = \frac{(\lambda_v + 2\mu_v)\mathrm{I}_d^2 - 4\mu_v \mathrm{II}_d + \rho h}{\mathrm{I}_0} \tag{13.1}$$

where I_d, II_d are invariants of **d** (see Ref. 5) and h is heat conduction per unit mass. In the case of a plane wave $v_x = v_0 \exp(ikx - i\omega t)$, we have

$$\mathrm{I}_d^2 = k^2 v_0^2$$

$$\mathrm{II}_d = 0. \tag{13.2}$$

When the heat source can be described by the process of heat conduction,

$$\rho h = (\chi T,_k),_k$$

$$\rho h = \frac{\chi}{T}(T,_k)(T,_k) + \frac{\chi}{T} T,_k),_k$$

$$\chi = \text{thermal conductivity}, \ \frac{N}{(^\circ K)\ sec} \tag{13.3}$$

When computing the average ρh (i.e. when integrating over the finite volume of sources) the last term in Eq. (13.3) reduces from a volume integral over the divergence of the quantity $(\chi T,_k)/T$ to the surface integral over the normal component of this quantity. By making the surface large enough we make $T,_k$ vanish so that

$$\rho h \simeq \frac{\chi}{T} |T,_k|^2. \tag{13.4}$$

Since I_0 for plane waves is given by $\rho c|v_0|^2$, the attenuation coefficient for plane waves is

$$2\alpha = \frac{k^2}{\rho c}\left[\lambda_v + 2\mu_v + \frac{\chi}{T}\frac{|T,x|^2}{k^2 v_0^2}\right] (\text{dimensions: } m^{-1}). \tag{13.5}$$

Assuming that the compression of the medium due to the passage of the wave is adiabatic, we write the incremental temperature rise as

$$dT = \left(\frac{\partial T}{\partial P}\right)_s dP,$$

or in terms of incremental quantities, $dT = T'$ and $dP = p$;

$$T' = \left(\frac{\partial T}{\partial P}\right)_s p.$$

Thus for plane waves in the x direction,

$$T'_{,x} = \left(\frac{\partial T}{\partial P}\right)_s k\rho c v_0 .$$

Using the thermodynamic relation that

$$\left(\frac{\partial T}{\partial P}\right)_s = (\gamma - 1)\,\frac{K_s}{\beta} ,$$

where

$$\gamma = \frac{C_p}{C_v}\,; \qquad K_s = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial P}\right)_s; \qquad \beta = -\,\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_P;$$

C_p, C_v = specific heat at constant pressure, volume

one arrives at

$$\frac{\chi}{T}\,|\text{grad } T'|^2 = \frac{\chi}{T}\,k^2\rho^2c^2v_0^2\left[(\gamma - 1)^2\,\frac{\chi_s^2}{\beta^2}\right].$$

Now for *any* material

$$\gamma - 1 = \frac{T\beta^2}{KC_p\rho} = \frac{C_p - C_v}{C_p} .$$

Hence,

$$\frac{\chi}{T}\,|\text{grad } T'|^2 = \frac{\chi|kv_0|^2(\gamma - 1)}{C_p}. \tag{13.6}$$

The classical absorption coefficient for plane waves due to viscosity and heat conduction is thus given by

$$\alpha = \frac{\omega^2}{2\rho c^3}\left[\frac{4}{3}\,\eta + \eta' + \chi\left(\frac{1}{C_v} - \frac{1}{C_p}\right)\right] = f\omega^2. \tag{13.7}$$

The relation between wave number and frequency in this one-dimensional case now becomes

$$k = \frac{\omega}{c} + if\omega^2. \tag{13.8}$$

These classical formulas for plane wave sound absorption are valid only if the absorption is small over one wavelength of propagation. They serve, however, as models for deriving attenuation coefficients for non-plane-wave fields.

14. RELAXATION THEORY

The bulk viscosity, (see Sec. 20) is associated with deformation of the medium in which volume changes occur. Since deformation is a thermodynamic change of state and states of thermodynamic equilibrium are reached reversibly only by quasistatic processes, changes of state in finite time (where equilibrium is not simultaneously achieved at all points of the medium) are reached in an irreversible manner, i.e., with the consumption of energy. A measure of energy consumption is given by the time required for the medium under deformation to reach thermodynamic equilibrium (relaxation time τ). We illustrate the concept τ by assuming that if the instantaneous pressure at a point has a time history $f(t)$, the equilibrium pressure at the same point has a time history $f(t + \tau)$. If $\tau \approx 0$, then equilibrium is restored almost instantaneously and the energy loss is very small. If τ is large, the equilibrium state is not reestablished immediately, and considerable energy is lost (i.e., an entropy increase occurs caused by the lag of the pressure relative to the change in volume, both varying at different rates in finite time). This dissipation of energy is associated with the second (or bulk) viscosity coefficient η'.

When the medium exhibits relaxation [6,7] during time-varying deformation the equation of state is made to contain an additional parameter ξ which is a thermodynamic coordinate characterizing the internal state of the fluid. Thus the equation of state has the form

$$p = p(\rho, s, \xi). \tag{14.1}$$

We consider first the case in which the entropy coordinate is negligible. Beginning with the identity at equilibrium,

$$\frac{\partial p}{\partial \rho} = \left(\frac{\partial p}{\partial \rho}\right)_{EQ} \equiv c_0^2, \tag{14.2}$$

we add a second term to each side which accounts for nonequilibrium changes in pressure caused by time-varying deformation. Defining the normalized time $t^0 = t/\tau$, we write

$$\frac{\partial p}{\partial \rho} + \frac{\partial}{\partial t^0}\left(\frac{\partial p}{\partial \rho}\right) = \left(\frac{\partial p}{\partial \rho}\right)_{EQ} + \frac{\partial}{\partial t^0}\left(\frac{\partial p}{\partial \rho}\right)_{\xi}. \tag{14.3}$$

For sinusoidal time (exp $(j\omega\tau)$,

$$\begin{aligned} \frac{\partial p}{\partial \rho} &= \frac{1}{1 + j\omega\tau}\left[\left(\frac{\partial p}{\partial \rho}\right)_{EQ} + j\omega\tau\left(\frac{\partial p}{\partial \rho}\right)_{\xi}\right] \\ &= \frac{1}{1 + j\omega\tau}\,[c_0^2 + j\omega\tau c_\infty^2]. \end{aligned} \tag{14.4}$$

Here the term $j\omega\tau c_\infty^2$ is an increment in acoustic velocity because of temporal nonequilibrium of the medium.* The symbol c_∞ is defined by

$$c_\infty^2 = \left(\frac{\partial p}{\partial \rho}\right)_\xi , \tag{14.5}$$

i.e., c_∞ is the velocity of sound for nonequilibrium processes. Now the *difference* in fluid pressure from nonequilibrium deformation changes is given by

$$\begin{aligned}\frac{\Delta p}{\Delta \rho} &= \frac{\partial p}{\partial \rho} - \left(\frac{\partial p}{\partial \rho}\right)_{EQ} \\ &= \frac{1}{1 + j\omega\tau}\,[c_0^2 + j\omega\tau c_\infty^2] - c_0^2 \\ &= \frac{-j\omega\tau}{1 + j\omega\tau}\,[c_0^2 - c_\infty^2].\end{aligned} \tag{14.6}$$

From the equation of continuity,

$$\frac{\partial}{\partial t}\,(\Delta\rho) = -\rho_0 \operatorname{div} \mathbf{v}$$

or

$$\Delta\rho = \frac{-\rho_0 \operatorname{div} \mathbf{v}}{j\omega} . \tag{14.7}$$

Thus,

$$\Delta p = \frac{\tau\rho_0(c_0^2 - c_\infty^2)}{1 + j\omega\tau} \operatorname{div} \mathbf{v}. \tag{14.8}$$

From the hydrodynamic equations of motion this difference of pressure is identified with the viscosity term

$$\zeta \operatorname{div} \mathbf{v}$$

so that

$$\zeta = \frac{\tau\rho_0(c_0^2 - c_\infty^2)}{1 + j\omega\tau} , \tag{14.9}$$

i.e., the viscosity coefficient ζ is associated with the concept of relaxation.

*Note that c_∞ in Meyers and Neumann [2] is the unrelaxed velocity, i.e., speed at $\omega \to \infty$.

When the constitutive relation for the medium has nonlinear terms, we modify the above procedure in the following way,

$$\frac{\partial p}{\partial \rho} + \frac{\partial}{\partial t^0}\left(\frac{\partial p}{\partial \rho}\right) = c_0^2 + \frac{1}{2}\left(\frac{\partial^2 p}{\partial \rho^2}\right)_\xi \rho' + \frac{\partial}{\partial t^0}\left[c_\infty^2 + \frac{\partial}{\partial \rho}\left(\frac{\partial p}{\partial \rho}\right)_\xi \rho'\right] \tag{14.10}$$

or

$$\Delta p = \frac{\tau \rho_0 \left\{c_0^2 + \frac{1}{2}\left(\frac{\partial^2 p}{\partial \rho}\right)_\xi \rho' - \left[c_\infty^2 + \frac{\partial}{\partial \rho}\left(\frac{\partial p}{\partial \rho}\right)_\xi \rho'\right]\right\} \operatorname{div} \mathbf{v}}{1 + j\omega\tau}. \tag{14.11}$$

If the time variation is not sinusoidal, then [6]

$$\Delta p + \frac{\partial}{\partial t^0}\Delta p = -\frac{\partial}{\partial t^0}(\Delta \rho)\left\{c_0^2 + \frac{1}{2}\left(\frac{\partial^2 p}{\partial \rho^2}\right)_\xi \rho' - \left[c_\infty^2 + \frac{\partial}{\partial \rho}\left(\frac{\partial p}{\partial \rho}\right)_\xi \rho'\right]\right\}. \tag{14.12}*$$

On a molecular scale the process of relaxation tells how an increase in internal energy of a volume of matter results in an increase in three forms of energy of its molecules, viz, translation, rotation, oscillation. When a fluid is suddenly compressed the work of compression increases its internal energy in the form of translation, which corresponds to an increase in pressure. Translational energy is then partially converted to both rotational and oscillatory forms. After a definite time (relaxation time) all three forms of energy reach equilibrium (i.e., no further internal increase or decrease of any form of energy). When equilibrium is reached the final pressure is less than the instantaneous initial pressure because translational energy is a lesser part of the total internal energy. When the fluid is "relaxed" it is less stiff, i.e., its sound speed is less. Hence the speed of sound in the fluid depends on the degree of relaxation (i.e., the fluid is dispersive because relaxation is a function of frequency). If it is "unrelaxed" its speed is c_∞. If it is completely relaxed its speed is $c_0 (< c_\infty)$. Defining the relaxation frequency as $\omega_R = \tau^{-1}$ we see that when the fluid is excited at $\omega << \omega_R$ its sound speed approaches c_0, and when $\omega >> \omega_R$, its sound speed approaches c_∞. The frequency width of the transition region from c_0 to c_∞ depends on the absorption. For gases the width is of the order of some three octaves and the attenuation per wavelength at ω_R is of the order of 8%. For $\omega << \omega_R$ the absorption in liquids is low, and it varies as the square of the frequency (classical absorption). Near ω_R the absorption is larger by factors of about 10 or more. For $\omega >> \omega_R$ the absorption falls again to low values.

By incorporating relaxation and viscosity effects into one formula and using empirical coefficients, one can write the absorption coefficient for seawater in the form [8]

$$\alpha = \frac{a\omega^2}{b^2 + \omega^2} + g\omega^2, \tag{14.13}$$

*See Appendix F for a mathematical formulation of a more general treatment of relaxation in the basic equations of state and propagation.

in which

a = constant [Ref. 3]

b = relaxation frequency

$$g = \left[\frac{4}{3}\ \eta + \zeta + \chi\left(\frac{1}{C_v} - \frac{1}{C_p}\right)\right] \Big/ 2\rho c^3 .$$

The first term accounts for relaxation attenuation associated with the chemical dissociation process of magnesium sulfate in seawater. The second term is the attenuation due to thermal conductivity and viscosity. The theory of relaxation (briefly touched on here) is very extensive. A detailed guide is provided in the book by Herzfeld and Litovitz [9].

15. DISPERSION

In relaxing media (as noted) the phase velocity of sinusoidal waves is a function of frequency (dispersion). Other examples of frequency dependence of phase velocity are found in the propagation of sinusoidal acoustic waves in tubes with soft walls and the propagation of flexural waves in elastic beams. Dispersion is mathematically accounted for as follows. Let $\mathcal{L}$ be the Lagrangian of the wave system and q a generalized coordinate (say displacement). We assume wave solutions of the form

$$q = Aq(\theta) \tag{15.1}$$

$$\theta = k_i t_i - \omega t, \qquad i = 1, 2, 3 \tag{15.2}$$

in which A is an amplitude function, $q(\theta)$ is a phase function, k_i is the wave number in the ith coordinate, and ω is the frequency. In the case of harmonic wave trains it may be directly shown [10] that the application of Hamilton's principle leads to a statement that the velocity of energy transport $\mathbf{U}$ has components U_m (m = 1, 2 ...) which satisfy the equations

$$\delta\omega = \sum_m U_m \delta k_m \tag{15.3}$$

$$U_m = \frac{\left\langle \sum_i \dot{q}_i \dfrac{\partial \mathcal{L}}{\partial \left(\dfrac{\partial q_i}{\partial k_m}\right)} \right\rangle}{\left\langle \sum_i \dot{q}_i \dfrac{\partial \mathcal{L}}{\partial q_i} - \mathcal{L} \right\rangle} \tag{15.4}$$

where the symbol $\langle\,\rangle$ signifies time average over harmonic period. The velocity $\mathbf{U}$ is the group velocity. It points in the direction of the flow of energy. In the general nonlinear, nonhomogeneous case, $A = A(a, k_i, \omega)$, i.e., A is a function of amplitude, wave number, and frequency. To account for dispersion we use the following method of Whitham [11].

For every x_i,t the phase θ has a specific value when a, k_i, ω are specified. Averaging $\mathcal{L}$ over θ (i.e., over x_i,t) yields the averaged Lagrangian density

$$\overline{\mathcal{L}} = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{L} d\theta, \qquad \overline{\mathcal{L}} = \overline{\mathcal{L}}(a, k_i, \omega) \tag{15.5}$$

Applying Hamilton's principle

$$\delta \int_T \overline{\mathcal{L}} d\tau = 0 \tag{15.6}$$

leads to the conservation laws

$$\frac{\partial \overline{\mathcal{L}}}{\partial a} = 0 \tag{15.7}$$

$$\frac{\partial}{\partial t} \frac{\partial \overline{\mathcal{L}}}{\partial \omega} - \frac{\partial}{\partial x_i} \frac{\partial \overline{\mathcal{L}}}{\partial k_i} = 0. \tag{15.8}$$

If there are n generalized coordinates (say n displacements), Eq. (15.7) is replaced by a system of n homogeneous equations in n unknowns a_n,

$$\frac{\partial \overline{\mathcal{L}}}{\partial a_n} = 0. \tag{15.9}$$

A solution of Eqs. (15.7) or (15.9) gives the characteristic equation of the system, i.e., defines ω as a function of k_i and a_i. These are the dispersion equations of the system.

Simple cases of nondispersive and dispersive wave trains can be illustrated by writing the one-dimensional propagation of waves governed by the operator equation

$$L\,\{u\} = 0, \qquad L = \text{differential-integral operator} \tag{15.10}$$

and choosing solutions in the form of plane progressive waves in one direction, i.e.,

$$u = \exp\,[i(kt - \omega t)]\,. \tag{15.11}$$

The dispersion relation (solution to (15.10) using (15.11), then has the form

$$k = f(\omega). \tag{15.12}$$

Since the frequency ω and wave speed c_ϕ of an elementary wave are related by $kc_\phi = \omega$, the (phase) speed of a single wave is given by

$$c_\phi = \frac{\omega}{f(\omega)}\,. \tag{15.13}$$

The group speed of a collection of elementary waves is

$$c_g = \frac{d\omega}{df(\omega)} . \tag{15.14}$$

Simple cases of waves trains are

1. Nondispersive,

$$L = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \; ; \qquad f = \frac{\omega}{c} \; ; \qquad c_\phi = c_g = \text{const.}$$

2. Dispersive,

$$L = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} + g$$

$$f(\omega) = \frac{\sqrt{\omega^2 + g}}{c}$$

$$c_\phi = c \frac{\omega}{\sqrt{\omega^2 + g}} \tag{15.15}$$

Thus nondispersive wave equations of given order in the differential operator can be made dispersive by adding homogeneous terms of different order. The simplest technique is to introduce complex attenuation coefficients. In the more general case, for a differential homogeneous equation of first or second order, dispersion can be introduced by insertion of third or higher order derivatives. This is illustrated by the Korteweg-deVries equation to be discussed in Sec. 44.

Dispersion has an important bearing on the transmission of pulse signals. If the medium is dispersive a solution of a pulse train in the form of a superposition of elementary progressive waves in one direction shows that different components are propagated at different speeds. Hence the composite wave will change its shape as it propagates. If the medium is also nonlinear, the concept of group velocity becomes cloudy and must be redefined. This has been done by Whitham [11]. Nonlinear dispersive wave trains will be discussed in Secs. 40 through 43 when we review the work of Whitham in greater detail. Dispersion also greatly disturbs the phase relation between a propagation fundamental and its daughter hormonics in a nonlinear medium. Where present its importance in this regard should not be overlooked.

16. THE $(\partial p/\partial s)_\rho$ s' TERM IN THE EQUATION OF STATE

As noted in Eq. (12.11), the equation of state can be written in a form which mathematically describes pressure as an expansion in powers of density and entropy. For purposes of physical measurement the entropy term $(\partial p/\partial s)\rho s'$ appearing in this expansion must be related to measurable quantities, viz., to thermal conductivity and specific heat. This is done in the following discussion.

The energy equation (see Eq. (22.7)),

$$\rho T\left(\frac{\partial s}{\partial t} + \mathbf{v}\cdot\nabla s\right) = \sigma'_{ik}\ \frac{\partial v_i}{\partial x_k} + \chi\nabla^2 T \tag{16.1}$$

may be linearized by noting that $T = T_0 + T'$, $\rho = \rho_0 + \rho'$, $s = s_0 + s'$, and that T', s', v, ρ' are of the same order of smallness. Thus to first order in this smallness one has

$$\rho_0 T_0\ \frac{\partial s'}{\partial t} = \chi\nabla^2 T'. \tag{16.2}$$

Now for an acoustic wave having a potential φ,

$$T' = \left(\frac{\partial T}{\partial p}\right)_s p' = -\rho_0\left(\frac{\partial T}{\partial p}\right)_s \frac{\partial \varphi}{\partial t} \tag{16.3}$$

and therefore

$$\nabla^2 T' = -\left(\frac{\partial T}{\partial p}\right)_s \rho_0\ \frac{\partial}{\partial t}\ \nabla\cdot\mathbf{v}, \qquad \nabla\varphi = \mathbf{v} \tag{16.4}$$

with the result that

$$s' = -\chi\ \frac{1}{T}\left(\frac{\partial T}{\partial p}\right)_s \nabla\cdot\mathbf{v}. \tag{16.5}$$

From thermodynamic relations [12]

$$\frac{1}{T}\left(\frac{\partial T}{\partial p}\right)_s\left(\frac{\partial p}{\partial s}\right)_\rho = \frac{1}{T}\left[(\gamma - 1)\ \frac{K_s}{\beta}\right]\left(\frac{\beta T}{K_T C_v}\right)$$

$$= \frac{1}{C_v} - \frac{1}{C_p} \tag{16.6}$$

in which

$$\gamma = \frac{C_p}{C_v}; \qquad K_T = K_s\gamma.$$

Hence

$$\frac{1}{T}\left(\frac{\partial T}{\partial p}\right)_s\left(\frac{\partial p}{\partial s}\right)_\rho s' = \left[\frac{1}{C_v} - \frac{1}{C_p}\right]\left(\frac{-\chi}{T}\right)\left(\frac{\partial T}{\partial p}\right)_s \nabla\cdot\mathbf{v} \tag{16.7}$$

or

$$\left(\frac{\partial p}{\partial s}\right)_\rho s' = -\chi\left[\frac{1}{C_v} - \frac{1}{C_p}\right]\nabla\cdot\mathbf{v}. \tag{16.8}$$

In linear theory, by definition of the pressure and particle velocity in terms of the potential φ, one has

$$\nabla \cdot \mathbf{v} = \frac{\omega \rho'}{\rho_0} . \tag{16.9}$$

Hence, the magnitude of the following ratio is determined;

$$\frac{\left(\frac{\partial p}{\partial s}\right)_\rho s'}{c^2 \rho'} = \left(\frac{\chi}{C_v} - \frac{\chi}{C_p}\right) \frac{\omega}{\rho_0 c^2}. \tag{16.10}$$

It is required next to form the ratio of the viscosity terms to $c^2\rho'$, i.e., the ratio

$$\left| \frac{\eta \nabla^2 \mathbf{v}}{\nabla c^2 \rho'} \right| . \tag{16.11}$$

Since

$$\nabla^2 v = \nabla \cdot \nabla \mathbf{v} \sim \frac{\omega}{\rho_0} \nabla \rho' \qquad \text{(Note: } \sim \text{ means "of the order of")} \tag{16.12}$$

it is seen that

$$\left| \frac{\eta \nabla^2 \mathbf{v}}{\nabla c^2 \rho'} \right| \sim \frac{\eta \omega}{\rho_0 c^2} . \tag{16.13}$$

For irrotational flow $\nabla\nabla \cdot \mathbf{v} = \nabla \cdot \nabla \mathbf{v}$. Following the same procedure used above, we see that

$$\left| \left(\frac{\eta}{3} + \eta'\right) \nabla \operatorname{div} \mathbf{v} \right| \sim \left(\frac{\eta}{3} + \eta'\right) \frac{\omega}{\rho_0 c^2} . \tag{16.14}$$

Since for most substances $\eta \approx \dfrac{\chi}{C_v}$,

$$\frac{\left(\frac{\partial p}{\partial s}\right)_\rho s'}{c^2 \rho'} \sim \left| \frac{\left(\frac{4}{3}\eta + \zeta\right) \nabla^2 \mathbf{v}}{(\nabla c^2 \rho')} \right| ; \tag{16.15}$$

and

1. $\left(\dfrac{\partial p}{\partial s}\right)_\rho s'$ is of order $\dfrac{\omega \eta}{\rho_0 c^2}$ (16.16)

2. $\left| \dfrac{\left(\frac{4}{3}\eta + \zeta'\right) \nabla^2 v}{\nabla c^2 p'} \right|$ is of order $\dfrac{\omega \eta}{\rho_0 c^2}$. (16.17)

Since the dissipation per wavelength is always considered small, we will take

$$\frac{\omega\eta}{\rho_0 c^2} = \frac{kv}{c} << 1, \qquad \omega = kc. \tag{16.18}$$

We next consider the ratio

$$\frac{(\text{Mach Number})^2}{\text{Reynolds Number}} = \frac{M^2}{R_e} = \frac{U^2}{c^2}\,\frac{\nu}{LU} = M\,\frac{\omega\eta}{\rho_0 c^2}\;. \tag{16.19}$$

To consider all cases of this ratio, we select three ranges for investigation;

$$\left.\begin{array}{ll} 1. & M << \\ 2.^{\dagger} & M \approx \\ 3. & M >> \end{array}\right\{ \frac{\omega\eta}{\rho_0 c^2} = \frac{\omega v}{c^2}\;. \tag{16.20}$$

Further discussion of these choices will be undertaken in Secs. 17-26, following a presentation of the basic hydrodynamic equations.

The constitutive relations for a viscoelastic material and the thermodynamic equation of state both present nonlinear terms. It has been found desirable to assemble these terms into a conveniently identifiable *parameter of nonlinear acoustics.* This can be readily done in the case of plane waves. The theory is presented in the next section.

17. THE DIMENSIONLESS PARAMETER OF NONLINEAR ACOUSTICS

The constitutive relation between acoustic pressure, density, and velocity for a heat conducting viscous fluid is given by

$$p' = c^2\rho' + \frac{1}{2}\left(\frac{\partial c^2}{\partial\rho}\right)_s \rho'^2 - \chi\left[\frac{1}{C_v} - \frac{1}{C_p}\right] \operatorname{div} \mathbf{v}$$

or

$$p' = c^2\rho'\left[1 + \rho'\left(\frac{\partial c}{\partial\rho}\right)_s\right] - \chi\left[\frac{1}{C_v} - \frac{1}{C_p}\right] \operatorname{div} \mathbf{v} \tag{17.1}$$

(see Eq. (12.12) and Sect. 16). Substituting this into the Eulerian form of the hydrodynamic equations of motion (see Sec. 20) we obtain

†Case 2 corresponds to a "weak shock" solution (Ref. 13, p. 298).

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] = -c^2\nabla\left\{\rho'\left[1+\frac{\rho'}{c}\left(\frac{\partial c}{\partial \rho}\right)_s\right]\right\} + \chi\left[\frac{1}{C_v} - \frac{1}{C_p}\right]\nabla(\nabla\cdot\mathbf{v})$$

$$+\eta\nabla\cdot(\nabla\mathbf{v}) + \left(\frac{\eta}{3}+\eta'\right)\nabla(\nabla\cdot\mathbf{v}). \qquad (17.2)$$

Restricting attention to irrotational fluid flow, we write

$$\nabla\cdot\nabla\mathbf{v} = \nabla(\nabla\cdot\mathbf{v}). \qquad (17.3)$$

(It will be shown later that curl v is at the least a second order quantity, i.e., fluid motion in the first approximation is potential in nature.) We next form the ratio

$$\mathcal{S} = \left|\frac{\left(\frac{4\eta}{3} + \eta' + \chi\left[\frac{1}{C_v} - \frac{1}{C_p}\right]\right)\nabla(\nabla\cdot\mathbf{v})}{c^2\nabla\left\{\rho'\left[1+\frac{\rho'}{c}\left(\frac{\partial c}{\partial\rho}\right)_s\right]\right\}}\right| . \qquad (17.4)$$

$\mathcal{S}$ states the relative importance of viscosity and thermal forces to the pressure forces. Applying this formula to plane waves for which the amplitudes of pressure, density, and particle velocity are given, by

$$p' = -i\omega\rho_0\Phi_0$$

$$\mathbf{v} = -\nabla\Phi_0$$

$$\Phi = \Phi_0 \exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t), \qquad (17.5)$$

we find that

$$\nabla\cdot\mathbf{v} = -\nabla^2\Phi_0 = \mathbf{k}\cdot\mathbf{k}\Phi_0 = k^2\Phi_0$$

$$\nabla\cdot\mathbf{v} = \frac{k^2 p'}{-i\omega\rho_0} . \qquad (17.6)$$

We now make the conventional linear approximation that $p' \approx c_0^2\rho'$, so that

$$\nabla\cdot\mathbf{v} = \frac{k^2c_0^2\rho'}{-i\omega\rho_0} = \frac{\omega\rho'}{-i\rho_0} . \qquad (17.7)$$

Forming the approximation

$$\nabla\left\{\rho'\left[1+\left(\frac{\rho'}{c}\frac{\partial c}{\partial\rho}\right)_s\right]\right\} \approx \left[1+\frac{\rho'}{c}\left(\frac{\partial c}{\partial\rho}\right)_s\right]\nabla\rho' \qquad (17.8)$$

shows that for plane waves

$$\mathcal{S} = \frac{\omega b}{c^2 \rho_0 \left[1 + \frac{\rho'}{c}\left(\frac{\partial c}{\partial \rho}\right)_s\right]}$$

and

$$b = \frac{4}{3}\eta + \eta' + \chi\left(\frac{1}{C_v} - \frac{1}{C_p}\right) \ . \tag{17.9}$$

We next compare $\mathcal{S}$ with the Mach number $M = U/c$, where U is an acoustic particle velocity, and choose $U = |v|_{max}$. (Note that Gol'dberg [14] defines a scaling factor $N = U_\tau/X$ and selects $X/\tau \sim U/c = M$.) We have the three cases

$$1. \quad \mathcal{S} >> M \qquad 2. \quad \mathcal{S} \approx M \qquad 3. \quad \mathcal{S} << M \tag{17.10}$$

The first case corresponds to very small-amplitude acoustics (i.e., the linear case). The second and third cases correspond to finite-amplitude cases. Particular attention is given to the second case, roughly, the condition under which a finite-amplitude wave will develop a discontinuity, i.e., a shock (see Gol'dberg [14]). Considering this case, we write

$$\mathcal{S} = \frac{\omega b}{c^2 \rho_0 \left[1 + \frac{\rho'}{c}\left(\frac{\partial c}{\partial \rho}\right)_s\right]} \sim \frac{U}{c} \ . \tag{17.11}$$

Since for a plane wave

$$\frac{U}{c} \approx \frac{\rho'}{\rho_0} = \frac{c^2 \rho'}{c^2 \rho_0} \approx \frac{p'_{max}}{c^2 \rho_0} \ , \tag{17.12}$$

the condition for the formation of a shock is

$$Q = \frac{p'_{max}\left[1 + \frac{\rho'}{c}\left(\frac{\partial c}{\partial \rho}\right)_s\right]}{\omega b} \sim 1. \tag{17.13}$$

One can define finite-amplitude acoustics by the statement that the maximum amplitude of the pressure wave in a viscous, thermally conducting medium obeys the relation

$$p'_{max} > \frac{\omega b}{\left[1 + \frac{\rho'}{c}\left(\frac{\partial c}{\partial \rho}\right)_s\right]} . \tag{17.14}$$

When viscosity and conductivity are negligible, i.e., when fluid is ideal, any pressure amplitude greater than zero defines finite-amplitude acoustics.

CONCLUSION TO SECTIONS 11-17

The assembly of material parameters, constitutive relations, and themodynamic equations of state has now been completed. We turn next to a description of the laws of conservation of mass density, momentum, and energy.

The basic hydrodynamic equations, as noted earlier, can be presented in two forms, Eulerian and Lagrangian. Although these forms describe the same physical process, their structures are very different, and as a consequence they yield different information. In addition, it is common experience of analysts that in specific cases of boundary-value or initial-value problems the preference of one set over another leads to more tractable methods of solution. We therefore continue by presenting the two forms sequentially.

18. EULERIAN AND LAGRANGIAN DESCRIPTIONS OF DEFORMATION

Introduction

The theory of nonlinear acoustics rests upon the theory of finite deformation of continua. In the latter, great care is given to the coordinate systems used to define states before and after deformation. With such care considerable ambiguity in theoretical results is avoided. We begin therefore with a statement of the description of appropriate coordinate systems.

Coordinate Systems

Before deformation the material particles of a continuum occupy material coordinates X_K (i.e., a trio of numbers serving as material point identification tags) which refer to the body but not to space. In the absence of the body the spatial coordinate system is x_k. When the continuum undergoes elastic deformation in time, the material coordinate system X_K (by definition) deforms with the body. In contrast the spatial coordinate system x_k (by definition) remains fixed. All material points in the deformed body occupy the positions relative to deformed X_K as they did in the undeformed body relative to the undeformed X_K. Hence in the X_K system the particles have not moved from their coordinate points, although the coordinate points have been separated or contracted relative to each other. This separation or contraction of X_K is measurable in the fixed x_k system. Thus displacement in X_K is describable by the identify $\mathrm{X}_K = \mathrm{X}_K(x_k, t)$. Similarly, displacement in x_k is measurable in X_K; i.e., $x_k = x_k(\mathrm{X}_{K,t})$.

An alternate statement, much used in the theory of continua [5], is this. Deformable material bodies occupying points in coordinates $\mathbf{X}$, 0, can be deformed to occupy points in coordinates $\mathbf{x}$, t, where $\mathbf{X}$, $\mathbf{x}$ are two local coordinate systems (with unit orthogonal vectors I_K, i_k) whose origins are separated by a vector distance $\mathbf{b}$. The process of deformation is stated in geometric terms to be $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$, which is to be read as "material initially at $\mathbf{X}$ reaches $\mathbf{x}$ in time t;" and $\mathbf{X} = \mathbf{X}(\mathbf{x}, t)$, which is read "material known to be at $\mathbf{x}$ at time t can be traced back to $\mathbf{X}$." The transformation of points in the neighborhood $\mathbf{x}_0$ from $\mathbf{x}$ to $\mathbf{X}$ is the unique inverse of the transformation of $\mathbf{X}$ to $\mathbf{x}$ in the neighborhood X_0, provided the Jacobian j (= $|\partial x_k/\partial X_K|$) does not vanish at $\mathbf{x}_0$ and at $\mathbf{X}_0$. If a and A are areas in the two systems, and if v and V are comparable volumes, a finite deformation (viz, mapping of points in one system with points in the other) leads to the statement*

$$da_k = j\left(\frac{\partial X_K}{\partial x_k}\right) dA_K \tag{18.1}$$

which is to be read, "an area initially dA_K is now da_k; and

$$dv = j dV \tag{18.2}$$

which reads, "a volume initially dV is now dv."

By current convention **X** system is called the material or Lagrangian coordinate, and **x** system is called the spatial or Eulerian coordinate. The description $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ is called the Lagrangian description of motion, i.e., the observer identifies a particle **X** and moves with it. The description $\mathbf{X} = \mathbf{X}(\mathbf{x}, t)$ is called the Eulerian description. The observer remains stationary and observes point $(\mathbf{x}, t)$ in space-time, ignorant of which particle occupies that point. Using these definitions, we can have two possible time derivatives of a fluid property F,

$$\frac{DF}{Dt} = \left[\frac{\partial F(\mathbf{x}, t)}{\partial t}\right]_{\mathbf{x}} \quad \text{spatial derivative} \tag{18.3}$$

$$\frac{DF}{Dt} = \left[\frac{\partial F(\mathbf{X}, t)}{\partial t}\right]_{\mathbf{X}} \quad \text{material derivative} \tag{18.4}$$

The *velocity vector* can be defined in two ways, i.e.,

$$\text{material:} \quad \mathbf{v} = \mathbf{V}(X, t) = \left[\frac{\partial \mathbf{x}(\mathbf{X}, t)}{\partial t}\right]_X \quad \text{identity of particle known} \tag{18.5}$$

$$\text{spatial:} \quad \mathbf{v} = \mathbf{V}(x, t) = \left[\frac{\partial \mathbf{X}(\mathbf{x}, t)}{\partial t}\right]_x \quad \text{identity of particle unknown} \tag{18.6}$$

(In convected (= X) coordinates the spatial velocity vanishes.) Similarly, the acceleration has two forms,

$$\text{spatial:} \quad \mathbf{a} = \mathbf{a}(x, t) \text{ or } a_k = \frac{\partial v_k}{\partial t} + v_{k,\ell} v_\ell = \left.\frac{\partial v_k(x, t)}{\partial t}\right|_x \tag{18.7}$$

$$\text{material:} \quad \mathbf{a} = \mathbf{A}(X, t) \text{ or } a_K = \frac{\partial V_K(X, t)}{\partial t}\;{}_X \tag{18.8}$$

(In convected coordinates the material acceleration vanishes). A similar terminology applies to all quantities which are respectively capitalized or uncapitalized. After deformation the material body has a set of stress vectors (t_k, T_K) such that the force over area

*Repeated subscripts signify summation on 1, 2, 3.

da_k is $t_k da_k$, and the force over area dA_K is $T_K dA_K$. Each stress vector is a sum of components over stress tensors $t_{k\ell}$, T_{KL},

$$t_k = t_{k\ell} i_\ell \text{ or } T_K = T_{KL} I_L. \tag{18.9}$$

The stress tensor $t_{k\ell}$ is to be read as component ℓ of stress vector at **x** computed relative to the deformed area da_k at **x**. Similarly, T_{KL} is the component L of the stress vector at **X** computed relative to the undeformed area dA_K at **X**.

The equations of motion during deformation can be written in x coordinates as

$$t_{k\ell,\ell}(\mathbf{x}, t) = \rho \dot{v}_k(\mathbf{x}, t), \qquad \rho_0 = \rho_j. \tag{18.10}$$

Here all area and stress tensor components are measured in the deformed state. The shape of the deformed body is not known a priori. Hence the (finite-amplitude) boundary conditions when expressed in coordinates **x** are ambiguous, and the problem is nonlinear. Ambiguity also arises from the expression of stresses at **x** in terms of the a priori unknown deformation area da_k at **x**. In contrast the equation of motion in **X** coordinates is

$$T_{KL,L}(\mathbf{X}, t) = \rho_0 \dot{v}_K(\mathbf{X}, t). \tag{18.11}$$

Here the initial shape of the undeformed body is known. The stresses are pictured, however, as computed relative to undeformed coordinates (i.e., T_{KL} is the stress computed in the X system relative to the undeformed area dA_K, both quantities related to the true location **x** of the stress and true area da_k through transformation rules, viz, $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$, etc.)

It is clearly desirable to take the forces to be at **x** and to compute stresses relative to areas in **X** because in this system the boundary conditions are unambiguously statable since the shape of the undeformed body is known. When so defined the stresses are Piola stresses [5a]. The equations of motion are then written

$$T_{Kk,K}(\mathbf{X}, t) = \rho \dot{v}_k(\mathbf{X}, t) \tag{18.12}$$

in which T_{Kk} is the Piola stress. The independent position variable is **X** which is not the point (namely **x**) where the actual stress (in the deformed body) appears, but is related to that point through $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$. Piola (or pseudo) stresses, while advantageous, may be difficult to write explicitly.

If a material point moves a vector distance U measured in the X system from initial (undeformed state) to final (deformed state), the strain calculated in the reference (or underformed) system of coordinates is

$$E_{KL} = \frac{1}{2} \frac{\partial U_K}{\partial X_L} + \frac{\partial U_L}{\partial X_K} + \delta_{MN} \frac{\partial U_M}{\partial X_K} \frac{\partial U_N}{\partial X_L} \tag{18.13}$$

or

$$E_{KL} = U_{(K,L)} + \frac{1}{2} (U_{(M,L)} + U_{[M,L]})(U_{(M,K)} + U_{[M,K]}) \tag{18.14}$$

in which the $U_{K,L}$ tensors are segregated into symmetric (,) and unsymmetric [,] parts. Similarly, if the vector deformation displacement is u (measured in the **x** system) then the strains in the **x** system are given by

$$e_{k\ell} = u_{(k,\ell)} - \frac{1}{2}\,(u_{(m,k)} + u_{[m,k]}) \cdot (u_{(m,\ell)} + u_{[m,\ell]}). \tag{18.15}$$

The cross-product terms originate in the procedure of obtaining deformation change in the element of length in oblique coordinates. Also, the strain tensors in both the **X** and **x** systems are nonlinear. In the general theory of finite deformation the strain tensors may therefore be reduced to the sum of one linear term and four nonlinear terms; thus, in the **X** system, one has,

Linear	Nonlinear
$U_{(K,L)}$	$U_{(M,L)}U_{(M,K)}$
	$U_{(M,L)}U_{[M,K]}$
	$U_{[M,L]}U_{(M,K)}$
	$U_{[M,L]}U_{[M,K]}$. (18.16)

Similarly, in the x system,

Linear	Nonlinear
$u_{(k,\ell)}$	$u_{(m,k)}u_{(m,\ell)}$
	$u_{(m,k)}u_{[m,\ell]}$
	$u_{[m,k]}u_{(m,\ell)}$
	$u_{[m,k]}u_{[m,\ell]}$. (18.17)

The Jacobian j can also be written in terms of a sum linear and nonlinear components, viz,

$$j = (1 - 2\mathrm{I}_e + 4\mathrm{II}_e - 8\mathrm{III}_e)^{-1/2}$$

$$j = (1 + 2\mathrm{I}_E + 4\mathrm{II}_E + 8\mathrm{III}_E)^{1/2}, \tag{18.18}$$

in which I_E, II_E, III_E are the invariants of the strain tensor in the **X** system and I_e, II_e, III_e are the invariants of the strain tensor in the **x** system. Explicitly [5b],

$$\begin{aligned} \mathrm{I}_E &= \text{sum of diagonal terms of } E_{KL}. \\ \mathrm{II}_E &= \text{sum of all } 2\times 2 \text{ determinants of } E_{KL}. \\ \mathrm{III}_E &= 3\times 3 \text{ determinant of } E_{KL}. \end{aligned}$$

All of these forms have nonlinear components. The deformation gradients $X_{K,k}$, $x_{k,K}$ in the theory of finite deformation may similarly be written in terms of symmetric and antisymmetric parts,

$$X_{K,k} = [\delta_{mk} - (u_{(m,k)} + u_{[m,k]})]\,\delta_{mK}$$

$$x_{m,K} = [\delta_{MK} + (U_{(M,K)} + U_{[M,K]})]\,\delta_{MK}. \tag{18.19}$$

Using these forms, one can express Lagrangian strains in terms of Eulerian strains [5c],

$$E_{KL} = e_{k\ell} \frac{\partial x_k}{\partial X_K} \frac{\partial x_\ell}{\partial X_L}$$

$$e_{k\ell} = E_{KL} \frac{\partial X_K}{\partial x_k} \frac{\partial X_L}{\partial x_\ell}. \tag{18.20}$$

These equations show that the interrelation between E_{KL} and $e_{k\ell}$ is nonlinear. While strain tensors may be described in the **X** or **x** system as needed, the stress tensor $t_{k\ell}$ is most properly defined in the **x** system, since that is the deformed state. However, the stress distribution in the deformed body may (as noted above) be described by two pseudostresses $T_{K\ell}$, T_{KL} (Piola stresses). The possible advantages in the use of these stresses has already been noted.

