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Electroacoustic Modeling of Magnetostrictive Shells and Rings:

Part 1 - Mathematical Modeling

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The basic canonical equations of a matrix of n linear coupled electroacoustic transducers are stated in matrix form; from them are derived by matrix inversion the electroacoustic parameters of performance (n-port voltages, currents, velocities, forces) with associated electrical and mechanical impedances. The problem of the mechanical vibration of a thin magnetostrictive shell is formulated in abstract operator form, then specialized to the case of a finite-length, axisymmetric, free-flooded magnetostrictive shell. Acoustic loading is accounted for by converting the differential equations of motion into a set of linear integral

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equations in unknown surface velocities, which are solved by inversion of a set of equivalent algebraic equations. The concepts of electromechanical modal mass and modal coupling are introduced in three alternative formulations. Details on the method of calculating the acoustic loading of a finite-length, axisymmetric, free-flooded magnetostrictive shell are then presented. In an appendix is a digest of the mathematical model of a free-flooded magnetostrictive shell of very short axial length, that is, a ring.

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ELECTROACOUSTIC MODELING OF MAGNETOSTRICTIVE SHELLS AND RINGS: PART 1, MATHEMATICAL MODELING

1. INTRODUCTION

Long-range active acoustic surveillance of the ocean for submerged hostile submarines requires a high-power, low-frequency source of acoustic energy. An important candidate for meeting such a requirement is the free-flooded cylindrical magnetostrictive shell of finite axial length. To judge proposed designs of these shells for antisubmarine warfare, designers need a reliable mathematical model. The mathematical modeling of such an electroacoustic structure is complicated by the multimode mechanical vibration of the elastic shell and by the difficulty of calculating the acoustic loading due to the finite length of shell and the free-flooded condition. Since elastic, magnetic, electric, and acoustic fields are all coupled to one another, an effort to assemble them in one complete whole and to convert the resultant model into a computer program for high-speed analysis of proposed designs poses a difficult problem. The approach adapted here is to solve the elastic problem by thin-shell theory, the acoustic problem by a Helmholtz integral formulation, and the combined elastic-electroacoustic problem by a set of linear integral equations. The result of this approach is to be published in three parts. The theoretical formulation is contained in this report as Part 1.

2. OPERATOR FORM OF THE CANONICAL EQUATIONS OF COUPLED MOTION AND OF FORCED HARMONIC DRIVE

Historically, the continuum analysis of shell structures and the electric field—velocity field coupling of piezoactive structures have each had separate theoretical developments. In the last decade, several articles [1-4] have treated both fields together. When they are so coupled, theoretical analysis is complicated by the addition of coupling factors connecting the elastic field (considered as a continuum) and the electromagnetic field. Conventional procedures of electroacoustic analysis [5] must be modified to handle the increased number of electrical and mechanical ports introduced by the coupling of multicomponent vector displacements with multiloop electric circuits and multipole acoustic radiation. To organize the complexities of these composite systems, we restate in this section the basic canonical equations in abstract matrix-operator notation.

Canonical Equations in Matrix-operator Form

Consider an electroacoustic transducer in the form of a multiport network ($\mathcal{N}$ electrical and $\mathcal{M}$ mechanical ports). The canonical matrix equations of the coupled currents I_i and linear velocity v_j fields in terms of applied voltages E_i and applied stresses p_j are

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$$E_i = \sum_{j=1}^{\mathfrak{N}} \mathcal{Z}_{ij}^{(b)} \{I_j\} + \sum_{j=1}^{\mathfrak{M}} \Psi'_{ij}\{v_j\}, \qquad i=1, \ldots, \mathfrak{N} \tag{2.1}$$

$$p_j = \sum_{i=1}^{\mathfrak{N}} \Psi_{ji}\{I_i\} + \sum_{i=1}^{\mathfrak{M}} \mathcal{Z}_{ji}^{(m)}\{v_i\}, \qquad j=1, \ldots, \mathfrak{M} \tag{2.2}$$

where

$\{\ \}$ = symbol of operand

$\mathcal{Z}_{ij}^{(b)}$ = matrix of blocked (electrical) impedance operators ($\mathfrak{M} \times \mathfrak{N}$)

Ψ'_{ij}, Ψ_{ji} = matrix of transduction operators ($\mathfrak{M} \times \mathfrak{N}$)

$\mathcal{Z}_{ji}^{(m)}$ = matrix of specific acoustic impedance operators ($\mathfrak{M} \times \mathfrak{M}$).

