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Equivalence of Two Criteria for Optimal Closed-Loop Control

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A family of optimal closed-loop control processes was defined, and two definitions of optimality were considered for this family. The two definitions are equivalent provided the admissible closed-loop controls satisfy a certain closure property. As a consequence of this equivalence, it suffices to compare candidates for optimality with admissible controls on a global rather than a pointwise basis.

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EQUIVALENCE OF TWO CRITERIA FOR OPTIMAL CLOSED-LOOP CONTROL

INTRODUCTION

In the general case the synthesis problem, that is, the problem of the existence of an optimal closed-loop control and of generating it, is unsolved. In the special case of linear time-optimal processes, the synthesis problem is solved, in principle, as a consequence of existence and uniqueness theorems [1]. The synthesis of time-optimal feedback controllers for second-order nonlinear systems having one degree of freedom is given in Boltyanskii [2] and Lee and Markus [3].

A recent note deals with the synthesis problem for general nonlinear processes [4]. There it is shown that optimal closed-loop control can be synthesized from optimal open-loop control provided three conditions are met. These conditions deal with the class properties of the admissible controls and with the existence and uniqueness of optimal control. The question of existence of an optimal closed-loop control even for a relatively small class of nonlinear processes is a difficult part of the synthesis problem.

Optimal closed-loop control is defined by an optimality criterion involving comparisons of candidates for the optimum. In this report two optimality criteria are investigated. In one criterion a candidate is compared with all closed-loop controls generating a terminating trajectory from at least one initial state of the state space. In the other criterion a candidate is compared with only those closed-loop controls generating terminating trajectories from all points of the state space. By a theorem, the two optimality criteria are shown to be equivalent for classes of admissible closed-loop controls that satisfy any one of three conditions that are defined in the next section. It is hoped that this equivalence may be of help in establishing existence of an optimal closed-loop control.

OPTIMAL CLOSED-LOOP CONTROL PROCESS

We consider an optimal control process whose dynamic behavior is governed by the state equation

$$\dot{x} = f(x, u) \tag{1}$$

where the state x is contained in E^n, the control value u belongs to E^m, and f is a Borel-measurable function with domain $E^n \times E^m$ and range in E^n. Note that the *nth* component of the state variable is considered to be time t; that is, if $x = (x_1, x_2, \ldots x_n)$, then $\dot{x}_n \equiv 1$ from $x_n \equiv t$. An additional dimension can be added to the original state equations whenever they do not have the form of Eq. (1) with $x_n \equiv t$.

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The state space, denoted by X, is a Borel-measurable subset of E^n. The terminal set θ is a closed set contained in the closure of X. Constraints on the control values u available to the controller are given implicitly by the set-valued function

$$U \; : \; X \rightarrow \text{set of compact subsets of } E^m . \tag{2}$$

A necessary condition for a Borel-measurable function

$$U \; : \; X \rightarrow E^m \tag{3}$$

to be an admissible closed-loop control policy is the satisfaction of two criteria:

(i) $U(x) \in U(x)$ for all $x \in X$

(ii) For some $x_0 \in X - \theta$, there exists an absolutely continuous function

$$\varphi_{x_0} \; : \; [t_0, t_{x_0}] \rightarrow X \cup \theta$$

such that the equation

$$\phi_{x_0}(t) = x_0 + \int_{t_0}^{t} f(\varphi_{x_0}(\tau), U(\varphi_{x_0}(\tau))) \, d\tau \tag{4}$$

is satisfied for all $t \in [t_0, t_{x_0}]$ where t_{x_0} is some time greater than t_0.

This necessary condition on admissibility implies that the class of admissible policies is a subclass of the class of all Borel-measurable policies. In this report we consider only classes of admissible closed-loop control policies that satisfy any one of the following conditions

Condition 1. If U_1 and U_2 are admissible closed-loop control policies and if B is a Borel-measurable subset of the state space X, then the function

$$U_B \; : \; X \rightarrow E^m$$

defined by

$$U_B(x) = \begin{cases} U_1(x) & \forall \, x \in B \\ U_2(x) & \forall \, x \in X - B \end{cases} \tag{5}$$

is also an admissible closed-loop control policy.

Condition 2. This condition is identical to Condition 1 with the exception that B is a closed subset of X rather than a general Borel-measurable subset.

Condition 3. If U_1 and U_2 are admissible closed-loop control policies and if for some $x_0 \in X$ *the* absolutely continuous function $\phi_{x_0} : [t_0, t_{x_0}] \to X \cup \theta$ corresponds to U_1 in Eq. (4), then the function

$$U_3 \; : \; X \to E^m$$

defined by

$$U_3(x) = \begin{cases} U_1(x) & \forall\, x \in \{\phi_{x_0}(t) \; : \; t \in [t_0, t_{x_0}]\} \\ \\ U_2(x) & \text{for all other} \quad x \in X \end{cases} \tag{6}$$

is also an admissible closed-loop control policy. Note that Condition 1 implies Condition 2 and that Condition 2 implies Condition 3.