The Eulerian and Lagrangian descriptions of finite deformation discussed above are essential tools in the building of a theory of nonlinear continua. They have been elucidated by examples based on the properties of elastic bodies. These descriptions of stresses and strains are now combined with the constitutive relations to form the conservation equations for mass density, momentum, and energy. By noting the origin of all component parts of these conservation equations one can surmise their limits of applicability in the cases of nonlinear motion, to be analyzed in later sections.

19. HYDRODYNAMIC EQUATIONS IN MATERIAL (LAGRANGIAN) VARIABLES

A physical quantity deformed by strain, stress, motion, etc., if referred to X_K, is labeled $L(X, t)$. The same physical quantity if referred to x_k is labeled $E(x, t)$. Consider a plane-wave deformation in Lagrangian coordinates,

$$x(X, t) = X + \xi(X). \tag{19.1}$$

It reads that a material point at X remains at X during a deformation but is carried to a point in x which is equal to X plus a displacement ξ which is a function of X. The two descriptions of the same physical process are related by

$$L(X, t) = E(X + \xi(X), t). \tag{19.2}$$

We choose next $\xi(X)$ to be small but finite, and expand the r.h.s. in a power series of ξ; i.e., after deformation [3],

$$L(X,\ t) = E(X,\ t) + \xi(X,\ t)\,\frac{\partial E(X,\ t)}{\partial X} + \frac{\xi^2}{2!}\,\frac{\partial^2 E}{\partial X^2} + \dots . \tag{19.3}$$

Now for a plane wave traveling in the X direction,

$$\frac{\partial E}{\partial X} = kE. \tag{19.4}$$

Since

$$\xi k = \frac{\xi\omega}{c} = \frac{v}{c} \tag{19.5}$$

and since we always choose $v/c \ll 1$, a necessary result is

$$\xi k \ll 1. \tag{19.6}$$

We thus approximate the series expansion by the statement

$$E(X,\ t) \approx L(X,\ t) - \xi(X,\ t)\,\frac{\partial E(X,\ t)}{\partial X}\ . \tag{19.7}$$

When the second term on the r.h.s. is small we can replace X by x in this approximate relation;

$$E(x,\ t) \approx L(x,\ t) - \xi(x,\ t)\,\frac{\partial E(x,\ t)}{\partial x}. \tag{19.8}$$

This is the (approximate) equation relating the E and L descriptions to second order, referred to the spatial variables x.

The Jacobian j of transformation $x = x(X)$ is the determinant

$$j = \left|\frac{\partial x_k}{\partial X_K}\right|. \tag{19.9}$$

For a plane-wave deformation,

$$j = 1 + \frac{\partial \xi(X,\ t)}{\partial X}\ . \tag{19.10}$$

The equation of continuity in material (body) variables is

$$\rho_0 = \rho j. \tag{19.11}$$

Hence, the density in the deformed state is

$$\rho = \frac{\rho_0}{j} = \frac{\rho_0}{1 + \dfrac{\partial \xi(X, t)}{\partial X}} \tag{19.12}$$

or

$$\rho = \rho_0 \left[1 - \frac{\partial \xi}{\partial X} + \left(\frac{\partial \xi}{\partial X}\right)^2 - \ldots\right]. \tag{19.13}$$

When $\partial\xi/\partial X$ is small (as we have chosen) we can truncate the series as shown. This material description of ρ has a counterpart in the material description of the velocity and acceleration, namely,

$$v(X) = \left.\frac{\partial \xi(X, t)}{\partial t}\right|_{X = \text{const.}} \tag{19.14}$$

$$a(X) = \left.\frac{\partial v(X)}{\partial t}\right|_{X = \text{const.}} \tag{19.15}$$

As noted earlier, the equations of motion in material variables are given by [5a]

$$\rho_0 \mathbf{a} = \mathbf{T}_{K,K}. \tag{19.16}$$

For a plane (acoustic) wave with viscous losses and zero vorticity, this reduces to

$$\rho_0 \left.\frac{\partial v(X, t)}{\partial t}\right|_X = -\left(\frac{\partial p}{\partial X}\right)_t + \left(\frac{4\eta}{3} + \eta'\right) \frac{\partial^2 v}{\partial X^2}\,(X, t). \tag{19.17}$$

Now from the constitutive relations,

$$p' \approx c^2 \rho' + \frac{1}{2}\left(\frac{\partial c^2}{\partial \rho}\right)_s \rho'^2 - \chi\left(\frac{1}{C_v} - \frac{1}{C_p}\right)\frac{\partial v}{\partial x}\,. \tag{19.18}$$

Thus,

$$\begin{aligned} p'(X) = {} & c^2 \rho_0 \left[1 - \frac{\partial \xi}{\partial X} + \left(\frac{\partial \xi}{\partial X}\right)^2\right] \\ & + \frac{1}{2}\left(\frac{\partial c^2}{\partial \rho}\right)_s \rho_0^2 \left[1 - \frac{\partial \xi}{\partial X} + \left(\frac{\partial \xi}{\partial X}\right)^2\right]^2 \\ & - \chi\left(\frac{1}{C_v} - \frac{1}{C_p}\right)\frac{\partial^2 \xi}{\partial t\,\partial x} \end{aligned} \tag{19.19}$$

and

$$\frac{\partial p'}{\partial X} = - c^2\rho_0 \frac{\partial^2 \xi}{\partial X^2}\left[1 + \frac{2}{2c^2}\left(\frac{\partial c^2}{\partial \rho}\right)_s \rho_0\right]$$

$$+ 2c^2\rho_0 \frac{\partial \xi}{\partial X}\frac{\partial^2 \xi}{\partial X^2}\left[1 + \frac{3}{2c^2}\left(\frac{\partial c^2}{\partial \rho}\right)_s \rho_0\right]$$

$$- \chi\left(\frac{1}{C_v} - \frac{1}{C_p}\right)\frac{\partial^3 \xi}{\partial t \partial x^2} . \tag{19.20}$$

The equations of motion thus become

$$\rho_0 \left.\frac{\partial^2 \xi(X, t)}{\partial t^2}\right|_X - c^2\rho_0 \frac{\partial^2 \xi}{\partial X^2}\left[1 + \frac{\rho_0}{c^2}\left(\frac{\partial c^2}{\partial \rho}\right)_s\right]$$

$$+ 2c^2\rho_0 \frac{\partial \xi}{\partial X}\frac{\partial^2 \xi}{\partial X^2}\left[1 + \frac{3}{2c^2}\left(\frac{\partial c^2}{\partial \rho}\right)_s \rho_0\right]$$

$$- \left[\frac{4\eta}{3} + \eta' + \chi\left(\frac{1}{C_v} - \frac{1}{C_p}\right)\right]\frac{\partial^3 \xi}{\partial t \partial X^2} = 0. \tag{19.21}$$

This is the equation of motion in material variables of a viscous heat-conducting medium. In comparison with the equation of motion in spatial variables, it is quite complicated.

The above derivation, restricted to plane waves, can be generalized to three dimensions. As before let x_i, X_i be the Cartesian components of the spatial point $\mathbf{x}$ and the particle identifier $\mathbf{X}$. We assume that a transformation from $\mathbf{x}$ to $\mathbf{X}$ is available, i.e.,

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t). \tag{19.22}$$

Let the Jacobian of this transformation be written in the abbreviated notation,

$$j = \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} = [x_1, x_2, x_3]$$

$$= \begin{vmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{vmatrix} \tag{19.23}$$

Now, it can be directly demonstrated that

$$\frac{\partial f}{\partial x_k} = \frac{\partial f}{\partial X_\alpha}\frac{\partial X_\alpha}{\partial x_k} = \frac{1}{j}\,[x_\ell, x_m, f], \tag{19.24}$$

in which (k, ℓ, m) signifies an even permutation of (1, 2, 3), and repeated subscripts are summed from 1 to 3. This is a formula for calculating derivatives with respect to x_k in terms of derivatives with respect to X_k. Using this definition one can find the divergence of velocity **u** and the Laplacian of a field function f;

$$\nabla \cdot \mathbf{u} = \frac{\partial u_\alpha}{\partial x_\alpha} = \frac{1}{j}\left[\left(\frac{\partial x_1}{\partial t}, x_2, x_3\right) + \left(x_1, \frac{\partial x_2}{\partial t}, x_3\right) + \left(x_1, x_2, \frac{\partial x_3}{\partial t}\right)\right]$$

$$= \frac{1}{j}\frac{\partial j}{\partial t}$$

$$\nabla_x^2 f = \frac{\partial}{\partial x_\alpha}\frac{\partial f}{\partial x_\alpha}$$

$$= [x_2, x_3, (x_2, x_3, f)] + [x_3, x_1, (x_3, x_1, f)] + [x_1, x_2, (x_1, x_2, f)]. \tag{19.25}$$

The Lagrangian formulation of the Navier-Stokes equations (for zero vorticity), in terms of the velocity u_k, pressure p, and viscosity ν,

$$\frac{du_k}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x_k} + \nu\nabla_x^2 u_k \tag{19.26}$$

is

$$\frac{\partial^2 x_k}{\partial t^2} = -\frac{1}{\rho}(x_\ell, x_m, p) + \nu\left\{\left[x_2, x_3, \left(x_2, x_3, \frac{\partial u_k}{\partial t}\right)\right.\right.$$
$$\left.\left. + \left[x_3, x_1, \left(x_3, x_1\ \frac{\partial x_k}{\partial t}\right)\right] + \left[x_1, x_2, \left(x_1, x_2, \frac{\partial x_k}{\partial t}\right)\right]\right\}. \tag{19.27}$$

This is the general form of the Navier-Stokes equation in three dimensional Lagrangian coordinates. As before, the equation of conservation of mass density is

$$\rho_0 = \rho j. \tag{19.28}$$

Initial conditions in Lagrangian form are

$$V(X, to) = \left.\frac{\partial x(X, t)}{\partial t}\right|_{t=to}. \tag{19.29}$$

An example of the Lagrangian form of the wave equation is given in Appendix A.

20. HYDRODYNAMIC EQUATIONS (EULERIAN COORDINATES)

The fluid state considered as a continuum is completely described by three components of hydrodynamic velocity (v_k, $k = 1, 2, 3$) and two thermodynamic variables (say, pressure p and density ρ [5]). The equations of hydrodynamics in Eulerian form are therefore five in number [5d, 5e],

$$\rho \frac{Dv_i}{Dt} = -\pi,_i + \mu_v v_{i,kk} + (\lambda_v + \mu_v) v_{k,ki} \tag{20.1}$$

$$\frac{\partial \rho}{\partial t} + (\rho v_i),_i = 0 \tag{20.2}$$

$$\rho T \frac{Ds}{Dt} = (\lambda_v + 2\mu_v) I_d^2 - 4\mu_v II_d + \rho h \tag{20.3}$$

in which

$D/Dt = \partial/\partial t + v_i \, \partial/\partial x_i$

π = thermodynamic pressure (N/m^2)

λ_v, μ_v = viscosity coefficients ($N\text{-}s/m^2$)

T = thermodynamic temperature ($^\circ K$)

s = entropy ($m^2/(^\circ K)s^2$)

$\mathrm{I}_d = v_{k,k}$ (first invariant of $d_{k\ell}$) (s^{-1})

$\mathrm{II}_d = \frac{1}{2}(d_{kk} d_{\ell\ell} - d_{k\ell} d_{\ell k})$ (second invariant of $d_{k\ell}$) (s^{-2})

h = heat source per unit mass (m^2/s^3)

$d_{k\ell}$ = deformation rate tensor = $\frac{1}{2}(v_{k,\ell} + v_{\ell,k})$. ($s^{-1}$)

The viscosity coefficients λ_v, μ_v are related to the shear viscosity η and the dilatational viscosity coefficient η' by the formulas

$$\lambda_v = \eta' - \frac{2}{3}\eta$$

$$\mu_v = \eta; \qquad \lambda_v + \frac{2}{3}\mu_v = \eta'$$

$$\lambda_v + \mu_v = \frac{\eta}{3} + \eta'. \tag{20.4}$$

If the heat flow is caused by conduction,

$$\rho h \equiv (\chi T_{,k})_{,k}. \tag{20.5}$$

An alternative form of the energy equation is therefore

$$\rho T \frac{Ds}{Dt} = \sigma'_{ik} v_{k,i} + (\chi T_{,k})_{,k} \tag{20.6}$$

in which

$$\chi = \text{coefficient of thermal conductivity } \frac{\text{N}}{(^\circ\text{K})\text{s}}$$

$$\sigma'_{ik} = \eta(v_{i,k} + v_{k,i} - \frac{2}{3} v_{e,e}\delta_{ik}) + \eta' v_{e,e}\delta_{ik} \quad \left(\frac{\text{N}}{\text{m}^2}\right). \tag{20.7}$$

The Eulerian hydrodynamic equations are nonlinear in three aspects: (a) the particle acceleration in Eq. (20.1) contains the term $v_i(\partial/\partial x_i)v_j$; (b) the product ρv_i in Eq. (20.2) may be written as $\rho_0 j^{-1} v_i$, which is nonlinear since j^{-1} is a function of v_i; (c) the products I_d^2, II_d in the energy Eq. (20.3) are nonlinear.

The acceleration field Dv_i/Dt contains rotational components. To make them visible, one may write Eq. (20.1) in the form

$$\rho \left[\frac{\partial v_i}{\partial t} + \frac{1}{2}(v^2)_{,i} - e_{i\ell m} v_m e_{\ell pq} v_{p,q}\right] =$$

$$- \pi_{,i} + (\lambda_v + 2\mu_v) v_{p,pi} - \eta e_{irs}(e_{r\ell m} v_{\ell,m})_{,s} \tag{20.8}$$

in which the rotational terms are identified by the permutation symbol e_{ijk}. Equation (20.1), as well as (20.8), describes the law of motion of a continuum. It is also convenient to formulate the law for the flux of momentum. If the momentum flux density tensor Π_{ik} is defined

$$\Pi_{ik} = p\delta_{ik} + \rho v_i v_k - \sigma_{ik}, \tag{20.9}$$

the Eulerian law governing momentum flux is

$$\frac{\partial \rho v_i}{\partial t} = - \frac{\partial \Pi_{ik}}{\partial x_k}. \tag{20.10}$$

Similarly, the law governing the flux of entropy (i.e., $D(\rho s)/Dt$) is obtained from Eq. (20.6) and Eq. (20.2). In integral form the result is

$$\frac{D}{Dt}\int \rho s dV = \int \frac{\chi(T_{,i})^2}{T^2}\, dV + \int \frac{\mu_v}{T}\, v_{i,k}\left(v_{i,k} + v_{k,i} - \frac{2}{3}\,\delta_{ik} v_{\ell,\ell}\right) dV$$

$$+ \int \frac{\left(\lambda_v + \frac{2}{3}\,\mu_v\right)}{T}\,(v_{i,i})^2 dV. \tag{20.11}$$

The first term on the r.h.s. is the rate of increase of entropy caused by heat gained per unit volume by thermal conduction, the second term is the rate of increase of entropy caused by internal shear friction in which the volume under stress does not change, and the third term is the rate of increase of entropy caused by internal viscous friction in which the shape of the elementary volume does not change (i.e., dilatational deformation).

Other forms of inhomogeneous wave equations are found in current literature. A prominent one is that of Lighthill, discussed in Appendix B.

The hydrodynamic equations contain terms which have different magnitudes. It is essential in the approximations that follow to define an appropriate scale of magnitude so that all terms of same order are included in the approximations. It is therefore very needful to write out nondimensional forms of the basic equations. This is done in the next section.

21. NONDIMENSIONAL FORM OF THE HYDRODYNAMIC EQUATIONS (EULERIAN VARIABLES)

Viscous fluid flow is characterized by the following four quantities.

$\nu = \dfrac{\eta}{\rho_0}$	, the kinematic viscosity	$\left(\dfrac{\mathrm{N}}{\mathrm{m}^2}\,\dfrac{\mathrm{m}^4}{\mathrm{Ns}^2} = \dfrac{\mathrm{m}^2}{\mathrm{s}}\right)$
U	, the flow velocity (or speed)	m/s
L	, the characteristic length	m
τ	, the characteristic time	s

From these four quantities one can construct two dimensionless numbers. These are

$$\mathrm{Re}\,(\nu) = \frac{UL}{\nu}\text{ , the Reynolds number}$$

and

$$S = \frac{U\tau}{L}\text{ , the Strouhal number.}$$

Since L and τ are arbitrary, one may choose them for a convenient purpose. Gol'dberg [14] selects L, τ to be such that the dimensionless derivative of any acoustic variable has the same order of magnitude, e.g.,*

*Asterisk means dimensionless.

$$\frac{\partial \rho^*}{\partial t^*} \approx \frac{\partial \rho^*}{\partial x^*} \approx \frac{\partial^2 \rho^*}{\partial x^{*2}} . \tag{21.1}$$

The two conditions required for this to be true can be surmised from the sinusoidal case. Let $\rho = \rho_{\max} \exp(ikx - i\omega t)$. Then, considering magnitude only, one has

$$\frac{\partial \rho}{\partial t} = - i\omega\rho_{\max}; \quad \frac{\partial \rho}{\partial x} = ik\rho_{\max} \tag{21.2}$$

Condition 1

$$\tau \frac{\partial \rho}{\partial t} \sim L \frac{\partial \rho}{\partial x}$$

or

$$\tau \sim \frac{L}{c}, \quad c = \frac{\omega}{k} \tag{21.3}$$

Condition 2

$$L^2 \frac{\partial^2 \rho}{\partial x^2} \sim L \frac{\partial \rho}{\partial x}$$

or

$$kL \sim k^2 L^2$$

or

$$kL \sim 1$$

or

$$L \sim \frac{\lambda}{2\pi} \tag{21.4}$$

Thus both L and τ are given explicit values.

To convert the hydrodynamic equations into nondimensional forms, one uses the following normalized quantities;

density	$\rho^* = \frac{\rho}{\rho_0}$
length	$x^* = \frac{x}{L}$; i.e., $\nabla^* = L\nabla$
time	$t^* = \frac{t}{\tau}$
velocity	$v^* = \frac{v}{U}$; $c^* = \frac{c\tau}{L}$
temperature	$\theta^* = \frac{\theta}{T_0}$
viscosity	$\nu^* = \frac{\eta}{\rho_0}$. (21.5)

The process of conversion is illustrated below. It consists in selecting each term of the hydrodynamic equation to be made nondimensional, and proceeding as follows:

1. $$\rho \frac{\partial v}{\partial t} \Rightarrow \left(\rho^* \frac{\partial v^*}{\partial t^*}\right) S \left(\frac{\rho_0 L}{\tau^2}\right) \text{; or } \frac{\partial}{\partial t} (\rho v) = \frac{\rho_0 L}{\tau^2} S \frac{\partial}{\partial t^*} (\rho^* V^*)$$

2. $$\rho(\mathbf{v}\cdot\nabla)\mathbf{v} \Rightarrow S^2 \left(\frac{\rho_0 L}{\tau^2}\right) \left\{\rho^*(\mathbf{v}^*\cdot\nabla^*)\mathbf{v}^*\right\} \text{; or } \nabla\cdot\rho\mathbf{v}\mathbf{v} = S^2 \left(\frac{\rho_0 L}{\tau^2}\right) \nabla^*\cdot\rho^*\mathbf{v}^*\mathbf{v}$$

3. $$-\nabla p = -\nabla \left[c^2\rho' + \frac{1}{2}\left(\frac{\partial c^2}{\partial \rho}\right)_s \rho'^2 - \chi\left(\frac{1}{C_c} - \frac{1}{C_p}\right)\nabla\cdot\mathbf{v}\right]$$

a. $$\nabla c^2\rho' = \nabla^*\rho^{*\prime}c^{*\prime}\left(\frac{\rho_0 L}{\tau^2}\right)$$

b. $$\nabla\left(\frac{\partial c^2}{\partial \rho}\right)\rho'^2 \Rightarrow \nabla^*\left(\frac{\partial c^{*2}}{\partial \rho^*}\right)_s \rho^{*\prime 2}\left(\frac{\rho_0 L}{t^2}\right)$$

c. $$\nabla\left(\frac{\chi}{C_v} - \frac{\chi}{C_p}\right)\nabla\cdot\mathbf{v} = (\nabla^*\nabla^*\cdot\mathbf{v}^*)\frac{S\omega}{\mathcal{C}^2}\left(\chi_v^* - \chi_p^*\right)\frac{\rho_0 L}{\tau^2}$$

in which $\chi_v^* = \dfrac{\chi}{\rho_0 C_v \omega\tau}$; $\chi_p^* = \dfrac{\chi}{\rho_0 C_p \omega\tau}$ (dimensions: m^2/s)

and $\mathcal{C}^2 = \dfrac{L^2}{\tau^2}$.

Alternatively, the term 3c can be given the nondimensional form

$$\nabla^*\nabla^*\cdot\mathbf{v}\,\frac{S^2}{\mathrm{Re}(\chi_v)}\,\frac{\rho_0 L}{\tau^2},$$

in which

$$\mathrm{Re}(\chi_v) = \frac{UL\rho_0}{\dfrac{\chi}{C_v}} \qquad \mathrm{Re}(\chi_p) = \frac{UL\rho_0}{\dfrac{\chi}{C_p}}$$

d. $$\eta\nabla^2 v = \nabla^{*2}v^*\,\frac{S^2}{\mathrm{Re}(\nu)}\,\frac{\rho_0 L}{\tau^2} = \nabla^{*2}v^* S\,\frac{\nu^*\omega}{2}\,\frac{\rho_0 L}{\tau^2}$$

in which

$$\mathrm{Re}(\nu) = \frac{UL\rho_0}{\eta} \quad ; \qquad \nu^* = \frac{\eta}{\rho_0 \omega \tau}$$

e.

$$\left(\frac{\eta}{3} + \zeta\right) \nabla \operatorname{div} \mathbf{v} = (\nabla^* \nabla^* \cdot \mathbf{v}^*) \frac{S^2}{\mathrm{Re}(\nu)} \left(\frac{1}{3} + \frac{\zeta}{\eta}\right) \frac{\rho_0 L}{\tau^2}$$

or

$$= S \frac{\nu^* \omega}{\mathcal{C}^2} \left(\frac{1}{3} + \frac{\zeta}{\eta}\right) \nabla^* \nabla^* \cdot \mathbf{v}^* \frac{\rho_0 L}{\tau^2}$$

Thus the nondimensionalized form of the hydrodynamic equation of motion (in Eulerian variables) is

$$S\rho^* \frac{\partial \mathbf{v}^*}{\partial t^*} + S^2 \rho^* (\mathbf{v}^* \cdot \nabla^* \mathbf{v}^*) = S \frac{\partial}{\partial t^*} (\rho^* \mathbf{v}^*) + S^2 \nabla^* \cdot (\rho^* \mathbf{v}^* \mathbf{v}^*)$$

$$= -c^{*2} \nabla^* \rho^{*\prime} - \frac{1}{2} \left(\frac{\partial c^{*2}}{\partial \rho^*}\right)_s \nabla^* \rho^{*\prime 2} + S^2 \left(\frac{1}{\mathrm{Re}(\chi_v)} - \frac{1}{\mathrm{Re}(\chi_p)}\right) \nabla^* \nabla^* \cdot \mathbf{v}^*$$

$$+ \frac{S^2}{\mathrm{Re}(\nu)} (\nabla^* \cdot \nabla^* \mathbf{v}^*) + \frac{S^2}{\mathrm{Re}(\nu)} (\nabla^* \nabla^* \cdot \mathbf{v}^*)$$

$$= -c^{*2} \nabla^* \rho^{*\prime} - \frac{1}{2} \left(\frac{\partial c^{*2}}{\partial \rho^*}\right)_s \nabla^* \rho^{*\prime 2} + S\,(\chi_v^* - \chi_p^*) \frac{\omega}{\mathcal{C}^2} (\nabla^* \nabla^* \cdot \mathbf{v}^*)$$

$$+ S \frac{\nu^* \omega}{\mathcal{C}^2} \nabla^* \cdot \nabla^* \mathbf{v}^* + S \frac{\nu^* \omega}{\mathcal{C}^2} \left(\frac{1}{3} + \frac{\zeta}{\eta}\right) \nabla^* \nabla^* \cdot \mathbf{v}^*. \tag{21.6}$$

The r.h.s. has several convenient forms which will serve later to distinguish magnitudes.

The nondimensional form of the equation of continuity reduces to

$$\frac{\partial \rho^*}{\partial t^*} + S \nabla^* \cdot \rho^* \mathbf{v}^* = 0. \tag{21.7}$$

22. NONDIMENSIONAL FORM OF THE ENERGY EQUATION (EULERIAN COORDINATES)

Although noted earlier, it will be illuminating to derive the equations of energy balance from a different point of view to emphasize inherent thermodynamic aspects. A fluid moving with velocity $\mathbf{v}$ and having an internal energy e per unit mass has a total energy $\mathcal{E}$ per unit volume, of magnitude,

$$\mathcal{E} = \rho\left(\frac{v^2}{2} + e\right). \qquad N/m^2 \qquad (22.1)$$

The stress Π acting on the surface of this unit volume has the form

$$\Pi = \Pi \ \left(\Pi_{ik}\right)$$

$$\Pi_{ik} = p\delta_{ik} + \rho v_i v_k - \sigma_{ik} \qquad (22.2)$$

in which p is the pressure and σ_{ik} is the viscous stress tensor. The intensity $\mathcal{I}$ of mechanical energy is

$$\mathcal{I} = \Pi \cdot \mathbf{v} \qquad N/ms$$

or

$$\mathcal{I}_i = \Pi_{ik} v_k. \qquad (22.3)$$

The intensity of thermal energy flux $\mathcal{H}_i$ is

$$\mathcal{H}_i = \chi \frac{\partial T}{\partial x_i} \qquad N/ms \qquad (22.4)$$

The intensity of internal energy flux $\mathcal{Q}_i$ is

$$\mathcal{Q}_i = v_i \mathcal{E}. \qquad (22.5)$$

Using all contributions, we can construct the equations of energy balance;

$$\frac{\partial \mathcal{E}}{\partial t} = -\frac{\partial}{\partial x_i}\left(v_i \mathcal{E}\right) - \frac{\partial}{\partial x_i}\left(\Pi_{ik} v_k\right) + \frac{\partial}{\partial x_i}\left(\chi \frac{\partial T}{\partial x_i}\right) + v_i \rho F_i. \qquad (22.6)$$

Here we have added a body force $\mathbf{F}$ per unit mass. By using known thermodynamic relations relating pressure, entropy, and density to internal energy, and by employing previously defined equations of balance of mass density and of momentum, one can convert this energy balance into an equation of entropy balance. Excluding body forces, one arrives at the alternate form of the energy equation,

$$\rho T\left(\frac{\partial s}{\partial t} + \mathbf{v}\cdot\nabla s\right) = \sigma'_{ik} \frac{\partial v_i}{\partial x_k} + \mathrm{div}\,(\chi \nabla T). \qquad (22.7)$$

Beginning with the Tds relation

$$Tds = C_v dT + \frac{\beta T d}{K}, \qquad (22.8)$$

where

$$\beta = \frac{1}{\mathscr{v}}\left(\frac{\partial \mathscr{v}}{\partial T}\right)_p$$

$$\mathcal{K} = -\frac{1}{\mathscr{v}}\left(\frac{\partial \mathscr{v}}{\partial p}\right)_T$$

where $\mathscr{v}$ is specific volume and noting [12, p. 149] that

$$\frac{\beta T}{\mathcal{K}} = \frac{C_p - C_v}{\mathscr{v}\beta}, \tag{22.9}$$

we employ the following relation between density and specific volume

$$\rho d\mathscr{v} = -\mathscr{v} d\rho \tag{22.10}$$

and reduce the Tds formula to

$$ds = C_v \frac{dT}{T} - \frac{(C_p - C_v)d\rho}{\beta \rho T}. \tag{22.11}$$

In incremental terms $T = T_0 + T'$, $s = s_0 + s'$, and $\rho = \rho_0 + \rho'$. Writing $s' = ds$, $T' = dT$, and $\rho' = d\rho$, we see that the Tds equation in incremental (i.e. acoustic) form reduces to

$$s' \approx \frac{C_v T'}{T_0} - \frac{(C_p - C_v)}{\beta T_0}\frac{\rho'}{\rho_0}. \tag{22.12}$$

Next we prove that

$$\frac{C_p - C_v}{\beta T_0} = \beta a^2, \qquad a^2 = \left(\frac{\partial p}{\partial \rho}\right)_T. \tag{22.13}$$

First, from thermodynamic relations for ideal substances,

$$C_p - C_v = \frac{\beta^2 T}{\mathcal{K}}. \tag{22.14}$$

Using the definition

$$\left(\frac{\partial p}{\partial \rho}\right)_T \equiv \frac{1}{\rho \mathcal{K}_T} \tag{22.15}$$

results in

$$\frac{\beta T}{C_p - C_v} = \frac{1}{\beta\left(\frac{\partial p}{\partial \rho}\right)_T} = \frac{1}{\beta a^2}. \tag{22.16}$$

Thus, the incremental entropy takes on the new form

$$s' = \beta a^2 \left(\frac{C_v}{\beta a^2} T^{*\prime} - \rho^{*\prime} \right) , \tag{22.17}$$

in which

$$T^{*\prime} = \frac{T'}{T_0} \quad ; \quad \rho^{*\prime} = \frac{\rho'}{\rho_0} \ . \tag{22.18}$$

The l.h.s. of the hydrodynamic equation is next written to first order in the incremental entropy; that is,

$$\text{l.h.s.} = \frac{C_v \rho T}{\alpha_1} \left[\alpha_1 \left(\frac{\partial T^{*\prime}}{\partial t} + \mathbf{v} \cdot \nabla T^{*\prime} \right) - \left(\frac{\partial \rho^{*\prime}}{\partial t} + \mathbf{v} \cdot \nabla \rho^{\prime *} \right) \right] . \tag{22.19}$$

By use of the following nondimensional quantities,

$$\tau t^* = t; \qquad \mathbf{v}^* U = \mathbf{v}; \qquad \nabla^* = L \nabla ; \qquad S = \frac{U\tau}{L} \ . \tag{22.20}$$

one can reduce the above equation to the form

$$\text{l.h.s.} = \frac{C_v \rho T}{\tau \alpha_1} \left[\alpha_1 \left(\frac{\partial T^{*\prime}}{\partial t^*} + S \mathbf{v}^* \cdot \nabla^* T^{*\prime} \right) - \left(\frac{\partial \rho^{*\prime}}{\partial t^*} + S \mathbf{v}^* \cdot \nabla^* \rho^{*\prime} \right) \right] \ . \tag{22.21}$$

The factor in brackets is the nondimensional form of the l.h.s. of the energy equation, premultiplied by $C_v \rho T / \tau \alpha$. The first term on the r.h.s. of the energy equation can be put in nondimensional form in the following way. First, define σ'_{ik} by the relation,

$$\sigma'_{ik} = \eta \sigma_{ik} = \eta \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \frac{\partial v_e}{\partial x_e} \delta_{ik} + \frac{\eta'}{\eta} \delta_{ik} \frac{\partial v_e}{\partial x_e} \right) . \tag{22.22}$$

Then proceed as already indicated for the l.h.s. A typical cluster of terms being made nondimensional is shown below. (Note that the premultiplier of the l.h.s. is shifted to the r.h.s.) The cluster is

$$\left(\frac{\alpha_1 \tau}{C_v \rho T} \eta \frac{\partial v_i}{\partial x_k} \frac{\partial v_i}{\partial x_k} \right) \frac{C_p}{C_p \left(\frac{C_v}{C_v} \right)} \frac{\partial v_i^* U\tau}{\partial x_k^* L\tau} \frac{\partial v_i^* U\tau}{\partial x_k^* L\tau} \frac{\omega}{w} = \frac{\nu \omega}{C^2} \alpha_2 S^2 \frac{\partial v_i^*}{\partial x_k^*} \frac{\partial v_i^*}{\partial x_k^*} , \tag{22.23}$$

in which

$$C^2 = a^2 \frac{C_p}{C_v}$$

$$\alpha_2 = \frac{C_p}{\beta T_0 C_v}$$

$$T_0 \approx T.$$

Now the first term on the r.h.s. when premultiplied as shown is reducible to the nondimensional form,

$$\frac{\nu\omega}{\mathcal{C}^2} \alpha_2 S^2 \sigma_{ik}^* \frac{\partial v_i}{\partial x_k^*} . \tag{22.24}$$

To eliminate dimensions in the second term on the r.h.s. of the energy equation, one writes

$$\text{div } [\chi \nabla T] = \frac{T_0}{\tau^2 \mathcal{C}^2} \chi \nabla^{*2} T^{*\prime}; \qquad \mathcal{C}^2 = \frac{L^2}{\tau^2} . \tag{22.25}$$

Here χ is treated as a constant (spatially). Transferring the premultiplier from the l.h.s. of the energy equation once more, reduce the above term to the new form given by

$$\frac{\alpha_1 \tau}{C_v \rho T} \left(\frac{T_0}{\tau^2 \mathcal{C}^2} \chi \nabla^{*2} T^{*\prime} \right) = \alpha_1 \frac{C_p}{C_v} \frac{\chi^* \omega}{\mathcal{C}^2} \nabla^{*2} T^{*\prime}. \tag{22.26}$$

in which

$$\chi^* = \frac{\chi}{\rho C_p \omega \tau}. \qquad \text{(dimensions: } \mathrm{m^2/s})$$

In summary, the nondimensional form of the energy equation is given by [14]

$$\alpha_1 \left(\frac{\partial T^{*\prime}}{\partial t^*} + S\mathbf{v}^* \cdot \nabla^* T^{*\prime} \right) - \left(\frac{\partial \rho^{*\prime}}{\partial t^*} + S\mathbf{v}^* \cdot \nabla^* \rho^{*\prime} \right)$$

$$= \frac{\nu\omega}{\mathcal{C}^2} \alpha_2 S^2 \sigma_{ik}^* \frac{\partial v_i^*}{\partial x_k^*} + \alpha_1 \frac{C_p}{C_v} \frac{\chi^* \omega}{\mathcal{C}^2} \nabla^{*2} T^{*\prime}. \tag{22.27}$$

23. FIRST AND SECOND ORDER APPROXIMATIONS TO THE HYDRODYNAMIC EQUATIONS (EULERIAN VARIABLES)

It is now convenient to expand the density and velocity in a power series of Strouhal numbers. Thus,

$$\rho^* = \sum_{n=0}^{\infty} S^n \rho_n^*$$

$$\mathbf{v}^* = \sum_{n=1}^{\infty} S^{n-1} \mathbf{v}_n^*$$

$$\rho^{*\prime} = \sum_{n=1}^{\infty} S^n \rho_n^* . \tag{23.1}$$

Note that the lowest order term of ρ^* is $\rho_0/\rho_0 = 1$, of $\mathbf{v}^*$ is $\mathbf{v}_1$, and of $\rho^{*\prime}$ is $S\rho_1^*$. Since $\mathbf{v}^* = \mathbf{v}_1/U$, both $\mathbf{v}_1^*$ and $S\rho_1^*$ are of the same order of magnitude. To judge comparable orders of magnitude in the hydrodynamic equations, we must estimate what magnitude order to assign to the factors

$$S\,\frac{\chi^*\omega}{\mathcal{C}^2} \text{ and } S\,\frac{\nu^*\omega}{\mathcal{C}^2} .$$

As before, we will always assume that $\chi^*\omega/\mathcal{C}^2$ is of same order as $\nu^*\omega/\mathcal{C}^2$. There are three choices:

$$1.\quad \frac{\chi^*\omega}{\mathcal{C}^2} \gg S; \qquad 2.\quad \frac{\chi^*\omega}{\mathcal{C}^2} \sim S; \qquad 3.\quad \frac{\chi^*\omega}{\mathcal{C}^2} \ll S$$

Case 1. Choose

$$\frac{\chi^*\omega}{\mathcal{C}^2} \gg S.$$

In expanded form the equations of hydrodynamics have the form

$$\text{I. } S\,\frac{\partial}{\partial t^*}\left(\sum_{n=0}^{\infty} S^n\rho_n^* \sum_{n=1}^{\infty} S^{n-1}\mathbf{v}_n^*\right) + S^2\nabla^*\cdot\left(\sum_{n=0}^{\infty} S^n\rho_n^* \sum_{n=1}^{\infty} S^{n-1}\mathbf{v}_n^* \sum_{n=1}^{\infty} S^{n-1}\mathbf{v}_n^*\right)$$

$$= -c^{*2}\nabla^*\left(\sum_{n=1}^{\infty} S^n\rho_n^*\right) - \frac{1}{2}\left(\frac{\partial c^{*2}}{\partial\rho^*}\right)_\rho \nabla^*\left(\sum_{n=1}^{\infty} S^n\rho_n^* \sum_{\ell=1}^{\infty} S^\ell\rho_\ell^*\right)$$

$$+ S \left(\chi_v^* - \chi_p^*\right) \frac{\omega}{\mathcal{C}^2} \nabla^* \nabla^* \cdot \left(\sum_{n=1}^{\infty} S^{n-1} \mathbf{v}_n^*\right)$$

$$+ S \frac{\nu^* \omega}{\mathcal{C}^2} \left[\nabla^* \cdot \nabla^* \sum_{n=1}^{\infty} S^{n-1} \mathbf{v}_n^* + \left(\frac{1}{3} + \frac{\zeta}{\eta}\right) \nabla^* \nabla^* \cdot \sum_{n=1}^{\infty} S^{n-1} \mathbf{v}_n^*\right] \tag{23.2}$$

$$\text{II.} \quad \frac{\partial}{\partial t^*} \left(\sum_{n=0}^{\infty} S^n \rho_n^*\right) + S \left(\nabla^* \cdot \sum_{n=0}^{\infty} S^n \rho_n^* \sum_{\ell=1}^{\infty} S^{l-1} \mathbf{v}_l^*\right) = 0. \tag{23.3}$$

By assuming that

$$T^{*\prime} \equiv \frac{T'}{T_0} \sim \frac{v}{C_0} = v^*,$$

we make the expansion

$$T^{*\prime} = \sum_{n=1}^{\infty} S^{n-1} T_n^*.$$

The energy equation then has the form

$$\text{III.} \quad \alpha_1 \left(\frac{\partial}{\partial t^*} \sum_{n=1}^{\infty} S^{n-1} T_n^* + S \sum_{n=1}^{\infty} S^{n-1} \mathbf{v}_n^* \cdot \nabla^* \sum_{n=1}^{\infty} S^{n-1} T_n^*\right)$$

$$- \left(\frac{\partial}{\partial t^*} \sum_{n=1}^{\infty} S^n \rho_n^* + S \sum_{n=1}^{\infty} S^{n-1} \mathbf{v}_n^* . \nabla^* \sum_{n=1}^{\infty} S^n \rho_n^*\right)$$

$$= \frac{\nu\omega}{\mathcal{C}^2} \alpha_2 S^2 \left(\frac{\partial}{\partial x_k^*} \sum_{n=1}^{\infty} S^{n-1} v_{i(n)}^* + \frac{\partial}{\partial x_i} \sum_{n=1}^{\infty} S^{n-1} v_{k(n)}^*\right.$$

$$\left. - \frac{2}{3} \frac{\partial}{\partial x_e^*} \sum_{n=1}^{\infty} S^{n-1} v_{e(n)}^* \delta_{ik} + \frac{\eta}{\eta'} \delta_{ik} \frac{\partial}{\partial x_e} \sum_{n=1}^{\infty} S^{n-1} v_{e(n)}\right) \left(\frac{\partial}{\partial x_k^*} \sum_{n=1}^{\infty} S^{n-1} v_{i(n)}^*\right)$$

$$+ \alpha_1 \frac{C_p}{C_v} \frac{\chi^* \omega}{\mathcal{C}^2} \nabla^{*2} \sum_{n=1}^{\infty} S^{n-1} T_n^* . \tag{23.4}$$

These equations contain infinite sums. By pairing equal powers in S (say S^n) on the l.h.s. and r.h.s. we can form an infinite sequence of hydrodynamic equations. Note that the nth order equation contains only terms in S^n, labeled here "of order S^n." This is to be distinguished from equations "to order S^n," which contain terms in S, S^2, ... up to S^n. With this distinction in mind the hydrodynamic equations of order S are given by

$$\text{I.}\quad \rho_0 \frac{\partial \mathbf{v}_1^*}{\partial t^*} = -C^{*2}\nabla^*\rho_1 + \rho_0(\chi_v^* - \chi_p^*)\frac{\omega}{\mathcal{C}^2}\nabla^*\nabla^*\cdot\mathbf{v}_1^*$$

$$+\rho_0\frac{\nu^*\omega}{\mathcal{C}^2}\left[\nabla^*\cdot\nabla^*\mathbf{v}_1^* + \left(\frac{1}{3}+\frac{\zeta}{\eta}\right)\nabla^*\nabla^*\cdot\mathbf{v}_1^*\right] \tag{23.5}$$

$$\text{II.}\quad \frac{\partial\rho_1}{\partial t^*} + \rho_0\nabla^*\cdot\mathbf{v}_1^* = 0. \tag{23.6}$$