All matrix operators are assumed to contain differential and integral components. Next assume that applied stress p_j is generated by acoustic processes, so that

$$p_j = p_j^{(g)} - \sum_{i=1}^{\mathfrak{M}} \mathcal{Z}_{ji}^{(R)}\{v_i\}, \qquad j=1, \ldots, \mathfrak{M} \tag{2.3}$$

where

$$p_j^{(g)} = p_j^{(0)} - p_j^{(D)} \tag{2.4}$$

and

$\mathcal{Z}_{ji}^{(R)}$ = matrix of radiation impedance operators ($\mathfrak{M} \times \mathfrak{M}$)

$p_j^{(0)}$ = local acoustic pressure due to acoustic field sources in the absence of the transducer

$p_j^{(D)}$ = diffraction field due to interaction of $p_j^{(0)}$ with the transducer considered rigid.

Substituting p_j of Eq. (2.3) into Eq. (2.2) and collecting terms in v_i, we again condense notation and write the mechanical equation of motion.

$$\sum_{i=1}^{\mathfrak{M}} \hat{Z}_{ji}\{v_i\} = p_j^{(0)} - p_j^{(D)} - \sum_{i=1}^{\mathfrak{N}} \Psi_{ji}\{I_i\}, \qquad j=1, \ldots, \mathfrak{M} \tag{2.5}$$

Where

$$\hat{Z}_{ji} = \mathcal{Z}_{ji}^{(m)} + \mathcal{Z}_{ji}^{(R)}, \quad (\mathfrak{M} \times \mathfrak{M}). \tag{2.6}$$

Here $\mathcal{Z}_{ji}^{(m)}$ includes internal mechanical damping operators. Solving Eq. (2.5) for v_i and Eq. (2.1) for I_i, we find

$$v_i = \sum_{j=1}^{\mathfrak{M}} (\hat{Z}_{ij})^{-1} \left[p_j^{(0)} - p_j^{(D)} - \sum_{k=1}^{\mathfrak{N}} \Psi_{jk} \{I_k\} \right], \qquad i=1, \ldots, \mathfrak{M} \tag{2.7}$$

$$I_p = \sum_{q=1}^{\mathfrak{N}} \left(\mathcal{Z}_{pq}^{(b)} \right)^{-1} \left[E_q - \sum_{s=1}^{\mathfrak{M}} \Psi'_{qs} \{v_s\} \right], \qquad p=1, \ldots, \mathfrak{N}. \tag{2.8}$$

An alternate form of the velocity equation is obtained by substituting Eq. (2.8) into Eq. (2.7):

$$v_s = \sum_{i=1}^{\mathfrak{M}} \sum_{j=1}^{\mathfrak{M}} (H_{si})^{-1} (\hat{Z}_{ij})^{-1} \left\{ p_j^{(0)} - p_j^{(D)} - \sum_{k=1}^{\mathfrak{N}} \sum_{q=1}^{\mathfrak{N}} \Psi_{jk} \left(\mathcal{Z}_{kq}^{(b)} \right)^{-1} \{E_q\} \right\}, \qquad s=1, \ldots, \mathfrak{M} \tag{2.9}$$

where

$$H_{is} = \delta_{is} - \sum_{j=1}^{\mathfrak{M}} \sum_{k=1}^{\mathfrak{N}} \sum_{q=1}^{\mathfrak{N}} (\hat{Z}_{ij})^{-1} \left[\Psi_{jk} \left(\mathcal{Z}_{kq}^{(b)} \right)^{-1} \Psi'_{qs} \right], \qquad i,s=1, \ldots, \mathfrak{M} \tag{2.10}$$

$(H_{si})^{-1} (\hat{Z}_{ij})^{-1}$ = velocity s due to pressure j.