Suppose a class of admissible closed-loop control policies has been selected satisfying Condition 1, 2, or 3. Denote this class by A. Such a class is said to satisfy a closure property because it contains all controls of the type described by Eqs. (5) and/or (6).

A solution of Eq. (1); i.e., one satisfying Eq. (4), for some admissible closed-loop control policy and given initial condition x_0 is called a *trajectory.* A trajectory ϕ_{x_0} : $[t_0, t_{x_0}] \to E^n$ is said to be *admissible* iff it lies entirely in the state space X for all times t contained in $[t_0, t_{x_0}]$. An admissible trajectory is called *terminating* iff $\phi_{x_0}(t_{x_0})$ belongs to θ. If ϕ_{x_0} is terminating, then the time t_{x_0} is denoted by t_f and is called the *terminating* time.

Let x_0 be contained in X. Let $T(x_0)$ denote the set of all policies $U \in A$ having at least one terminating admissible trajectory emanating from x_0. For $U \in T(x_0)$ let $T(x_0, U)$ denote the set of all terminating admissible trajectories emanating from x_0 and corresponding to U.

Let $T(X)$ denote the set of all policies $U \in A$ having for each $x \in X$ at least one terminating admissible trajectory emanating from x. Of course, $T(X)$ is a subset of $T(x_0)$ for all $x_0 \in X$. The assumption is made that $T(X)$ is not empty.

The state is to be transferred from some $x_0 \in X$ *to the terminal set* θ. The performance index

$$J(x_0, U, \phi_{x_0}) = \int_{t_0}^{t_f} f_0(\phi_{x_0}(\tau), U(\phi_{x_0}(\tau))) \, d\tau \tag{7}$$

is to be minimized where $U \in T(x_0)$, $\phi_{x_0} \in T(x_0, U)$, and t_f is the terminating time of ϕ_{x_0}. Here, the function f_0 is a real-valued, bounded, Borel-measurable function with domain $E^n \times E^m$.

Consider the following two definitions of optimality.

Optimality I. An admissible closed-loop control policy $U^* \in T(X)$ is optimal iff for all $x_0 \in X$, for all $U \in T(x_0)$, and for all $\phi_{x_0} \in T(x_0, U)$ the inequality

$$J(x_0, U^*, \phi^*_{x_0}) \leqslant J(x_0, U, \phi_{x_0}) \qquad \forall \phi^*_{x_0} \in T(x_0, U^*) \tag{8}$$

is met.

Optimality II. A policy $U^* \in T(X)$ is optimal iff for all $U \in T(X)$ the inequality

$$J(x_0, U^*, \phi^*_{x_0}) \leqslant J(x_0, U, \phi_{x_0})$$

$$\forall x_0 \in X, \qquad \forall \phi_{x_0} \in T(x_0, U), \qquad \forall \phi^*_{x_0} \in T(x_0, U^*) \tag{9}$$

is met.

The main difference between these two types of optimality is that the optimality of Type I compares U^* with a larger set of policies than that of Type II: In Type I, U^* is compared with all $U \in T(x_0)$ for all $x_0 \in X$; and in Type II, U^* is compared only with $U \in T(X)$. Thus, with respect to the state space X, Type II is a comparison made globally between candidates for optimality and admissible policies, whereas Type I is a pointwise comparison.

EQUIVALENCE OF THE TWO OPTIMALITY CRITERIA

In defining optimal closed-loop control it suffices to compare the optimal policy (candidate) with only those policies having a terminating admissible trajectory from all states, provided any one of Conditions 1, 2, or 3 is satisfied. We state this as a theorem.

Theorem. A closed-loop control policy U* *contained in* T(X) *is optimal of Type II if and only if it is optimal of Type I.*

Proof. For the "if" part of the theorem consider a policy $U^* \in T(X)$ that is optimal of Type I. Because U^* satisfies inequality (8) for all $x_0 \in X$, for all $U \in T(x_0)$, and for all $\phi_{x_0} \in T(x_0, U)$, and because

$$T(X) = \bigcap_{x_0 \in X} T(x_0),$$

it follows that inequality (9) is met for all $U \in T(X)$. Consequently, U^* *is* optimal of Type II.

Now, for the "only if" part, suppose $U^* \in T(X)$ is optimal of Type II. Further suppose that U^* is not optimal of Type I. This implies that there exist $x_0 \in X$, $U \in T(x_0)$, and $\phi_{x_0} \in T(x_0, U)$ such that

$$J(x_0, U^*, \phi^*_{x_0}) > J(x_0, U, \phi_{x_0}) \tag{10}$$

for some $\phi^*_{x_0} \in T(x_0, U^*)$.

Let $C = \{\phi_{x_0}(t) : t \in [t_0, t_f]\}$. Define the function $U_3 : X \to E^m$ by

$$U_3(x) = \begin{cases} U(x) & \forall x \in C \\ \\ U^*(x) & \forall x \in X - C. \end{cases} \tag{11}$$

Because Condition 1, 2, or 3 must hold, it follows that Condition 3 holds. Thus, U_3 is an admissible closed-loop control policy.