The energy equation of zero order in S is

$$\text{III.}\quad \frac{\partial T_1^*}{\partial t^*} = \frac{C_p}{C_v}\frac{\chi^*\omega}{\mathcal{C}^2}\nabla^{*2}T_1^*. \tag{23.7}$$

The hydrodynamic equations of order S^2 are given in I and II below.

$$\text{I.}\quad \frac{\partial\rho_1^*\mathbf{v}^*}{\partial t^*} + \rho_0^*\frac{\partial\mathbf{v}_2^*}{\partial t^*} + \nabla^*\cdot\rho_0\mathbf{v}_1^*\mathbf{v}_1^*$$

$$= -C^{*2}\nabla^*\rho_2^* - \frac{1}{2\rho_0^*}\left(\frac{\partial C^{*2}}{\partial\rho^*}\right)_s\nabla^*\rho_1^{*2} + \rho_0^*[\chi_v^* - \chi_p^*]\frac{\nabla^*\nabla^*\cdot\mathbf{v}_2^*}{\mathcal{C}^2}$$

$$+\rho_0^*\frac{\nu^*\omega}{\mathcal{C}^2}\left[\nabla^*\cdot\nabla^*\mathbf{v}_2^* + \left(\frac{1}{3}+\frac{\zeta}{\eta}\right)\nabla^*\nabla^*\cdot\mathbf{v}_2^*\right] \tag{23.8}$$

$$\text{II.}\quad \frac{\partial\rho_2^*}{\partial t^*} + \nabla^*\cdot\rho_1^*\mathbf{v}_1^* + \nabla^*\cdot\rho_0^*\mathbf{v}_2^* = 0 \tag{23.9}$$

The energy relation of order S gives

$$\text{III.}\quad \alpha_1\left(\frac{\partial}{\partial t^*}T_2^* + \mathbf{v}_1^*\cdot\nabla^*T_1\right) - \frac{\partial\rho_1}{\partial t^*}$$

$$= \alpha_1\frac{C_p}{C_v}\frac{\chi^*\omega}{\mathcal{C}^2}\nabla^{*2}T_2^*. \tag{23.10}$$

The energy equation of order S^2 gives

$$\alpha_1 \left(\frac{\partial}{\partial t^*} T_3^* + \mathbf{v}_2^* \cdot \nabla^* T_1^* + \mathbf{v}_1^* \cdot \nabla^* T_2^*\right)$$

$$- \left(\frac{\partial \rho_2^*}{\partial t^*} + \mathbf{v}_1^* \cdot \nabla^* \rho_1^*\right)$$

$$= \frac{\nu\omega}{\mathcal{C}^2} \alpha_2 \sigma_{ik}^* \bigg|_{v_1^* = \text{const}} \frac{\partial \mathbf{v}_{1(i)}^*}{\partial x_k^*} + \alpha_1 \frac{C_p}{C_v} \frac{\chi\omega}{\mathcal{C}^2} \nabla^{*2} T_3^* . \tag{23.11}$$

Case 2. We next choose

$$\frac{\chi^* \omega}{\mathcal{C}^2} \left(\text{or } \frac{\nu^* \omega}{\mathcal{C}^2}\right) \sim S.$$

The hydrodynamic equations of order S are given by

$$\text{I.} \qquad \rho_0^* \frac{\partial \mathbf{v}_1^*}{\partial t^*} = - c^{*2} \nabla^* \rho_1^* \tag{23.12}$$

$$\text{II.} \qquad \frac{\partial \rho_1^*}{\partial t^*} + \nabla^* \cdot \rho_0 \mathbf{v}_1^* = 0. \tag{23.13}$$

Similarly, appropriate equations of order S^2 are given by

$$\text{I.} \qquad \frac{\partial}{\partial t^*} \rho_1^* \mathbf{v}_1^* + \rho_0^* \frac{\partial \mathbf{v}_2^*}{\partial t^*} + \nabla^* \cdot \rho_0 \mathbf{v}_1^* \mathbf{v}_1^*$$

$$= - c^{*2} \nabla^* \rho_2^* - \frac{1}{2} \left(\frac{\partial c^{*2}}{\partial \rho^*}\right)_s \nabla^* \rho_1^{*2}$$

$$+ (\chi_v^* - \chi_p^*) \frac{\omega}{\mathcal{C}^2} (\nabla^* \nabla^* \cdot \mathbf{v}_1^*)$$

$$+ \frac{\nu^* \omega}{\mathcal{C}^2} \left[\nabla^* \cdot \nabla^* \mathbf{v}_1^* + \left(\frac{1}{3} + \frac{\zeta}{\eta}\right) \nabla^* \nabla^* \cdot \mathbf{v}_1^*\right] \tag{23.14}$$

II. $$\frac{\partial \rho_2^*}{\partial t^*} + \nabla^* \cdot \rho_1^* \mathbf{v}_1^* + \rho_0^* \nabla^* \cdot \mathbf{v}_2^* = 0. \tag{23.15}$$

Case 3. Finally we choose

$$\frac{\chi^* \omega}{\mathcal{C}^2} \left(\text{or } \frac{\nu^* \omega}{\mathcal{C}^2} \right) << S.$$

This means that viscosity and thermal conduction effects are negligible in the first and second approximations. The hydrodynamic equations of order S are given by

I. $$\rho_0^* \frac{\partial \mathbf{v}_1^*}{\partial t^*} = - c^{*2} \nabla^* \rho_1^* \tag{23.16}$$

II. $$\frac{\partial \rho_1^*}{\partial t^*} + \nabla^* \cdot \rho_0 \mathbf{v}_1^* = 0. \tag{23.17}$$

Similarly, the equations of order S^2 reduce to the set

I. $$\frac{\partial}{\partial t^*} \rho_1^* \mathbf{v}_1^* + \rho_0 \frac{\partial}{\partial t^*} \mathbf{v}_2 + \nabla^* \cdot \rho_0^* \mathbf{v}_1^* \mathbf{v}_1^*$$

$$= - c^{*2} \nabla^* \rho_2^* - \frac{1}{2} \left(\frac{\partial c^{*2}}{\partial \rho} \right)_s \nabla^* \rho_1^{*2} \tag{23.18}$$

II. $$\frac{\partial \rho_2^*}{\partial t^*} + \nabla^* \cdot \rho_1^* \mathbf{v}_1^* + \nabla^* \cdot \rho_0 \mathbf{v}_2^* = 0. \tag{23.19}$$

24. DEFINITION OF ORDERS OF ACOUSTIC PRESSURE

The equation of state for a thermoviscous fluid in nondimensional form is given by

$$p'^* = c^{*2} \rho'^* + \frac{1}{2} \left(\frac{\partial c^{*2}}{\partial \rho^*} \right)_s (\rho'^*)^2 - \frac{\omega S}{\mathcal{C}^2} (\chi_v^* - \chi_p^*) \nabla^* \cdot \mathbf{v}^* \tag{24.1}$$

in which

$$p'^* = p' \Big/ \left(\frac{\rho_0 L^2}{\tau^2} \right); \qquad S = \frac{U\tau}{L}\ ; \qquad \mathcal{C}^2 = \frac{L^2}{\tau^2}.$$

We expand $p^{*\prime}$, $\rho^{*\prime}$, v^* in power series of the Mach number S as follows,

$$\rho'^* = \sum_{n=1}^{\infty} S^n \rho_n^*$$

$$\mathbf{v}^* = \sum_{n=1}^{\infty} S^{n-1} \mathbf{v}_n^*$$

$$p'^* = \sum_{n=1}^{\infty} S^n P_n^* .$$

Hence, to order S^2,

$$SP_1 + S^2 P_2 + \ldots = Sc_0^{*2}\rho_1^* + S^2 \left[c_0^{*2}\rho_2^* + \frac{1}{2}\left(\frac{\partial c^{*2}}{\partial \rho^*}\right)_s \rho_1^{*2} \right]$$
$$- \frac{\omega S}{\mathcal{C}^2} (\chi_v^* - \chi_p^*)(\nabla^* \cdot \mathbf{v}_1^* + S\,\nabla^* \cdot \mathbf{v}_2^*) + \ldots . \tag{24.2}$$

Equating coefficients of equal power in S yields

order S:
$$P_1 = c_0^{*2}\rho_1^* - \frac{\omega}{\mathcal{C}^2} (\chi_v^* - \chi_p^*)\nabla^* \cdot \mathbf{v}_1^* \tag{24.3}$$

order S^2:
$$P_2^* = c_0^{*2}\rho_2^* + \frac{1}{2}\left(\frac{\partial c^{*2}}{\partial \rho^*}\right)_s \rho_1^{*2} - \frac{\omega}{\mathcal{C}^2} (\chi_v^* - \chi_p^*)\, \nabla^* \cdot \mathbf{v}_2^* \tag{24.4}$$

Now

order S:
$$\nabla^* \cdot \mathbf{v}_1^* = -\frac{1}{\rho_0^*}\frac{\partial \rho_1^*}{\partial t^*} \approx \frac{1}{\rho_0^*}\frac{\partial}{\partial t^*}\frac{P_1}{c_0^{*2}}$$

order S^2:
$$\nabla^* \cdot \mathbf{v}_2^* = -\frac{1}{\rho_0^*}\frac{\partial \rho_2^*}{\partial t^*} - \frac{\rho_1^*}{\rho_0^*}\nabla^* \cdot \mathbf{v}_1^* - \frac{\mathbf{v}_1^* \cdot \nabla^* \rho_1^*}{\rho_0^*} \tag{24.5}$$

In linear approximation

$$|v_1| = \frac{c_0 \rho_1}{\rho_0} \; ; \qquad |v_2| = \frac{c_0 \rho_2}{\rho_0} . \tag{24.6}$$

Setting $\rho_0^* = 1$, we have

$$P_1^* = c_0^{*2}\rho_1^* + \frac{\omega}{\mathcal{C}^2}\,[\chi_v^* - \chi_p^*]\,\frac{\partial P_1^*}{c_0^{*2}\partial t^*} \tag{24.7}$$

$$P_2^* = c_0^{*2}\rho_2^* + \frac{1}{2}\left(\frac{\partial c^{*2}}{\partial \rho}\right)_s \rho_1^{*2} - \frac{\omega}{\mathcal{C}^2}\,(\chi_v^* - \chi_p)$$

$$\times\left(-\frac{\partial \rho_2^*}{\partial t^*} + \rho_1^*\,\frac{\partial \rho_1^*}{\partial t^*} - \mathbf{v}_1^*\cdot\nabla^*\rho_1^*\right) \tag{24.8}$$

Thus, of order S, we have

$$\rho_1^* = \frac{P_1^*}{c_0^{*2}} - \frac{\omega}{\mathcal{C}^2}\,(\chi_v^* - \chi_p^*)\,\frac{1}{c_0^{*4}}\,\frac{\partial P_1^*}{\partial t^*}\,. \tag{24.9}$$

of order S^2,

$$\rho_2^* = \frac{P_2^*}{c_0^{*2}} - \frac{1}{2c_0^{*2}}\left(\frac{\partial c^{*2}}{\partial \rho}\right)_s \rho_1^{*2} - \frac{\omega}{c_0^{*2}\mathcal{C}^2}\,(\chi_v^* - \chi_p^*)$$

$$\times\left(\frac{\partial \rho_2^*}{\partial t^*} - \rho_1^*\,\frac{\partial \rho_1^*}{\partial t^*} + \mathbf{v}_1^*\cdot\nabla^*\rho_1^*\right). \tag{24.10}$$

We consider the thermal and viscosity effects to be negligible and write

$$\rho_1^* = \frac{P_1^*}{c_0^{*2}}\,;\qquad \rho_2^* = \frac{P_2^*}{c_0^{*2}} - \frac{1}{2}\left(\frac{\partial c^{*2}}{\partial \rho}\right)_s \frac{P_1^{*2}}{c_0^{*4}}\,. \tag{24.11}$$

The equations of motion of order S^2 are

I.
$$\frac{\partial}{\partial t^*}\,\rho_1^*\mathbf{v}_1^* + \frac{\partial}{\partial t^*}\,\rho_0\mathbf{v}_2^* + \nabla^*\cdot\rho_0^*\mathbf{v}_1^*\mathbf{v}_1^*$$

$$= -c^{*2}\nabla^*\rho_2^* - \frac{1}{2\rho_0^*}\left(\frac{\partial c^{*2}}{\partial \rho^*}\right)_s \nabla^*\rho_1^{*2}\,. \tag{24.12}$$

The equations of continuity of order S^2 are

II.
$$\frac{\partial \rho_2^*}{\partial t^*} + \nabla^*\cdot\rho_1^*\mathbf{v}_1^* + \nabla^*\cdot\rho_0^*\mathbf{v}_2^* = 0. \tag{24.13}$$

Taking $\partial/\partial t^*$ of II and $\nabla^* \cdot$ of I and eliminating appropriate terms result in the formula

$$\Box^{*2} \frac{P_2^*}{c_0^{*2}} = - \nabla^* \cdot \nabla^* \cdot (\rho_0 \mathbf{v}_1^* \mathbf{v}_1^*) - \frac{1}{2} \left(\frac{\partial c^{*2}}{\partial \rho^*} \right)_s \frac{1}{c_0^{*6}} \frac{\partial^2 P_1^{*2}}{\partial t^{*2}}$$

$$\Box^{*2} \equiv - \frac{\partial^2}{\partial t^{*2}} + c_0^{*2} \nabla^{*2} . \tag{24.14}$$

Now for irrotational flow at any point in the field

$$\nabla \cdot \nabla \cdot (\mathbf{v}_1 \mathbf{v}_1) \approx (\nabla \cdot \mathbf{v}_1)^2 + \mathbf{v}_1 \cdot [\nabla \cdot \nabla \mathbf{v}_1] + \nabla^2 \frac{1}{2} (\mathbf{v}_1 \cdot \mathbf{v}_1)$$

$$\nabla \cdot \nabla (\mathbf{v}_1 \cdot \mathbf{v}_1) = 2 \nabla \cdot (\mathbf{v} \cdot \nabla) \mathbf{v} \tag{24.15}$$

(see Ref. 15, page 47). In the near field these formulas are very complicated. Selecting only the radiation, or far field ($|x| = R$), for consideration, we write

$$\frac{\partial}{\partial x_i} \approx - \frac{\partial}{\partial R} \frac{\partial R}{\partial x_i} = \left(\frac{-\partial}{c_0 \partial t} \right) \frac{x_i}{|x_i|} \; ; \qquad p_1 = \frac{\rho_0 c_0}{|\mathbf{v}_1|} . \tag{24.16}$$

Thus, in the far field,

$$- \nabla^* \cdot \nabla^* \cdot (\rho_0^* \mathbf{v}_1^* \mathbf{v}_1^*) \approx - \frac{\partial^2}{c_0^{*2} \partial t^{*2}} \rho_0^* |\mathbf{v}_1^*|^2 = \frac{-1}{c_0^{*4} \rho_0^*} \frac{\partial^2 P_1^{*2}}{\partial t^2} . \tag{24.17}$$

The equation of motion in nondimensional variables therefore becomes

$$\left(- \frac{\partial^2}{c_0^{*2} \partial t^{*2}} + \nabla^{*2} \right) P_2^* = \frac{-1}{c_0^{*4} \rho_0^*} \frac{\partial^2 P_1^{*2}}{\partial t^{*2}} \left[1 + \left(\frac{\partial c^{*2}}{\partial \rho^*} \right)_s \frac{\rho_0^*}{2 c_0^{*2}} \right] . \tag{24.18}$$

If

$$q^* = \frac{1}{\rho_0^{*2} c_0^{*4}} \left[1 + \frac{1}{2} \left(\frac{\partial c^{*2}}{\partial \rho} \right)_s \frac{\rho_0^*}{c_0^{*2}} \right] \frac{\partial P_1^{*2}}{\partial t^*} \tag{24.19}$$

the above equation is reduced to

$$\Box^{*2} P_2^* = - \rho_0^* \frac{\partial q^*}{\partial t^*} . \tag{24.20}$$

Upon removal of the asterisks, one finds the formula of Westervelt [16]; that is,

$$\Box^2 P_2 = -\rho_0 \frac{\partial q}{\partial t} \ . \tag{24.21}$$

This is the equation of motion of second order. Here q (dimensions: $s^{-1} = (m^3/s)/m^3)$) represents a simple source strength density. In the approximation shown (i.e. second order in S, negligible viscosity, and far field) the source strength depends on the square of the first (or linear) order acoustic pressure. Note that the contributions to nonlinearity are two types, one due to the double divergence of $\rho_0 \mathbf{v}_1 \mathbf{v}_1$ and the other due to the nonlinear terms in the constitutive relation $p = p(\rho)$. These two effects are independent of each other. When viscosity effects are included we modify the above calculation as follows. For irrotational fluid motion the hydrodynamic equations to first order in S in nondimensional variables have the form

I. $$\rho_0^* \frac{\partial \mathbf{v}_1^*}{\partial t^*} = -c^{*2} \nabla^* \rho_1^* + \rho_0^* \frac{\nu\omega}{\mathcal{C}^2} \left[\left(\frac{4}{3} + \frac{\zeta}{\eta}\right)\right] \nabla^* \cdot \nabla^* \mathbf{v}_1^* \tag{24.22}$$

II. $$\frac{\partial \rho_1^*}{\partial t^*} + \rho_0 \nabla^* \cdot \mathbf{v}_1^* = 0 \tag{24.23}$$

where $\omega\nu/\mathcal{C}^2 >> S$. Operating on I with $\nabla^* \cdot$ and on II with $\partial/\partial t^*$, and eliminating terms leads to

$$-\frac{\partial^2 \rho_1^*}{\partial t^{*2}} + c^{*2} \nabla^{*2} \rho_1^* = \rho_0^* \frac{\omega\nu}{\mathcal{C}^2} \left[\left(\frac{4}{3} + \frac{\zeta}{\eta}\right) \nabla^* \cdot \nabla^{*2} \mathbf{v}_1^*\right] \tag{24.24}$$

Consider only the radiation (or far) field; we write the gradient operator (negative toward x_i) as

1. $$\frac{\partial}{\partial x_i^*} = -\frac{x_i^*}{|x_i^*|} \frac{\partial}{c_0^* \partial t^*} \ ;$$

2. $$P_1^* = \rho_0^* c_0^* |\mathbf{v}_1^*|; \qquad \rho_1^* = \frac{P_1^*}{c_0^{*2}} \ . \tag{24.25}$$

Then

$$\nabla^* \cdot \nabla^{*2} \mathbf{v}_1^* = -\frac{\partial}{c_0^* \partial t^*} \nabla^{*2} |\mathbf{v}_1^*| = \frac{-1}{\rho_0^* c_0^{*2}} \nabla^{*2} \frac{\partial P_1^*}{\partial t^*} \ . \tag{24.26}$$

Hence, to first order in S, one has the following equation on converting back to dimensional variables,

$$-\frac{1}{c_0^2}\frac{\partial^2 P_1}{\partial t^2} + \nabla^2 P_1 = \frac{-\nu}{c_0^2}\left(\frac{4}{3} + \frac{\zeta}{\eta}\right)\nabla^2 \frac{\partial P_1}{\partial t} \tag{24.27}$$

in which, as before $\nu = \eta/\rho_0$. With second order (S^2) terms only the dimensional form of the equation of motion reduces to

$$-\frac{\partial^2 \rho_2}{\partial t^2} + \nabla\cdot\nabla\cdot\rho_0 \mathbf{v}_1\mathbf{v}_1 = -c^2\nabla^2\rho_2 - \frac{1}{2}\left(\frac{\partial c^2}{\partial\rho}\right)\nabla^2\rho_1^2$$

$$+ \nabla\cdot(\nabla\cdot\nabla\mathbf{v}_2)\left(\eta\left[\frac{4}{3} + \frac{\zeta}{\eta}\right] + \left(\frac{\chi}{c_v} - \frac{\chi}{c_p}\right)\right) \ . \tag{24.28}$$

Here we have neglected the thermal term in the constitutive relation. We next assume that

$$\frac{|\mathbf{v}_2|}{c_0} = \frac{\rho_2}{\rho_0} \ ,$$

so that for plane waves

$$|\mathbf{v}_2| = \frac{c_0}{\rho_0}\left[\frac{P_2}{c_0^2} - \frac{1}{2c_0^2}\left(\frac{\partial c^2}{\partial\rho}\right)_s \frac{P_1^2}{c_0^4}\right] . \tag{24.29}$$

With this assumption the equation of motion in second order quantities reduces to

$$-\frac{\partial^2 P_2}{c_0^2\partial t^2} + \nabla^2\left\{P_2 + \left[\frac{\eta}{\rho_0 c_0^2}\left(\frac{4}{3} + \frac{\zeta}{\eta}\right) + \left(\frac{\chi}{c_v} - \frac{\chi}{c_p}\right)\frac{1}{\rho_0 c_0^2}\right]\frac{\partial P_2}{\partial t}\right\}$$

$$= -\frac{1}{\rho_0 c_0^4}\frac{\partial}{\partial t}\left\{\frac{\partial P_1^2}{\partial t} - \frac{1}{c_0^2}\left(\frac{\partial c^2}{\partial\rho}\right)_s \nabla^2\rho_1^2\left[\eta\left(\frac{4}{3} + \frac{\zeta}{\eta}\right) + \left(\frac{\chi}{c_v} - \frac{\chi}{c_p}\right)\right]\right\}$$

$$-\frac{1}{2}\left(\frac{\partial c^2}{\partial\rho}\right)_s \frac{\nabla^2 P_1^2}{c_0^4} \ . \tag{24.30}$$

CONCLUSION TO SECTION 18-24

The approximate equations derived in the previous sections by perturbation methods provide convenient starting points in the mathematical analysis of nonlinear acoustic processes. When the effects of viscosity, heat conduction, and relaxation on finite-amplitude waves are to be investigated explicitly, it is desirable to have a general approach that embodies all these factors in a single nonlinear differential equation. Such general equations are difficult to obtain when an arbitrary type of nonlinear acoustic wave is under consideration. If, however, the wave is relatively weak and is unidirectional, then one may derive a general equation. This is the Burgers' equation. A thorough exploration of its properties greatly enhances the understanding of the combined effects mentioned above and provides a convenient mathematical model for an otherwise very complex physical process. We discuss Burgers' equation in the next sequence of sections.

25. DERIVATION OF BURGERS' EQUATION

The hydrodynamic equations can be cast in a useful form by transformation of variables [9]. One defines a parameter w (dimensions: m^2/s^2) in terms on the nonlinear speed c, by the relation,

$$\rho(\mathbf{x}, t) = \rho_0 exp \left(\frac{w(\mathbf{x}, t)}{c^2}\right) ; \tag{25.1}$$

i.e.,

$$w = c^2 \ell n \frac{\rho}{\rho_0} . \tag{25.2}$$

Using the relation between p_s and ρ' where p_s is defined by

$$p_s \equiv p_0 + c_0^2 \rho' + \frac{1}{2}\left(\frac{\partial c^2}{\partial \rho}\right)_s \rho'^2, \tag{25.3a}$$

one has

$$w = \int_{p_0}^{p} \frac{dp_s}{\rho} . \tag{25.3b}$$

If the new variable w is substituted in the equation of continuity,

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot v + \nabla \rho \cdot v = 0,$$

one obtains

$$\frac{\partial w}{\partial t} + \nabla w \cdot \nabla \varphi + c^2 \nabla^2 \varphi = 0. \qquad v = \nabla \varphi \tag{25.4}$$

Furthermore,

$$\frac{dp_s}{\rho} = \left[c_0^2 + \left(\frac{\partial c^2}{\partial \rho}\right)_s \rho' \ \frac{d\rho}{\rho}\right]. \tag{25.5a}$$

Defining γ such that

$$\gamma - 1 = \left(\frac{\partial c^2}{\partial \rho}\right)_s \frac{\rho_0}{c_0^2} \tag{25.5b}$$

and noting that local wave speed c in the nonlinear medium is defined by

$$\frac{dp_s}{d\rho} \equiv c^2, \tag{25.5c}$$

we see from Eq. (25.5a) that

$$c^2 = c_0^2 + \frac{c_0^2}{\rho_0}(\gamma - 1)\rho'. \tag{25.5d}$$

Now to a one-term approximation,

$$w \approx \int c_0^2 \frac{d\rho}{\rho} \approx \frac{c_0^2}{\rho_0} \int_{\rho_0}^{\rho} d\rho = \frac{c_0^2}{\rho_0}\rho' \approx c_0^2 \left(\frac{u}{c_0}\right);$$

hence,

$$c^2 \approx c_0^2 + (\gamma - 1)w. \tag{25.5}$$

Here, the second term is a correction to first order in the acoustic mach number. As noted previously, the hydrodynamic equation of motion in Eulerian variables has the form

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] = -\nabla p_s + b\nabla^2\mathbf{v}.$$

Noting again that

$$\nabla p_s = c_0^2\nabla\rho' + \left(\frac{\partial c^2}{\partial \rho}\right)\rho'\nabla\rho' = c_0^2\nabla\rho'\left[1 + \left(\frac{\partial c^2}{\partial \rho}\right)_s \frac{\rho'}{c_0^2}\right]$$

$$\nabla\rho' = \nabla(\rho - \rho_0) = \nabla\rho,$$

so that

$$\nabla p_s = [c_0^2 + (\gamma - 1)w]\,\frac{\nabla w}{c^2}\,e^{w/c^2} = \rho_0\nabla w e^{w/c^2},$$

we next set

$$\nabla^2\varphi \approx \frac{\rho}{\rho_0}\nabla^2\varphi = e^{w/c^2}\nabla^2\varphi, \qquad \rho' << \rho_0.$$

When all substitutions are made, the equation of motion reduces to

$$\rho_0 e^{w/c^2}\nabla[\dot{\varphi} + (\nabla\varphi\cdot\nabla\varphi)] = e^{w/c^2}\nabla(-w\rho_0 + b\nabla\cdot\nabla\varphi) \tag{25.6}$$

or

$$w = -\dot{\varphi} - \nabla\varphi\cdot\nabla\varphi + \frac{b}{\rho_0}\nabla\cdot\nabla\varphi. \tag{25.7}$$

We next consider waves in one dimension (coordinate r). Substituting the formulas for w and c^2 into the equation of continuity leads to

$$-\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial\varphi}{\partial r}\frac{\partial^2\varphi}{\partial r\partial t} + \frac{b}{\rho_0}\frac{\partial^3\varphi}{\partial t\partial r^2} + \left[\frac{\partial\varphi}{\partial r}\left(-\frac{\partial^2\varphi}{\partial t\partial r} - \frac{\partial\varphi}{\partial t}\frac{\partial^2\varphi}{\partial r^2} + \frac{b}{\rho_0}\frac{\partial^3\varphi}{\partial r^3}\right)\right]$$

$$+ (c_0^2 + (\gamma - 1)w)\frac{\partial^2\varphi}{\partial r^2} = 0. \qquad (25.8)$$

Now for plane waves

$$w \approx \frac{c_0^2\rho'}{\rho_0} = \frac{p'}{\rho_0} = \frac{\rho_0 c_0 \frac{\partial\varphi}{\partial r}}{\rho_0} = c_0\frac{\partial\varphi}{\partial r}$$

$$\frac{\partial}{\partial t} = -c_0\frac{\partial}{\partial r}.$$

Writing

$$-2\frac{\partial\varphi}{\partial r}\frac{\partial^2\varphi}{\partial r\partial t} = 2c_0\frac{\partial\varphi}{\partial r}\frac{\partial^2\varphi}{\partial r^2}$$

$$\frac{b}{\rho_0}\frac{\partial^3\varphi}{\partial r^3}\frac{\partial\varphi}{\partial r} << \frac{b}{\rho_0}\frac{\partial^3\varphi}{\partial t\partial r^2}$$

and neglecting terms of order 3 in Mach number, that is, neglecting

$$\left(\frac{\partial\varphi}{\partial r}\right)^2\frac{\partial^2\varphi}{\partial r^2},$$

we see that for *plane waves*,

$$c_0^2\frac{\partial^2\varphi}{\partial r^2} - \frac{\partial^2\varphi}{\partial t^2} + \frac{b}{\rho_0}\frac{\partial^3\varphi}{\partial t\partial r^2} + c_0(\gamma + 1)\frac{\partial\varphi}{\partial r}\frac{\partial^2\varphi}{\partial r^2} = 0, \qquad (25.9)$$

the transformation

$y = t - \frac{r}{c_0}$ to replace r

r to replace t

leads to the formulas

$$\frac{\partial}{\partial r} = -\frac{\partial}{c_0\partial y} = -\frac{\partial}{c_0\partial t}$$

and

$$\frac{\partial \varphi}{\partial t} = -c_0 v, \qquad \frac{\partial^2 \varphi}{\partial t^2} = -c_0 \frac{\partial v}{\partial y}.$$

Thus,

$$-2c_0 \frac{\partial v}{\partial y} - \frac{b}{c_0 \rho_0} \frac{\partial^2 v}{\partial y^2} - (\gamma + 1) v \frac{\partial v}{\partial y} = 0,$$

or

$$\frac{\partial v}{\partial r} - \left(\frac{\gamma + 1}{2}\right) \frac{v}{c_0^2} \frac{\partial v}{\partial y} = \frac{b}{2c_0^3 \rho_0} \frac{\partial^2 v}{\partial y^2} \quad [\text{dimensions: } \mathrm{s}^{-1}]\,. \tag{25.10}$$

One next selects a length L such that

$$\frac{\omega \mathfrak{N}}{c^2} = 1\,; \qquad \mathfrak{N} = \frac{\gamma + 1}{2} v_0 L, \qquad v_0 = |v|_{\max}.$$

Multiplying through by L and writing $\sigma = r/L$, $u = v/v_0$ result in

$$\frac{\partial u}{\partial \sigma} - u \frac{\partial u}{\partial(\omega y)} = \frac{\partial^2 u}{\partial(\omega y)^2} \left(\frac{\pi b}{\rho_0} \frac{2}{(\gamma + 1) \lambda v_0}\right). \tag{25.11}$$

For a plane wave,

$$v_0 = \frac{p_0'}{\rho_0 c_0}.$$

Hence

$$\frac{\partial u}{\partial \sigma} - u \frac{\partial u}{\partial(\omega y)} = \frac{b\omega}{(\gamma + 1) p_0'} \frac{\partial^2 u}{\partial(\omega y)^2}. \tag{25.12}$$

The form of this equation is the same as that derived by Burgers [17] in another context.

This form of Burgers' equation has the following features. On its l.h.s. we have a group of symbols that can be interpreted as a statement of conservation in the amplitude u. It contains a nonlinear term (second term). On the r.h.s. we have a group of symbols which represent diffusion (i.e., attenuation of amplitude). The total equation states how the amplitude of wave motion in one direction (l.h.s.) is distorted by nonlinearity and is attenuated by diffusion (r.h.s.).

The above derivation, while complete, does not display certain features associated with the classic concepts of Riemann invariants. Since these concepts have always served to illuminate the physical processes of nonlinear propagation of acoustic waves, it will be useful to present an alternative derivation of Burgers' equation in which Riemann invariants appear.

26. ALTERNATIVE DERIVATION OF BURGERS' EQUATION FOR NONLINEAR ACOUSTIC WAVES

We first define a total acoustic density ρ_A;

$$\rho_A \equiv \rho' \left[1 + \frac{1}{2c^2}\left(\frac{\partial c^2}{\partial \rho}\right)_s \rho'\right]. \tag{26.1}$$

The total fluid density is $\rho = \rho_0 + \rho_A$. Excluding the effects of relaxation and inhomogeneity in the medium, we have

$$\nabla \rho_A = \nabla \rho; \qquad \nabla^2 \rho_A = \nabla^2 \rho.$$

The nondimensional form of the hydrodynamic equations that include all orders of magnitude up to the second in the Strouhal number S are thus given by

$$S\rho^* \frac{\partial \mathbf{v}^*}{\partial t^*} + S^2 \rho^* (\mathbf{v}^* \cdot \nabla \mathbf{v}^*) = - c^{*2} \nabla^* \rho^* + S \left[\frac{\nu\omega}{\mathcal{C}^2}\left(\frac{4}{3} + \frac{\zeta}{\eta}\right) + \frac{(\chi_v^* - \chi_p^*)}{\mathcal{C}^2}\,\omega\right] \nabla^{*2} \mathbf{v}^*. \tag{26.2}$$

Here

$$\frac{\nu\omega}{\mathcal{C}^2} \sim \frac{\chi_v^* - \chi_p^*}{\mathcal{C}^2}\,\omega >> S.$$

In dimensional form this equation reduces to

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = - \frac{c^2}{\rho} \nabla \rho + \frac{1}{\rho}\left[\eta\left(\frac{4}{3} + \frac{\zeta}{\eta}\right) + \frac{\chi}{c_v} - \frac{\chi}{c_p}\right] \nabla^2 \mathbf{v}. \tag{26.3}$$

We now restrict attention to one dimension (coordinate r). Thus,

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} + \frac{c^2}{\rho} \frac{\partial \rho}{\partial r} = \delta \frac{\partial^2 v}{\partial r^2} \tag{26.4}$$

$$\delta = \frac{1}{\rho}\left[\eta\left(\frac{4}{3} + \frac{\zeta}{\eta}\right) + \chi\left(\frac{1}{c_v} - \frac{1}{c_p}\right)\right] \qquad \text{[dimensions: m}^2\text{/s]}. \tag{26.5}$$

Similarly, the dimensional form of the equation of continuity which includes zero and first order terms has the form

$$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial r} + \rho \frac{\partial v}{\partial r} = 0. \tag{26.6}$$

It is now convenient to consider the fluid a perfect gas and to change variables by writing

$$c^2 = a^2 = a_0^2 \left(\frac{\rho}{\rho_0}\right)^{\gamma-1}, \qquad \gamma = \frac{c_p}{c_v}, \tag{26.7}$$

in which a is the speed of an isentropic infinitesimal sound wave at the local values of pressure and density, and a_0 is this speed at the equilibrium pressure and density. In linear infinitesimal theory $a = a_0$, since $\rho/\rho_0 = (\rho_0 + \rho')/\rho_0 = 1 + \rho'/\rho_0$, and $\rho'/\rho_0 \ll 1$. Now

$$\frac{d\rho}{\rho} = \frac{2}{\gamma - 1} \frac{da}{a}. \tag{26.8}$$

Hence, Eq. (26.4) reduces to

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} + \frac{2}{\gamma - 1} a \frac{\partial a}{\partial r} = \delta \frac{\partial^2 v}{\partial r^2}. \tag{26.9}$$

Similarly, since

$$da = \frac{1}{2} (\gamma - 1) a \frac{d\rho}{\rho},$$

we see that

$$\frac{\partial a}{\partial t} + v \frac{\partial a}{\partial r} + \left(\frac{\gamma - 1}{2}\right) a \frac{\partial v}{\partial r} = 0. \tag{26.10}$$

The two equations (26.9) (26.10) were combined by Riemann [18] through the introduction of two invariants,

$$\mathfrak{R} = \frac{a}{\gamma - 1} + \frac{1}{2} v; \qquad \mathcal{S} = \frac{a}{\gamma - 1} - \frac{1}{2} v. \tag{26.11}$$

The new set of equations now becomes

$$\frac{\partial \mathfrak{R}}{\partial t} + (a + v) \frac{\partial \mathfrak{R}}{\partial r} = \frac{1}{2} \delta \left(\frac{\partial^2 \mathfrak{R}}{\partial r^2} - \frac{\partial^2 \mathcal{S}}{\partial r^2}\right) \tag{26.12}$$

$$\frac{\partial \mathcal{S}}{\partial t} - (a - v) \frac{\partial \mathcal{S}}{\partial r} = \frac{1}{2} \delta \left(\frac{\partial^2 \mathcal{S}}{\partial r^2} - \frac{\partial^2 \mathfrak{R}}{\partial r^2}\right), \tag{26.13}$$

in which

$$a + v = \frac{1}{2} (\gamma + 1) \mathfrak{R} + \frac{1}{2} (\gamma - 3) \mathcal{S} \tag{26.14}$$

$$a - v = \frac{1}{2} (\gamma - 3) \mathfrak{R} + \frac{1}{2} (\gamma + 1) \mathcal{S}. \tag{26.15}$$

Now to a first approximation Eq. (26.12) can be solved by using

$$\mathcal{S} = \frac{\delta}{4a_0} \frac{\partial \mathfrak{R}}{\partial x} .$$

This may be seen by neglecting products in v and third derivatives in R, and thus writing

$$\frac{\partial \mathcal{S}}{\partial t} - a_0 \frac{\partial \mathcal{S}}{\partial r} = -\frac{\delta}{2} \frac{\partial^2 \mathfrak{R}}{\partial r^2} ,$$

since

$$\frac{\partial}{\partial t} \approx -a_0 \frac{\partial}{\partial x} .$$

This approximation for $\mathcal{S}$ provides an order of magnitude estimate in Eq. (26.14). To justify omitting products of v, we note that

$$\begin{aligned} v &= 2\mathfrak{R} - \frac{2a}{\gamma - 1} \\ &= \frac{2a}{\gamma - 1} - 2\mathcal{S} \end{aligned}$$

and

$$\begin{aligned} \frac{v}{a_0} &= 2 \frac{\mathfrak{R}}{a_0} - \frac{a}{a_0} \left(\frac{2}{\gamma - 1}\right) \\ &= \frac{2a}{a_0(\gamma - 1)} - 2 \frac{\mathcal{S}}{a_0} . \end{aligned}$$

In nondimensional coordinates,

$$\frac{\mathcal{S}}{a_0} = \frac{\delta}{4a_0^2} \frac{\partial \mathfrak{R}}{\partial r} = \frac{1}{4} \left(\frac{\delta \omega}{a_0^2}\right) S \frac{\partial \mathfrak{R}^*}{\partial r^*} \left(\frac{1}{\omega \tau}\right);$$

i.e., $\mathcal{S}/a_0$ is of order $(\nu\omega/a^*)S$. In contrast,

$$\frac{\mathfrak{R}}{a_0} = \frac{\mathfrak{R}^* S}{a_0} \frac{L}{\tau} ;$$

i.e., $\mathfrak{R}/a_0$ is of order S. In incremental terms Eq. (26.12) is written

$$\frac{\partial(\mathfrak{R} - \mathfrak{R}_0)}{a_0 \partial t} + a_0 \frac{(a + v)}{a_0} \frac{\partial(\mathfrak{R} - \mathfrak{R}_0)}{a_0 \partial r} = \frac{1}{2} \delta \left[\frac{\partial^2(\mathfrak{R} - \mathfrak{R}_0)}{a_0 \partial r^2} - \frac{\partial^2(\mathcal{S} - \mathcal{S}_0)}{a_0 \partial r^2}\right] . \quad (26.16)$$

The relative magnitudes of each term are now explicit. Since

$$\frac{\mathcal{S} - \mathcal{S}_0}{a_0} \sim \left(\frac{\nu\omega}{a^2}\right) S,$$

one can write to order S,

$$\mathcal{S} \approx \mathcal{S}_0.$$

Thus, Eq. (26.12) takes on the simplified form

$$\frac{\partial \Re}{\partial t} + \left[\frac{1}{2}(\gamma + 1)\Re + \frac{1}{2}(\gamma - 3)\mathcal{S}_0\right] \frac{\partial \Re}{\partial r} = \frac{1}{2}\,\delta\,\frac{\partial^2 \Re}{\partial r^2}. \tag{26.17}$$

One next introduces a substitution of variables by defining the excess wavelength velocity u, where

$$u = a + v - a_0,$$

and by defining a coordinate system X moving with the wave at speed a_0;

$$X = r - a_0 t.$$

Now we set

$$ut = (a + v)t - a_0 t$$

so that

$$X = ut; \qquad r = (a + v)t.$$

Hence,

$$u\,\frac{\partial}{\partial X} = \frac{\partial}{\partial t}; \qquad dX = (a + v - a_0)dt$$

$$\frac{\partial}{\partial r} = \left(\frac{1}{a + v}\right)\frac{\partial}{\partial t} = \frac{1}{a + v}\,u\,\frac{\partial}{\partial X}.$$

Since $a + v = u + a_0$ we see that Eq. (26.12) reduces to

$$u\,\frac{\partial \Re}{\partial X} + \frac{\partial \Re}{\partial t} = \frac{\delta}{2}\left[\frac{1}{(a + v)^2}\,\frac{\partial^2 \Re}{\partial t^2}\right]. \tag{26.18}$$

The r.h.s. may have another form if we substitute

$$\left.\frac{\partial}{\partial r}\right|_{t=\text{const.}} \approx \frac{\partial}{\partial X}\,;$$

$$\Re = \frac{u}{2} - \left[\frac{a - a_0 - \dfrac{2a}{\gamma - 1}}{2}\right].$$

Thus

$$u\,\frac{\partial u}{\partial X} + \frac{\partial u}{\partial t} = \frac{\delta}{2}\,\frac{\partial^2 u}{\partial X^2} \qquad [\text{dimensions: } \mathrm{m/s^2}]\,. \tag{26.19}$$

We next define $\mathfrak{U}$ by the relation

$$u = \frac{1}{2}\,(\gamma + 1)\mathfrak{U}\,; \qquad \mathfrak{U} = \Re - \Re_0.$$

Hence,

$$\beta\,\mathfrak{U}\,\frac{\partial \mathfrak{U}}{\partial X} + \frac{\partial \mathfrak{U}}{\partial t} = \frac{\delta}{2}\,\frac{\partial^2 \mathfrak{U}}{\partial X^2}\,, \tag{26.20}$$

in which $\beta = (1/2)(\gamma + 1)$. Equations (26.19) and (26.20) are Burgers' equations for perfect gases. If the medium is a fluid,

$$\beta = \frac{\gamma + 1}{2} = 1 + \frac{B}{2A} = 1 + \frac{1}{2}\,\frac{\rho_0}{c_0^2}\left(\frac{\partial c^2}{\partial \rho}\right)_s. \tag{26.21}$$

In this way a large number of fluidlike media can be analyzed for finite-amplitude effects. Burgers' equation, however, is restricted to one-dimensional problems in which the propagation is in one direction (simple waves). The significance of the equation is this: it shows the conflict of the convective effect, $u\,\partial u/\partial X$, with the diffusive effect, $(\delta/2)(\partial^2 u/\partial X^2)$. It is a key equation in the theory of weak shocks.