Note the following significant groupings of symbols:

$$\Psi_{ji} \left(\mathcal{Z}_{iq}^{(b)} \right)^{-1} \Psi'_{qs} = \text{pressure } j \text{ per current } i \text{ per voltage } q \text{ per velocity } s \tag{2.11}$$

$$(H_{si})^{-1} (\hat{Z}_{ij})^{-1} = \text{velocity } s \text{ per velocity } i \text{ per pressure } j \tag{2.12}$$

$$\Psi_{ji} \left(\mathcal{Z}_{iq}^{(b)} \right)^{-1} = \text{pressure } j \text{ per current } i \text{ per voltage } q. \tag{2.13}$$

From Eqs. (2.5) and (2.9), we identify the following significant quantities:

$$\hat{Z}_{ij} = \text{open-circuit specific acoustic impedance} \tag{2.14}$$

$$\sum_{i=1}^{\mathfrak{N}} (H_{si})^{-1} (\hat{Z}_{ij})^{-1} = \text{short-circuit specific acoustic admittance.} \tag{2.15}$$

The acoustic pressure at the surface is seen by Eq. (2.3) to be

$$p_j = -\sum_{i=1}^{\mathfrak{M}} \mathcal{Z}_{ji}^{(R)}\{v_i\}, \qquad i=1,\ldots,\mathfrak{M}. \tag{2.16}$$

In the steady state, radiated power W will be determined from normal surface velocity v_j and acoustic pressure p_j:

$$W = \mathrm{Re}\sum_{j}^{\mathfrak{M}} \oint v_j^* p_j dS_j = \mathrm{Re}\sum_{i,j=1}^{\mathfrak{M}} \oint v_j^* \mathcal{Z}_{ji}^{(R)}\{v_i\} dS_j \tag{2.17}$$

in which rms quantities are implied. The total electrical impedance when $p_j^{(0)} - p_j^{(D)}$ vanishes is

$$\mathcal{Z}_{ij}^{(e)} = \mathcal{Z}_{ij}^{(b)} - \sum_{l,q=1}^{\mathfrak{M}} \Psi'_{il}(\hat{Z}_{lq})^{-1}\Psi_{qj}. \tag{2.18}$$

The meaning of the significant grouping on the r.h.s. is

$$\sum_{l,q=1}^{\mathfrak{M}} \Psi'_{il}(\hat{Z}_{lq})^{-1}\Psi_{qj} = \text{motional impedance}$$

$$= \text{voltage } i \text{ per velocity } l \text{ per pressure } q \text{ per current } j. \tag{2.19}$$

To obtain a receiving response, let $p_j^{(D)}$ and I_i vanish to give

$$v_i = \sum_{j=1}^{\mathfrak{M}} (\hat{Z}_{ij})^{-1} p_j^{(0)}, \qquad i=1,\ldots,\mathfrak{M}. \tag{2.20}$$

Substituting this into Eq. (2.1) and assuming that the receiver transducer is electrically terminated in an impedance $\mathcal{Z}_{ij}^{(te)}$,

$$E_i = -\sum_{i=1}^{\mathfrak{N}} \mathcal{Z}_{iq}^{(te)}\{I_q^{(te)}\}, \qquad i=1,\ldots,\mathfrak{N} \tag{2.21}$$

or

$$I_q^{(te)} = -\sum_{i=1}^{\mathfrak{N}} \left(\mathcal{Z}_{qi}^{(te)}\right)^{-1}\{E_i\}, \qquad q=1,\ldots,\mathfrak{N}. \tag{2.22}$$

Hence, in Eq. (2.1),

$$E_i = \sum_{j,p=1}^{\mathfrak{N}} \mathfrak{Z}_{ij}^{(b)} \left(-\mathfrak{Z}_{jp}^{(te)} \right)^{-1} \{E_p\} + \sum_{j,q=1}^{\mathfrak{M}} \Psi'_{ij} (\hat{Z}_{jq})^{-1} \left\{ p_q^{(0)} \right\}$$

or

$$E_p = \sum_{i=1}^{\mathfrak{N}} \left[\delta_{ip} + \sum_{j=1}^{\mathfrak{N}} \mathfrak{Z}_{ij}^{(b)} \left(\mathfrak{Z}_{jp}^{(te)} \right)^{-1} \right]^{-1} \sum_{j,q=1}^{\mathfrak{M}} \Psi'_{ij} (\hat{Z}_{jq})^{-1} \left\{ p_q^{(0)} \right\} . \tag{2.23}$$

When the terminal impedance is infinite, the open-circuit voltage is

$$(E_p)_{\text{o.c.}} = \sum_{j,q=1}^{\mathfrak{M}} \Psi'_{pj} (\hat{Z}_{jq})^{-1} \left\{ p_q^{(0)} \right\} . \tag{2.24}$$