It is clear from Eq. (11) that $U_3 \in T(x_0)$. We claim that U_3 also belongs to $T(X)$. To see this, suppose U_3 is not contained in $T(X)$. Then there is some $x \in X$ for which U_3 does not provide a corresponding terminating admissible trajectory emanating from x. This cannot happen for $x = x_0$ because $U \in T(x_0)$ and $U_3(x) = U(x)$ for all $x \in C$. So, suppose $x \neq x_0$.

Because $U^* \in T(X)$, it follows that U^* has a terminating admissible trajectory, say $\phi^*_x \in T(x, U^*)$, emanating from x. Suppose that ϕ^*_x is defined on the time interval $[t, t^*_f]$ where t corresponds to the state x. Let t_a be the smallest time contained in $[t, t^*_f]$ such that $\phi^*_x(t_a) \in C$. If no such time t_a exists, then ϕ^*_x belongs to $T(x, U_3)$ because $U_3(x) = U^*(x)$ for all $x \in X - C$. The nonexistence of t_a contradicts the supposition that U_3 does not provide a terminating admissible trajectory emanating from x. Hence, t_a must exist; we let $\widetilde{x} = \phi^*_x(t_a)$.

Now, let t_b be the time such that $\phi_{x_0}(t_b) = \widetilde{x}$. Note that $t_b = t_a$ because the nth component of $\widetilde{x}$ is time. Define the absolutely continuous function ϕ_x by

$$\phi_x(\tau) = \begin{cases} \phi^*_x(\tau) & \text{for all } t \leqslant \tau \leqslant t_a \\ \\ \phi_{x_0}(\tau) & \text{for all } t_a < \tau \leqslant t_f. \end{cases} \tag{12}$$

Observe that ϕ_x belongs to $T(x, U_3)$ because ϕ_x is a terminating admissible trajectory, it corresponds to U_3, and it satisfies Eq. (4). This is again a contradiction to the supposition that U_3 does not provide a terminating admissible trajectory emanating from x. This implies that U_3 belongs to $T(X)$.

Note that

$$J(x_0, U_3, \phi_{x_0}) = J(x_0, U, \phi_{x_0}) \tag{13}$$

because $U_3(x) = U(x)$ for all $x \in C$.

Finally, because U_3 belongs to $T(X)$, it follows from inequalities (9) and (10) together with Eq. (13) that U^* is not optimal of Type II as asserted. This contradiction implies that U^* is indeed optimal of Type I.

REMARKS

Remark 1. Condition (i) following Eq. (3) is a statement of the effect that admissible closed-loop control policies satisfy the control constraints of the process. Condition (ii) deals with the existence of a solution of the state equation for some initial state. Thus, policies are not considered admissible unless they satisfy control constraints and produce at least one solution. Requiring condition (i) to be necessary for admissibility agrees with the usual definitions of admissible controls; e.g., Pontryagin et al. [1] and Leitmann [5]. In the theory of ordinary differential equations there is no existence theorem that applies to the entire family of optimal control processes considered earlier. Thus, it was necessary to focus our attention on those control policies admitting solutions.

Remark 2. A closed-loop control policy is assumed to exist such that all initial states of the state space are transferable to the terminal set; i.e., the set of policies $T(X)$ is nonempty.

Remark 3. The optimalities of Types I and II imply that for a fixed initial state, all terminating admissible trajectories of an optimal control policy yield equal values of the performance index. This follows from inequalities (8) and (9) by inserting the optimal policy on both sides of the inequality sign. It is possible to redefine optimality for the case in which such trajectories are not required to render equal costs. This can be accomplished by considering optimal pairs (U^*, Φ^*) where Φ^* is a set of such trajectories, the set containing one and only one member for each initial state. That is, for each $x_0 \in X$, Φ^* contains one and only one member of $T(x_0, U^*)$. In general, in more formal terms, for each $U \in T(X)$, we define

$$T(U) = \{\Phi : \text{for each } x_0 \in X, \ \Phi \text{ contains one and only one member of } T(x_0, U)\}.$$

This results in the following definitions.

Optimality I′. An admissible closed-loop control policy pair (U^*, Φ^*) with $U^* \in T(X)$ and $\Phi^* \in T(U^*)$ is optimal iff for all $x_0 \in X$, for all $U \in T(x_0)$, and for all $\phi_{x_0} \in T(x_0, U)$, the inequality

$$J(x_0, U^*, \phi^*_{x_0}) \leqslant J(x_0, U, \phi_{x_0}) \tag{14}$$

is met.

Optimality II'. A policy pair (U^*, Φ^*) with $U^* \in T(X)$ and $\Phi^* \in T(U^*)$ is optimal iff for all $U \in T(X)$, the inequality

$$J(x_0, U^*, \phi^*_{x_0}) \leqslant J(x_0, U, \phi_{x_0})$$

$$\forall x_0 \in X, \qquad \forall \Phi \in T(U), \;\; \phi_{x_0} \in \Phi \tag{15}$$

is met.

The trajectory $\phi^*_{x_0}$ in inequalities (14) and (15) is the unique member of Φ^* that corresponds to the initial state x_0. The trajectory ϕ_{x_0} in inequality (15) is the unique member of Φ associated with x_0.

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