This derivation of Burgers' equation and the one in Sect. 25 do not exhaust the number that have been found (Appendix C). It is deemed worthwhile here to consider one more derivation in the hope of showing the close connection between Burgers' equation and the basic hydrodynamic equations noted earlier. This is the derivation of Kuznetsov.

27. KUZNETSOV'S EQUATION OF NONLINEAR ACOUSTICS AND BURGERS' EQUATION

Kuznetsov [19] considers a viscous, heat-conducting fluid and seeks to derive a single nonlinear wave equation in the scalar potential $\mathbf{u} = -\nabla\phi$ based on perturbation of the basic hydrodynamic equations of motion and continuity, and retention of terms to second order only. To assign orders of magnitude to perturbation terms he introduces two quantities, μ and δ. The quantity μ is analogous to a sonic Mach number and serves to order the acoustic variables. Thus by assumption,

$$\frac{p - p_0}{p_0} \sim \mu; \qquad \frac{\rho - \rho_0}{\rho_0} \sim \mu; \qquad \frac{u}{c_0} \sim \mu; \qquad \mu < 1. \tag{27.1}$$

The second quantity δ, defined by

$$\delta \equiv \frac{k}{\rho_0 c_0} b_1 \qquad b_1 = \frac{4}{3}\eta + \zeta, \tag{27.2}$$

serves to order the viscosity and heat-conducting terms. In this construction, therefore, second order terms are of magnitudes μ^2, $\mu\delta$, or δ^2. Now, if vortex terms are omitted the equations of motion and continuity have the form.

$$\rho \frac{\partial \mathbf{u}}{\partial t} = -\frac{\rho}{2} \nabla u^2 - \nabla p + b_1 \nabla^2 \mathbf{u} \tag{27.3}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0. \tag{27.4}$$

By omitting viscosity terms (of order $\mu\delta^2$) and scattering of sound by temperature inhomogeneities (i.e., $\mathbf{u} \cdot \nabla s$), one can write the equation of heat influx in the form

$$\frac{\partial s}{\partial t} = \frac{\chi}{p} \nabla^2 \left(\frac{p}{\rho}\right), \tag{27.5}$$

where χ is the coefficient of thermal conductivity. Note that in this approximation $\partial s/\partial t$ is of second order. Now taking the medium as a perfect gas, the constitutive relation is

$$p \approx f(\rho, s).$$

To terms of second order, this is

$$\frac{p - p_0}{p_0} = \frac{\rho_0 c_0^2}{p_0}\left(\frac{\rho - \rho_0}{\rho_0}\right) + \frac{\gamma - 1}{2} \frac{\rho_0 c_0^2}{p_0}\left(\frac{\rho - \rho_0}{\rho_0}\right)^2 + \frac{c_0^2 \rho_0 s_0}{p_0 \gamma c_v}\left(\frac{s - s_0}{s_0}\right) + \ldots \tag{27.6}$$

in which

$$c_0^2 = \frac{\gamma p_0}{\rho_0}; \qquad \gamma = \frac{c_p}{c_v}.$$

Elimination of p from the hydrodynamic equations and energy equations, then differentiation of the equation of motion with respect to time, together with use of the energy equation to eliminate the entropy, and finally introduction of the scalar potential leads to a single equation, good to second order (as defined above). This is Kuznetsov's equation,

$$\frac{\partial^2 \phi}{\partial t^2} - c_0^2 \nabla^2 \phi = \frac{\partial}{\partial t}\left[b \nabla^2 \phi + (\nabla \phi)^2 + a \left(\frac{\partial \phi}{\partial t}\right)^2\right], \tag{27.7}$$

in which

$$a \equiv \frac{\gamma - 1}{2c_0^2} \quad (s^2/m^2);$$

$$b = \frac{1}{\rho_0}\left[\frac{4}{3}\eta + \zeta + \chi\left(\frac{1}{c_v} - \frac{1}{c_p}\right)\right], \quad (\mathrm{m}^2/\mathrm{s}) \tag{27.8}$$

In Eq. (27.7) the l.h.s. is a statement of the operation of the linear D'Alembertian operator on the potential. The r.h.s. contains second order terms in the form of "forcing functions," each term dependent on the first order potential. Within the limitations of this formula Kuznetsov proceeds to derive the acoustic form of Burgers' equation. This is done as follows.

Case I. In one dimension,

$$\frac{\partial^2 \phi}{\partial t^2} - c_0^2 \frac{\partial^2 \phi}{\partial t^2} = \frac{\partial}{\partial t}\left[b \frac{\partial^2 \phi}{\partial x^2} + \left(\frac{\partial \phi}{\partial x}\right)^2 + a \left(\frac{\partial \phi}{\partial t}\right)^2\right]. \tag{27.9}$$

Here the r.h.s. has all second order terms. Now one assumes that the medium is weakly nonlinear, i.e., the effects of the medium on the waveform are finite (or noticeable) only at large distances. Changing the scale of x such that x is always large, one defines x' by the formula

$$x = \frac{x'}{\nu}, \quad \nu < 1.$$

Choosing solutions in the form of traveling waves, one also defines the retarded time,

$$\tau = t - \frac{x}{c_0} = t - \frac{x'}{\nu c_0},$$

i.e.,

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \nu c_0 \frac{\partial}{\partial x'}; \quad \frac{\partial}{\partial x} = \nu \frac{\partial}{\partial x'}.$$

Now

$$\frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial \tau} + \nu c_0 \frac{\partial \phi}{\partial x'}$$

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial \tau^2} + 2\nu c_0 \frac{\partial}{\partial \tau}\left(\frac{\partial \phi}{\partial x'}\right) + \nu^2 c_0^2 \frac{\partial^2 \phi}{\partial x'^2} .$$

Since $\partial^2\phi/\partial t^2$ is first order and $\nu^2 c_0^2 \partial^2\phi/\partial x'^2$ cancels $- c_0^2 \partial^2\phi/\partial x^2$, a change of scale and a transformation of variables have left a residual second order term on the l.h.s., namely $2c_0 \partial/\partial\tau\ (\partial\phi/\partial x)$. On the r.h.s. the first two terms are of the order of ν^2; hence, we take $\partial/\partial t \approx \partial/\partial\tau$ and $dx_0 = c_0 d\tau$. The third term is expanded so that

$$\left(\frac{\partial \phi}{\partial t}\right)^2 = \left(\frac{\partial \phi}{\partial \tau}\right)^2 + 2\nu c_0 \frac{\partial \phi}{\partial \tau}\frac{\partial \phi}{\partial x'} + \nu^2 c_0^2 \left(\frac{\partial \phi}{\partial x'}\right)^2 .$$

By approximation,

$$\left(\frac{\partial \phi}{\partial t}\right)^2 \approx \left(\frac{\partial \phi}{\partial \tau}\right)^2 .$$

Thus, if only second order terms are retained,

$$2c_0 \frac{\partial \phi}{\partial x} = \frac{b}{c_0^2}\frac{\partial^2 \phi}{\partial \tau^2} + \frac{1}{c_0^2}\left(\frac{\partial \phi}{\partial \tau}\right)^2 + a\left(\frac{\partial \phi}{\partial \tau}\right)^2 \tag{27.10}$$

Operating on this equation with $- \partial/\partial x$, and noting that $u = - \partial\phi/\partial x$, one reduces this last equation to

$$\frac{\partial u}{\partial x} - \alpha u \frac{\partial u}{\partial \tau} = b' \frac{\partial^2 u}{\partial \tau^2} \tag{27.11}$$

in which

$$\alpha = \frac{\gamma + 1}{2c_0^2} \; ; \quad b' = \frac{b}{2c_0^3} .$$

This is the acoustic equivalent of Burgers' equation. For one-dimensional waves in cylindrical and spherical coordinates, the comparable equation is

$$\frac{\partial u}{\partial r} + \frac{u}{nr} - \alpha u \frac{\partial u}{\partial \tau} = b' \frac{\partial^2 u}{\partial \tau^2} , \tag{27.12}$$

in which $n = 1, 1/2$, for spherical and cylindrical waves, respectively [20].

Case II. For problems in two and three dimensions it is of some interest to investigate the propagation of nonlinear waves on the assumption that the waveform varies slowly in the x direction, and more slowly in the y and z directions. If the wave travels n units in the x direction, a change of scale can make this appear as νn units ($\nu < 1$), thus slowing down the change in waveform. Similarly, if the wave travels q units in the y direction and r units in the z direction, change of scale can make these appear as $(\nu q)^{1/2}$ units and $(\nu r)^{1/2}$ units, respectively, slowing down the change in waveform even more. Thus, one seeks a plane-wave solution of the form

$$\phi\left(\tau \equiv t - \frac{x}{c_0}\right); \qquad x' \equiv \nu x; \qquad y' \equiv (\nu y)^{1/2}; \qquad z' \equiv (\nu z)^{1/2}. \tag{27.13}$$

By proceeding as in Case I, one arrives at

$$\frac{\partial^2 \Phi}{\partial \tau \partial x} - \frac{c_0}{2} \nabla^2_{y,z} \Phi = \frac{\partial}{\partial \tau}\left(b' \frac{\partial^2 \Phi}{\partial \tau^2} + \frac{\alpha}{2} \frac{\partial \Phi^2}{\partial \tau}\right)$$

$$\Phi(x, y, z, t) = \frac{\partial \phi}{\partial \tau}$$

$$\alpha \equiv \frac{\gamma + 1}{2c_0^2}; \qquad b' = \frac{b}{2c_0^3}. \tag{27.14}$$

Kuznetsov explains: When $b' = 0$, the resultant equation is that of Zabolotskaya and Khokhlov [21] for a three-dimensional, quasi-planar, nonlinear wave. If $b' = 0$ and $a = 0$, the resultant equation,

$$\frac{\partial^2 \Phi}{\partial \tau \partial x} = \frac{c_0}{2} \nabla^2_{y,z} \Phi, \tag{27.15}$$

is a basic equation of diffraction theory in the diffusion approximation.

We return now to Kuznetsov's development to make additional comments. Kuznetsov's equation, (27.7), is closely allied to several other prominent formulations. We note here in particular an alternate form of Eq. (25.9), which by simple manipulation reduces to

$$\frac{\partial^2 u}{\partial r^2} - \frac{1}{c_0^2} \frac{\partial^2 u}{\partial t^2} + \frac{b}{\rho_0 c_0^2} \frac{\partial^3 u}{\partial t \partial r^2} - \left(\frac{\gamma + 1}{2}\right) \frac{\partial^2}{\partial x \partial t} \left[\left(\frac{\partial \phi}{c_0 \partial x}\right)^2\right] = 0, \tag{27.16}$$

in which we have assumed that

$$c \frac{\partial \varphi}{\partial r} = -\frac{\partial \varphi}{\partial t}; \qquad \frac{\partial \varphi}{\partial r} = u. \tag{27.17}$$

Now, to a first approximation, we relate the particle displacement ξ to ϕ by using

$$\frac{\partial \varphi}{c_0 \partial x} = \frac{u}{c_0} \approx - \frac{\partial \xi}{\partial x} \tag{27.18}$$

(see Eq. (19.3), with $\rho/\rho_0 = 1 - \partial\xi/\partial X = 1 + \rho'/\rho_0 = 1 + u/c_0$). Thus Eq. (27.16) reduces to

$$\frac{\partial^2 u}{\partial r^2} - \frac{1}{c_0^2}\frac{\partial^2 u}{\partial t^2} + \frac{b}{\rho_0 c_0^2}\frac{\partial^3 u}{\partial t \partial r^2} = \left(\frac{\gamma + 1}{2}\right)\frac{\partial^2}{\partial x \partial t}\left[\left(\frac{\partial \xi}{\partial x}\right)^2\right]. \tag{27.19}$$

The nonlinear term now appears as a forcing function (r.h.s. term) and the viscosity term appears as a damping term. We compare this equation with Eq. (27.9) of Kuznetsov and note that they are equivalent to within the limits of the linear substitutions made. Note also that in Eq. (27.19) the velocity u can be written as a sum of harmonic components,

$$u = \sum_n^\infty u_n.$$

Burgers' equation, as derived in several ways above, does not accommodate several other aspects of wave motion. Two of the most important of these are medium relaxation and spherical spreading. We note in the next section an attempt to modify Burgers' equation so as to incorporate these features.

28. EQUATIONS OF MARSH ET AL EXTENDING BURGERS' EQUATION

By measuring rise time of pulses from explosive sources in seawater, Marsh, et al. [8] found nonacoustic behavior of the medium far beyond distances where linear theory should prevail. This experimental result is attributed by them to finite-amplitude effects. To account analytically for these experiments they have constructed an equation which specifically accommodates spherical spreading, dissipation, and finite amplitude. It is

$$\left(q + \frac{\partial}{\partial \tau}\right)\left(\frac{\partial y}{\partial \sigma} - \frac{y}{\sigma}\frac{\partial y}{\partial \tau}\right) = \frac{\gamma - 1}{4\mathcal{B}p_0 c_0}\frac{\partial^2 y}{\partial \tau^2} + \frac{\mathcal{T}}{4\mathcal{B}p_0 c_0}\frac{\partial^2 y}{\partial \tau^2} \tag{28.1}$$

in which

q = heat source, or radiation coefficient

τ = local time = $t - r/c_0$, r = radial distance

$r = \sigma p/p_0$

σ = scaled range = 2 $p_0 r$

$\mathcal{B} = (1 + B/2A)/2\rho_0 c_0^3$; B and A are defined from

$$p - p_0 = A \frac{\rho - \rho_0}{\rho_0} + \frac{\beta}{2}\left(\frac{\rho - \rho_0}{\rho_0}\right)^2 + \ldots$$

$\gamma = c_p / c_v$

$\mathcal{T} = (2\eta + \eta')/\rho_0 c_0^2$.

Equation (28.1) is a Burgers' type of equation. A plane-wave formulation which incorporates this type of dissipation has also been constructed by Blackstock [22]. Other forms, such as those of Polyakova, et al [6] have appeared in the literature. These improvements over the simple Burgers' equation derived above may form the basis for new developments. It is clear, however, that every increase in complexity of the basic Burgers' equation increases the difficulty of obtaining solutions. We turn next to classic solutions of the (relatively) simple form of Burgers' equation (say Eq. (26.20)). Their extraordinarily complex form should convince the reader that nonlinear differential equations governing realistic processes in nonlinear media come close to being intractable.

29. SOLUTION OF BURGERS' EQUATION

We begin with Burgers equation, (26.20), and rewrite it in the form

$$V \frac{\partial V}{\partial y} + \frac{\partial V}{\partial \sigma} = \Gamma \frac{\partial^2 V}{\partial y^2} . \tag{29.1}$$

According to Hopf [23] and Cole [24], the following procedure is useful in obtaining a particular solution of this form of Burgers' equation. Let there be a function $\phi(y, \sigma)$ such that

$$\frac{\partial \phi}{\partial y} = - V; \quad \frac{\partial \phi}{\partial \sigma} = \frac{V^2}{2} - \frac{1}{\Gamma} \frac{\partial V}{\partial y} ; \quad y = x - c_0 t; \quad \sigma = t. \tag{29.2}$$

Then, by Eqs. (29.1) and (29.2),

$$\frac{\partial \phi}{\partial \sigma} = \frac{1}{2}\left(\frac{\partial \phi}{\partial y}\right)^2 + \frac{1}{\Gamma} \frac{\partial^2 \phi}{\partial y^2} . \tag{29.3}$$

Now, substitute $\phi = 2/\Gamma \log \psi$ in the above equations, with

$$\frac{\partial \phi}{\partial \sigma} = \frac{2}{\Gamma} \frac{1}{\psi} \frac{\partial \psi}{\partial \sigma}$$

$$\frac{\partial \phi}{\partial y} = \frac{2}{\Gamma} \frac{1}{\psi} \frac{\partial \psi}{\partial y}$$

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{2}{\Gamma} \left[\frac{1}{\psi} \frac{\partial^2 \psi}{\partial y^2} - \frac{1}{\psi^2} \left(\frac{\partial \psi}{\partial y} \right)^2 \right] . \tag{29.4}$$

After reduction, we see that by this choice of new variables, Burgers' equation (29.1), is reduced to a classical diffusion-type equation; i.e.,

$$\frac{\partial \psi}{\partial \sigma} = \frac{1}{\Gamma} \frac{\partial^2 \psi}{\partial y^2} , \qquad \psi = \psi(y, \sigma), \tag{29.5}$$

in which

$$\psi = \exp \left(\frac{2}{\Gamma} \int_{-\infty}^{V} V dy \right) ; \qquad V = - \frac{2}{\Gamma} \frac{1}{\psi} \frac{\partial \psi}{\partial y} . \tag{29.6}$$

Laplace solved the diffusion equation by assuming that at $\sigma = 0$ there is a distribution of sources $\Psi(Y, 0)dY$, and that each source contributes to the field $\psi(y, \sigma)$ an amount proportional to

$$\frac{1}{\sqrt{\frac{4\pi\sigma}{\Gamma}}} \Psi(Y, 0)dY \exp \left[- \frac{(y - Y)^2}{\frac{4\sigma}{\Gamma}} \right] . \tag{29.7}$$

Now, at a fixed point in space (at y) and a fixed retarded time (σ), all contributions of initial sources are added to give a particular solution,

$$\psi(y, \sigma) = \frac{1}{\sqrt{\frac{4\pi\sigma}{\Gamma}}} \int_{-\infty}^{\infty} \psi(Y, 0) \exp \left[\frac{(y - Y)^2}{\frac{4\sigma}{\Gamma}} \right] dY. \tag{29.8}$$

The range of integration that covers all possible distributions can be reduced in special cases. Furthermore, since $V = f(\psi)$ (as noted above),

$$V(y, \sigma) = \frac{\int_{-\infty}^{\infty} \left(\frac{y - Y}{\sigma}\right) \exp\left[\frac{1}{\delta} \int_{Y}^{\infty} V(Y, 0) dY - \frac{(y - Y)^2}{2\sigma}\right] dY}{\int_{-\infty}^{\infty} \exp \frac{1}{\delta} \left[\int_{Y}^{\infty} V(Y, 0) dY - \frac{(y - Y)^2}{2\sigma}\right] dY}, \tag{29.9}$$

where $\delta = 2/\Gamma$. We rewrite this integral in the following convenient form,

$$u(X, t) = \int_{-\infty}^{\infty} f(X, Y, t) \exp\left[\frac{1}{\delta} F(X, Y, t)\right] dY \Big/ \int_{-\infty}^{\infty} \exp\left(\frac{F}{\delta}\, dY\right). \tag{29.10}$$

The following procedure for solution has been devised by Whitham [25]. This integral has a pronounced peak at $Y_m = Y_{max}$, which makes F greatest, and is relatively negligible elsewhere. For small δ (i.e., for weak diffusity),

$$\begin{aligned} u(X_1 t) &= f(X, Y_m t) \qquad \text{at } F = F_{max} = F(X, Y_m, t) \\ &= \frac{X - Y_m}{t}. \end{aligned} \tag{29.11}$$

Now

$$F(X, Y, t) = \int_{Y}^{\infty} u(Y, 0) dY - \frac{(X - Y)^2}{2t} \tag{29.12}$$

and

$$\left(\frac{\partial F}{\partial Y}\right)_{Y = Y_m} = 0 = u(Y_m, 0) = \frac{X - Y_m}{t}. \tag{29.13}$$

Thus for any given position X, and time t, there is a unique value of waveform velocity. In the region of the shock, however, there are many pairs of positions (Y_1, Y_2) where F is maximum, since the mathematical development is multivalued in the shock region. Thus, for each pair,

$$X = u(Y_1, 0)t + Y_1$$

and

$$X = u(Y_2, 0)t + Y_2$$

give two distinct locations in the shock wave where

$$F_{max}(X, Y_1, t) = F_{max}(X, Y_2, t). \tag{29.14}$$

This equation provides a relation between Y_1 and Y_2. Using the definition of F, we deduce that

$$\int_{Y_1}^{Y_2} u(Y,0)dY = \frac{Y_1 - Y_2}{2}\,[u(Y_1,0) - u(Y_2,0)]\,. \tag{29.15}$$

Geometrically, the formula states that the area under the curve $u(Y, 0)$ between Y_1 and Y_2 is equal to the area under the chord joining Y_1 and Y_2. This can be true only if lobes above this chord are equal to lobes below. Thus, the pair of points Y_1, Y_2 is located by drawing a straight line across the curve $u(Y, 0)$ vs Y such that equal-area lobes (above and below) are cut off by the chord between Y_1 and Y_2. If such a chord is found, then Y_1, Y_2 constitute a pair of points which (a) fall on the curve $u(X, 0)$ vs X for the condition $\delta \to 0$ and which (b) are nonphysical (i.e., represent state of shock). The slope of this chord is

$$\frac{u(Y_1,0) - u(Y_2,0)}{Y_1 - Y_2} = -\frac{1}{t}. \tag{29.16}$$

Thus to find $u(Y, t)$ in the vicinity of shocks one must proceed as follows. On the curve $u(Y, 0)$ vs Y one must search for chords which have negative slope and which cut off equal-lobe areas (as described above). The first of such negative-slope chords is evidently at the first point of inflection of $u(Y, 0)$. Corresponding to it, the time t (equal to the reciprocal of the slope) is the time of onset of the shock wave. The intersection of the chord (extended if needed) with the X axis gives the point in (relative or moving origin) space where the shock forms. As time increases chords with more negative slope develop, with corresponding diminished magnitude of slope. Each pair of end points of these chords gives a double value to the field and therefore constitutes a shock wave front occurring at a later time, given by the value of its reciprocal slope. At successive times, pairs of points on the initial curve $u(Y, 0)$ near an inflection point reach the shock point, enter into it, and vanish. Shock velocity for each pair is given by $1/2\ [u(Y_1, 0) + u(Y_2, 0)]$. If the initial curve $u(Y, 0)$ has several negative inflection points, several shocks propagating at differing speeds are formed. Faster shocks overtake slower shocks, coalescing with them so that eventually, if sufficient energy was originally in $u(Y, 0)$, only one shock remains.

The above sequence of events is illustrated by the (Whitham) graph, Fig. 29.1, of relative time t/t_0 vs relative space X. Here t_0 is taken as the time corresponding to an arbitrarily selected equal-lobe chord, say, an early one.

At time $t/t_0 = 0$, we locate 11 points on the curve $u(Y, 0)$ in space X (relative or moving origin). Points 4 and 8 are points of negative inflection. As time increases, point 1 shifts its relative position in $u(Y, t)$ along the line $1 - J$, moving ahead. The slope of this line is $i/u(1, 0)$. Point 11 shifts its relative position back in $u(Y, t)$ along line $11 - J$. At some specific time, both 1 and 11 coincide at J, which is therefore one point on a shock wave. Two shock waves are initially formed (4-G-I and 8-H-I), which coalesce at a particular t/t_0 and form the single shock $I - J$. Point I differs from other points on shock waves in that it is a point of confluence of three initial points (2, 6 and 10). Any initial point A moving AB, which is parallel to IJ, is never absorbed into a shock.

The graphical manipulation of the plot of the wave train at $t = 0$ to find its subsequent time history has been exploited by several authors [26], particularly in applications to gravity waves on the surface of fluids. It constitutes one species of solution of Burgers' equation. Intensive efforts have been applied [27] to find and improve new deductions from the basic solutions proposed by Hopf and Cole. In the next section we review the most significant of such efforts.

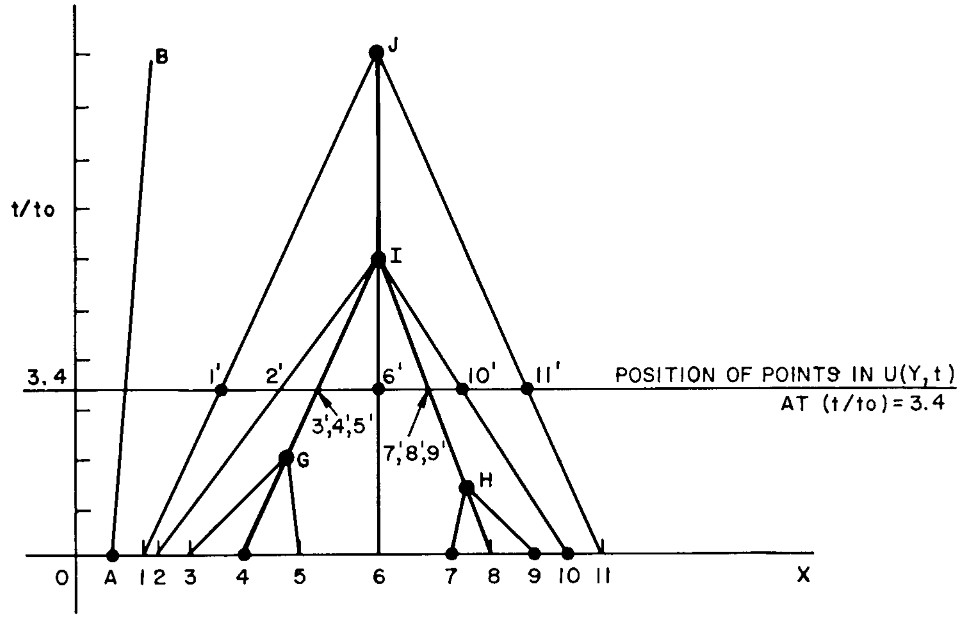


Fig. 29.1a—Formation of shock waves

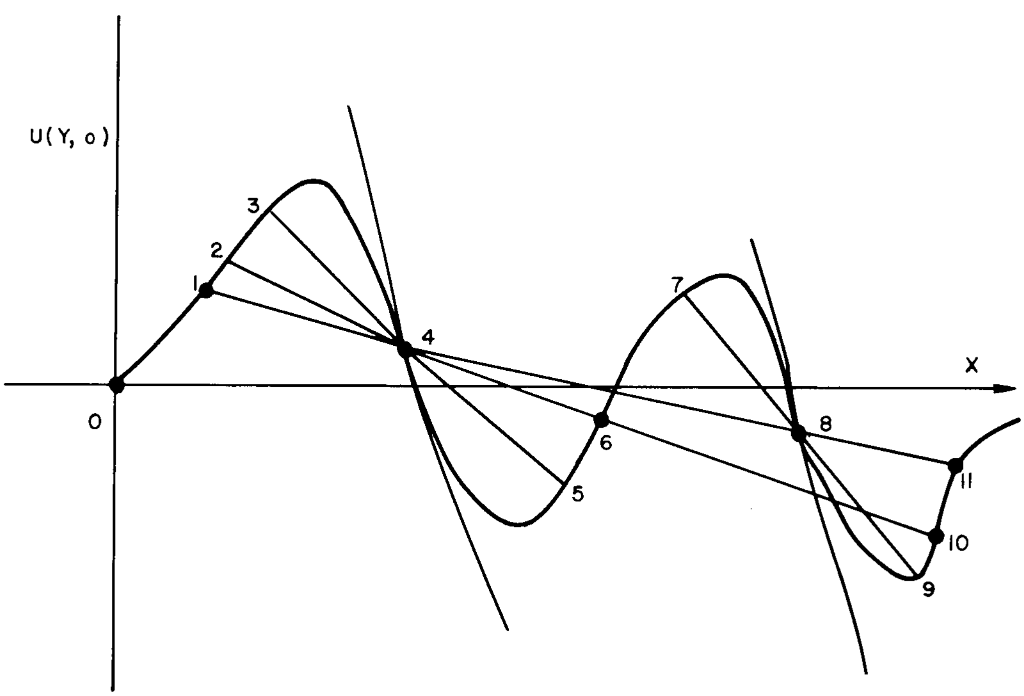


Fig. 29.1b—Whitham's graphical technique for finding conditions of shock formation

30. FURTHER DEVELOPMENTS IN THE SOLUTION OF BURGERS' EQUATION AND COMPARISON WITH EQUATIONS OF FAY, FUBINI, BLACKSTOCK, ETC.

We consider again the solution of Burgers' equation advanced by Hopf and Cole. Referring to Eq. (29.1), or (26.20), we see that it is a solution to an initial value problem, that is, the waveform is specified at $\sigma = 0$ (i.e., $t = 0$) at any location Y, and the problem is to find the waveform at a subsequent time and place. In many cases of interest the initial value problem must be replaced by a boundary value problem. To transform the equations from initial value to boundary value form we begin with Eq. (26.19) and change the time scale by introducing a time delay t' such that the time required for a wave to travel to position Z at speed c_0 is delayed by t' sec, viz,

$$\frac{Z}{c_0} = t - t'. \tag{30.1}$$

With this change in time scale,

$$\frac{\partial}{\partial X} = -\frac{\partial}{c_0 \partial t'} \quad \text{at} \quad t = \text{const.}$$

$$\frac{\partial}{\partial t} = c_0 \frac{\partial}{\partial Z} \quad \text{at} \quad t' = \text{const.}$$

Hence, in Eq. (26.19) we see that

$$-\frac{u \partial u}{c_0 \partial t'} + c_0 \frac{\partial u}{\partial Z} = \frac{\delta}{2} \frac{\partial^2 u}{c_0^2 \partial t'^2} \tag{30.2}$$

or

$$c_0^3 \frac{\partial u}{\partial Z} - c_0 u \frac{\partial u}{\partial t'} = \frac{\delta}{2} \frac{\partial^2 u}{\partial t'^2} \quad [\text{dimensions: } m^3/s^4] \tag{30.3}$$

in which

$$u = u(Z, t').$$

Now one defines new variables $\mathcal{Z}$ (seconds) and $\mathcal{T}$ (meters) such that

$$\frac{\partial}{\partial Z} \equiv \frac{\partial}{c_0 \partial \mathcal{Z}}; \quad \frac{\partial}{\partial t'} = c_0 \frac{\partial}{\partial \mathcal{T}},$$

so that

$$\frac{\partial u}{\partial \mathcal{Z}} - u \frac{\partial u}{\partial \mathcal{T}} = \frac{\delta}{2} \frac{\partial^2 u}{\partial \mathcal{T}^2}. \quad [\text{dimensions: } m/s^2] \tag{30.4}$$

To reduce this still more one uses the method of Hopf and defines a potential such that

$$\frac{\partial\phi}{\partial\mathcal{T}} = u\,; \qquad \frac{\partial\phi}{\partial\mathcal{Z}} = \frac{u^2}{2} + \frac{\delta}{2}\frac{\partial u}{\partial\mathcal{T}}\,; \qquad \phi = \delta\log\psi \qquad \begin{matrix}[\text{dimensions of } \phi\text{: } \mathrm{m^2/s}] \\ [\text{dimensions of } \psi\text{: none}]\end{matrix}$$

Thus,

$$\frac{\partial\phi}{\partial\mathcal{Z}} = \frac{1}{2}\left(\frac{\partial\phi}{\partial\mathcal{T}}\right)^2 + \frac{\delta}{2}\frac{\partial^2\phi}{\partial\mathcal{T}^2}\,,$$

and so Eq. (30.4) reduces to

$$\frac{\partial\psi}{\partial\mathcal{Z}} = \frac{\delta}{2}\frac{\partial^2\psi}{\partial\mathcal{T}^2} \qquad [\text{dimensions: } 1/\mathrm{s}^2] \tag{30.5}$$

or

$$\frac{\partial\psi}{\partial Z} = \alpha\,\frac{\partial^2\psi}{\partial t'^2}, \qquad [\text{dimensions: } 1/\mathrm{m}] \tag{30.6}$$

where

$$\alpha = \frac{\delta}{2c_0^3} \qquad [\text{dimensions: } \mathrm{s^2/m}]$$

and

$$\psi = \psi(Z, t')\,.$$

Since

$$u = \frac{\partial\phi}{\partial\mathcal{T}} = \frac{1}{c_0}\frac{\partial\phi}{\partial t'} = \frac{\delta}{c_0}\frac{\partial}{\partial t'}\log\psi\,,$$

then

$$\log\psi = \frac{c_0}{\delta}\int_{-\infty}^{t'} u(\tau)\,d\tau$$

and

$$\psi(t') = \exp\left(\frac{c_0}{\delta}\int_{-\infty}^{t'} u(\mu)\right). \tag{30.7}$$

Equation (30.6) is a diffusion-type equation; it has a known solution, viz,

$$\psi(Z, t') = \frac{1}{\sqrt{4\pi Z\alpha}}\int_{-\infty}^{\infty}\psi_0(o,\tau)\exp\left[-\frac{(t'-\tau)^2}{4\alpha Z}\right]d\tau\,, \qquad [\alpha\text{: } \mathrm{s^2/m}] \tag{30.8}$$

in which $\psi_0(o, \tau)$ is the distribution of ψ on the boundary $Z = 0$ at time τ. When $\tau = t'$, there is the condition that

$$\psi(t') = \exp\left(\frac{c_0}{\delta} \int_0^{t'} u(\mu)\, d\mu\right), \tag{30.9}$$

in which $u(\mu)$ has been taken such that

$$u(\mu) = 0, \qquad \mu < 0$$

or

$$\psi(t') = 1, \qquad \mu < 0. \tag{30.10}$$

An important case is that in which the boundary undergoes sinusoidal motion. Let the boundary velocity be given by a step-function sinusoid,

$$u_0(t) = u_0 \sin \omega(t) H(t). \tag{30.11}$$

Then

$$\psi(t') = \exp\left[\frac{c_0}{\delta} \frac{u_0}{\omega} (1 - \cos \omega t')\right]. \tag{30.12}$$

Similarly, when $t' > 0$,

$$\psi(Z, t') = \frac{-e^{\mu}}{\sqrt{\pi}} \int_{t'}^{\infty} \exp\left[-\lambda^2 - \mu \cos \omega\left(t' - \lambda\sqrt{4\alpha Z}\right)\right] d\lambda, \qquad t' > 0 \tag{30.13}$$

where

$$\lambda = (t' - \tau)\Big/\sqrt{4\alpha Z}\,; \qquad \mu = c_0 u_0/\delta\omega \qquad \text{[nondimensional]}. \tag{30.14}$$

Both of these components of the response represent transients. There is, in addition, a steady state response to the input waveform $u(t)$. It is

$$\psi_{\text{s.s.}}(Z, t') = \frac{-e^{\mu}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left[-\lambda^2 - \mu \cos \omega\left(t' - \lambda\sqrt{4\alpha Z}\right)\right] d\lambda \tag{30.15}$$

or

$$\psi_{\text{s.s.}}(Z, t') = \frac{e^{\mu}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-\xi^2 - \mu \cos \theta\right) d\xi, \tag{30.16}$$

in which

$$\theta(\xi, Z, t') = \omega\left(t' + \xi\sqrt{4\alpha Z}\right), \qquad \xi = -\lambda.$$

Following Blackstock [27], one writes

$$e^{-\mu\cos\theta} = \sum_{n=0}^{\infty} \epsilon_n(-1)^n \mathrm{I}_n(\mu)\cos n\theta$$

$$\epsilon_0 = 1, \qquad \epsilon_n = 2, \quad \text{for} \quad n \neq 0.$$

Since the range of ξ is $-\infty$ to $+\infty$ and $\cos\theta$ is an even function in the variable ξ, we write

$$\psi_{\text{s.s.}}(Z, t') = \frac{e^{\mu}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi^2} \left[\sum_n \epsilon_n(-1)^n \mathrm{I}_n(\mu)\cos n\omega t' \cos n\omega\xi\sqrt{4\alpha Z}\right] d\xi. \tag{30.17}$$

To obtain an alternate form, it is convenient to set

$$e^{\mu}\xi = s \qquad m = \sqrt{4\alpha Z}$$

and write

$$\psi_{\text{s.s.}}(Z, t') = \frac{e^{-\mu}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left[\exp\left(-s^2 e^{-2\mu}\right) e^{-\mu\cos m\omega e^{-\mu}\left(s + \frac{t}{me^{-\mu}}\right)}\right] ds \tag{30.18}$$

or

$$\psi_{\text{s.s.}}(Z, t') = \frac{e^{-\mu}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-s^2 e^{-2\mu}\right) \left\{\sum_{n=0}^{\infty} \epsilon_n(-1)^n \mathrm{I}_n(\mu)\cos\left[nm\omega e^{-\mu} \times \left(s + \frac{t'}{me^{-\mu}}\right)\right]\right\} ds.$$

Using Gradshteyn and Ryzhik [28, p. 480], the integration is directly performed. The result is

$$\psi_{\text{s.s.}}(Z, t') = \sum_{n=0}^{\infty} \epsilon_n(-1)^n \mathrm{I}_n(\mu) e^{-n^2\omega^2\alpha Z}\cos n\omega t' \tag{30.19}$$

This agrees with Blackstock [27] if we note that

$$\alpha\omega^2 \text{ (this report)} \equiv \alpha \text{ (Blackstock)}.$$

When $\psi_{\text{s.s.}}(Z, t')$ is determined by use of these formulas, the acoustic velocity is obtainable by

$$u = k_c \frac{\partial \phi}{\partial y} = k_c \delta \frac{1}{\psi} \frac{\partial \psi}{\partial y}; \qquad y = \omega t'. \tag{30.20}$$

Since ψ is an infinite series, this formula predicts a very inconvenient division of one infinite series by a second infinite series. To avoid this difficulty, Blackstock [27] proposes a direct Fourier sine series for V;

$$\frac{\psi}{k_c \delta} V = \frac{\partial \psi}{u_0 \partial y}, \quad V \equiv \frac{u}{u_0}.$$

$$V(Z, t') = \sum_{n=1}^{\infty} B_n \sin ny. \tag{30.21}$$

The result is an infinite set of linear simultaneous equations,

$$\sum_{q=0}^{\infty} \epsilon_q (-1)^q \mathrm{I}_q(\mu) e^{-q^2 \omega^2 \alpha Z} \cos qy \sum_{n=1}^{\infty} B_n \sin ny$$

$$= -\frac{k_c \delta}{u_0} \sum_{r=1}^{\infty} \epsilon_r (-1)^r \mathrm{I}_r(\mu) e^{-r^2 \omega^2 \alpha Z} r \sin ry. \tag{30.22}$$

The objective here is to solve for B_1, B_2, etc., for each choice of δ (and μ). This may be done by a high-speed computer that matches left- and right-hand sides at each value of $\sin qy$, $q = 1, 2, 3, \ldots$. The relative magnitudes of B_n show the development of harmonics due to nonlinearity. Writing the normal small-signal attenuation as $\exp(-\alpha\omega^2 Z)$, Blackstock forms the ratio

$$R = B_1 / \exp(-\alpha \omega^2 Z). \tag{30.23}$$

At the origin ($Z = 0$) the value of this ratio is B_1 (the fundamental). As Z increases the fundamental decreases due to nonlinearity. At the same time the wave amplitude decreases because of normal thermoviscous attenuation. A plot of $-20 \log_{10} R$ vs Z with δ as a parameter can be interpreted as an extra loss (in decibels) due to nonlinearity above the normal loss due to thermoviscous attenuation. To illuminate the conflicting effects of distance Z and attenuation $\alpha\omega^2$, it is convenient [21] to define the following quantities

$$E = \beta S k_c \quad \text{[dimensions: } m^{-1}\text{]}$$

$$k_c = \frac{\omega}{c_0} = \text{characteristic wave number}$$

$$\sigma = EZ;$$

$$\Gamma = \frac{E}{\alpha \omega^2} \quad \text{[dimensions: m}^{-1}/\text{m}^{-1} = \text{none]}. \tag{30.24}$$

Thus,

$$\alpha\omega Z = \frac{\sigma}{\Gamma}. \tag{30.25}$$

From Sect. 17 it is known that the shock formation distance L is given by

$$L = \frac{c}{u_0}\,\frac{2}{k(\gamma+1)} = \frac{1}{\beta S k_c}. \tag{30.26}$$

Thus the distance Z for shock formation is given by

$$\sigma = 1, \qquad \text{or} \qquad Z = \frac{1}{E} = \frac{1}{\beta S k_c}. \tag{30.27}$$

By substituting previously defined values for $\alpha\omega^2$ and E, it is seen that

$$\Gamma = \frac{\beta S k_c 2c_0^3}{\left[\frac{\eta}{\rho}\left(\frac{4}{3} + \frac{\zeta}{\eta}\right) + \frac{\chi}{\rho}\left(\frac{1}{C_v} - \frac{1}{C_p}\right)\right]\omega^2}, \tag{30.28}$$

or

$$\frac{\Gamma}{2} = \frac{\beta p_{\max}}{b\omega} \tag{30.29}$$

in which

$$b = \frac{4}{3}\eta + \zeta + \chi\left(\frac{1}{C_v} - \frac{1}{C_p}\right).$$

According to Gol'dberg [14] the condition for shock formation is

$$\frac{\beta p_{\max}}{b\omega} > 1; \qquad \beta = 1 + \frac{\rho_0}{c_0}\left(\frac{\partial c}{\partial \rho}\right)_s. \tag{30.30}$$

Hence the range $(\Gamma/2) > 1$ defines the condition for shock formation. Note that Blackstock [27] states the condition as $\Gamma > 1$. However, this condition does not agree with Gol'dberg [14]. The discrepancy arises from Blackstock's definition of δ, which is 1/2 of that defined in this article. Thus when Blackstock writes

$$\Gamma = \frac{\beta c_0 u_0}{\delta\omega} = \frac{\beta\rho c_0 v_0}{\frac{b}{2}\omega} = 2\,\frac{\beta p_{\max}}{b\omega} \tag{30.31}$$

and

$$\beta = 1 + \frac{B}{2A}$$

(see Eqs. 3n-91 and 3n-60b of Ref. 27), his condition $\Gamma = 1$ is half the condition given by Gol'dberg.