The electrical admittance Y_{ij} upon transmission is obtained from Eqs. (2.1) (2.7) by letting $p_j^{(0)}$ and $p_j^{(D)}$ both vanish. Thus,

$$E_i = \mathfrak{Z}_{ij}^{(b)} \{I_j\} + \sum_{j=1}^{\mathfrak{M}} \sum_{q=1}^{\mathfrak{M}} \sum_{s=1}^{\mathfrak{N}} \Psi'_{ij} (\hat{Z}_{jq})^{-1} \{ -\Psi_{qs} \{I_s\} \}$$

or

$$E_i = \sum_{j=1}^{\mathfrak{N}} \left(\mathfrak{Z}_{ij}^{(b)} - \sum_{k=1}^{\mathfrak{M}} \sum_{q=1}^{\mathfrak{M}} \Psi'_{ik} (\hat{Z}_{kq})^{-1} \Psi_{qj} \right) \{I_j\}$$

or

$$E_i = \sum_{j}^{N} L_{ij} \{I_j\}$$

so that

$$Y_{ij} = (L_{ij})^{-1} = \left[\mathfrak{Z}_{ij}^{(b)} - \sum_{q,l=1}^{M} \Psi'_{il} (\hat{Z}_{lq})^{-1} \Psi_{qj} \right]^{-1} . \tag{2.25}$$

Equations (2.1) to (2.25) complete the statement in abstract matrix notation of the canonical equations and electromechanical behavior of a multiported collection of electroacoustic transducers loaded by acoustic forces. Formulation of these matrix relations in the case of magnetostriction transduction is briefly discussed in Appendix C.

3. GENERAL FORMULATION OF THE PROBLEM OF A VIBRATING MAGNETOSTRICTIVE SHELL

The formulas derived in Sec. 2 provide a basis for applications to specific shell structures and specific piezoactivity. We now choose the geometric configuration of the shell to be of such uniformity and regularity that its surface can be described by a smooth function of two coordinates and its piezoactivity can be considered linear in the mechanical strains and the applied electromagnetic fields.

Equations of Forced Harmonic Drive of Shells in Operator Form

Let the time dependence of forced drive of shells be $\exp(j\omega t)$. The equations of motion in operator form can then be written

$$\sum_j \mathfrak{L}_{ij} U_j(\mathbf{x}, \omega) = P_i(\mathbf{x}, \omega), \qquad i = 1, 2, \ldots, \mathfrak{M} \tag{3.1}$$

where

i = index number of mechanical ports

$\mathfrak{L}_{ij}$ = dyadic elastodynamic operator which converts displacement into force per unit area (dimensions, N/m^3)

U_j = shell displacement vector. In a single cylindrical shell U_j has components u_1 (axial), u_2 (tangential), w (radial) (dimensions, m)

P_i = surface force per unit area (dimensions, N/m^2) in the ith mechanical port.

Operator $\mathfrak{L}_{ij}$ can be written as a sum of stiffness $\mathfrak{K}_{ij}$ and inertial $\mathfrak{M}_{ij}$ operators,

$$\mathfrak{L}_{ij} = \mathfrak{K}_{ij} + \mathfrak{M}_{ij} \tag{3.2}$$

where $\mathfrak{M}_{ij}$ is generally defined to cover the case of composite motion in all degrees of freedom, including rigid body motion. Internal mechanical losses will be discussed later in this analysis.

First assume that Eq. (3.1) can be solved by an infinite series of normal modes $Y_i(q;\mathbf{x})$ of the unloaded shell:

$$U_j(x_1, x_2, \omega) = \sum_{q=1}^{\infty} A(q;\omega) Y_j(q; x_1, x_2), \qquad j=1, \ldots, \mathfrak{M}. \tag{3.3}$$

Since the normal modes are orthogonal with weight ζ over the reference area of the shell, it is obvious that

$$A(q;\omega) = \frac{\sum_j \left(U_j(x_1, x_2; \omega), \zeta(x_1; x_2)\, Y_j(q; x_1, x_2)\right)}{\sum_j \left(Y_j(q; x_1, x_2), \zeta(x_1; x_2)\, Y_j(q; x_1, x_2)\right)} \tag{3.4}$$

in which (,) denotes inner product integration over x_1, x_2, i.e., over shell area. In the numerator of the r.h.s. of Eq. (3.4) U_j is unknown. To find it we multiply (take the inner product of) Eq. (3.1) by Y_i and integrate over the area of the shell. The result is

$$\left(\sum_j \mathfrak{L}_{ij} U_j(x_1, x_2; \omega),\ Y_i(q; x_1, x_2)\right) = \left(P_i(x_1, x_2; \omega),\ Y_i(q; x_1, x_2)\right). \tag{3.5}$$

To solve this equation we assume that component operator $\mathfrak{K}_{ij}$ in $\mathfrak{L}_{ij}$ is self-adjoint and has the eigenvalues $-\Lambda^2(q)$ (dimensions, $\sec^{-2}$); that is, we assume $\mathfrak{K}_{ij}$ to be such that

$$\sum_j \mathfrak{K}_{ij} Y_j(q; x_1, x_2) = -\Lambda^2(q;\omega)\zeta(x_1, x_2) Y_i(q; x_1, x_2) \tag{3.6}$$

and

$$\left(\sum_j \mathfrak{K}_{ij} U_j,\ Y_i\right) = \left(U_j,\ \sum_i \mathfrak{K}_{ji} Y_i\right) = -(U_j,\ \Lambda^2 \zeta Y_j). \tag{3.7}$$

In addition, in the first approximation we assume

$$\mathfrak{M}_{ij} = m_E \delta_{ij} \omega^2 \qquad (\text{dimensions of } m_E,\quad \text{N-sec}^2/\text{m}^3)\ ;$$

that is, we neglect rotary inertia and rigid body motion and take the shell to be homogeneous. Using Eqs. (3.6) and (3.7) in Eq. (3.5) leads to the statement that

$$(U_i, m_E \omega^2 Y_i) - (U_i, \zeta \Lambda^2 Y_i) = (P_i, Y_i). \tag{3.8}$$

Restricting attention to a thin shell, we know (Krauss [6] page 365) that the weighting function $\zeta = m_E = \rho b$, where b = shell thickness. Thus,

$$(U_i(x_1, x_2; \omega), Y_i(q; x_1, x_2)) = -\frac{(P_i(x_1, x_2; \omega), Y_i(q; x_1, x_2))}{m_E[\Lambda^2(q) - \omega^2]}. \tag{3.9}$$

Equation (3.4) then becomes,

$$A(q;\omega) = -\sum_j \frac{(P_j(x_1, x_2; \omega), Y_j(q; x_1, x_2))}{m_E[\Lambda^2(q) - \omega^2]N(q)} \tag{3.10}$$

where

$$N(q) = \sum_j (Y_j(q; x_1, x_2), Y_j(q; x_1, x_2)). \tag{3.11}$$

To include localized forcing functions (P_j), the integration required in the numerator of Eq. (3.10) can be restricted to a particular area of the shell, say

$$dS = dS(\xi_{\Delta 1}, \xi_{\Delta 2}).$$

Define an average (forcing) force per unit area $\overline{p}$ and a forcing integral Y° for that particular area, by the relation

$$\overline{p}Y^\circ(q; \xi_{\Delta_1}, \xi_{\Delta_2}; \omega) = \sum_j \left(P_j(\xi_{\Delta_1}, \xi_{\Delta_2}; \omega), Y_j(q; \xi_{\Delta_1}, \xi_{\Delta_2})\right). \tag{3.12}$$

Thus, Eq. (3.3) becomes

$$U_k(x_1, x_2 | \xi_{\Delta_1}, \xi_{\Delta_2}; \omega) = -\sum_{q=1}^{\infty} \frac{\overline{p}(\omega; \xi_{\Delta_1}\xi_{\Delta_2})Y^\circ(q; \xi_{\Delta_1}, \xi_{\Delta_2})Y_k(q; x_1, x_2)}{m_E[\Lambda^2(q) - \omega^2]N(q)}. \tag{3.13}$$

This is the k component of displacement at $\mathbf{x}$ due to a forcing function at ξ ($\xi_{\Delta_1}, \xi_{\Delta_2}$). The symbol $\overline{p}$ can be understood to be a surface force of either nonacoustic or acoustic origin. In the latter case, we can define an acoustic surface radiation operator $\mathcal{G}$ such that

$$\overline{p}_{(k)}(\omega; \xi_{\Delta_1}, \xi_{\Delta_2}) = \mathcal{G}(\xi_{\Delta_1}, \xi_{\Delta_2} | \xi'_{\Delta_1}, \xi'_{\Delta_2}; \omega) j\omega U_k(\xi'_{\Delta_1}, \xi'_{\Delta_2}) \tag{3.14}$$

where U_k is a normal displacement. Substitution of Eq. (3.14) into Eq. (3.13) thus provides a solution to Eq. (3.1) in the case of acoustic loading.