A plot of $-20 \log R$ vs Z (or σ) for choices of α (or Γ) shows each curve tending to an asymptotic value in which the extra attenuation becomes independent of Mach number (i.e., signal amplitude). This occurs when $\alpha\omega^2 Z >> 1$ (or $\sigma/\Gamma >> 1$). For this range, ψ can be approximated by

$$\psi \approx \mathrm{I}_0 - 2\mathrm{I}_1 e^{-\omega^2 \alpha Z} \cos y \tag{30.32}$$

and

$$V = \frac{2k_c\delta}{u_0} \frac{\mathrm{I}_1(\mu)}{\mathrm{I}_0(\mu)} e^{-\omega^2 \alpha Z} \sin y, \qquad \mu = \frac{u_0}{k_c \delta}.$$

When the amplitude u_0 becomes large (i.e., when μ is large)

$$\mathrm{I}_0 \sim \mathrm{I}_1 \sim e^{\mu}/\sqrt{2\pi\mu}$$

and

$$V \to \frac{2}{\mu} e^{-\omega^2 \alpha Z} \sin y \tag{30.33}$$

$$R \to -20 \log_{10}\left(\frac{2}{\mu}\right) = 20 \log_{10} \frac{u_0}{k_c \delta} - 20 \log_{10}^{2}.$$

Thus in the region $(\sigma/\Gamma) >> 1$ (as Blackstock [27] has noted), if one doubles u_0, one also doubles the extra loss due to nonlinearity. Hence for $(\sigma/\Gamma) >> 1$, high-intensity sound "saturates" the medium and no additional increase in source amplitude u_0 will increase the signal at field points far from the source. Asymptotically, one can write for μ large,

$$u \sim 2k_c \delta e^{-\alpha\omega^2 Z} \sin y, \tag{30.34}$$

in which once again it is seen that u is independent of u_0 (i.e., the medium saturates). The statement "μ large" is here given the meaning

$$\mu \left(= \frac{k_c S}{2\alpha\omega^2}\right) \text{ is large}.$$

To include nonlinearity of the medium in this definition, Blackstock defines a particle velocity

$$u = \beta U, \qquad \beta = 1 + B/2A.$$

$$u_0 = \beta U_0$$

With this definition, "μ large" means

$$\frac{\beta k_c S}{2\alpha\omega^2}\left(=\frac{\Gamma}{2}\right) \text{ is large}.$$

Numerically we choose

$$\Gamma > 50 \quad \left(\text{i.e., } \frac{\beta k_c S}{2\alpha\omega^2} > 25\right) \tag{30.35}$$

as a good dividing line between "large" and "not large" [27]. For fixed n and μ large, $I_n(\mu)$ may be developed in an asymptotic series

$$I_n(\mu) \sim \frac{e^\mu}{\sqrt{2\pi\mu}}\left[1 - \frac{4n^2 - 1^2}{1!\,8\mu} + \frac{(4n^2-1^2)(4n^2-3^2)}{2!\,(8\mu)^2} - \dots\right] \tag{30.36}$$

[see Ref. 29, page 377].

When $I_n(\mu)$ can be replaced by $e^\mu/\sqrt{2\pi\mu}$ (the 1st term approximation), then, for $\alpha\omega^2 Z >> 1$,

$$\psi \rightarrow \frac{e^\mu}{\sqrt{2\pi\mu}}\left[1 + \sum_{n=1}^{\infty} 2(-1)^n e^{-n^2\alpha\omega^2 Z}\cos n\omega t'\right]. \tag{30.37}$$

As noted by Cole [24], this infinite series closely resembles a very similar series $\vartheta_4(u, q)$ (theta functions) which appear in the theory of elliptic functions. From Morse and Feshbach (Ref. 30, page 431),

$$\vartheta_4(u, q) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2nu$$

and

$$\vartheta_4(u+\pi, q) = \vartheta_4(u, q).$$

Hence for μ large in the first term approximation of $I_n(\mu)$ one has

$$\psi(Z, t') \rightarrow \frac{e^\mu}{\sqrt{2\pi\mu}}\,\vartheta_4\left(\frac{\omega t'}{2}, e^{-\alpha\omega^2 Z}\right) \tag{30.38}$$

A second approximation for $I_n(\mu)$ which includes $4n^2/8\mu$ is

$$I_n(\mu) \sim \frac{e^\mu}{\sqrt{2\pi\mu}}\,e^{-n^2/2\mu}.$$

Thus

$$\psi(Z, t') \rightarrow \frac{e^{\mu}}{\sqrt{2\pi\mu}} \vartheta_4 \left[\frac{\omega t'}{2}, e^{-(\alpha\omega^2 Z + 1/(2\mu))}\right]. \tag{30.39}$$

Hence,

$$V = \frac{k_c\delta}{u_0} \frac{1}{\psi} \frac{\partial\psi}{\partial\omega t'} = \frac{k_c\delta}{u_0} \frac{\partial}{\partial\omega t'} (\log_{10} \psi).$$

From the *Handbook of Mathematical Functions* [29], page 577, one may obtain the logarithmic derivative of the theta function. The result is

$$V = \frac{k_c\delta}{u_0} \sum_{n=1}^{\infty} \frac{\sin n\omega t'}{\sinh\left[n\left(\alpha\omega^2 Z + \frac{1}{2\mu}\right)\right]}, \qquad \mu = \frac{u_0}{k_c\delta}. \tag{30.40}$$

This result was first derived by Fay [31].

By including terms up to μ^2 Blackstock has obtained still more improved solutions (see Ref. 27).

Fay's solution has been numerically calculated and compared with the solution noted above beginning with Eq. 30.22. The result is

$$V = \sum_{n=1}^{\infty} B_n \sin n\omega t', \tag{30.41}$$

which was obtained by solving a truncated "infinite series" of simultaneous equations (harmonic series solution). For $\sigma > 3.3$ Fay's value of B_1 is accurate to 1% or better. Near the source (i.e., when $\sigma = EZ < 1$) all approximate solutions fail even when μ is large. Thus an analysis is needed in the range $\sigma \rightarrow 0$.

When $\mu \rightarrow 0$,

$$\frac{\partial V}{\partial(k_c SZ)} - V \frac{\partial V}{\partial(\omega t')} = 0. \tag{30.42}$$

Let

$$\Phi = \omega t' + k_c SZ \sin\Phi; \tag{30.43}$$

then $V = \sin\Phi$ is a solution of the above equation for $\mu \rightarrow 0$. Now following Blackstock, let there be two new independent variables,

$$\begin{aligned} \sigma^* &= k_c SZ \\ \omega t' &= \Phi - \sigma^* \sin\Phi. \end{aligned} \tag{30.44}$$

Thus,

$$\frac{\partial V}{\partial(k_c SZ)} = \frac{\partial V}{\partial\sigma^*}\frac{\partial\sigma^*}{\partial(k_c SZ)} + \frac{\partial V}{\partial\Phi}\frac{\partial\Phi}{\partial(k_c SZ)}$$

$$= \frac{\partial V}{\partial\sigma^*} + \frac{\partial V}{\partial\Phi}\frac{\sin\Phi}{(1-\sigma^*\cos\Phi)}$$

$$\frac{\partial V}{\partial(\omega t')} = \frac{\partial V}{\partial\Phi}\frac{1}{1-\sigma^*\cos\Phi}.$$

Writing Burger's equation in the form

$$\frac{\partial V}{\partial k_c SZ} - V\frac{\partial V}{\partial\omega t'} = \frac{1}{2\mu}\frac{\partial^2 V}{\partial(\omega t')^2}$$

shows that in terms of the new variables this equation reduces to

$$(1-\sigma^*\cos\Phi)\frac{\partial V}{\partial\sigma^*} - \frac{\partial V}{\partial\Phi}(V-\sin\Phi) = \frac{1}{2\pi}\frac{\partial}{\partial\Phi}\left(\frac{\frac{\partial V}{\partial\Phi}}{1-\sigma^*\cos\Phi}\right). \tag{30.45}$$

for small values of σ^* (i.e., near the source) Blackstock proposes a solution of this equation in the form

$$V = V^{(0)} + \frac{V^{(1)}}{2\mu} + \frac{V^{(2)}}{(2\mu)^2}. \tag{30.46}$$

Now when μ is very large, the zero order solution is clearly $V^{(0)} = \sin\Phi$ as before. In the first order solution one retains terms containing μ^{-1} and omits terms in $V^{(0)}$, μ^{-2}, etc. It is directly proven by substitution that

$$V^{(1)} = \frac{-\sigma\sin\Phi}{(1-\cos\Phi)^2} \tag{30.47}$$

is a first order solution.

Since the wave is periodic in time the solution V can be expanded in a harmonic series in $\omega t'$. Since V is an odd function,

$$V = V^0 + \frac{V^{(1)}}{2\mu} = \sum_{n=1}^{\infty} B_n\sin n\omega t', \tag{30.48}$$

in which $\omega t' = \Phi - \sigma^*\sin\Phi$. The coefficients B_n are given by

$$B_n = \int_0^{2\pi} V \sin n\omega t' d(\omega t')$$

$$= \int_0^{2\pi} V \sin n(\Phi - \sigma^* \sin \Phi)(d\Phi - \sigma^* \cos \Phi \, d\Phi). \tag{30.49}$$

By use of the theory of Bessel functions the integral can be explicitly carried out [27, 32], with the result that

$$V = \sum_{n=1}^{\infty} \left\{ \frac{2}{n\sigma} J_n(n\sigma) - \frac{1}{n\mu} \sum_r b_r [J_{n-r}(n\sigma) + J_{n+r}(n\sigma)] \right\} \sin n(\omega t')$$

$$b_r = r(1-\sigma)^{-1/2} \sigma^{-r} \left[1 - (1-\sigma^2)^{1/2}\right]^r . \tag{30.50}$$

Here σ is "small" and μ is "large." Selecting the fundamental B_1 from this series and forming the ratio $R = B_1/\exp(-\alpha\omega^2 Z)$, one can plot the extra loss above thermoviscous dissipation (as before) vs distance with $\mu > 25$ as a parameter [27]. Conclusions drawn from these plots are similar to those quoted earlier (viz, saturation effects., etc.). The series obtained by including only the first term in the braces is called the Fubini series [33]. It will be discussed below.

Fay's solution, already noted, is also a Fourier expansion. By additional manipulation (i.e., expanding sinh nf^{-1} in an infinite series and summing the resultant double series term by term), it is possible [27] to write Fay's solution in the form

$$V = \frac{1}{1 + \beta S k_c Z} \left[\pi - \omega t' + 2\pi \sum_k \frac{(-1)^k \sinh\left[\frac{2k(\pi - \omega t')}{\Delta}\right]}{\sinh \frac{2\pi k}{\Delta}} \right] \tag{30.51}$$

where

$$\Delta = \frac{2}{\pi} \left(\alpha\omega^2 Z + \frac{1}{2\mu} \right).$$

In the region $1 < \sigma < 3.5$ there is a transition between the Fubini solution and the asymptotic Fay solution. Blackstock proposes the following formulation to cover this range. In the integral

$$B_n = \int V \sin n(\omega t') d(\omega t'), \tag{30.52}$$

one selects the range $0 < y < \pi$ which, because of symmetry, is adequate. The basic solution $\Phi = \sin^{-1}(u/u_0)$ is used, and again one sets $\omega t' = \Phi - \sigma \sin \Phi$. Integration by parts leads to

$$B_n = \frac{2}{n\pi}\left(-\sin\Phi\cos ny\Big|_{y=0}^{y=\pi} + \int_{y=0}^{\pi}\cos ny\cos\Phi\, d\Phi\right).$$

Interest is focused on the first term. At $y = \pi$, $\Phi = \pi$, so that the upper limit vanishes. At $y = 0$, $\Phi = 0$ when $0 \leqslant \sigma < 1$. But for $\sigma > 1$ $\Phi = \Phi_{\min} = \Phi_b$ where b means "just behind the shock wave." Thus the lower limit is $\sin\Phi_b$ $(=V_b = u_b/u_0)$. Here, for "large σ," $V_b = \pi/(1+\sigma)$, and for $\sigma < 1$, $V_b = 0$. The second term provides B_n for the Fubini solution if $\sigma < 1$, and a small correction for $\sigma > 1$. Thus, in the region $1 < \sigma < 3.5$,

$$V = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[V_b + \frac{2}{n\pi\sigma} J_n(n\sigma)\right] \sin n(\omega t') \qquad \sigma > 1. \tag{30.53}$$

But $\Sigma\, n^{-1} \sin n\omega t'$ is a sawtooth wave. (The theory associated with the sawtooth wave in lossless media is called weak shock theory.) Therefore Eq. (30.53) is a sum of the modified Fay (or sawtooth) solution and the Fubini extended beyond $\sigma = 1$. A finite-amplitude wave originally sinusoidal develops harmonics in accordance with this formula. If μ is very large the above formula becomes

$$\begin{aligned} V &= \frac{1}{1+\beta S k_c Z}\left[\pi - \omega t' + \pi\tanh\left(\frac{y}{\Delta}\right) - \pi\tanh\left(\frac{2\pi - y}{\Delta}\right)\right] \\ &\approx \frac{1}{1+\beta S k_c Z}\left[-\omega t' + \pi\tanh\left(\frac{\omega t'}{\Delta}\right)\right] \qquad \omega t \neq 2\pi \\ &\approx \frac{1}{1+\beta S Z k_c}\left[-\omega t' + 2\pi - \pi\tanh\left(\frac{2\pi - y}{\Delta}\right)\right] \qquad \omega t' \approx 2\pi. \end{aligned} \tag{30.54}$$

All of the above formulas that give the particle velocity $V = V(\omega t', Z)$ have specific regions of validity. These will be discussed in the next section.

31. SUMMARY OF FORMULAS FOR PARTICLE VELOCITY IN NONLINEAR ACOUSTICS

The initiation of the propagation of finite-amplitude waves is treated as a boundary value problem in which the motion of a piston is given by

$$u(0, t) = (u_0 \sin\omega t)H(t). \tag{31.1}$$

Considering only the steady state, we summarize formulas derived in the literature which give the value of $u(Z, t)$, where Z is contained in σ, t in $\omega t'$, etc. (see Sect. 30).

The Fubini Solution [33]

$$u(\sigma, \omega t') = u_0 \sum_{n=1}^{\infty} \frac{2}{n\sigma} J_n(n\sigma) \sin n(\omega t'). \tag{31.2}$$

This solution is valid in lossless fluids near the piston source in the shock-free region (i.e., $\sigma < 1$).

The Fay Solution [31]

$$u(\sigma, \omega t') = \frac{u_0}{\mu} \sum_{n=1}^{\infty} \frac{\sin n(\omega t')}{\sinh \left[\frac{n(1+\sigma)}{2\mu}\right]} . \tag{31.3}$$

This solution is valid in a viscous fluid in a region where shock waves have been already formed. When $\mu \to \infty$, the fluid becomes lossless, and Fay's solution becomes

$$u(\sigma, \omega t') = u_0 \sum_{n=1}^{\infty} \frac{2}{n(1+\sigma)} \sin n(\omega t') . \tag{31.4}$$

This represents a sawtooth wave. It is generally taken that Fay's solutions are valid when $\sigma > 3.5$ (approximately).

The Blackstock Transition Solution [27]

$$u = u_0 \sum_{n=1}^{\infty} B_n \sin n(\omega t')$$

$$B_n = \frac{2}{n\pi} V_b + \frac{2}{n\pi\sigma} \int_{\Phi_{\min}}^{\pi} \cos n(\Phi - \sigma \sin \Phi) d\Phi . \tag{31.5}$$

This solution is valid in the case of lossless fluid over the range $0 \leqslant \sigma < \infty$ for an initially sinusoidal wave (i.e., an infinite wave train). When the fluid is lossy there is a maximum distance at which the sawtooth solution is valid. According to Blackstock [27] and Gol'dberg [14], this maximum distance is given by comparing the diminution of the fundamental B_1 due to finite amplitude with the loss from attenuation in small-signal wave theory. When these two losses are made equal, the resultant distance $Z_{\max}$ is given by

$$1 + Z_{\max} \beta S k_c = \frac{\beta S k_c}{\alpha \omega^2} = \Gamma \tag{31.6}$$

or, $Z_{\max} \approx 1/\alpha\omega^2$ is the maximum distance at which the sawtooth solution is valid.

Comment: The many formulas described above require considerable knowledge and experience in their use. Occasionally, simpler formulas which conveniently display some special aspect of the nonlinear process are in great demand. We therefore review in the following sections a group of formulas that may be derived from a very simple physical basis and an equally simple kinematic construction. These are the formulas based on the method of Riemann, Earnshaw [34], and later followers.

Since they are closely allied to the concepts of geometric acoustics and the method of characteristics in the theory of partial differential equations, we consider these subjects first in the next sections.

32. NONLINEAR PROPAGATION ILLUSTRATED BY CHARACTERISTICS

Nonlinear propagation of waves received its earliest (and classic) treatment by the method of characteristics, i.e., geometrical acoustics [35]. Although this method is applicable to all types of fluid flow, it is illuminating to consider one-dimensional fluid motion in a pipe fitted with two pistons, one at $x = 0$ and the other at $x = \infty$. Let the tube walls and piston constitute a cavity extending, say, indefinitely to the right. At arbitrary time $t = 0$, the pistons are made to undergo an acceleration. In Fig. 32.1 we show three types of piston motion on an xt plot. In Fig. 32.1a the piston at $x = \infty$ is quiescent while the piston at $x = 0$ accelerates in the negative x direction; in Fig. 32.1b the same piston accelerates in the positive direction; and in Fig. 32.1c both pistons accelerate in opposite directions with identical motion. Cases a and b define *simple* waves, while case c defines a *compound* wave. If the disturbances in the fluid caused by piston motion are modest they are propagated at a certain speed, $dx/dt = 1/dt/dx$. In the general case this speed varies with position in the tube. Let c be the speed of sound relative to the fluid and let v be the velocity imparted to the fluid by the piston. First consider waves propagated to the right and plot points of equal total velocity ($c \pm v$). In the case of simple waves, points of equal velocity fall on straight lines (Fig. 32.2a). In Fig. 32.2a the piston accelerates toward the left while the disturbance moves toward the right. The velocity relative to a fixed coordinate system is therefore less than the velocity relative to the fluid. All equal-velocity lines therefore diverge. In Fig. 32.2b the piston accelerates toward the right in the direction of the disturbance so that the velocity relative to the fixed coordinate systems is greater than the velocity relative to the fluid. The lines of equal velocity converge and form an envelope. On the envelope the wave velocity is multivalued and thus not physical. This envelope defines the onset of shock waves. Thus a compression wave, case b, must always form a shock wave if it propagates indefinitely in the tube. In contrast a rarefaction wave, case a, forms shock waves only when the curvature of the xt plot reverses sign.

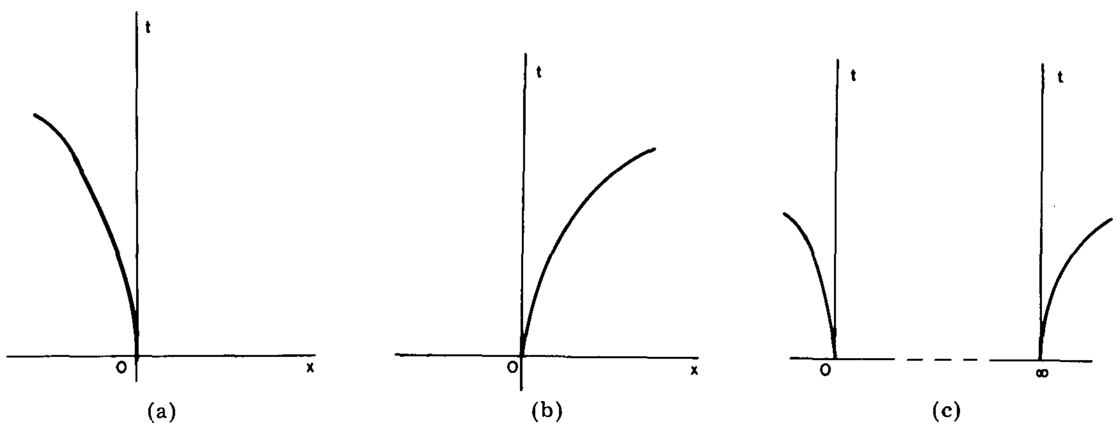
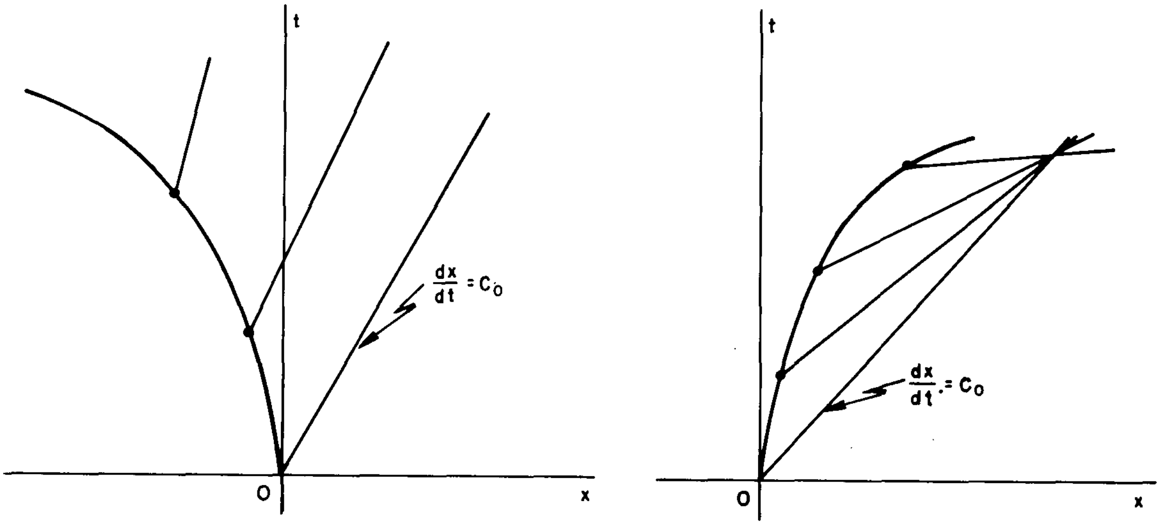


Fig. 32.1—Three types of piston motion on xt plot



(a) (b)

Fig. 32.2—Characteristics of plane propagating waves

The lines of equal velocity on the *xt* plane constitute the *characteristics* of the flow. For a stationary observer compression waves propagating toward the right are described by a family of characteristics whose differential equation is

$$\left(\frac{dx}{dt}\right)_0 = v + c. \tag{32.1}$$

Similarly rarifaction waves traveling toward the left form a second family of characteristics whose differential equation is $(dx/dt)_0 = v - c$. If the observer moves with the wave he observes a third family of characteristics, namely, $(dx/dt)_0 = v$.

When compound waves are present (Fig. 32.1c), the disturbance propagates simultaneously to the right and left as viewed by a stationary observer, and in the direction v, as viewed by an observer moving with the fluid. For such arbitrary motion it is desirable to separate mathematically the components of flow into parts each of which propagates along one characteristic alone. To illustrate such a separation of parts it is simplest to use the conservation equations for isentropic one-dimensional gas flow. After combination shows components v, p unseparated. Now by choosing two new functions

$$\frac{\partial v}{\partial t} \pm \frac{1}{\rho c}\frac{\partial p}{\partial t} + \left(\frac{\partial v}{\partial x} \pm \frac{1}{\rho c}\frac{\partial p}{\partial x}\right)(v \pm c) = 0, \tag{32.2}$$

the plus (minus) sign designating positive (negative) direction of propagation. This equation shows all components unseparated. Now by choosing two new functions

$$\mathfrak{R} = v + \int \frac{dp}{\rho c}, \qquad \mathfrak{S} = v - \int \frac{dp}{\rho c} \tag{32.3}$$

and substituting them into this equation, one derives the separated forms

$$\left[\frac{\partial}{\partial t} + (v+c)\,\frac{\partial}{\partial x}\right]\,\mathfrak{R} = 0\,, \qquad \left[\frac{\partial}{\partial t} + (v-c)\,\frac{\partial}{\partial x}\right]\,\mathcal{S} = 0\,. \tag{32.4}$$

This shows that $\mathfrak{R}$ is constant along the characteristic $v + c =$ const. and $\mathcal{S}$ is constant along $(v - c) =$ const. The now separated components $\mathfrak{R}$ and $\mathcal{S}$ are the Riemann invariants of the flow. Similarly the component of flow along the characteristic $v =$ const. is a solution of the adiabatic equation

$$\left(\frac{\partial}{\partial t} + v\,\frac{\partial}{\partial x}\right)s = 0\,, \qquad s = \text{entropy}\,. \tag{32.5}$$

The concept of characteristic curves illustrated above has wide significance in the theory of partial differential equations. Since the conservation laws of nonlinear acoustics form a system of partial differential equations, it is important to have methods of solution. The theory of characteristics provides one method. We discuss this theory briefly in the next section.

33. METHOD OF CHARACTERISTICS

We consider the conservation laws of continuity and momentum and write the two unknown variables (say density and velocity) as u_1, u_2. Then, in general, these laws read [36],

$$a_{11}\,\frac{\partial u_1}{\partial x} + b_{11}\,\frac{\partial u_1}{\partial t} + a_{12}\,\frac{\partial u_2}{\partial x} + b_{12}\,\frac{\partial u_2}{\partial t} = R_1 \tag{33.1}$$

$$a_{21}\,\frac{\partial u_1}{\partial x} + b_{21}\,\frac{\partial u_1}{\partial t} + a_{22}\,\frac{\partial u_2}{\partial x} + b_{22}\,\frac{\partial u_2}{\partial t} = R_2\,. \tag{33.2}$$

This system is quasi-linear, i.e., it is linear in the first derivatives of u_1, u_2 but may be nonlinear in the coefficients a_{ij}, b_{ij}. We now search for new coordinates (α, β) such that in Eq. (33.1) u_1, u_2 are functions of α only, and in Eq. (33.2), u_1, u_2 are functions of β only. The partial differential equations in x, t then become ordinary differential equations in α, β. This search for α, β is equivalent to finding particular curves in the xt plane along which the differentiation is "interior," i.e., along a *line* for a problem in two variables, a *plane* in three variables, etc. To find these curves in the case of two variables one multiplies Eq. (33.1) by an undetermined factor α_1 and Eq. (33.2) by another factor α_2. Then the equations are added. In the resultant single equation, one next groups the factors in the form

$$A\,\frac{\partial u_1}{\partial x} + B\,\frac{\partial u_1}{\partial t} + C\,\frac{\partial u_2}{\partial x} + D\,\frac{\partial u_2}{\partial t} = \alpha_1 R_1 + \alpha_2 R_2\,. \tag{33.3}$$

If $\partial u_1/\partial x$, $\partial u_1/\partial t$ are components of the vector differentiation, and A, B are the components of a vector direction, then the first two terms in Eq. (33.3) constitute a directional derivative of u_1.

$$du_1 = \frac{\partial u_1}{\partial x} dx + \frac{\partial u_1}{\partial t} dt,$$

then $A{:}B = dx{:}dt$. Similarly,

$$\frac{C}{D} = \frac{dx}{dt}.$$

We therefore have two expressions for dx/dt. In order for all directional derivatives to be "interior," i.e., along a line, we require dx/dt $(=\tau)$ to be the same for each. Thus, we require τ such that A, B, C, D satisfy the set

$$\begin{aligned} A - B\tau &= 0 \\ C - D\tau &= 0 \end{aligned} \tag{33.4}$$

or

$$\begin{aligned} (a_{11} - \tau b_{11})\alpha_1 + (a_{21} - \tau b_{21})\alpha_2 &= 0 \\ (a_{12} - \tau b_{12})\alpha_1 + (a_{22} - \tau b_{22})\alpha_2 &= 0. \end{aligned} \tag{33.5}$$

For nontrivial α_1, α_2, the determinant of Eq. (33.5) must vanish. This occurs for two distinct values of τ for each point x, t; i.e.,

$$\begin{aligned} \tau_1 &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \\ \tau_2 &= \frac{-b - \sqrt{b^2 - 4ac}}{2a} \\ b^2 - 4ac &> 0, \end{aligned} \tag{33.6}$$

in which

$$a = b_{11}b_{22} - b_{12}b_{21}$$

$$b = a_{21}b_{12} + b_{21}a_{12} - b_{11}a_{22} - b_{22}a_{11}$$

$$c = a_{11}a_{22} - a_{12}a_{21}.$$

The curves $\tau_1(x, t)$, $\tau_2(x, t)$ constitute two curves C_1, C_2 in the xt plane. With successive values of τ one can form the ratios $(\alpha_2/\alpha_1)_1$ using τ_1, and $(\alpha_2/\alpha_1)_2$ using τ_2. Substituting $(\alpha_2/\alpha_1)_1$ into Eq. (33.5), then $(\alpha_2/\alpha_1)_2$ results in two equations of the form

$$\begin{aligned} E \frac{du_1}{d\alpha} + F \frac{du_2}{d\alpha} &= GR_1 + HR_2 \\ J \frac{du_1}{d\beta} + K \frac{du_2}{d\beta} &= LR_1 + MR_2, \end{aligned} \tag{33.7}$$

with the definitions

$$\frac{d}{d\alpha} = \frac{1}{\sqrt{1 + \tau_1^2}} \left(\tau_1 \frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)$$

$$\frac{d}{d\beta} = \frac{1}{\sqrt{1 + \tau_2^2}} \left(t_2 \frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right). \tag{33.8}$$

Here, $d\alpha$ is an increment along the curve C_1, and the differentiation $d/d\alpha$ is the directional derivative. Similarly, $d\beta$ is an increment along C_2, and $d/d\beta$ is the corresponding directional derivative. The forms of E, F, G, H. J, K, L, M can be found in Ref. 36. Thus, two curves $C_1 : dx/dt = \tau_1$; $C_2 : dx/dt = \tau_2$, along which the differentiation is "interior," have been found. These curves in the xt plane are the characteristic curves of the system. They possess the special property that differentiation of the dependent variables u_1, u_2 along their arcs is ordinary, thus permitting solution by quadrature. Because of this property the characteristic curves are lines along which the higher partial derivatives of u_1, u_2 are indeterminate, given the boundary condition that u_1, u_2 (and/or their first derivatives) are specified on them. Thus boundary value problems in which the boundary conditions coincide with characteristic curves of the parent partial differential system are indeterminate, i.e., the higher order partials cannot be determined uniquely from the data. Such indeterminacy also has a physical meaning for the case when the system describes wave motion. The observer, who ordinarily can observe the physical process of wave motion from a fixed coordinate system, must (under the condition that the boundary conditions coincide with a characteristic) ride with the wave, i.e., he can observe only events that occur relative to his (convected) location in the wave.

Characteristic curves possess a third property whose formulation shows immediate connection to the two others noted above. This is the property that jump discontinuities in the first and higher order derivatives shown in Eq. (33.1) exist only across characteristic curves. In mathematical terms, if $x = x(\lambda)$, $t = t(\lambda)$ defines a characteristic curve C in the xt plane, then the condition on u_1, u_2 that their, say, first derivatives undergo a jump (symbol [ ]) across C is

$$[du_1] = \left[\frac{\partial u_1}{\partial x}\right] \frac{dx}{d\lambda} + \left[\frac{\partial u_1}{\partial t}\right] \frac{dt}{d\lambda} = 0$$

$$[du_2] = \left[\frac{\partial u_2}{\partial x}\right] \frac{dx}{d\lambda} + \left[\frac{\partial u_2}{\partial t}\right] \frac{dt}{d\lambda} = 0. \tag{33.9}$$

Here it is assumed that (a) u_1, u_2 are continuous; (b) all differentiation along C is continuous; and (c) all differentiation elsewhere is discontinuous. From these assumptions, one can modify Eq. (33.1) to read

$$a_{11} \left[\frac{\partial u_1}{\partial x}\right] + b_{11} \left[\frac{\partial u_1}{\partial t}\right] + a_{12} \left[\frac{\partial u_2}{\partial x}\right] + b_{12} \left[\frac{\partial u_2}{\partial t}\right] = 0 \tag{33.10}$$

Continued

$$a_{21}\left[\frac{\partial u_1}{\partial x}\right] + b_{21}\left[\frac{\partial u_1}{\partial t}\right] + a_{22}\left[\frac{\partial u_2}{\partial x}\right] + b_{22}\left[\frac{\partial u_2}{\partial t}\right] = 0 . \tag{33.10}$$

The four equations given in Eqs. (33.9) and (33.10) form a system whose solution is identical with Eq. (33.6) and subsequently Eq. (33.7). Thus discontinuities in first order derivatives of u_1, u_2 occur only across characteristic curves. Similarly, it may be shown that discontinuities in second order derivatives occur only across characteristic curves [36]. Hence waves containing jumps (in first or second order derivatives) propagate along characteristic curves in *space-time.*

When the assumption is made that u_1, u_2 themselves undergo a jump across a line C then it is shown in Ref. 3.6, that line C is a characteristic curve along which the jump wave propagates (in space-time) if the parent system of partial differential equations is linear. If the parent system is nonlinear, then jump waves in u_1, u_2 do not propagate along characteristic curves. Furthermore, in the linear system the propagation velocity of the discontinuity is $\tau = dx/dt$, as found above. In nonlinear systems the propagation velocity U of discontinuities does not define a characteristic curve in the xt plane.

34. WEAK SOLUTIONS

The hydrodynamic equations of fluid motion can be written as a set of conservation laws [37] of general form .

$$\frac{\partial P}{\partial t} + \frac{\partial Q}{\partial x} + R = 0 ; \qquad P, Q, R = (P, Q, R)(x, t, u) . \tag{34.1}$$

Let there be a smooth test function w which is finite inside a region $\mathcal{V}$ of the xt plane and which vanishes on the surface of and outside this region. Multiplying (Eq. 34.1) by w, and integrating over $\mathcal{V}$ leads to

$$\iint_{\mathcal{V}} \left(w \frac{\partial P}{\partial t} + w \frac{\partial Q}{\partial x} + wR\right) dxdt = 0 \tag{34.2}$$

or

$$\iint_{\mathcal{V}} \left(\frac{\partial(wP)}{\partial t} + \frac{\partial(wQ)}{\partial t}\right) dxdt - \iint_{\mathcal{V}} \left(P \frac{\partial w}{\partial t} + Q \frac{\partial w}{\partial x} - Rw\right) dxdt = 0 . \tag{34.3}$$

Since w vanishes on the surface of V, and since V is finite, then (by Gauss's theorem) the first term in Eq. (34.3) vanishes. Thus, Eq. (34.2) implies that

$$\iint_{\mathcal{V}} \left(P \frac{\partial w}{\partial t} + Q \frac{\partial w}{\partial x} - Rw\right) dxdt = 0 . \tag{34.4}$$

If Eq. (34.4) holds for any solution u of Eq. (34.1) then u is designated to be a 'weak solution.' Now let u be discontinuous across a curve $\mathcal{C}$ in the xt plane, and delimit a finite region $\mathcal{O}$ containing a section of $\mathcal{C}$, such that $\mathcal{C}$ divides $\mathcal{O}$ into two parts. By applying Eq. (34.2) in turn to each part and using Eq. (34.4) we see that the jump condition across $\mathcal{C}$ is

$$\int w\{[P]dx - [Q]dt\} = 0 \tag{34.5}$$

(see Ref. 37). Thus the jump conditions are given by

$$[P]U = [Q] \tag{34.6}$$

where

U = velocity of propagation of the jump

$P = P(x, t, u(x, t)$

$Q = Q(x, t, u(x, t))$.

As noted earlier, U is the slope of a characteristic in the xt plane only if the parent system is linear.

35. METHOD OF CHARACTERISTICS FOR THE CASE OF THREE DIMENSIONS

We consider a system of three partial differential equations of first order in three variables. A convenient example of such a system is the propagation of circular cylindrical waves which, though governed by a linear wave equation of second order can be converted by introduction of new variables to a set of three partial differential equations of first order. To apply the method of characteristics [36] we multiply each member of the set respectively by α_1, α_2, α_3 and add them together to form a single equation. By regrouping terms, we can show this single equation to be the sum of three directional derivatives of the new variables in directions of certain space vectors $A1$, $A2$, $A3$ having α_1, α_2 or α_3 as scale factors (lengths). The method of characteristics sets the requirement that the three directions of differentiation be in a common plane, specified by its normal λ. This means that conditions are sought under which all differentiation in a space of three variables is "interior," i.e., restricted to two dimensions. Mathematically, the requirement is

$$\mathbf{A}_1(\alpha_i) \cdot \boldsymbol{\lambda} = 0; \quad \mathbf{A}_2(\alpha_i) \cdot \boldsymbol{\lambda} = 0; \quad \mathbf{A}_3(\alpha_i) \cdot \boldsymbol{\lambda} = 0 \tag{35.1}$$

where

$$\boldsymbol{\lambda} = \lambda_1 \hat{e}_1 + \lambda_2 \hat{e}_2 + \lambda_3 \hat{e}_3.$$

Eq. (35.1) is a set of three homogeneous equations which has a nontrivial solution only when the corresponding determinantal equation (in λ_i) vanishes. The components so found constitute a one-parameter family of planes (characteristic planes) passing through a specific point in the 3-space of the three variables of the parent system. The envelope of all these planes is a double (i.e., itself and its mirror image) cone (characteristic cone). Every generator of this double cone is a characteristic direction along which differentiation of the dependent variable is "interior," that is, in terms of two independent variables (differentiation in a plane) instead of three. When the λ_i are found, one can form with them the ratios α_3/α_1, α_2/α_1 which are then inserted in the single (linear sum) equation described above to yield a compatibility equation, i.e., an equation which restricts solution of the original dependent variable to characteristic planes. Compatibility equations can be converted to finite-difference equations by available techniques. The latter are directly solvable by the use of numerical methods.

The characteristic cone (or envelope of characteristic planes) has the following significance in numerical work associated with, say, the propagation of circularly symmetric cylindrical waves. Let the velocity potential be a convenient field quantity. The single linear wave equation can then be replaced by three partial differential equations in the two partials of the velocity potential with respect to space and the one partial with respect to time. By application of the method of characteristics a compatibility equation is readily derived (i.e., the one-parameter family of planes defined by the unit vector λ is obtained). On the line of intersection of the characteristic planes and the cone (i.e., on the "bicharacteristic" of the cone) the unit vector is a normal vector. Two other vectors, β in the direction of the bicharacteristic and γ in the direction tangent to the base of the cone, are easily established as an orthogonal trio of working variables. The compatibility equation, by derivation restricted to the characteristic cone, is reformulated in terms of this trio. The numerical problem then reduces to this: to find the three partials of the field at the apex of the cone knowing their respective values at the base. Since the height of the apex above the base is made to represent time, the numerical problem is restated to be: find the partials of the field at time $t_0 = \Delta t$ when the partials are known at time $t = 0$, (alternatively at time $t_0 - \Delta t$). To do this requires auxiliary equations (generally continuity equations for the partials) to supplement the compatibility equation. The solution (when found) gives one point in space-time of the parent system. Repeated solutions give all mesh points in a space-time region as a function of data at the boundary of the region. Thus, by numerical calculation, the original partial differential equation (in this case, of the hyperbolic type) is solved point by point at the apices of characteristic cones whose axes represent time and whose base represents space. This procedure (method of characteristics) has been applied to dynamic problems of continua [36] (solids, fluids, etc.) of both linear and nonlinear type and represents a very tractable approach to what in many cases is an exceptionally difficult analytic task. However, since the procedure is numerical, each solution represents only one choice of a set of parameters of the field. The effect of a change in parameters, so readily visible in an analytic approach, must here be obtained only step by step by repeated numerical solution of the entire field.

As noted earlier, in the case of simple waves, the lines of characteristics are straight lines on an xt plot. In Fig. (32.1) characteristic lines are seen issuing from the xt plot of piston motion, each portion of which (motion) represents a disturbance. Thus at $x = -x_1$, say, the disturbance occurs at time $t = t_1$ and travels along its characteristic (in space-time) to reach $x = x_2$ at time t_2, etc. The wave in space actually travels from point to point of the projection of the points t = const. on the space plane. If the piston motion undergoes a discontinuity of velocity, or derivative of velocity, the jump of this discontinuity occurs across a characteristic (by construction). In the case of a two-dimensional wave the space-time characteristic is a surface (in, say, xyt space). The projection of the lines t = const. onto the space plane constitutes a family of lines in the parameter t. These lines represent successive wave fronts propagating normal to themselves with the local speed of sound. The wave fronts in space are also called characteristics.

The concept of boundaries in motion generating simple waves was one of the earliest in the history of nonlinear propagation to be given analytic treatment. We now take up this (historically based) treatment to illustrate the very satisfactory solutions achieved by Earnshaw, Riemann, and later followers to idealized fluids in one-dimensional, unidirectional motion.

36. ANALYSIS OF LOSSLESS SIMPLE WAVES BY WEAK SHOCK THEORY

We begin with Eq. (30.3) and write it in the form

$$c_0^3 \frac{\partial V}{\partial Z} - c_0 V \frac{\partial V}{\partial t'} = \frac{\delta}{2} \frac{\partial^2 V}{\partial t'^2}. \tag{36.1}$$

For the lossless case, $\delta \equiv 0$. Let u be a Riemann invariant. Then $V = \beta u$, and the above equation reduces to

$$c_0^2 \frac{\partial u}{\partial Z} - \beta u \frac{\partial u}{\partial t'} = 0; \qquad \beta = \frac{\gamma + 1}{2} = 1 + \frac{B}{2A}. \tag{36.2}$$

If the boundary condition is

$$\begin{aligned} u(0, t)|_{Z=0} &= g(t) \\ g(t) &= 0 \qquad t < 0, \end{aligned} \tag{36.3}$$

then the solution of the boundary value problem, obtained by neglecting the finite displacement of the boundary, is [18, 22, 34]

$$u(Z, t) = g(\phi), \tag{36.4}$$

in which the new time variable ϕ is discussed below. This solution is valid for low Mach numbers (say, <0.1) and for all points not in the immediate vicinity of the moving boundary. From it we see that if the (moving) boundary condition is a known function of time $g(t)$ the solution at any field point ($Z > 0$) is $g(\phi)$. For this to be true it is required that ϕ have the special functional dependence given by

$$\phi = t' - \frac{Z}{c_0} \left[1 + \frac{\beta}{c_0} g(\phi)\right]^{-1}, \tag{36.5}$$

in which β is the parameter of nonlinearity. (Note that when $\beta = 0$, $\phi = t' = t - Z/c_0$, i.e., ϕ is the retarded time.) For $\beta g/c_0 << 1$ (namely, the product of the nonlinearity parameter and the acoustic Mach number is small),

$$t' = \phi - \frac{\beta g(\phi) Z}{c_0^2}. \tag{36.6}$$

Let $\bar{t}$ be the instant of shock formation as measured by the arrival of the initial zero crossing at the distance $\bar{Z}$ of shock formation, and let $\underline{v}$ be the speed of shock-wave propagation. Then the time of arrivals at point Z beyond $\bar{Z}$ is given by

$$t = \bar{t} + \int_{\bar{Z}}^{Z} \frac{1}{v(\lambda)}\, d\lambda, \qquad \lambda = \text{distance}. \tag{36.7}$$

To first order,

$$v = c_0 + \frac{1}{2}\beta(u_a + u_b) \tag{36.8}$$

where u_a, u_b are the particle velocities just ahead (a) and just behind (b) the shock. Thus,

$$t' = \bar{t}' + \int_{\bar{Z}}^{Z} \frac{d\lambda}{c_0\left[1 + \frac{\beta}{2}(u_a + u_b)\right]} \tag{36.9}$$

or

$$\frac{dt'}{dz} \approx -\frac{\beta}{2}\frac{1}{c_0^2}(u_a + u_b).$$

Eliminating ϕ between

$$u = g(\phi); \qquad t' \doteq \phi - \frac{\beta Z g(\phi)}{c_0^2}$$

yields the general formula

$$t' = g^{-1}(u) - \frac{\beta Z u}{c_0^2}. \tag{36.10}$$

Let $t_s'(Z)$ be the time of arrival of the shock at Z. Then just ahead of the shock,

$$t_s' \doteq g^{-1}(u_a) - \frac{\beta Z u_a}{c_0^2}, \tag{36.11}$$

and just behind the shock,

$$t_s' \doteq g^{-1}(u_b) - \frac{\beta Z u_b}{c_0^2}. \tag{36.12}$$

Equations (36.9), (36.11) and (36.12) can be solved simultaneously to find u_a, u_b, and t_s'. When these are known one can use

$$\phi = t_s' + \frac{\beta Z}{c_0^2} g(\phi) \tag{36.13}$$

to find ϕ, and hence to find $u = g(\phi)$. The pressure p is given by

$$p = p_0 \pm \rho_0 c_0 u . \tag{36.15}$$

In this weak shock theory the shock discontinuity is a mathematical line. This line is assumed present at all distances. At great distances, however, it is known that dissipation washes out the sharp discontinuities. Hence, weak shock theory does not apply at great distances. A theory of shock propagation in the presence of dissipation based on Burgers' equation allows for a shock formation which is not limited to short distances.

Weak shock theory can be used to predict the shape and amplitude of discontinuous waves as a function of time and distance. An important application is the N-wave whose functional form is given by

$$\begin{aligned} u(0,t) &= \frac{-u_0 t}{T_0} \qquad -T_0 < t < T_0 \\ &= 0 \qquad\qquad\quad t > |T_0| . \end{aligned} \tag{36.16}$$

Accordingly,

$$\begin{aligned} u(Z,t) &= g(\phi) \\ g(\phi) &= \frac{-u_0 \phi(Z,t)}{T_0} \end{aligned} \tag{36.17}$$

in which ϕ is given by

$$\phi = t - \frac{Z}{c_0\left[1 + \dfrac{\beta}{c_0} g(\phi)\right]} \tag{36.18}$$

or

$$\phi(Z,t) = \frac{t'}{1 + bZ}, \qquad b = \frac{\beta u_0}{c_0^2 T_0} . \tag{36.19}$$

Thus,

$$u(Z,t) = \frac{-u_0 t'}{T_0(1 + bZ)}, \qquad -T < t' < T . \tag{36.20}$$

Note that whereas $u(0,t)$ is defined in the range $-T_0$, $+\,T_0$, $u(Z,t)$ is defined in the range $-T$, $+T$. It is therefore required to find a relation between T and T_0. This is done by applying Eq. (36.9) to the head shock in which

$$\frac{dt_s'}{dZ} = -\frac{1}{2}\frac{\beta}{c_0^2}u_b, \tag{36.21}$$

$$= -\frac{1}{2}\frac{\beta}{c_0^2}\left[\frac{-u_0 t_s'}{T_0(1+bZ)}\right] \quad \text{at} \quad t_s' = T.$$

It is seen that by integration,

$$-t_s' = T = C(1+bZ)^{1/2}. \tag{36.22}$$

When $Z = 0$, the shock begins at time $t_s' = -T_0$. Thus $C = T_0$, and the desired relation between T, T_0 becomes

$$-t_s' = T = T_0(1+bZ)^{1/2}.$$

The N-wave is therefore predicted to spread out along the Z-axis with distance. The wave amplitude at time $t_s' = -T_0(1+bZ)^{1/2}$ is

$$u_b = \frac{u_0}{(1+bZ)^{1/2}}.$$

When $bZ >> 1$, $u_b \sim Z^{-1/2}$.

A second application is a sawtooth wave. At the jump discontinuity the negative wave amplitude equals the positive wave amplitude in magnitude and is opposite in sign. Hence

$$v = c_0 \pm \frac{1}{2}(u_a + u_0) = c_0, \tag{36.23}$$

which means that all shock waves travel at sonic speed. Applying weak shock theory again, one has

$$u(0,t) = U_0\left(1 - \frac{t}{T_0}\right); \qquad g(\phi) = U_0\left(1 - \frac{\phi}{T_0}\right)$$

$$t' = \phi\left(1 + \frac{\beta Z U_0}{c_0^2 T_0}\right) - \frac{\beta Z U_0}{c_0^2} \tag{36.24}$$

$$\frac{\phi}{T_0} = \frac{\frac{t'}{T_0} + bZ}{1 + bZ}; \qquad b = \frac{\beta U_0}{c_0^2 T_0}$$

$$u(Z,t') = U_0\left(\frac{1 - \frac{t'}{T_0}}{1 + bZ}\right).$$

Since

$$\frac{dt_s'}{dZ} = 0, \qquad t_s' = \text{const.} = 0 \ \ (\text{say}),$$

one finds

$$u(Z, t_s') = \frac{U_0}{1 + bZ}. \tag{36.25}$$

Now

$$bZ = \frac{\beta SkZ}{\omega T_0} = \frac{\beta SkZ}{\pi}, \qquad \omega T_0 = \pi; \qquad k = \frac{\omega}{C}.$$

Thus,

$$u(Z, t_s') = \frac{\pi U_0}{\pi + \beta SkZ}. \tag{36.26}$$

From this the decay of the sawtooth wave is seen to be much more rapid than that of the N-wave.

The propagation of the isolated sine-wave cycle may also be treated by weak shock theory. The results agree with the more accurate theory that stems from the use of Burgers' equation.

37. DISTANCE TO FORMATION OF A SHOCK

Let c be the speed of sound in a fluid and T the instantaneous temperature. Then [13]

$$c_0 \propto \sqrt{T_0}, \qquad c \propto \sqrt{T}, \qquad \theta = \Delta T \propto \delta T$$

$$\delta c \propto \frac{1}{2}\frac{\delta T}{\sqrt{T}} = \frac{\theta}{2\sqrt{T}} \times \frac{\sqrt{T_0}}{\sqrt{T_0}} \propto \frac{c_0 \theta}{2T}. \tag{37.1}$$

Thus we take the total velocity at any instant to be

$$c = c_0 + \frac{c_0 \theta}{2T_0}. \tag{37.2}$$

For small-amplitude plane waves the particle velocity u can be written

$$\frac{u}{c_0} = \frac{\theta}{T_0(\gamma - 1)}, \qquad \gamma = \frac{C_p}{C_v}. \tag{37.3}$$

Since $c_T = c + u$,

$$c_T = c_0 + \frac{(\gamma - 1)u}{2} + u = c_0 + \frac{(\gamma + 1)u}{2}. \tag{37.4}$$

Now for a time increment δt,

$$u\delta t = t\delta u$$

$$\delta x' = \delta x + \frac{(\gamma + 1)\delta ut}{2}, \tag{37.5}$$

where

$$\delta x' = c_T \delta t,$$

$$\delta x = c_0 \delta t.$$

A state of shock is reached at time t_S when $\delta x' = 0$; i.e.,

$$t_S = \frac{-\delta x}{\frac{(\gamma + 1)\delta u}{2}} = \frac{-2}{(\gamma + 1) \left.\frac{du}{dx}\right|_{t=0}}. \tag{37.6}$$

The distance L_S that the wave travels from $t_0 = 0$ to $t = t_S$ is $c_0 t_S$. For a plane harmonic wave, $u = u_0 \exp(ikx - i\omega t)$, and

$$\left|\frac{du}{dx}\right|_{t=0} = ku_0 . \tag{37.7}$$

We assume initially that this is the largest *negative* gradient in the wave. Then,

$$L_S = c_0 t_S = \frac{2c_0}{(\gamma + 1)ku_0} = \frac{2\gamma}{(\gamma + 1)k} \frac{P_0}{p_0} = \frac{2\rho c^2}{(\gamma + 1)kp_0}, \tag{37.8}$$

in which P_0 is the static pressure ($= \rho c^2/\gamma$), p_0 is the sound pressure ($= \rho c u_0$), and infinitesimal theory is used. Note that for gases $\gamma = C_p/C_v$, and that the local speed of sound is

$$c_0 + \beta u \tag{37.9}$$

where

$$\beta = \frac{\gamma + 1}{2}. \tag{37.10}$$

For fluids,

$$\beta = 1 + \frac{B}{2A}. \tag{37.11}$$

Hence for nondispersive fluids the distance to shock formation in a plane wave is

$$L_S = \frac{c_0^2}{\left(1 + \frac{B}{2A}\right) \omega u_0} . \tag{37.12}$$

38. LIMITATIONS OF WEAK SHOCK THEORY

In mathematical terms a shock wave is a discontinuous function of coordinates. A theory of shock waves must define the discontinuity and, in particular, its thickness. Weak shock theory does not address itself to this problem. Also, in physical terms a discontinuity in a wave is a reflector of waves. The theory of simple (i.e., undirectional) nonlinear waves does not account for the reflection (therefore backward travel) of individual wavelets in a nonlinear wave train. Thus weak shock theory is valid only for waves of modest amplitude at "short" distances.

CONCLUSION OF REVIEW OF BASIC EQUATIONS AND SOLUTIONS

The many forms of the basic equations currently in use that were discussed in the above sections underscore the very serious difficulty in formulating physical processes in terms of nonlinear partial differential equations which turn out to be nearly intractable when *analytic* methods of solution are tried. Many of the formulas derived are very special solutions. None of them is general enough to include all relevant features of the physical process they are intended to describe. This state of knowledge of nonlinear processes touches upon every degree of satisfaction, from the very satisfying results in the case of simple waves to the completely unsatisfactory results in the case of three-dimensional wave trains (regular and statistical) in nonhomogeneous media. The limited capability of purely analytic methods has lead to new approaches in graphical and numerical procedures and in their combination with analytical methods. Just as in the theory of nonlinear ordinary differential equations which has used graphical methods over the course of a century, and thereby achieved very useful new results, the theory of nonlinear partial differential equations in the last decade has entered a new stage by combining analytical, graphical, and numerical techniques in a new effort to overcome the very considerable problems presented by apparently simple-looking basic equations. We present in the following sections a new approach by Prof. G. B. Whitham, designed to overcome past difficulties in the nonlinear mechanics of continua. Its key feature is use of an averaging process to convert a nonlinear partial differential equation into a set of nonlinear ordinary differential equations, which in turn opens all solution techniques based on graphical and numerical procedures.

39. WHITHAM'S TECHNIQUE

Whitham, in a series of articles [11, 38], has developed a technique for handling nonlinear wave trains of very general description. Following the Krylov-Bogoliubov averaging procedure for solving ordinary differential equations found in vibration theory, Whitham applies averaging to the partial differential equations of propagation of waves in fluid continua. This application enables him to solve otherwise difficult nonlinear partial differential equations in an advantageous manner. While Whitham's principal examples are in gravity waves, his method itself is general enough to be of great potential use in nonlinear acoustics. We present in the following sections a partial account of his procedures, emphasizing the aspects of (a) uniform wave trains of large amplitude and the associated dispersion of phase velocities among the component wavelets, (b) conservation equations, (c) characteristic (group) velocities for nonlinear wave trains, and (d) shock wave theory.

Since an elucidation of Whitham's theory is quite involved, it will be convenient first to consider a brief critique of the method by Lighthill, then to summarize the essential features of the theory, and finally to present some detail on applications.

40. LIGHTHILL'S CRITIQUE OF WHITHAM'S THEORY

Lighthill [11] has summarized Whitham's theory as follows: If the parameters that characterize a wave train vary gradually on a scale of wavelengths, then locally the wavelets of the train must closely approximate to plane periodic waves (i.e., their determination should involve the solution only of ordinary differential equations). In particular when the waves are *nonlinear*, exhibiting gradual variation over a scale of wavelengths, plane periodic solutions still exist, although nonsinusoidal in form. However a basic difficulty arises in applications (both linear and nonlinear), namely, solutions that are one-valued in certain regions of space-time become many-valued in others.

To explore Whitham's theory Lighthill takes the example of gravity waves over deep water. Since the height of the wave above mean is a Lagrangian variable, one can formulate the Lagrangian energy function $\mathcal{L}$. The independent variables are taken as ω and k (frequency and wave number). According to the method of plane periodic waves, ω and k are related to a phase function $\theta(x, t)$, viz,

$$\theta(x, t) = kx - \omega t; \qquad \omega = -\frac{\partial\theta}{\partial t}; \qquad k = \frac{\partial\theta}{\partial x}. \tag{40.1}$$

Using simple (Hamilton) variational principle and Euler's solutions, one quickly obtains the statement that $\delta \iint \mathcal{L}(\omega, k)dtdx = 0$ which is solved by the Euler equations.

$$\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\omega}\right) = \frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial k}. \tag{40.2}$$

This is a nonlinear equation in θ. By introduction of new variables ϕ, t, x defined by

$$t = -\frac{\partial\phi}{\partial\omega}; \qquad x = \frac{\partial\phi}{\partial k}; \qquad \phi(k, \omega) = kx - \omega t - \theta(x, t), \tag{40.3}$$

one can linearize the Euler equation so that it reads

$$\mathcal{L}_{\omega\omega}\phi_{kk} - 2\mathcal{L}_{\omega k}\phi_{\omega k} + \mathcal{L}_{kk}\phi_{\omega\omega} = 0. \tag{40.4}$$

The *characteristic condition* which makes this equation hyperbolic (i.e., which describes wave trains) is [37]

$$\mathcal{L}_{\omega\omega}d\omega^2 + 2\mathcal{L}_{\omega k}d\omega dk + \mathcal{L}_{kk}dk^2 = 0. \tag{40.5}$$

If the relation between ω and k is $\omega = f(k)$, then the above equation *is* hyperbolic if [11]

$$[\omega - f(k)] f''(k) > 0. \tag{40.6}$$

To apply Whitham's theory we consider Eq. (40.1) and visualize a wave train modulated in (a) frequency or (b) amplitude. On the basic assumption that *the modulation is slow* the solution in plane periodic waves begins with a statement that

$$\begin{aligned} &\text{Case 1:} \quad -\frac{\partial \theta}{\partial t} = \omega + \text{frequency modulation} \\ &\text{Case 2:} \quad \frac{\partial \theta}{\partial x} = k + \text{wave number modulation.} \end{aligned} \tag{40.7}$$

To illustrate (and check) Whitham's theory Lighthill chooses case 2 and applies it to gravity waves in which the modulation is introduced at the gravity wave maker (i.e., at the origin of coordinates). Thus the phase function $\theta(x, t)$ has the boundary conditions

$$\theta = -\omega_0 t, \qquad \frac{\partial \theta}{\partial x} = k_0 + \epsilon\, e^{i\alpha t}, \qquad x = 0. \tag{40.8}$$

Thus a case is chosen in which the frequency is fixed at ω_0 for all t, but the amplitude (i.e., wave number) is modulated. Solution of Eq. (40.4), subject to Eq. (40.8), gives

$$\theta = -\omega_0 t + k_0 x + \epsilon\, e^{i\alpha\left(t - \frac{x}{c_0}\right)} \frac{\sinh \beta x}{\beta}, \tag{40.9}$$

in which

$$c_0 = -\frac{\mathcal{L}_{kk_0}}{\mathcal{L}_{\omega k_0}} = \text{effective group velocity} \tag{40.10}$$

$$\beta = \alpha(\mathcal{L}_{\omega\omega_0}\mathcal{L}_{kk_0} - \mathcal{L}^2_{\omega k_0})^{1/2}/\mathcal{L}_{kk_0} = \text{rate of exponential increase of modulation}, \tag{40.11}$$

and subscript zero in $\mathcal{L}_{\omega\omega_0}$, etc., means "value of $\mathcal{L}_{\omega\omega}$, etc., at $x = 0$." Here, β is real. If imaginary, β means "the splitting of group velocity," so that the modulation would proceed at two speeds, viz,

$$\frac{\alpha}{c_1} = \frac{\alpha}{c_0} - |\beta|\,; \qquad \frac{\alpha}{c_2} = \frac{\alpha}{c_0} + |\beta|. \tag{40.12}$$

For the case of gravity waves, explicit values of $\mathcal{L}$ are available, namely,

$$\mathcal{L} = \left(\frac{\rho g}{k^2}\right) \frac{1}{8} \left[(z-1)^2 - (z-1)^3 - (z-1)^4\right] \qquad (\text{N} - \text{m/m}^2) \tag{40.13}$$

$$z \approx 1 + \frac{\pi H^2}{\lambda},$$

where

H = depth of wave from peak to trough

λ = wavelength.

Upon calculating c_0, β from Eq. (40.13) it is found that they vary strongly with powers of boundary phase velocity, i.e., with powers of ω_0/k_0 at $x = 0$ (or alternatively, they vary with ratio H/λ). A plot of $c_0/(\omega_0/k_0)$ and $\beta/(\alpha/2\pi c_0)$ vs H/λ (or $\omega_0^2/k_0^2 g$) shows that β is zero at $H/\lambda = 0$ and $H/\lambda > 0.108$, and peaks in between (i.e., amplitude modulation is vigorous in this region). Thus between these limits β is real, so that there is one propagation speed α/c_0. For $H/\lambda > 0.108$, β is imaginary and the curve $c_0/(\omega_0/k_0)$ splits into two branches $c_1/(\omega_0/k_0)$, $c_2/(\omega_0/k_0)$. Amplitude modulation no longer occurs. A comparison of these results using Whitham theory with a more accurate analysis of Benjamin and Feir [11] who do not assume the modulation to be slow, shows that Whitham's theory is inaccurate at low values of H/λ but is reasonably good at high values.

From the nature of Lighthill's example it is evident that Whitham's theory is very involved. More elucidation will be provided by later examples. However, it will serve a useful purpose first to summarize Whitham's technique in the form of a list of what is to be solved and how solutions are formed:

1. A wave train which is frequency or amplitude modulated is to be analyzed.

2. The appropriate field equations are nonlinear partial differential equations.

3. The modulation over distances of several wavelengths is slow.

4. A solution is posed in the form of plane periodic waves (nonsinusoidal) by first integrating all variations of field with distance, the integration domain being a few wavelengths. This integration leads to a set of ordinary differential equations with amplitude and frequency as variables, whose solution in plane periodic waves is then performed by standard methods.

5. Dispersion formulas (i.e., $\omega = f(k)$) are directly obtainable if there is (a) a Langrangian variable available so that (b) a Lagrangian energy density can be formulated, which can (c) be applied in a variational principle to find the corresponding Euler formulas (conservation laws), for which (d) one can assume plane periodic waves, subject to boundary conditions, resulting in (e) dispersion formulas.

6. Whitham's theory provides approximate solutions in cases where the modulation is not slow.

The above summary, while not complete, will serve as an introduction to the complexities of Whitham's theory. Before we enter on specific applications, it will be useful to state some analytical tools for handling wave crests, since these figure prominently in theoretical solutions based on plane periodic waves.

41. WAVE CRESTS

A small-amplitude wave train $f(\mathbf{x}, t)$ finite in x and t can be described as $F(\mathbf{k}, \omega)$ by Fourier transformation [39]. At fixed t the train is composed of a range of wave numbers $\Delta\mathbf{k}$, and at fixed $\mathbf{x}$ it is correspondingly composed of a range of frequencies $\Delta\omega$. For a homogeneous and conservative medium, periodicity in space (describable by $\mathbf{k}$) is related to periodicity in time (namely ω), by a constant phase velocity c; that is,

$$\omega = |\mathbf{k}|c\,; \qquad |\mathbf{k}| = \left(k_x^2 + k_y^2 + k_z^2\right)^{1/2}. \tag{41.1}$$

A more general type of medium consists of individual patches $\Delta\mathbf{x}$ in each of which k is constant in direction and magnitude but differs from patch to patch. Then, $k = k(\Delta x, \Delta t)$ and $\omega = \omega(\Delta\mathbf{x}, \Delta t)$. The most general medium is a continuum of $\mathbf{k} = \mathbf{k}(x, t)$ and $\omega = \omega(\mathbf{x}, t)$. Now at fixed time we consider a mixed description,

$$\omega = \overline{W}(\mathbf{k}, \mathbf{x}). \tag{41.2}$$

This is the dispersion relation, namely, the variation of periodicity in time as a function of wave number and spatial location. The group velocity vector $\mathbf{c}(\mathbf{k}, x)$ is given by

$$\mathbf{c} = \frac{\partial\overline{W}}{\partial k_x}\,\hat{\imath} + \frac{\partial\overline{W}}{\partial k_y}\,\hat{\jmath} + \frac{\partial\overline{W}}{\partial k_z}\,\hat{k}. \tag{41.3}$$

This gives the change in time periodicity due to a change in spatial periodicity at fixed $\mathbf{x}$. Now at any point in a regular wave train (i.e., one in which adjoining wave numbers differ little from each other, thus excluding jumps and discontinuities), the wave crest is conserved, by which is meant that a time change of the spatial periodicity is related to a spatial change in time periodicity by the equation

$$\frac{\partial\mathbf{k}}{\partial t} + \nabla\omega = 0.$$

Since

$$\nabla\overline{W} = \left(\frac{\partial\overline{W}}{\partial x} + \frac{\partial\overline{W}}{\partial k}\frac{\partial k}{\partial x}\right)\hat{\imath} + \ldots,$$

we see that

$$\nabla\omega = \nabla\overline{W} + (\mathbf{c}\cdot\nabla)\mathbf{k}.$$

The conservation of wave crests thus requires that

$$\frac{\partial k}{\partial t} + (\mathbf{c}\cdot\nabla)\mathbf{k} = -\nabla\overline{W}. \tag{41.4}$$

This equation states that $\mathbf{k}$ at point $\mathbf{x}$ in a medium moving with the group velocity $\mathbf{c}$ of the wave train is changing in time in the manner described by this formula. Each constituent wave travels in a curved path. If the medium is homogeneous (i.e., if $\nabla\overline{W} = 0$), then $\omega = \overline{W}(\mathbf{k})$, $\mathbf{c} = \mathbf{c}(\mathbf{k})$, and each constituent wave travels in a straight line. Lines of propagation for several k's are (in general) not parallel.

The kinematics of wave crests, described in the preceding paragraphs, forms a basic element in Whitham's procedure for solving nonlinear wave problems. This will follow.

42. WHITHAM'S EXPOSITION OF HIS METHOD WITH EXAMPLES

Whitham [40] develops a general theory for studying changes in a propagating wave train governed by nonlinear partial differential equations. His method is best introduced by an example. Let there be a partial differential equation of the type

$$\phi_{tt} - \phi_{xx} + V'(\phi) = 0 \tag{42.1}$$

in which $V'(\phi)(= \partial V/\partial\phi)$ is a nonlinear term. In the physical embodiment of this equation there is always a characteristic length (say, λ_0) and time (say, τ_0). These have been eliminated in Eq. (42.1) by changing x to x/λ_0, t to t/τ_0. We first try solutions in the form of uniform wave trains; i.e., we assume the dependent variables are functions of $X = x - Ut$, where U is the velocity of translation of phase. Thus the elementary steady profile is obtained by taking $\phi = \Phi(X)$. Then Eq. (42.1) becomes

$$(U^2 - 1)\Phi_{XX} + V'(\Phi) = 0. \tag{42.2}$$

This can be integrated to yield

$$\frac{1}{2}(U^2 - 1)\Phi_X^2 + V(\Phi) = A, \tag{42.3}$$

in which A is a constant of integration. Now if $\kappa = (U^2 - 1)^{-1}$ and $\alpha = 2A$, then

$$\Phi = \alpha \cos \kappa X$$

is a solution of Eq. (42.3) in the linear case $V(\Phi) = \Phi^2/2$. The frequency $\omega = U\kappa = \sqrt{\kappa^2 + 1}$. In the nonlinear case one can solve Eq. 42.3 for X, i.e.,

$$X = \sqrt{\frac{1}{2}(U^2 - 1)} \oint \frac{d\Phi}{\sqrt{A - V(\Phi)}}. \tag{42.4}$$

One can say that in the nonlinear case,

$$\begin{aligned} \phi &= \Phi(X; U, A_i) \\ \Phi_X^2 &= F(\Phi; U, A_i); \end{aligned} \tag{42.5}$$

that is, ϕ is a function of X, and implicitly (through X) is a function of U and A_i, where A_i are constants of integration. Now $U(x, t)$ and $A_i(x, t)$ are assumed to be slowly varying functions of space and time. A significant change (in space) of both occurs in no shorter distance than, say, L, which compares with a wavelength λ_0 of a periodic solution according to the formula, $L >> \lambda_0$. This means that many oscillations of the wave train occur in the distance L. This assumption leads to the key approach of Whitham, namely to average the wave train quantities over several oscillations in a space of distance ξ in order to obtain differential equations for the variation of amplitude, frequency, etc. (i.e., of U, A_i) over the much longer distance L. One chooses ξ such that

$$L >> \xi >> \lambda_0 . \tag{42.6}$$

Hence to (space) average any function $F(x, t)$ one writes

$$\widetilde{F}(x, t) = \frac{1}{2\xi} \int_{x-\xi}^{x+\xi} F(x', t)dx' \tag{42.7}$$

and chooses ξ according to Eq. (42.6). If this integration includes a number of oscillations, then we can say that U, A_i in the solutions V are constant over ξ, and so we can average over X by integration, leaving U, A_i constant. Thus the procedure of averaging yields a differential equation in U, A_i by eliminating short-range variations over X. For example, let us assume that $\mathcal{P}(x, t)$ is some function of Φ, where Φ satisfies V. Then the average

$$\bar{P}(U, A_i) = \frac{1}{2\xi} \int_{x-\xi}^{x+\xi} \mathcal{P}[\Phi(X; U, A_i)]dX \tag{42.8}$$

yields a function of U, A_i as required, provided Eq. (42.6) is satisfied. Averages can be made over Φ, as well as over X. For example, one can average in Eq. (42.5) by writing

$$\begin{aligned} W(U, A_i) &= \int \Phi_X d\Phi \\ &= \int \sqrt{F(\Phi; U, A_i}\, d\Phi . \end{aligned} \tag{42.9}$$

This average is very important in Whitham's method.

Having averaged functions of ϕ over X (or Φ), one can proceed to the next step in Whitham's approach, which is to insert the averaged quantities into the generic equations of conservation. Let P and Q be two functions of ϕ and its derivatives. Then the conservation equations can be written in two ways,

$$\frac{\partial \widetilde{P}}{\partial t} + \frac{\partial \widetilde{Q}}{\partial x} = 0 \tag{42.10}$$

and

$$\frac{\partial}{\partial t} \bar{P}(U, A_i) + \frac{\partial}{\partial x} \bar{Q}(U, A_i) = 0 . \tag{42.11}$$

Equation (42.10) is exact since nothing is said of U, A_i. Equation (42.11) is approximate since U, A_i are held constant over the range of integration. The functions P, Q which are found in the conservation equations may be difficult to deduce in general. A good attack is to use the invariance properties of the Lagrangian density with respect to translations of x and t. Since U, A_i constitute $i + 1$ unknowns, the number of conservation equations must equal the number of variables U, A_i in the wave train solutions of Eq. (42.5). If this requirement is satisfied, the resultant set of equations in U, A_i can then be solved to obtain amplitude, frequency, etc., variations in the wave train as it moves through space.

As a typical example we consider Eq. (42.1) again. Here the number of variable U, A_i is 2. Thus, there are two conservation equations. They are

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\,\phi_t^2 + \frac{1}{2}\,\phi_x^2 + V(\phi)\right) + \frac{\partial}{\partial x}\,(-\phi_t\phi_x) = 0 \tag{42.12}$$

and

$$\frac{\partial}{\partial t}\,(-\phi_t\phi_x) + \frac{\partial}{\partial x}\left(\frac{1}{2}\,\phi_t^2 + \frac{\phi_2^2}{2} - V(\phi)\right) = 0\,. \tag{42.13}$$

To calculate mean values (as called for by the method) Whitham uses the auxiliary function W, where

$$\begin{aligned} W &= (U^2 - 1) \oint \Phi_X\,dX \\ &= \sqrt{\{2(U^2 - 1)\}} \oint \sqrt{A - V(\Phi)^2}\; d\Phi\,. \end{aligned} \tag{42.14}$$

As a good short-range distance over which averaging is to occur, he chooses the wavelength λ, defined by

$$\begin{aligned} \lambda &= \int_0^{\lambda} dX = \sqrt{\left\{\frac{1}{2}\,(U^2 - 1)\right\}} \int \frac{d\Phi}{\sqrt{A - V(\Phi)}} \\ &= \frac{\partial W}{\partial A}\,. \end{aligned} \tag{42.15}$$

Since $dX = d\Phi/\Phi_X$, the average over X of any quantity $\mathcal{P}\,(\phi)$ is

$$\overline{\mathcal{P}\,(\phi)} = \frac{1}{\lambda}\int_0^{\lambda} \mathcal{P}\{\Phi\}dX = \kappa \oint \frac{\mathcal{P}\{\Phi\}d\Phi}{\Phi_X}\,, \qquad \kappa = \frac{1}{\lambda}. \tag{42.16}$$

Thus

$$\overline{\frac{1}{2}\,\phi_t^2 + \frac{1}{2}\,\phi_x^2} = \frac{\kappa}{2}\,\frac{U^2+1}{U^2-1}\,W$$

$$\overline{V(\phi)} = A - \frac{\kappa}{2}\,W$$

$$-\overline{\phi_t\phi_x} = \frac{\kappa U}{U^2-1}\,W. \qquad (42.17)$$

As a result of averaging, Whitham shows that the conservation Eqs. (42.12), (42.13) reduce to the set

$$\frac{D}{Dt}\,W_A - W_A\,\frac{\partial U}{\partial X} = 0$$

$$\frac{D}{Dt}\,W_U - W_A\,\frac{\partial A}{\partial x} = 0, \qquad (42.18)$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + U\,\frac{\partial}{\partial x}.$$

Defining $G(A)$ as

$$G(A) \equiv \sqrt{2}\oint \sqrt{\{A - V(\Phi)\}}\; d\Phi \qquad (42.19)$$

and using it in Eq. (42.18), Whitham shows that the characteristic form of the hyperbolic set (Eq. (42.18)) is

$$\frac{dU}{U^2-1} \mp \sqrt{\frac{-G''}{G}}\; dA = 0 \qquad (42.20)$$

on the characteristic curves $C\pm$,

$$C + \frac{dx}{dt} = \frac{1 \pm \sqrt{-GG''/G'^2}}{U \pm \sqrt{-GG''/G'^2}}. \qquad (42.21)$$

(It is to be noted again that all variables in these equations are nondimensional.) Equations (42.21) are the characteristic velocities of the system. They are the nonlinear generalization of the group velocities which appear in linear theory. In nonlinear theory the characteristic velocities of Eq. (42.18) are not equal. A physical consequence is this. Let there be a wave train that is initially uniform outside some finite region. At the boundaries of the region, we have $A = A_0$, U_0. After it propagates into the region and interaction among its components occur, the wave train will separate into two simple (i.e., unidirectional) waves traveling at different speeds. One simple wave travels on the $C+$

characteristic of Eq. (42.21). The relation between U and A on the curve $C+$ is given by the Riemann invariant (from Eq. (92.20)),

$$\int_{U_0}^{U} \frac{dU}{U^2 - 1} + \int_{A_0}^{A} \sqrt{\frac{-G''}{G}}\, dA = 0. \tag{42.22}$$

The other simple wave travels on curve $C-$, and U and A are related on it by an equation like Eq. (42.22) but with opposite sign.

Whitham's procedures also include a treatment of shock waves. Using standard theory one begins by noting that for each conservation equation of the form

$$\frac{\partial f}{\partial t} + \frac{\partial g}{\partial x} = 0$$

there is a corresponding shock condition

$$[g] = V[f] \tag{42.23}$$

in which $[f]$, $[g]$ are discontinuities, and V is the velocity of the shock. A physical system in n unknowns has at least $(n + 1)$ conservation equations. However, since only n shock conditions must be selected, the correct choice is the unique one which corresponds to the real physical situation. These rules are applied by Whitham to Eqs. (42.12) and (42.13), and he then arrives at the jump equations,

$$\begin{aligned}
[\kappa U(UW_U + AW_A - W) - UA] &= V[\kappa(UW_U + AW_A - W) - W] \\
[\kappa UW_U - A] &= V[\kappa W_U] \\
[kU] &\neq V[k]\,.
\end{aligned} \tag{42.24}$$

An important conclusion is this. The frequencies of wave components of the wave train always increase as the waves cross the shock. This frequency jump is irreversible, even though the original equations for ϕ are conservative and reversible. Thus Whitham proves theoretically that discontinuities (analogous to shock waves) can occur in reversible systems which have no dissipation. In such jump-type situations frequency increase across the wave is analogous to entropy increase across shock waves where dissipation is present.

The example of nonlinear gravity waves treated above by Whitham's theory raises new problems in interpretation of results. An additional example, this time the Korteweg-deVries equation, will serve to elucidate the method even further.

43. APPLICATION OF WHITHAM'S APPROACH TO THE KORTEWEG—deVRIES EQUATION FOR WATER WAVES

In a layer of water, at undisturbed depth h_0, let there be generated a train of water waves of typical amplitude a and displacement η about h_0. Then, for relatively long waves Korteweg and deVries [41] derived the equation

$$\eta_t + \sqrt{gh_0}\left[1 + \frac{3}{2}\left(\frac{\eta}{h_0}\right)\right]\eta_x + \frac{1}{6}\sqrt{gh_0}\, h_0^2\, \eta_{xxx} = 0. \tag{43.1}$$

If λ is a typical wavelength, then it is assumed in this derivation that

$$\frac{a}{h_0}, \left(\frac{h_0}{\lambda}\right)^2$$

are comparable small quantities. By adjustment of constants Eq. (43.1) is transformed to

$$\eta_t + 6\,\eta\eta_x + \eta_{xxx} = 0. \tag{43.2}$$

A solution in the form of a uniform wave train is obtained by setting $\eta = \eta\ X$ and $X = x - Ut$. Then

$$\begin{aligned} &1. \quad \eta_{XXX} = U\eta_X - 6\eta\eta_X \\ &2. \quad \eta_{XX} = B + U\eta - 3\eta^2 \\ &3. \quad \frac{1}{2}\eta_X^2 = -A + B\eta + \frac{1}{2}U\eta^2 - \eta^3 \end{aligned} \tag{43.3}$$

Here, 2 and 3 have been obtained by successive integration and A, B are integration constants. Since there are three parameters (U, A, B), three conservation equations are needed. To obtain these one returns to Eq. (43.2) and manipulates them to find

$$\frac{\partial}{\partial t}P_i + \frac{\partial}{\partial x}Q_i = 0 \qquad i = 1, 2, 3, \tag{43.4}$$

in which

$$P_1 = \eta \qquad\qquad Q_1 = 3\eta^2 + \eta_{xx}$$

$$P_2 = \frac{\eta^2}{2} \qquad\qquad Q_2 = 2\eta^3 + \eta\eta_{xx} - \frac{1}{2}\eta_x^2$$

$$P_3 \equiv \eta^3 - \frac{1}{2}\eta_x^2 \qquad\qquad Q_3 = \frac{9}{2}\eta^4 + 3\eta^2\,\eta_{xx} + \frac{1}{2}\eta_{xx}^2 + \eta_x\eta_t.$$

As before, Whitham intorduces the auxiliary functions

$$1. \quad W(U, A, B) = -\oint \eta_x d\eta$$

and

$$2. \quad \frac{\partial W}{\partial A} = \lambda = \frac{1}{\kappa}. \tag{43.5}$$

Using Eq. (43.5) Whitham averages Eq. (43.3) over distance λ and writes the resultant set of averaged conservation equations in the form

$$\frac{\partial}{\partial t}\overline{P}_\iota + \frac{\partial}{\partial x}\overline{Q}_\iota = 0$$

$$\begin{aligned}
\overline{P}_1 &= \kappa W_B & \overline{Q}_1 &= \kappa U W_B - B \\
\overline{P}_2 &= \kappa W_U & \overline{Q}_2 &= \kappa U W_U - A \\
\overline{P}_3 &= \kappa(AW_A + BW_B + UW_U - W) & \overline{Q}_3 &= \kappa U(AW_A + BW_B + UW - W) - \frac{1}{2}B^2 - AU.
\end{aligned} \tag{43.6}$$

To find the characteristic velocities one redefines the auxiliary function W in terms of the zeros α, β, γ, δ of $\eta^3 - 1/2\ U\eta - B\eta + A = 0$. After considerable manipulation Whitham derives the following formulas for the three characteristic velocities;

$$U - \frac{4aK}{K - E}, \qquad U - \frac{4a(1 - s^2)K}{E - (1 - s^2)K}, \qquad U + \frac{4a(1 - s^2)K}{\rho^2 E}, \tag{43.7}$$

in which

$$a = \frac{\alpha - \beta}{2}; \qquad s^2 = \frac{\alpha - \beta}{\alpha + \beta}; \qquad U = 2(\alpha + \beta + \gamma)$$

$$K(s^2); \qquad E(s^2)$$

are complete elliptic integrals of the first and second kind. The wave number is

$$\kappa = \frac{1}{\dfrac{\partial W}{\partial A}} = \frac{a^{1/2}}{2sK}. \tag{43.8}$$

It is directly provable that s is a function of a/κ^2. Thus the characteristic velocities have the form

$$U + f\left(\frac{a}{\kappa^2}\right) \tag{43.9}$$

The noteworthy feature here is that the characteristic (phase) velocities are functions of amplitude a and wave number k. The comparable linear case is obtainable by dropping $\eta\eta_x$ and integrating to find

$$\eta_{xx} - U\eta = B.$$

Setting $B = 0$ and $\eta = \exp\,(i\,2\pi\kappa X)$ leads to the characteristic equation

$$U + \kappa^2 = 0.$$

Setting $U = \omega/2\pi\kappa$, one obtains the group velocities (two of them, both equal),

$$\frac{d\omega}{d\kappa} = -3(2\pi\kappa)^2 . \tag{43.10}$$

In the limit $a/\kappa^2 \to 0$, Eq. (43.7) agrees with Eq. (43.10). When the dispersion formulas are set up, Whitham proceeds with a discussion of shock waves which is obtainable from the formulas for the conservation equations. For a discussion of Whitham's theory of shock waves see Sec. 29 of this report.

The theory of Whitham presented above has been further refined by him and by several others. These refinements bear directly on the problem of finding appropriate methods of solving nonlinear partial differential wave equations. We therefore continue our presentation of Whitham's methods in the following section on the use of the variational method. As noted earlier, the variational method is directly applicable if a Lagrangian energy density for the field can be found, i.e., if a Lagrangian variable is available. This availability is assumed in the next section.

44. WHITHAM'S USE OF THE VARIATIONAL METHOD

The Hamilton variational principle using the Lagrangian $\mathfrak{L}$ as applied to a dynamic mechanical system is

$$\delta \int_{t_1}^{t_2} \int_{\mathbf{x}} \mathfrak{L}\, d\mathbf{x}\, dt = 0 . \tag{44.1}$$

To find wave train solutions Whitham [38, 39] introduces a wave-crest phase function,

$$\theta = \mathbf{k}\cdot\mathbf{x} - \omega t \qquad \mathbf{k} = \nabla\theta; \qquad \omega = -\frac{\partial\theta}{\partial t} \tag{44.2}$$

in which ω expresses the periodicity of the sought-for wave train, and $\mathbf{k}$ expresses the corresponding wave number of the train. The general solution of the wave train is a function $\phi(\mathbf{x}, t)$, and it is assumed that there is a function β (pseudo wave number) and a function γ (pseudofrequency) such that

$$\phi(\mathbf{x}, t) = \Phi(\theta) + \beta\cdot\mathbf{x} - \gamma t . \tag{44.3}$$

The two pseudofunctions are related to a "potential" ψ by relations

$$\nabla\psi = \beta; \qquad -\frac{\partial\psi}{\partial t} = \gamma . \tag{44.4}$$

If the Lagrangian $\mathcal{L}$ varies slowly with θ over a wavelength (i.e., over a unit change of phase, $0 \leqslant \theta \leqslant 1$), one can replace it with a phase-averaged form $\overline{\mathfrak{L}}$, where

$$\overline{\mathfrak{L}} = \int_0^1 \mathcal{L} d\theta; \qquad \overline{\mathfrak{L}} = \overline{\mathfrak{L}}(\omega, \mathbf{k}, a \mid \gamma, \beta, b) . \tag{44.5}$$

Here $\overline{\mathfrak{L}}$ is constructed to depend not only on ω, $\mathbf{k}$, γ, β but also on the amplitude a of the wave train, and the pseudoamplitude b. The physical significance of pseudoquantities is to be sought in the possibility that when $\Phi(\theta) \equiv 0$ there will still be a solution ϕ. From the assumed parameter dependence of $\overline{\mathfrak{L}}$ one can derive the following continuity equations.

Equation of Energy

$$\frac{\partial}{\partial t}\left(\omega\frac{\partial\overline{\mathfrak{L}}}{\partial\omega} + \gamma\frac{\partial\overline{\mathfrak{L}}}{\partial\gamma} - \overline{\mathfrak{L}}\right) - \frac{\partial}{\partial x_i}\left(\omega\frac{\partial\overline{\mathfrak{L}}}{\partial k_i} + \gamma\frac{\partial\overline{\mathfrak{L}}}{\partial\beta_i}\right) = 0 \tag{44.6}$$

Equations of Momentum

$$-\frac{\partial}{\partial t}\left(kj\frac{\partial\overline{\mathfrak{L}}}{\partial\omega} + \beta j\frac{\partial\overline{\mathfrak{L}}}{\partial\gamma}\right) + \frac{\partial}{\partial x_i}\left(kj\frac{\partial\overline{\mathfrak{L}}}{\partial k_i} + \beta j\frac{\partial\overline{\mathfrak{L}}}{\partial\beta_i} - \overline{\mathfrak{L}}\,\delta ij\right) = 0, \qquad j = 1, 2, 3 \tag{44.7}$$

Euler Equation for Pseudoquantities

$$-\frac{\partial}{\partial t}\frac{\partial\overline{\mathfrak{L}}}{\partial\gamma} + \frac{\partial}{\partial x_i}\frac{\partial\overline{\mathfrak{L}}}{\partial\beta_i} = 0 \tag{44.8}$$

Conservation Laws for Wave Crests

$$\begin{aligned} &\frac{\partial\mathbf{k}}{\partial t} + \nabla\omega = 0, \qquad \operatorname{curl}\mathbf{k} = 0 \\ &\frac{\partial\beta}{\partial t} + \nabla\gamma = 0, \qquad \operatorname{curl}\beta = 0 \end{aligned} \tag{44.9}$$

Since the study of all possibilities is tedious, it will be convenient to set pseudoquantities to zero (i.e., $\gamma = \beta = b = 0$). Then, taking $\partial/\partial\omega$ of the energy equation and $\partial/\partial k_i$ of the momentum equations and using $k_i = \nabla_i\theta$, $\omega = -\theta_t$, one can combine all equations and arrive at the reduced wave-train equations

$$\overline{\mathfrak{L}}_{\omega\omega}\,\theta_{tt} - 2\overline{\mathfrak{L}}_{\omega k_i}\,\theta_{tx_i} + \overline{\mathfrak{L}}_{k_ik_j}\theta_{x_ix_j} = 0 \qquad \text{Sum on } i, j = 1, 2, 3. \tag{44.10}$$

This is an equation of second degree in the phase function θ with nonlinear coefficients. It now remains to find appropriate forms for $\overline{\mathfrak{L}}$. A direct approach is to assume the case that amplitudes are small but finite. The $\overline{\mathfrak{L}}$ can be expanded in powers of a^2 beginning with a^4 since the Lagrangian here is averaged over a complete cycle of phase and hence disappears in the linear (i.e., a^2) approximation. Thus,

$$\overline{\mathfrak{L}}(\omega, k, a) = a^4\,\overline{\mathfrak{L}}_1(\omega, k) + a^6\,\overline{\mathfrak{L}}_2(\omega, k) + \ldots . \tag{44.11}$$

Now, for the phase function chosen (i.e., θ), there is a dispersion equation between ω and k. In nonlinear waves $\omega = \omega(k, a)$, i.e., the dispersion relation is a function of amplitude. For small but finite amplitude one can expand ω in a power series in a^2,

$$\omega = f(k) + a^2 \omega_1(k) + a^4 \omega_2(k) + \dots . \tag{44.12}$$

This is an even function of a since odd terms are equivalent to sign changes in phase and it is decided to avoid phase change in this approximation. Now the deviation τ from linear frequency (i.e., from ω) is given by

$$\tau = \omega - f(k) - a^2 \omega_1(k) - \dots . \tag{44.13}$$

Solving for a^2 (and noting that to this a^2 approximation, $\omega = f(k)$) one can write

$$a^2 \approx \frac{-\tau}{\omega_1(k)} \qquad \begin{bmatrix}\text{dimensions: } \omega_1 = \mathrm{m^2 s} \\ f = \mathrm{s^{-1}}\end{bmatrix} \tag{44.14}$$

Thus, the averaged $\bar{\mathfrak{L}}$ is

$$\bar{\mathfrak{L}} = \frac{\tau^2}{\omega_1^2(k)} \bar{\mathfrak{L}}_1 - \frac{\tau^3}{\omega_1^3(k)} \bar{\mathfrak{L}}_2 \tag{44.15}$$

or

$$\bar{\mathfrak{L}} = \frac{g(k)}{2!} \tau^2 + \frac{h(k)}{3!} \tau^3 \tag{44.16}$$

where

$$g(k) = \frac{2\bar{\mathfrak{L}}_1}{\omega_1^2(k)}; \qquad h(k) = -3!\, \frac{\bar{\mathfrak{L}}_2}{\omega_1^3(k)} . \tag{44.17}$$

Since $\bar{\mathfrak{L}}$ is a function of τ and k, it will be efficient to change variables x, t into variables k, τ. This is done by a Legendre transformation from θ to ϕ, where

$$\phi = \mathbf{k} \cdot \mathbf{x} - \omega t - \theta; \qquad x = \frac{\partial \phi}{\partial k}, \qquad t = -\frac{\partial \phi}{\partial \omega} . \tag{44.18}$$

The differential equation in ϕ is

$$\bar{\mathfrak{L}}_{\omega\omega} \phi_{kk} - 2\bar{\mathfrak{L}}_{\omega k} \theta_{k\omega} + \bar{\mathfrak{L}}_{kk} \phi_{\omega\omega} = 0 . \tag{44.19}$$

From the equation for τ it is seen that because of the change of variables the following replacements are needed

$$\frac{\partial}{\partial \omega} \to \frac{\partial}{\partial \tau}; \quad \frac{\partial}{\partial k} \to \frac{\partial}{\partial k} - f'(k)\frac{\partial}{\partial \tau} . \tag{44.20}$$

The differential equation in τ, k thus becomes

$$\overline{\mathfrak{L}}_{\tau\tau}\,\phi_{kk} \;-\; 2\overline{\mathfrak{L}}_{\tau k}\,\phi_{k\tau} \;+\; \overline{\mathfrak{L}}_{kk}\,\phi_{\tau\tau} \;-\; f''(k)\,(\overline{\mathfrak{L}}_{\tau}\,\phi_{\tau\tau} + \overline{\mathfrak{L}}_{\tau\tau}\,\phi_{\tau}) \;=\; 0. \tag{44.21}$$

Using the power series expansion of $\overline{\mathfrak{L}}\,(k,\,\tau)$ to order τ^3 in the above equation one arrives at the perturbation approximation

$$\begin{gathered}(g + 2h)\,(\phi_{kk} - f''\phi_{\tau}) \sim 2g'\tau\phi_{\phi k} - f''g\tau\phi_{\tau\tau} = 0 \\ g' = \frac{dg}{dk}\,; \qquad f''(k) = \frac{d^2f}{dk^2}\,.\end{gathered} \tag{44.22}$$

To order τ^2, this equation can be recast into the form

$$\phi_{kk} - \frac{2g'}{g}\,\tau\phi_{\tau k} - f''(\tau\phi_{\tau\tau} + \phi\tau) = 0\,. \tag{44.23}$$

If the waveform changes little in a few wavelengths, then $g'\tau/g$ is negligibly small, and the differential equation for ϕ reduces to

$$\phi_{kk} - f''(\tau\phi_{\tau\tau} + \phi_{\tau}) = 0\,. \tag{44.24}$$

The curves of characteristics (i.e., the curves $d\tau/dk$) of Eq. (44.23) and Eq. (44.34) are, respectively,

$$\begin{gathered}\frac{d\tau}{dk} = \frac{g'\tau}{g} \pm \sqrt{\frac{g'^2\tau^2}{g} + \tau f''(k)} \\ \frac{d\tau}{dk} = \pm\sqrt{f''(k)}\,.\end{gathered} \tag{44.25}$$

If one chooses a new variable,

$$r = 2\sqrt{\frac{\tau}{f''(k)}}\,, \qquad \text{[dimensions: } r\text{:m}^{-1}] \tag{44.26}$$

then the wave equations reduces to

$$\phi_{kk} - \phi_{rr} - \frac{1}{r}\phi_r = 0\,, \tag{44.27}$$

in which terms proportional to r are neglected. Since

$$x = \frac{\partial\phi}{\partial k} \qquad t = -\frac{\partial\phi}{\partial\omega}\,, \qquad [\phi \text{ dimensionless}] \tag{44.28}$$

it is seen that

$$
\begin{aligned}
x &= \phi_k - \frac{2f'}{rf''}\,\phi_r \\
t &= -\frac{2}{rf''}\,\phi_r\,.
\end{aligned}
\tag{44.29}
$$

Equation (44.27) is an axisymmetric form of the wave equation. Equations (44.29) in combination yield the trajectory on the x, t (or ω, k) plane of a constituent wave as a function of (ω, k). The slope $d\mathbf{x}/dt$ gives the velocity of each constituent wave at the spatial point $\mathbf{x}$ and at time t.

A typical use of Whitham's approach is contained in the following example Schmid [39]. We consider the axisymmetric case and assume a source of waves at $x = x_0$ whose amplitudes vary in the same way as the variable r,

$$
r = \alpha t; \quad \frac{r_1}{\alpha} \leqslant t \leqslant \frac{r_2}{\alpha} \qquad \begin{matrix}[\text{dimensions} \\ \alpha{:}\mathrm{m}^{-1}\ \mathrm{s}^{-1} \\ r{:}\mathrm{m}^{-1}\end{matrix}
\tag{44.30}
$$

At x_0 there is a wave train at constant frequency whose constituent waves initially have uniform wave lengths (i.e., linear relations hold). It is proposed to study dispersion among the wavelengths in the region $x > x_0$ due to amplitude variation in time. At $x = x_0$, with a reference wave number k_0, and reference velocity u_0, the boundary conditions are

$$
\begin{aligned}
k &= k_0 \\
\phi &= 0 \\
\phi_k &= -\frac{u_0}{\alpha} r = -u_0 t\,, \qquad t_1 \leqslant t \leqslant t_2\,.
\end{aligned}
\tag{44.31}
$$

The axisymmetric potential solution $\phi\,(k, r)$ to Eq. (44.24) with these boundary conditions and the transformations

$$
\begin{aligned}
\frac{\partial}{\partial \tau} &\to \frac{2}{rf''(k)}\frac{\partial}{\partial r} \\
\frac{\partial}{\partial k} &\to \frac{\partial}{\partial k} - \frac{r}{2}\frac{f'''(k)}{f''(k)}\frac{\partial}{\partial r}
\end{aligned}
\tag{44.32}
$$

is

$$
\phi(k, r) - \frac{u_0}{4\alpha}\left[r^2 + 2(k - k_0)^2 \ \arcsin\frac{k - k_0}{r} \right.
$$

$$
\left. + \ 3(k - k_0)\ \sqrt{r^2 - (k - k_0)^2}\,\right]. \tag{44.33}
$$

Since k_0 is so defined that

$$\begin{aligned} \omega_0 &= f(k_0) \\ u_0 &= f'(k_0) \\ \mu &\equiv f''(k_0), \end{aligned} \tag{44.34}$$

one can use the Eqs. (44.29) to find x and t for each choice of k and r, i.e.,

$$\begin{aligned} t &= \frac{u_0}{\alpha\mu}\left(\arcsin\frac{k-k_0}{r}\right) + \frac{k-k_0}{r^2}\sqrt{r^2 - (k-k_0)^2} \\ x &= u_0 t - \frac{u_0}{\alpha r^2}\left(r^2 - (k-k_0)^2\right)^{3/2}. \end{aligned} \tag{44.35}$$

By plotting x vs t a velocity field is constructed for each choice of r, i.e., of amplitude (noting that k is a function of r). The curves dx/dt for specific r constitute the characteristics (and characteristic velocities) of the system along which waves of wavenumber k propagate. Since the wave train is bounded in time, the characteristics are confined to a bounded region in the $x - t$ plane. In the above case the boundaries are straight lines which intersect the ordinate $x_0 = 0$ at two points and which intersect each other at

$$x_{\max} = \frac{\pi}{2}\frac{u_0^2}{\alpha\mu}. \tag{44.36}$$

All characteristics run together at $x_{\max}$, thus making $x_{\max}$ a distance at which possible phase jumps (or shock discontinuities) occur.

Whitham's method of analyzing nonlinear waves is a "geometric optical" construction of solutions of hydrodynamic equations using a Lagrangian. It deals with the dispersion relations among the constituent wave numbers of a temporally finite wave train. It distinguishes between frequency dispersion and amplitude dispersion, emphasizing that the latter is a different mechanism of wave-front distortion that is peculiar to nonlinear waves. The solution technique is by perturbation using numerical and graphical construction.

45. ADIABATIC INVARIANT FOR WAVE PROPAGATION IN A MOVING MEDIUM

In Whitham's theory the variational equation (in the linear case) appears in the form

$$\frac{\partial \mathcal{L}\omega}{\partial t} - \frac{\partial}{\partial x_i}\mathcal{L}_{k_i} = 0. \tag{45.1}$$

Since the group velocity $\mathbf{C}g = -\, d\omega/d\mathbf{k}$ (the minus sign enters for plane waves traveling to the right),

$$\mathfrak{L}_{k_i} = C_{g_i}\, \mathfrak{L}_{\omega}, \tag{45.2}$$

so that

$$\frac{\partial \mathfrak{L}\omega}{\partial t} + \nabla \cdot (C_{g_i} \mathfrak{L}_{\omega}) = 0. \tag{45.3}$$

Now the energy density of a wave whose Lagrangian density is $\mathfrak{L}$ is

$$E = \omega \frac{\partial \mathfrak{L}}{\partial \omega} - \mathfrak{L} \tag{45.4}$$

or

$$\frac{\partial \mathfrak{L}}{\partial \omega} = \frac{E}{\omega} = \text{“adiabatic invariant,”}$$

since $\mathfrak{L} \equiv 0$ for nondissipative systems. Thus, the conservation law for energy is

$$\frac{\partial}{\partial t}\left(\frac{E}{\omega}\right) + \nabla \cdot \left(\mathbf{C}_g \frac{E}{\omega}\right) = 0. \tag{45.5}$$

When the propagation medium is in motion with velocity $\mathbf{U}$ which changes slowly over a wavelength, then the frequency ω of the wave train is

$$\omega = \Omega + \mathbf{U} \cdot \mathbf{k} \tag{45.6}$$

in which Ω is the frequency relative to the medium. For relatively small-amplitude waves under slow change of wave number, the conservation law for energy is given by

$$\frac{\partial}{\partial t}\left(\frac{E}{\Omega}\right) + \nabla \cdot \left\{\frac{E}{\Omega}(\mathbf{U} + \mathbf{C}_g)\right\} = 0 \tag{45.7}$$

(Garrett [11]). In the theory of acoustics of a nonhomogeneous medium, Blokhintsev [42] found the energy equation to be

$$\frac{\partial E'}{\partial t} + \nabla \cdot \left\{E'(\mathbf{U} + \mathbf{C}_g)\right\} = 0. \tag{45.8}$$

The total energy E' is found to be

$$E' = E\frac{\omega}{\Omega}. \tag{45.9}$$

However, the law of conservation of wave crests is

$$\frac{\partial \omega}{\partial t} + \left\{(\mathbf{U} + \mathbf{C}_g) \cdot \nabla\right\}\omega = 0. \tag{45.10}$$

Thus Eq. (45.8) is seen to hold for sound waves.

46. CONSERVATION LAWS AND PERTURBATION EXPANSIONS

We consider a system of n conservation laws [43] in the n variables and assume the system can be put in the form

$$\mathbf{B}\,\mathbf{u}_t + \mathbf{C}(\mathbf{u}) \cdot \mathbf{u}_x = \sum_{r=1}^{m} \mu_\kappa \mathfrak{D}(\mathbf{u}, x, t). \tag{46.1}$$

Here, **B**, **C** are (in general) $N \times N$ matrixes, $\mathfrak{D}$ is a vector-valued matrix, and $\mu_\kappa (\kappa = 1, 2 \ldots m)$ are *small* parameters. To illustrate, we take the system $\mathbf{u}(u_1, u_2)$;

$$\begin{aligned} (u_1)_t - \sigma(u_2)_x &= \mu_1 u_{1xx} \\ (u_2)_t - (u_1)_x &= 0. \end{aligned} \tag{46.2}$$

Expanded, this reads

$$\begin{aligned} &\frac{\partial u_1}{\partial t} + 0 \cdot \frac{\partial u_2}{\partial t} - 0 \cdot \frac{\partial u_1}{\partial x} - \frac{\partial \sigma}{\partial u_2}\frac{\partial u_2}{\partial x} = \mu_1 \mathfrak{D}_1 \\ &0 \cdot \frac{\partial u_1}{\partial t} + \frac{\partial u_2}{\partial t} - \frac{\partial u_1}{\partial x} - 0 \cdot \frac{\partial u_2}{\partial x} = 0, \end{aligned} \tag{46.3}$$

so that

$$\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \mathbf{C} = \begin{pmatrix} 0 & -\dfrac{\partial \sigma}{\partial u_2} \\ -1 & 0 \end{pmatrix}; \quad \mathfrak{D} = \begin{pmatrix} \dfrac{\partial^2 u_1}{\partial x^2} \\ 0 \end{pmatrix} \tag{46.4}$$

If all small parameters are negligible we assume the solution of (46.2) is $\mathbf{u}_c$. We next let system (46.1) be perturbed and seek solutions in the form of a perturbation series in smallness parameter ϵ, viz,

$$\begin{aligned} \mathbf{u} = \mathbf{u}_c + \epsilon\{&\mathbf{u}_0 + \epsilon \mathbf{u}_1 + \epsilon^2 \mathbf{u}_2 + \ldots + \mu_1 \mathbf{u}_2 + \mu_2 \mathbf{u}_3 + \ldots + \mu_m \mathbf{u}_{m+1} \\ &+ \mu_1^2 u_{m+2} + \mu_2^2 \mathbf{u}_{m+3} + \ldots\}. \end{aligned} \tag{46.5}$$

Substituting (46.5) into $C(\mathbf{u})$ leads to a definition of $\mathbf{c}_0$, $\mathbf{c}_1$, etc.

$$\begin{aligned} \mathbf{C}(\mathbf{u}) &= \mathbf{C}_c + \epsilon \mathbf{C}_1(\mathbf{u}_0, \mathbf{u}_c) + \dots . \\ \mathbf{C}_c &= \mathbf{C}_c(\mathbf{u}_c). \end{aligned} \tag{46.7}$$

Now let λ represent speed of an elementary progressive wave, $\mathbf{u} = \mathbf{r}U(X)$, $X = x - \lambda t$, so that

$$\frac{\partial u}{\partial t} = - \lambda \frac{\partial u}{\partial x}. \tag{46.8}$$

Then the matrix $\mathbf{C}_c$ satisfies an eigenvalue equation,

$$(- \lambda \mathbf{B} + \mathbf{C}_c) \cdot \mathbf{r}U_X = 0. \tag{46.9}$$

The eigenvalues λ_i satisfy the characteristic equation

$$|- \lambda \mathbf{B} + \mathbf{C}_c| = 0, \qquad \lambda = \lambda_1, \dots \lambda_N. \tag{46.10}$$

The matrix B can be combined with $\mathbf{C}_c$ to define a new operator $\mathbf{C}_0$, such that

$$\mathbf{C}_0 \cdot \mathbf{r} = \lambda_i \mathbf{r}_i, \tag{46.11}$$

in which $\mathbf{r}$ is a "right eigenvector." The operator $\mathfrak{D}_\kappa$ is assumed to have the property that

$$\mathfrak{D}_\kappa(\mathbf{u}_c + \epsilon \mathbf{u}_0, x, t) = \epsilon \mathfrak{D}_\kappa(\mathbf{u}_c, u_0, x, t) + \dots . \tag{46.12}$$

By substitution of Eqs. (46.5) (46.6) (46.7) and (46.12) into Eq. (46.1) and equating terms of equal powers ϵ, one can obtain an infinite set of equations in which solutions u_n act as forcing functions to obtain solutions u_{n+1}.

As noted in Appendix D these equations are not valid unless the time $t << \mu^{-1}$ where $\mu = \max\{\epsilon, \mu_1, \dots \mu_m\}$. To avoid secular terms, we once again allow τ_0 to be "fast time," τ_1 to be slow time corresponding to ϵ, τ_2 slow time corresponding to μ_1, etc. The time derivative is then interpreted to be

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau_0} + \epsilon \frac{\partial}{\partial \tau_1} + \sum_{\kappa=1}^{m} \mu_\kappa \frac{\partial}{\partial \tau_{\kappa+1}} . \tag{46.13}$$

Transforming to coordinates $X = x - \lambda t$, we see that equations in $\mathbf{u}_1$ and $\mathbf{u}_{\kappa=1}$, $\kappa = 1, 2 \dots$, take on the form

$$\frac{\partial u_1}{\partial \tau_0} + (\mathbf{c}_0 - \lambda \mathbf{I}) \cdot \frac{\partial \mathbf{u}_1}{\partial X} = - \mathbf{c}_1 \cdot \frac{\partial \mathbf{u}_0}{\partial X} - \frac{\partial \mathbf{u}_0}{\partial \tau_1} \tag{46.14}$$

$$\frac{\partial \mathbf{u}_{\kappa+1}}{\partial \tau_0} + (\mathbf{c}_0 - \lambda \mathbf{I}) \cdot \frac{\partial \mathbf{u}_{\kappa+1}}{\partial X} = \mathfrak{D}_\kappa(u_c, u_0, x, t) - \frac{\partial \mathbf{u}_0}{\partial \tau_{\kappa+1}} . \tag{46.15}$$

To avoid secularity we premultiply Eqs. (46.14) (46.15) by the left eigenvector ℓ corresponding to the eigenvalue λ; that is,

$$\ell \cdot (\mathbf{c}_0 - \lambda \mathbf{I}) = 0$$

and set the r.h.s. respectively to zero;

$$\ell \cdot \mathbf{c}_1 \cdot \frac{\partial \mathbf{u}_0}{\partial X} + \ell \cdot \frac{\partial \mathbf{u}_0}{\partial \tau_1} = 0 \tag{46.16}$$

$$\ell \cdot \mathfrak{D}_K - \ell \cdot \frac{\partial \mathbf{u}_0}{\partial \tau_{K+1}} = 0 . \tag{46.17}$$

The solution of these equations sets a requirement on $\mathbf{u}_0$ which avoids secularity in $\mathbf{u}_1$, on $\mathbf{u}_1$ which avoids secularity in $\mathbf{u}_2$, etc.

Two examples will illustrate the procedures. In gas dynamics the conservation laws can be written so that,

$$\mathbf{c}_0 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{c}_1 = U \begin{pmatrix} 1 & 1 & 0 \\ \gamma - 2 & 1 & \gamma - 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\lambda_1 = 1, \quad \lambda_2 = -1, \quad \lambda_3 = 0$$

$$\mathbf{r} = \text{right eigenvectors:} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\ell = \text{left eigenvectors: } (1, 10),\ (1, -1, 0),\ (-1, 0, 1)$$

$$\mathbf{D}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{4}{3} + 2 & 0 \\ \frac{\gamma - 1}{P_r} & 0 & \frac{\gamma}{P_r} \end{pmatrix} \mathbf{u}_{xx} \tag{46.18}$$

Thus,

$$\ell \cdot \mathbf{D}_1 = \left(\frac{4}{3} + \nu + \frac{\gamma - 1}{P_r} \right) U_{XX} \tag{46.19}$$

$$\ell \cdot \mathbf{r} = 2$$

$$\ell \cdot \mathbf{c}_1 \cdot \mathbf{r} = (\gamma + 1)\, U$$

Substituting Eq. (46.19) into (46.16) and (46.17) leads to

$$\frac{\partial U}{\partial \tau_1} = -\frac{1}{2}(\gamma+1)\, UU_X \tag{46.20}$$

$$\frac{\partial U}{\partial \tau_2} = \frac{1}{2}\left(\frac{4}{3} + \nu + \frac{\gamma-1}{P_r}\right) U_{XX}\,. \tag{46.21}$$

Now, from Eq. (46.13)

$$\frac{\partial U}{\partial t} = \frac{\partial U}{\partial \tau_0} + \epsilon \frac{\partial U}{\partial \tau_1} + \mu_1 \frac{\partial U}{\partial \tau_2}\,.$$

To reconstitute the basic equation of motion we set $\partial U/\partial \tau_0 = 0$, and using Eqs. (46.20) (46.21), arrive at Burgers' equation,

$$Ut + \epsilon\left(\frac{\gamma+1}{2}\right) UU_X = \frac{1}{2}\left(\frac{4}{3} + \nu + \frac{\gamma-1}{P_r}\right) \mu_1\, U_{XX}, \tag{46.22}$$

in which $\mu_1 = \mathrm{R}_e^{-1}$.

The second example applies to shallow-water waves, in which

$$\mathbf{c}_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \mathbf{c}_1(\mathbf{u}) = \begin{pmatrix} u_2 & u_1 \\ 0 & u_2 \end{pmatrix}$$

$$D_1(\mathbf{u}) = \begin{pmatrix} \frac{1}{3} & u_{2\,xxx} \\ u_{2\,xxt} & \end{pmatrix}$$

$$\lambda_1 = 1 \qquad \lambda_2 = -1$$

$$\mathbf{r}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \mathbf{r}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\ell_1 = (1, 1) \qquad \ell_2 = (1, -1)$$

Applying Eq. (46.13) as before and reconstituting the basic equation leads to the Kortewig–deVries equation,

$$U_t + \frac{3}{2}\epsilon\, UU_X + \frac{\mu_1}{3} U_{XXX} = 0. \tag{46.24}$$

47. PROPAGATION OF FINITE-AMPLITUDE WAVES IN TUBES

The mathematical description of the propagation of finite-amplitude waves in tubes exhibiting wall loss is very complicated. A tractable model, based on the work of Coppens [44] and earlier investigators [45] is constructable as follows. Let the tube effective radius R, cross-sectional area A, and length L contain a liquid medium in which a finite-amplitude wave $u(x, t)$, generated by a proving piston, is propagating. Assume first that the boundary-layer thickness is much smaller than the tube radius, and that the sonic particle velocity is uniform over the mainstream (cross-sectional area minus the boundary layers). The problem is to calculate the viscous force in the boundary layer and find an appropriate way to insert this force in the Navier-Stokes equation. Coppens solves this problem by (a) calculating the wall forces by standard procedures, (b) distributing these wall forces over the entire cross section of the tube to act as body forces. This method results in accounting for wall dissipation by means of a compound absorption coefficient α_1 and phase speed $c(\alpha_1)$, both functions of frequency. If the tube is closed by a rigid wall, the absorption is modified to read $\alpha_1' = \alpha_1(1 + R/3L)$. To obtain a tractable model one assumes that (a) the wall losses are dominant and (b) Equation (27.19) is valid with $b \equiv 0$. This implies that the absorption coefficient is much smaller than the wave number of the fundamental, and that the acoustic Mach number is small. The Mach number is defined either by $M = u_0/c_0$ or by $M = p_1/\rho_0 c_0^2$, where u_0 is the peak amplitude of piston velocity and p_1 is the peak acoustic pressure of the fundamental. With these assumptions, Coppens, as noted above, constructs a distributed body force per unit viscosity, $D_n u_n$, where

$$D_n = \delta_1 \left(\frac{1}{\omega n^{3/2}} \frac{\partial}{\partial t} \frac{\partial^2}{\partial x^2} - \frac{1}{n^{1/2}} \frac{\partial^2}{\partial x^2}\right), \tag{47.1}$$

in which δ (the nondimensional dissipation factor) is given by

$$\delta_1 = \frac{2\alpha_1}{\kappa_1} = \frac{2}{R}\left(\frac{\nu}{2\omega}\right)^{1/2}\left[1 + \frac{\gamma - 1}{P_r{}^{1/2}}\right], \tag{47.2}$$

where P_r = Prandtl number, ν = kinematic viscosoty, k_1 = wave number (of fundamental). The first term on the r.h.s. of Eqs. (47.1) gives the wall loss (in units of m^{-2}) per unit of particle velocity. The second term describes the dispersion of the phase speed for each harmonic in terms of the harmonic number n. Inserting Eq. (47.1) into Eq. (27.19) yields the formulation,

$$\sum_{n=1}^{\infty}\left(\frac{\partial^2 u_n}{\partial r^2} - \frac{1}{c_0^2}\frac{\partial u_n}{\partial t^2} + \mathrm{D}_n u_n\right) = \frac{\gamma + 1}{2} \frac{\partial^2}{\partial t \partial x}\left[\left(\frac{\partial \xi}{\partial x}\right)\right]^2 . \tag{47.3}$$

To solve this equation Coppens recommends the following procedure. Noting that, to a linear plane-wave approximation,

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{1}{c_0^2}\frac{\partial^2 \xi}{\partial x^2} = \frac{1}{c_0^2}\left[\frac{1}{2}\frac{\partial^2 \xi}{\partial t^2} + \frac{1}{2}\frac{\partial^2 \xi}{\partial t^2}\right]$$

$$\frac{\partial \xi}{c_0 \partial t} = -\frac{\partial \xi}{\partial x} \tag{47.4}$$

$$\frac{\partial \xi}{\partial x} \approx \frac{u}{c_0},$$

one arrives at

$$\frac{\partial^2 \xi}{\partial x^2} \approx \frac{1}{2}\left(\frac{1}{c_0}\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\frac{u}{c_0}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \xi}{\partial x}\right)^2 \approx -\frac{1}{2}\left(\frac{1}{c_0}\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\left(\frac{u}{c_0}\right)^2 . \tag{47.5}$$

substituting Eqs. (47.4) and (47.5) into Eq. (47.3), and then assuming a traveling-wave solution of the form,

$$u(x, t) = \sum_{n=1}^{\infty} U_n(x) \sin\left[n\omega\left(t - \frac{x}{c_0}\right) + \phi_n(x)\right], \tag{47.6}$$

Coppens arrives at a coupled set of first order differential equations in the sine and cosine terms for each harmonic component. These solutions are amenable to analysis on a digital computer. Similarly, to obtian standing wave solutions, one can assume a Fourier series solution of the form

$$\frac{p(x, t)}{\rho_0 c_0^2} = \frac{u_0}{c_0}\sum_{n=1}^{\infty} R_n \cos\left[n\pi\left(1 - \frac{x}{L}\right)\right] \sin\,(n\omega t + \phi_n \quad . \tag{47.7}$$

The resultant solution is also found to be amenable to numerical analysis using a high-speed computer.

In all cases computed by him Coppens found that computer results compare favorably with the results achieved by other approaches. However, the limitations of his theory are again to be emphasized:

1. The loss associated with the boundary-layer absorption varies with harmonic number n as $n^{1/2}$, while the bulk absorption loss in the fluid varies as n^2. Hence when n is high enough, the bulk losses predominate, contrary to the initial assumption that wall losses predominate.

2. The boundary layer loss as written is valid only when the wavelength is much larger than the layer thickness. Above a certain frequency this requirement fails, and the formula for this loss is no longer valid.

3. The harmonic content of the wave field excites neighboring tube modes. These "cross modes" are not accounted for in the theory. Hence the calculated solution may not be tenable at frequencies corresponding to cross (or tube) modes.

Experiments on piston-generated standing waves in closed tubes shows that at high amplitudes and at frequencies near and at tube resonance the waveform exhibits cusps. A detailed study of these is found in Ref. 45.

48. RESONANT INTERACTIONS, THEORY OF PHILLIPS

The theory of nonlinear waves in resonant cavities [46] can serve as an introduction to the theory of resonant interactions. When a cavity is force driven at a frequency of a natural mode it exhibits cumulative standing-wave fields. The forcing frequency may itself be the result of the nonlinear interaction of two nonresonant drive frequencies in a nonlinear medium. Progressive wave fields, under special (nonlinear) conditions, exhibit similar resonant interactions. A brief discussion of the latter resonant interactions is presented here.

We consider first a linear operator $\mathfrak{L}$ and a field $p(\mathbf{x}, t)$ (capillary, surface wave, elastic stress in beams, plasma waves, etc.) The homogeneous equation

$$\mathfrak{L}\{p\} = 0 \tag{48.1}$$

is assumed to allow solutions in the form of progressive waves,

$$p = a \exp [i(\mathbf{k} \cdot \mathbf{x} - \omega t)]. \tag{48.2}$$

When Eq. (48.2) is substituted in Eq. (48.1) a dispersion formula $\omega = \omega(\kappa)$ is obtained. Under special conditions the curve $\omega(\kappa)$ vs κ exhibits a character which permits *resonant interactions.* When this occurs the operator $\mathcal{L}$ is (here) called special. We now consider a second (possibly nonlinear) operator $\mathfrak{N}$, and a small parameter ϵ such that (48.1) is replaced by

$$\mathfrak{L}\{p\} = \epsilon\, \mathfrak{N}\{p\}. \tag{48.3}$$

Here ϵ is important if the amplitude of p is large and is negligible for waves of infinitesimal amplitude. Eq. (48.3) may be interpreted as an equation of forced motion with the r.h.s. acting as a forcing term. If $\mathfrak{N}\{p\}$ has a simple sinusoidal time dependence (= exp ift) and $\mathcal{L}$ is special then a solution of (48.3) for selected f will exhibit resonance, that is, a buildup of p with time. Now in lieu of Eq. (48.2) we choose solutions p which are sums of simple wave trains, viz,

$$p = \sum_i a_i \exp iX_i\,; \quad X_i = \mathbf{k}_i \cdot \mathbf{x} - \omega_i t, \qquad a_i = a_i(\epsilon t). \tag{48.4}$$

In addition, the operator $\mathfrak{N}$ is assumed to posess quadratic nonlinearity. When (48.4) is substituted in (48.3) and $\mathfrak{L}$ is special there will be generated on the r.h.s. many sum and difference phases, labeled X_n due to the nonlinearity of $\mathfrak{N}$. The amplitude of p remains small due to ϵ except at a particular pair X_m which satisfies the condition

$$X_1 + X_2 + \ldots + X_n = 0, \quad \begin{cases} \omega_1 \pm \omega_2 \pm \ldots \omega_m = 0 \\ \kappa_1 \pm \kappa_2 \pm \ldots \kappa_n = 0 \end{cases} \tag{48.5}$$

In this case, the term exp (iX_1) on the l.h.s. matches the term with exp $[i(X_2 + \ldots + X_n)]$ on the r.h.s. Then X_1 is in "resonance" with $X_2 + \ldots + X_n$; that is, the spatial and temporal variations of these phase function are the same. Using Eq. (48.5) one may construct a set of equations in amplitudes a_1, a_2, ... a_n, with interaction coefficients c_1, c_2, ... c_n, initially unknown, but obtainable in several ways, particularly by use of Whitham's averaging technique (see Sec. 42 and Ref. 47. The solutions of the resonant interaction problem (namely finding c_1, c_2, ... c_n, etc), thus obtained allow the following directions:

1. At least one coefficient (say c_1) is negative. This means that if energy density in component 1 increases at some instant, the combined energy density of all remaining components must decrease.

2. The wave amplitudes are generally periodic, with period ϵ^{-1} times the period of the characteristic wave associated with it.

3. Growth of one component by transfer of energy from a second component can proceed until the latter is exhausted. However, if interaction continues (in time), the process is reversed and the exhausted component begins to build up again until initial conditions between components are again reached.

4. If a stable wave train X_q is exposed to new waves X_r such that Eq. (48.5) holds, a small perturbation of the wave train removes stability and resonant interactions begin.

5. When the wave shape of a pure sinusoid is distorted by any mechanism the nth harmonic developed corresponds to a free wave mode traveling at the same speed as the primary but at n times the primary wave number. Resonant interaction between primary and its harmonics can then occur.

These deductions and many others form an introduction to the rapidly growing field of resonant interactions in connection with nonlinear waves. A convenient summary is provided by Ref. 46. Important concepts are discussed in Ref. 48.

CONCLUSIONS ON WHITHAM'S THEORY

Whitham's theory of nonlinear dispersive waves provides fully nonlinear solutions to a wave field described by nonlinear partial differential equations. They are a class of solutions based on the concept of wave trains in which the modulation of amplitude and frequency is slow on the scale of wavelengths. Assuming plane periodic waves in which frequency and wave number are related through a phase function, Whitham inserts the phase function into an expression for the Lagrangian density and then averages this density over some judiciously chosen spatial distance (or phase interval). By use of a variational principle a set of Euler (conservation) equations is obtained involving these averaged Lagrangians. Among them is an explicit form of dispersion relation. These conservation equations show how the first derivatives of the Lagrangian energy density with respect to frequency and wave number (called adiabatic invariants) change as the energy is transferred slowly to different parts of the wave. The conservation equations, and in particular, the dispersion relation, form a set of differential equations in which wave number and amplitude are coupled. For certain values of field parameters the equations are hyperbolic. For other values the equations are elliptic. In the hyperbolic case the two characteristic velocities, which are identical in linear theory, split into two separate velocities in nonlinear theory. In the elliptic case the characteristic velocity is complex, and the modulations grow exponentially, i.e., the wave train is unstable. Again, in the hyperbolic case certain solutions eventually become multivalued. These may, in certain application, be the physically required solution (namely the "shock" solution). Near the shock the assumption of slow modulation breaks down. However, weak solutions can be formed by matching parameters across discontinuities using the conservation equations. Alternatively multivalued solutions may simply represent two wave trains passing through each other. In the nonlinear case, complex interaction between these two trains will occur.

The application of Whitham's theory to nonlinear acoustics remains yet to be done. Nonlinear partial differential equations appear in these acoustic problems, and slow modulation does represent important physical situations. The chief difficulty will be to find the Lagrangian density, the adiabatic invariants associated with the density, and the elucidation of the shock wave condition.

CONCLUSION OF PART 1

For realistic continua the basic equations of hydrodynamics and the constitutive relations are nonlinear, dissipative, dispersive, and relaxing. Thus dynamic problems in four variables (three space and one time) or less, both for steady state and for transients, are mathematically described by nonlinear partial differential equaitons of first and higher orders. The solutions of these equations in the general case are unknown. Analysts of acoustics problems usually begin with approximate equations to describe particular cases. Approximate equations can be obtained by perturbation theory in which orders of magnitude of successive terms are determined by the acoustic Mach number or Strouhal number. Solutions of certain (simple) types of approximate equations lead to Westervelt's equation. Different approximation are found in the acoustic equivalent of Burgers' equation and the Korteweg-deVries equation. Solutions of Burgers' equation lead to much ramification into equations of nonlinear harmonic buildup, attenuation with distance, shock wave theory, etc. Comparison of these equations with the classic equations of Fay, Fubini, Riemann, Earnshaw, etc. have been made. A complete study of the effects of varying parameters in these equations, of numerical consequences, of applications, etc. has been deferred to Part II of this review.

The very serious difficulty of finding solutions to nonlinear partial differential equations with (possible) nonlinear boundary conditions has been discussed from the point of view of two approaches, namely, by the method of characteristics and by the method of Whitham. Each of these is very involved in both derivation and analytic and numerical manipulation, and is limited by special assumptions. Whitham's method, however, leads to a procedure for handling dispersion in nonlinear wave trains. The method of characteristics lends itself to numerical calculation of space-time field in dynamic problems.

It is concluded here that the study of nonlinear acoustics is currently in an unsatisfactory state in regard to the following items: (a) In contrast to the situation in linear acoustics, a general nonlinear wave equation is not available in tractable form. Several approximate equations do exist. They form the corpus of "ready-to-use solutions" but are very restrictive from birth. There is, consequently, a need for further research in the derivation of more general equations to describe realistic acoustic situations including such processes as reflection, diffraction, duct propagation, etc. This is particularly true for nonlinear waves in two or three dimensions. (b) The methods of solving nonlinear partial differential equations are few and extremely complex. Further development of such methods is a vital need if the mathematical modeling of nonlinear acoustic process is to prosper. (c) Some realistic continua, viz, the ocean, are media which are inhomogeneous, layered, have rough boundaries, exhibit statistical fluctuations, and contain pockets of noise. If we add to these the property of nonlinearity *under intense (or high-amplitude) acoustic signaling* it is clear that the challenge of formulating a realistic nonlinear model of the acoustics of such media, and in particular the ocean, remains to be met.

REFERENCES

1. (a) T. G. Muir, editor, *Nonlinear Acoustics*, Proceedings of a conference held at APL, University of Texas, Austin, Texas, Nov. 10-11, 1969; (b) U.S. Navy *J. Underwater Acoust.* 22, No. 2, Apr. 1972 (Confidential).
2. E. Meyer and E. G. Neumann, *Physical and Applied Acoustics*, Academic Press, New York, 1972.
3. L. D. Rozenberg, editor, *High Intensity Ultrasonic Fields*, Plenum Press, New York, 1971.
4. A. C. Eringen, *Nonlinear Theory of Continuous Media*, McGraw-Hill, New York, 1962; (a) p. 341, (b) p. 343, (c) p. 340, (d) p. 346, (e) p. 347, (f) p. 348, (g) p. 352, (h) p. 354, (i) p. 356, (j) p. 168, (k) p. 169.
5. A. C. Eringen, *Mechanics of Continua*, McGraw-Hill, New York, 1967; (a) p. 109, (b) p. 35, (c) p. 13, (d) p. 240, (e) p. 237.
6. A. L. Polyakova, S. I. Soluyan, and R. V. Khokhlov, *Sov. Phys.-Acoust.* 8, 78-82 (1962).
7. L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, Pergamon Press, London: Addison-Wesley, Reading, Mass., 1959.
8. H. W. Marsh, R. H. Mellen, and W. L. Konrad, "Anomalous Absorption of Pressure Waves from Explosions in Sea Water," *J. Acoust. Soc. Amer.* 38, 326-338 (1965).
9. K. F. Herzfeld and T. A. Litovitz, *Absorption and Dispersion of Ultrasonic Waves*, Academic Press, New York, 1959.
10. I. Tolstoy, *Wave Propagation*, McGraw-Hill, New York, 1973.
11. "A Discussion on Nonlinear Theory of Wave Propagation in Dispersive Systems," organized by M. J. Lighthill, *Proc. Roy. Soc.* A299, 2-145 (1967).
12. F. W. Sears, *An Introduction to Thermodynamics*, 2d ed., Addison-Wesley, Cambridge, Mass., 1953, p. 149.
13. M. J. Lighthill, *Surveys in Mechanics*, G. K. Batchelor and R. M. Davies, editors,

Cambridge University Press, Cambridge, England, 1956, pp. 250-351.

14. Z. A. Gol'dberg, *Sov. Phys.-Acoust.* **2**, 346-350 (1956); **3**, 340-347 (1957).

15. H. Lass, *Vector and Tensor Analysis*, McGraw-Hill, New York, 1950.

16. P. J. Westervelt, *J. Acoust. Soc. Amer.* **35**, 535-537 (1963).

17. J. M. Burgers, *Advance. Appl Mech.* **1**, 171 (1948).

18. B. Riemann, Abhandl. Gcs. Wiss. Göttingen, *Math-Physik. K1* **8**, 43 (1860) or *Gesammelte Mathematische Worke*, 2d ed., H. Weber, editor, Dover Publications, New York, 1953, pp. 146-175.

19. V. P. Kuznetsov, "Equations of Nonlinear Acoustics," *Sov. Phys.-Acoust.* **16**, 467-470 (1971).

20. R. V. Khokhlov and S. I. Soluyan, *Acoustics* **14**, 5, 248-253 (1964).

21, E. A. Zabolotskaya and R. V. Khokhlov, *Sov. Phys.-Acoust.* **15**, 35-40 (1969).

22. D. T. Blackstock, "*Nonlinear Acoustics (Theoretical)*," *American Institute of Physics Handbook*, 3d ed., D. E. Gray, editor, McGraw-Hill, New York, 1972, pp. 3-128—3-205.

23. E. Hopf, *Commun. Pure Appl. Math.* **3**, 201 (1950).

24. J. D. Cole, *Quart. Appl. Math.* **9**, 225 (1951).

25. G. B. Whitham, "The Flow Pattern of a Supersonic Projectile," *Commun. Pure Appl. Math.* **5**, 301-348 (1952).

26. (a) M. S. Howe, *J. Fluid Mech.* **30**, 497-512 (1967); **32**, 779-789 (1968); (b) G. B. Whitham, *J. Fluid Mech.* **9**, 347-352 (1960; **22**, 273-283 (1965); **27**, 399-412 (1967); *Proc. Roy. Soc.* A292, 403 (1966).

27. D. T. Blackstock, "Thermoviscous Attenuation of Plane, Periodic Finite Amplitude Sound Waves," *J. Acoust. Soc. Amer.* **36**, 534-542 (1964).

28. I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products*, 4th ed., Academic Press, New York, 1965.

29. M. Abramowitz and I. A. Stegun, editors, *Handbood of Mathematical Functions*, NBS AMS 55, U. S. Government Printing Office, Washington, D.C., 1970.

30. P. M. Morse and H. Feshback, *Methods of Theoretical Physics*, McGraw-Hill, New York, 1953.

31. G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2d ed., Cambridge University Press, Cambridge, England, 1966 (reprint).

32. R. D. Fay, *J. Acoust. Soc. Amer.* **3**, 222 (1931).

33. E. Fubini Ghiron, *Alta Frequenza* **4**, 530-581 (1935).

34. S. Earnshaw, *Phil. Trans. Roy. Soc.* (London) **150**, 133-148 (1860).

35. F. G. Friedlander, *Sound Pulses*, Cambridge University Press, Cambridge, England, 1958.

36. P. C. Chou and A. K. Hopkins, editors, *Dynamic Response of Materials to Intense Impulsive Loading*,

37. R. Courant and D. Hilbert, *Methods of Mathematical Physics,* vol 2, Interscience, New York, 1952.

38. G. B. Whitham, *J. Fluid Mech.* 22, 273 (1965).

39. J. M. Schmid, "Theory of Nonlinear Waves," Scientific Report 74, UNICP-SR74, Oct. 15, 1970, Institute of Theoretical Physics, University of Morsbruck, Innrain 52, A-6020, Innsbruck, Austria.

40. G. B. Whitham, *Proc. Roy. Soc.* A283, 238 (1965).

41. D. J. Korteweg and G. de Vries, *Phil. Mag.* 39 (5), 422 (1895).

42. D. I. Blokhintsev, "Acoustics of a Nonhomogeneous Moving Medium," N.A.C.A.T.M. 1399 (1956).

43. S. Leibovich and A. R. Seebass, editors, *Nonlinear Waves,* Ch. 4 by S. Leibovich and A. R. Seebass, Cornell University Press, Ithaca, N.Y., 1974.

44. *Nonlinear Acoustics,* T. B. Muir, editor, Proceedings of a conference held at APL, University of Texas, Austin, Texas, No. 10-11, 1960, p. 99.

45. W. Chester, *J. Fluid Mech.* 18, 44-64 (1964).

46. O. M. Phillips, "Wave Interactions," Ch. VII of *Nonlinear Waves,* S. Leibovich and A. R. Seebass, editors, Cornell University Press, Ithaca, N.Y., 1974.

47. W. F. Simmons, *Proc. Roy. Soc.* A309, 551-575 (1969).

48. L. F. McGoldrick, *J. Fluid Mech.* 21, 305-331 (1965).

Appendix A

EULERIAN AND LAGRANGIAN DESCRIPTION OF PLANE WAVE MOTION

The local equation of continuity is customarily written in spatial form in two equivalent ways, viz,*

$$\frac{\partial \rho}{\partial t} + (\rho v_k)_{,k} = 0 \qquad \rho = \rho(\mathbf{x}, t); \qquad v = v(\mathbf{x}, t) \tag{A1}$$

$$\rho_0 = \rho j, \qquad j = \text{Jacobian.} \tag{A2}$$

In the case of plane wave motion the spatial x and material X coordinates are related by the equation

$$x(X, t) = X + \xi(X, t).$$

From this, we derive

$$\rho_0 = \rho \left[1 + \frac{\partial \xi}{\partial X}(X, t)\right] \tag{A3}$$

(see Sect. 19), and

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial x} + \frac{\partial \rho}{\partial x} u = 0 \tag{A4}$$

(see Sect. 20). Thus Eq. (A3) is a statement of the local conservation of mass in Lagrangian coordinates, while Eq. (A4) is the analogous statement in Eulerian coordinates.

The principle of balance of local momenta in spatial form is written as

$$\mathbf{t}_{k,k} - \rho \mathbf{a} = 0 \qquad (\text{i.e., } t_{\ell k,\ell} - \rho a_k = 0). \tag{A5}$$

In a lossless, fluid medium for one-dimensional propagation,

$$\frac{\partial t_{\ell x}}{\partial x_\ell} = -\frac{\partial p}{\partial x}, \tag{A6}$$

where p is the acoustic pressure. The fluid acceleration in spatial form is given by

$$a_k = \frac{\partial v_k}{\partial t} + v_{k,\ell} v_\ell,$$

*A. C. Eringen, *Nonlinear Theory of Continuous Media,* Wiley, New York, 1967.

for which, in one dimension,

$$a = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} u. \tag{A7}$$

Hence, the momentum equation is given by

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} u + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0. \tag{A8}$$

In material form the equation of motion has the form*,

$$\mathbf{T}_{K,K} - \rho_0 \mathbf{a} = \mathbf{0} \qquad (\text{or } T_{KL,K} - \rho_0 a_L = 0), \tag{A9}$$

in which for a lossless medium in one dimension,

$$\frac{\partial T_{KX}}{\partial X_K} = -\frac{\partial p}{\partial X}, \qquad p = p(X, t) \tag{A10}$$

$$a_L = \frac{\partial u(X, t)}{\partial t} = \frac{\partial^2 \xi}{\partial t^2}(X, t) \tag{A11}$$

Thus, Eq. (A9) becomes

$$\frac{\partial^2 \xi}{\partial t^2} + \frac{1}{\rho_0} \frac{\partial p(X, t)}{\partial X} = 0. \tag{A12}$$

The above equations can be put in a more symmetric form. Following the methods of Blackstock† and Landau and Lifshitz‡, one defines a speed λ such that

$$\frac{d\lambda}{c} = \frac{d\rho}{\rho} \qquad c^2 = \frac{dp}{d\rho} \tag{A13}$$

or

$$d\lambda = \frac{c^2 d\rho}{\rho c} = \frac{dp}{\rho c}, \qquad \phi = p(\rho) \tag{A14}$$

or

$$\lambda = \int_{\rho_0}^{\rho} \frac{c\,d\rho}{\rho}.$$

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*A. C. Eringen, *Nonlinear Theory of Continuous Media*, McGraw-Hill, New York, 1967, p. 109.
†D. T. Blackstock, "Nonlinear Acoustics (Theoretical)," American Institute of Physics Handbook, 3d ed., D. E. Gray, editor, McGraw-Hill, New York, 1972, pp. 3-128—3-205.
‡L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, Pergamon Press, London; Addison-Wesley, Reading, Mass., 1959.

Using these definitions one may write the basic equations in the forms

Eulerian form

$$\begin{cases} \dfrac{\partial \lambda}{\partial t} + u \dfrac{\partial \lambda}{\partial x} + c \dfrac{\partial u}{\partial x} = 0 & \text{(A16a)} \\[2ex] \dfrac{\partial u}{\partial t} + u \dfrac{\partial u}{\partial a} + c \dfrac{\partial \lambda}{\partial x} = 0 & \text{(A16b)} \end{cases}$$

Lagrangian form

$$\begin{cases} \dfrac{\partial \lambda}{\partial t} + \dfrac{\rho}{\rho_0} c \dfrac{\partial u}{\partial x} = 0 & \text{(A17a)} \\[2ex] \dfrac{\partial u}{\partial t} + \dfrac{\rho}{\rho_0} c \dfrac{\partial \lambda}{\partial x} = 0 . & \text{(A17b)} \end{cases}$$

Appendix B

INHOMOGENEOUS WAVE EQUATION (LIGHTHILL'S EQUATION)

The theory of the generation of aerodynamic sound* is based on an inhomogeneous wave equation with nonlinear terms. It will be useful to note this equation here. Let $\rho(x_k, t)$ be mass density, $v_i(x_k, t)$ be local velocity, and ζ (dimensions: s^{-1}) be the local rate of production of a field source. The conservation equations for mass density and momentum are then written as

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) = \rho\zeta \tag{B1}$$

$$\frac{\partial \rho v_i}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_i v_j + p_{ij}) = \rho f_i + \rho\zeta v_i, \tag{B2}$$

in which p_{ij} is the internal stress tensor (containing thermoviscous terms) defined on a unit element of the continuum and f_i is an external force per unit mass. By elimination of ρv_i between Eqs. (B1) and (B2) and rearrangement of terms one arrives at the exact formula

$$\frac{\partial^2 \rho}{\partial x_j^2} - \frac{1}{c^2}\frac{\partial^2 \rho}{\partial t^2} = -\frac{1}{c^2}\left\{\frac{\partial \rho\zeta}{\partial t} - \frac{\partial}{\partial x_i}(\rho f_i + \rho\zeta v_i) + \frac{\partial^2}{\partial x_i \partial x_j}(\rho v_i v_j + p_{ij} - c^2\rho\delta_{ij})\right\}. \tag{B3}$$

Since the r.h.s. contains true sources (i.e., $\rho\zeta$, ρf_i, $\rho v_i\zeta$) and fictitious sources $(\rho v_i v_j + p_{ij} - c^2\rho\delta_{ij})$, it may be said† that Eq. (A3) describes a fictitious material of uniform sound speed c in which all mass density relations are considered as due to *equivalent* acoustic waves. Practical use of Eq. (A3) requires detailed knowledge of these sources. Equation (A3) has been used by several authors as a starting point in the analysis of nonlinear acoustic phenomena‡.

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*M. J. Lighthill, *Proc. Roy. Soc.* **A211**, 564 (1952).

†P. E. Doak, "Multipole Analysis of Acoustic Radiation," paper K56, Fifth Congress on International Acoustics, Liege, Belgium, Sept. 7-14, 1965.

‡See, for example, P. J. Westervelt, J. *Acoust. Soc. Amer.* **35**, 535-537 (1963).

Appendix C

HEURISTIC DERIVATIONS OF BURGERS' EQUATION AND ITS ALLIES

The linear wave equation for a lossless fluid can be written in the form

$$\left(\frac{\partial}{c\partial t} - \frac{\partial}{\partial x}\right)\left(\frac{\partial}{c\partial t} + \frac{\partial}{\partial x}\right) u = 0 \tag{C1}$$

where u = velocity potential, pressure, particle velocity, etc. Thus, for a simple wave with infinitesimal amplitude traveling in one direction, one may choose either factor on the l.h.s. (say, the second) and write a progressive wave equation in the form

$$\frac{\partial u}{\partial t} + c_0 \frac{\partial u}{\partial x} = 0 \tag{C2}$$

in which c_0 is the local velocity of sound. This is a type of conservation equation (see Sects. 40-44 on Whitham's methods). Now let the wave have finite amplitude. Then one can surmise that the speed of the wave depends on amplitude (i.e., in lieu of c_0 we should have $c_0 + \beta u$). Equation (C2) is therefore modified to read

$$\frac{\partial u}{\partial t} + (c_0 + \beta u) \frac{\partial u}{\partial x} = 0, \tag{C3}$$

(Poisson's eq.)

in which β is a constant of nonlinearity, where

$$\begin{aligned} \beta &= \frac{\gamma + 1}{2} && \text{for gases} \\ \beta &= 1 + \frac{B}{2A} && \text{for fluids.} \end{aligned} \tag{C4}$$

Here $\gamma = C_p/C_v$ and B, A are constants in the pressure-density relation, i.e., $p = p_0 + A[(\rho - \rho_0)/\rho_0] + (B/2)\,[(\rho - \rho_0)/\rho_0]^2 + \ldots$. Finite-amplitude effects are not fully understood unless viscous attenuation is considered. Viscosity effects (in gases) were introduced (Stokes) by writing Eq. (C1) as

$$u_{tt} - c_0^2 u_{xx} = 2\delta u_{xxt}. \tag{C5}$$

The factor δ is defined as

$$\delta = \frac{\nu \mho}{2}; \quad \nu = \frac{\mu}{\rho_0}; \quad \mho = \frac{4}{3} + \frac{\mu_B}{\mu} \quad \text{(for gases)}$$

$$\delta = \frac{1}{2}\,\nu\left[\mathcal{V} + \frac{\gamma - 1}{P_r}\right]; \qquad \kappa = \frac{\chi}{\rho C_p} \qquad \text{(for liquids)}$$

where ν = kinematic viscosity, μ = coefficient of shear viscosity, μ_B = coefficient of bulk viscosity and P_r = Prandtl number (ν/κ). Since an operator solution of (C1) implies that

$$\frac{\partial}{\partial t} \to -c_0\,\frac{\partial}{\partial x}, \tag{C6}$$

V can be integrated (with respect to time) to yield an equation of the form*

$$u_t + c_0 u_x = \delta u_{xx}. \tag{C7}$$

Now if finite amplitude is again considered, then the appropriate equation is

$$u_t + (c_0 + \beta u)u_x = \delta u_{xx}. \tag{C8}$$

By introducing the coordinate transformation

$$x' = x - c_0 t$$

$$t = t,$$

we reduce Eq. (C8) to

$$u_t + \beta u u_{x'} = \delta u_{x'x'}. \tag{C9}$$

This is a form of Burgers' equation. By use of (C6) in (C8), one obtains (approximately) a second form of Burgers' equation,

$$u_t + c_0 u_x - \frac{\beta}{c_0}\,u u_t = \frac{\delta}{c_0^2}\,u_{tt}. \tag{C10}$$

Under the transformation

$$x = x$$

$$t' = t - \frac{x}{c_0}, \tag{C11}$$

Eq. (C10) becomes

$$u_x - \frac{\beta}{c_0^2}\,u u_{t'} = \frac{\delta}{c_0^3}\,u_{t't'}. \tag{C12}$$

Other forms of this equation are discussed by Blackstock.

*D. T. Blackstock, *Nonlinear Acoustics*, T. G. Muir, editor, Proceedings of conference held at Applied Physics Laboratory, University of Texas, Austin, Texas, Nov. 10-11, 1969, pp. 3-23.

The above heuristic derivations of Burgers' equation and its several allies must be more rigorously derived in order to show the intimate connections between the forms and the basic hydrodynamic equations. This was done in Sects. 25-28.

Appendix D

EXTENDING THE RANGE OF VALIDITY OF PERTURBATION EQUATIONS BY INTRODUCING TWO TIME SCALES

We consider a single one-dimensional conservation law for the field u in the form

$$u_t + L\{u\} = \epsilon M\{u\}, \qquad u = u(x,t), \tag{D1}$$

in which L is a linear operator and M a second operator, possibly nonlinear. If $\bar{u}$ is the spatial Fourier transform of u, the operation $L\{\bar{u}\}$ is defined as $i\omega(k)\bar{u}$. The symbol ϵ is a small parameter. For reasons explained later, the time t is considered a variable possessing two scales, a fast scale τ_0, and a slow scale τ*. Thus the field variable u is written more precisely as U, where

$$u(x,t;\epsilon) = U(x,\tau_0,\tau;\epsilon). \tag{D2}$$

The time derivative is understood to be

$$\frac{\partial}{\partial \tau} = \frac{\partial}{\partial \tau_0} + \epsilon \frac{\partial}{\partial \tau}. \tag{D3}$$

To solve Eq. (D1) we take the Fourier transform $\bar{U}$ of U in spatial coordinates using the transform pair kx. Then it is assumed that $\bar{U}$ can be expanded in a power series in ϵ,

$$\bar{U}(\tau_0,\tau;k;\epsilon) = \sum_{n=0}^{\infty} \epsilon^n \bar{U}_n(\tau_0,\tau;k). \tag{D4}$$

(Note that the operation $\overline{MU_0}$ is taken to be $im(k)\bar{U}_0$, where $m(k)$ is a function resulting from the operation M, analogous to the function $\omega(k)$ resulting from the operation L.) Substituting Eq. (D4) into the transform of Eq. (D1) and equating equal powers of ϵ leads to the first two equations of the perturbation series,

$$\bar{U}_{0_{\tau_0}} + i\omega(k)\bar{U}_0 = 0 \tag{D5}$$

and

$$\bar{U}_{1_{\tau_0}} + i\omega(k)\bar{U}_1 = im(k)\bar{U}_0 - \bar{U}_{0_\tau}. \tag{D6}$$

For the initial condition $\bar{U}_0(0;k) = \bar{f}(k)$ the solution of Eq. (D5) is

*S. Leibovich and A. R. Seebass, Ch. IV of Nonlinear Waves, S. Leibovich and A. R. Seebass, editors, Cornell University Press, Ithaca, N.Y., 1974.

$$\bar{U}_0 = \bar{f}(k)A_0(\tau;k)e^{-i\omega(k)\tau_0}. \tag{D7}$$

Here A_0 is an amplitude function explicitly dependent on slow time with the restriction $A_0(0;k) = 1$. Substituting Eq. (D7) into the r.h.s. of (D6) leads to a forcing function

$$\bar{f}(k)e^{-i\omega(k)t}\left[im(k)A_0(\tau;k) - \frac{\partial A_0}{\partial \tau}\right]. \tag{D8}$$

Now if

$$A_0 = b(k)e^{i\tau m(k)}, \qquad b(k) = 1, \tag{D9}$$

then (D8) will vanish. The *form* of the solution of Eq. (D6) is then identical to Eq. (D7). The first two terms of Eq. (D4) then read,

$$U = \bar{f}(k)\exp\{-i[\tau_0\omega(k) - \tau m(k)\} + \epsilon A_1(\tau;k)e^{-i\tau_0\omega(k)}. \tag{D10}$$

If, on the other hand, slow-time variables are all negligible, then $A_0(\tau;k)$ is unity for all τ, and the substitution of (D7) into (D6) leads to the equation

$$\bar{U}_{1_{\tau_0}} + i\omega(k)\bar{U}_1 = i\bar{f}(k)m(k)e^{-i\omega(k)\tau_0}, \tag{D11}$$

the solution of which is

$$\bar{U}_1 = i\bar{f}(k)m(k)\tau_0 e^{-i\omega(k)\tau_0}. \tag{D12}$$

The first two terms of Eq. (D4) are then given by

$$\bar{U}(\tau_0;k;\epsilon) = \bar{f}(k)e^{-i\omega(k)\tau_0}[1 + i\epsilon\tau_0 m(k) + \ldots]. \tag{D13}$$

From this equation it is seen that the series (D4) is invalid when $\epsilon\tau_0$ is of the order of unity, i.e., for time of the order $0(\epsilon^{-1})$. Comparing (D12) with (D10) shows that the introduction of slow time in addition to fast time has made the amplitude of the first two terms of expansion (D4) independent of time (i.e., removed secularity), and has made the entire expansion (D4) valid to time $0(\epsilon^{-1})$.

Appendix E

PARKER'S ASYMPTOTIC METHODS IN NONLINEAR ACOUSTICS

Parker* has derived a set of equations by extension of Whitham's method. His procedure is as follows. If $\mathbf{u} = \mathbf{u}(s, \mathbf{x}, t)$, $\rho = \rho(s, \mathbf{x}, t)$, $p = p(\rho)$ are fluid velocity, density and pressure respectively, and $s = s(x, t) =$ const. defines the phase front of the propagating wavelets, then for the selected form

$$s(\mathbf{x}, t) = \mathbf{k} \cdot \mathbf{x} - \Omega t, \quad \frac{\partial s}{\partial x_i} = k_i, \quad \frac{\partial s}{\partial t} = -\Omega \tag{E1}$$

it is seen that the equations of continuity and momentum of a viscous fluid are given by

$$(\Omega - \mathbf{k} \cdot \mathbf{u}) \frac{\partial \rho}{\partial s} - \rho k_i \frac{\partial u_i}{\partial s} = -\left\{\frac{\partial \rho}{\partial t} + \rho \frac{\partial u_i}{\partial x_i} + u_i \frac{\partial \rho}{\partial x_i}\right\} \tag{E2a}$$

$$-k_i \rho'(\rho) \frac{\partial \rho}{\partial s} + \rho(\Omega - \mathbf{k} \cdot \mathbf{u}) \frac{\partial u_i}{\partial s} = -\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) - \tag{E2b}$$

$$-p'(\rho) \frac{\partial \rho}{\partial x_i} + \mu\left\{k_j \frac{\partial}{\partial s}\left(K_j \frac{\partial u_i}{\partial s}\right) + \frac{1}{3} k_j \frac{\partial}{\partial s}\left(k_i \frac{\partial u_i}{\partial s}\right)\right.$$

$$\left. + \text{negligible order terms}\right\}. \tag{E3}$$

Since the operation $\partial^2/\partial t \partial x_i$ must be independent of the order of taking the t or x_i derivative, a set of compatibility conditions are imposed by Eq. (E1), namely,

$$\left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial s} \frac{\partial s}{\partial x_i}\right) \frac{\partial s}{\partial t} = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s} \frac{\partial s}{\partial x_i}\right) \frac{\partial s}{\partial x_i}$$

$$\frac{\partial k_i}{\partial x_j} + \frac{\partial k_j}{\partial s} \frac{\partial s}{\partial x_i} = \frac{\partial k_j}{\partial x_i} + \frac{\partial k_i}{\partial s} \frac{\partial s}{\partial x_j}. \tag{E4}$$

If the r.h.s. of both equations of (E2) are neglected (i.e., viscous damping and signal modulation are neglected), then the solutions which satisfy Eq. (E2) are

*D. F. Parker, Proc. 1973 Symposium, Copenhagen; Ed. L. Bjorno, IPC Science and Technology Press Ltd. Surrey, England.

$$(\Omega - \mathbf{k} \cdot \mathbf{u}) = kc(\rho) \quad , \quad k = |\mathbf{k}|, c^2(\rho) = \frac{\partial p}{\partial \rho} \tag{E5}$$

$$\rho \frac{\partial u_i}{\partial s} = n_i c(\rho) \frac{\partial \rho}{\partial s}, \quad k_i = kn_i \tag{E6}$$

Eq. (E5) defines the speed of propagation at a fixed space-time point in the wavefront and Eq. (E6) shows how u_i and ρ are related at this point.

It is next assumed that the fluid is subject to an acoustic perturbation such that for a small quantity δ one has

$$\rho = \rho_0(\mathbf{x}, t) + \delta\hat{\rho}(s, \mathbf{x}, t)$$

$$\mathbf{u} = U(\mathbf{x}, t) + \delta\hat{u}(s, \mathbf{x}, t) \tag{E7a}$$

$$k^{-1} = k^{-1}(\mathbf{x}, t)[1 + \hat{k}(s, \mathbf{x}, t)] \tag{E7b}$$

in which $\hat{\rho}$, $\hat{u}$, $\hat{K}$ are acoustic perturbation to the equilibrium density, velocity and propagation constant. The nonlinearily induced sound speed is

$$c(\rho) = c_0(\mathbf{x}, t) + \delta\hat{c}(s, \mathbf{x}, t), \quad c_0 = c(\rho_0). \tag{E7c}$$

To first order in δ, it is known that

$$\frac{\mathbf{u}}{c_0} = \mathbf{n} \frac{\hat{\rho}(s, \mathbf{x}, t)}{\rho_0(\mathbf{x}, t)}, \quad \mathbf{n} = \text{unit normal vector}. \tag{E8}$$

Substitution of the acoustic approximations given above into Eqs. (E2a) and (E2b) leads to left hand sides which are of order δk. Multiplying Eq. (A5.2a) by c and Eq. (E2b) by n_i, then adding, causes all terms in $\partial/\partial s$ on the r.h.s. to vanish. The result is a "transport equation,"

$$2\left[\frac{\partial \hat{\rho}}{\partial t} + (U_i + c_0 n_i) \frac{\partial \hat{\rho}}{\partial x_i}\right] + \hat{\rho} \frac{\rho_0}{c_0} \left[\frac{\partial}{\partial t}\left(\frac{c_0}{\rho_0}\right)\right.$$

$$\left. +(U_i + c_0 n_i) \frac{\partial}{\partial x_i}\left(\frac{c_0}{\rho_0}\right)\right] + \hat{\rho}\left[c_0 \frac{\partial n_i}{\partial x_i} + \frac{\partial U_i}{\partial x_i}\right.$$

$$\left. +n_i\left(2 \frac{\partial c_0}{\partial x_i} + n_j \frac{\partial U_i}{\partial x_j}\right)\right] = \frac{4}{3} \frac{\mu k}{\rho_0} \frac{\partial}{\partial s}\left(k \frac{\partial \hat{\rho}}{\partial s}\right) + O(\delta, \mu k). \tag{E9}$$

Combination of Eqs. (E4) imply that

$$\frac{\partial k_i}{\partial t} + (U_j + c_0 n_j)\frac{\partial k_i}{\partial x_j} = -k\left(\frac{\partial c_0}{\partial x_i} + n_j\frac{\partial U_j}{\partial x_i}\right). \tag{E10}$$

The last two transport equations define rays which are determined by the ratios

$$d\mathbf{x}:dk_i:dt = \mathbf{U} + c_0\mathbf{n}: - k[\nabla c_0 + n\cdot(\nabla \mathbf{u})]:1 \tag{E11}$$

Using this last equation it is seen that Eqs. (E9) and (E10) reduce to

$$2\frac{D\hat{\rho}}{Dt} + \hat{\rho}\left[\frac{k\rho_0}{c_o}\frac{D}{Dt}\left(\frac{c_0}{k\rho_0}\right) + \nabla\cdot(\mathbf{U} + c_0\mathbf{n})\right] = \frac{4}{3}\frac{\mu k}{\rho_0}\frac{\partial}{\partial s}\left(K\frac{\partial\hat{\rho}}{\partial s} + O(\delta, \mu k)\right) \tag{E12}$$

$$\frac{D\hat{k}}{Dt} = -k\delta\left[\frac{c_0}{\rho_0} + c'(\rho_0)\right]\frac{\partial\hat{\rho}}{\partial s} + O(\delta) \tag{E13}$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} \neq (U_j + c_0 n_j)\frac{\partial}{\partial x_j}.$$

Thus, an acoustic signal, propagating through an inhomogeneous fluid having velocity $\mathbf{U}(\mathbf{x}, t)$, density $\rho_0(\mathbf{x}, t)$ and sound speed $c_0(\mathbf{x}, t)$ is found by solving Eq. (E11) for rays, and then proceeding to solve Eqs. (E13) and (E14) interatively for $\hat{\rho}$ and $\hat{k}$.

Let c_1 be the variable velocity along a ray, and l be the propagation distance along the ray, then $dl = c_1 dt$. Now assume that

$$\frac{D\hat{\rho}}{Dt} \approx \frac{\partial\hat{\rho}}{\partial t} \tag{E14}$$

i.e., assume that the time variation of $\hat{\rho}$ due to the convection of the gradient of $\hat{\rho}$ is negligible compared to the time variation of $\hat{\rho}$ in situ. Then Eqs. (E12) and (E13) take on the form

$$\frac{\partial\hat{\rho}}{\partial l} + \frac{\hat{\rho}}{\alpha^{-1}} = \frac{2/3}{\frac{\rho_0 c_1}{\mu k}}\frac{\partial}{\partial s}\left(k\frac{\partial\hat{\rho}}{\partial s}\right) \tag{E15a}$$

$$\frac{\partial\hat{k}}{\partial l} = -\frac{1}{\rho_0(k\delta)^{-1}}\left[1 + \frac{\rho_0 c'(\rho_0)}{c_0}\right]\frac{\partial\hat{\rho}}{\partial s} \tag{E15b}$$

$$\alpha \equiv \left[\frac{k\rho_0}{c_0}\frac{D}{Dt}\left(\frac{c_0}{k\rho_0}\right) + \nabla\cdot(\mathbf{U} + c_0\mathbf{n})\right]\frac{1}{2c_1} \tag{E15c}$$

$$c_1 = |\mathbf{U} + c_0\mathbf{n}|. \tag{E15d}$$

There are three characteristic lengths:

(1) α^{-1} describes the length scale of unhomogeneity, convection and spreading

(2) $\dfrac{\rho_0 c_0}{\mu k^2}$ describes the length scale of viscous damping

(3) $(k\delta)^{-1}$ describes the length scale of amplitude dispersion.

Discussion: If $k\delta$ is significantly smaller than α or $\mu k^2/\rho_0 c_0$ then nonlinearity of the medium is unimportant in the propagation process. If $k\delta$ is much larger than $\mu k^2/\rho_0 c_0$ then viscous attenuation is negligible, and if $k\delta$ is also much larger than α then the propagation reduces to that of a simple wave. If a signal is damped only by viscosity (i.e., if α vanishes) then

$$\frac{\partial \hat{\rho}}{\partial l} = \frac{2/3}{\dfrac{\rho_0 c_1}{\mu k}} \frac{\partial}{\partial s}\left(k \frac{\partial \hat{\rho}}{\partial s}\right). \tag{E16}$$

Now the amplitude dispersion relation Eq. (E15b) can be used as a compatibility condition for a function $\beta(l, s)$. Noting that

$$\frac{\partial^2 \beta}{\partial l \partial s} = \frac{\partial^2 \beta}{\partial s \partial l}.$$

One can deduce that

$$\frac{\partial \beta}{\partial l} = -\frac{\delta}{\rho_0}\left[1 + \frac{\rho_0 c'(\rho_0)}{c_0}\right] \hat{\rho}(l, s) \tag{E17a}$$

$$\frac{\partial \beta}{\partial s} = -\frac{(1 + \hat{k})}{k}. \tag{E17b}$$

Defining a new function $w(l, \beta)$ such that

$$w(l, \beta) = -\frac{\delta}{\rho_0}\left[1 + \frac{\rho_0 c'(\rho_0)}{c_0}\right] \hat{\rho}(l, s) \tag{E18}$$

it is directly seen that (E16) reduces to a form of Burgers' equation.

$$\frac{\partial w}{\partial l} + w \frac{\partial w}{\partial \beta} = v \frac{\partial^2 w}{\partial \beta^2}, \qquad v = \frac{2}{3} \frac{\mu}{\rho_0 c_0} \tag{E19}$$

$$k \frac{\partial}{\partial s} \equiv -\frac{\partial}{\partial \beta}.$$

Here, β is a distance normal to the wavefront.

An important new feature of Parker's method is the development of the theory of nonlinear wave propagation by means of acoustic rays (geometrical acoustic analogy). The theory is thus asymptotic with respect to wavelength. Nonlinearity appears as a modulation of a simple wave, i.e., of a wave whose field parameters (pressure, particle velocity, density, etc.) are functions of a single phase variable s.

Appendix F

GENERAL NONLINEAR ACOUSTIC EQUATION WITH IMPROVED RELAXATION TERM

When the medium in the state of acoustic excitation undergoes relaxation an approximate equation of state may be written in the form

$$p'(\mathbf{x}, t) = c_0^2\rho' + \frac{1}{2}\left(\frac{\partial^2 p}{\partial \rho^2}\right)\rho'^2 + mc_0^2 \int_{-\infty}^{t} \frac{\partial \rho'}{\partial t'} e^{-\frac{t-t'}{T}} dt' - \chi\left(\frac{1}{C_v} - \frac{1}{C_p}\right) \operatorname{div} \mathbf{v} \quad \text{(F1)}$$

in which T is the characteristic relaxation time, $m = c_\infty^2 - c_0^2/c_0^2$, and all other symbols have been defined in the text. The propagation of a sound beam of limited cross-sectional dimension is then derived from the complete system of hydrodynamical equations in the cylindrical coordinate $\mathbf{x} = \mathbf{x}(r(y, z), x)$ where x is the direction of propagation. Its form is,

$$\frac{\partial}{\partial t}\left[\frac{\partial \rho'}{\partial x} - \frac{\epsilon}{c_0\rho_0}\rho'\frac{\partial \rho'}{\partial t} - \frac{b}{2c_0^3\rho_0}\frac{\partial^2 \rho'}{\partial \tau^2} - \frac{m}{2c_0}\frac{\partial}{\partial \tau}\int_{-\infty}^{\tau}\frac{\partial \rho'}{\partial \tau'} e^{-\frac{(\tau-\tau')}{T}} d\tau'\right] = \frac{c_0}{2}\left(\frac{\partial^2 \rho'}{\partial r^2} + \frac{1}{r}\frac{\partial \rho'}{\partial r}\right) \quad \text{(F2)}$$

in which $\tau = t - x/c_0$, $\epsilon = (\gamma + 1)/2$, $b = \zeta + 4/3\ \eta + \chi((1/c_v) - (1/c_p))$. Equation (F2) includes effects of dissipation, relaxation, dispersion and diffraction. However all these processes, together with nonlinearity, are considered weak relative to the acoustic amplitude of the basic propagating wave. For additional details see Rudenko et al. Sov. Phys. — Acoust. **20** (3), 271-275 (1974).