An alternative procedure for handling acoustic excitation is to return to Eq. (3.10) and expand the acoustic forcing function ($P_{i(A)}$) in the modes of the free-free shell. This procedure is an approximation in which the normal force loading on the ends of the shell is considered negligible compared to the loading on the sides of the shell (thus satisfying the free-free condition). Thus

$$P_{i(A)}(\xi_{\Delta_1}, \xi_{\Delta_2}; \omega) = \sum_{s=1}^{\infty} j\omega \mathcal{G}(\xi_{\Delta_1}, \xi_{\Delta_2} | \xi'_{\Delta_1}, \xi'_{\Delta_2}; \omega) Y_i(s; \xi'_{\Delta_1}, \xi'_{\Delta_2}) A(s; \omega) \tag{3.15}$$

in which A is the modal expansion constant. Equation (3.10) then takes on the following appearance (for the acoustic case):

$$A(q;\omega) = \frac{-\left(j\omega \mathcal{G}(\xi_{\Delta_1}, \xi_{\Delta_2} | \xi'_{\Delta_1}, \xi'_{\Delta_2}; \omega) \sum_{s=1}^{\infty} Y_i(s; \xi'_{\Delta_1}, \xi'_{\Delta_2}) A(s;\omega),\ Y_i(q; \xi_{\Delta_1}, \xi_{\Delta_2})\right)}{m_E[\Lambda^2(q) - \omega^2]N(q)} . \tag{3.16}$$

(For simplicity we have taken only one direction (j) to be the important normal direction.) This equation shows that the expansion coefficient for each q mode (namely $A(q;\omega)$) is to be determined from a knowledge of the expansion coefficients of all the (infinite number of) s modes (namely $A(s;\omega)$). A solution of Eq. (3.16) therefore requires a solution of a set (theoretically infinite, but practically finite) of simultaneous equations in expansion coefficients A, because the numerator eigenfunction products are not orthogonal.

A third procedure for including acoustic excitation is to assume that the pressure distribution on the shell exactly matches the shell velocity distribution and to define from both of them a set of modal radiation impedances. Since this matching can be true only for a very limited number of geometrical configurations of the shell (at most, only for surfaces separable in the Helmholtz operator) this method is very restricted in application (see Appendix B for details).

We next consider the case where the vibration of the shell structure is damped internally. To account for modal damping we allow the operator $\mathfrak{L}_{ij}$ to have an additive term $\mathfrak{D}_{ij}(\omega)$, and define a three symbol damping quantity $R_{im}(q; s; \omega)$ by the relation

$$\sum_j \mathfrak{D}_{ij}(\omega)[Y_j(q; x_1, x_2)] = j \sum_{m=1}^{3} \sum_{s=1}^{\infty} R_{im}(q; s; \omega) Y_m(s; x_1, x_2) . \tag{3.17}$$

This equation expresses the hypothesis that all the s modes $Y_m(s)$ are coupled by R_{im} to contribute to the q mode, and that in any one mode (viz., when $q = s$) the damping entity R_{im} couples all orthogonal displacements (i.e., couples Y_1, Y_2, and Y_3). It is conventional, however, to consider each mode to be damped only by a modally defined constant $\eta(q)$ and to omit any effect of intermodal coupling due to damping. This convention is formulated in the following way:

$$\sum_j \mathfrak{D}_{ij}(\omega)[Y_j(q; x_1, x_2)] = j\eta(q;\omega) Y_i(q; x_1, x_2) . \tag{3.18}$$

The eigenvalues of the stiffness operator (i.e., Λ^2) are thus converted by damping from real numbers to complex numbers. In place of Λ^2 we then write Λ_D^2 (damped eigenvalues), in which

$$\Lambda_D^2 = \Lambda^2(q;\omega)[1 + j\eta(q;\omega)] . \tag{3.19}$$

In the theory of the vibration of elastic structures a complex eigenvalue may be interpreted as a complex stiffness (a complex Young's modulus).

The elastic structures used in efficiently designed electroacoustic devices are always selected to have small internal damping. We assume small internal damping in all subsequent analysis. As a corollary to this assumption we shall consider that all mode shapes with damping do not differ in the first approximation from mode shapes without damping; i.e., a first-order change in the eigenvalue results in a second-order change (at most) of the eigenfunction.

In the next section the arbitrary number of ports in canonical Eq. (2.2) and mechanical Eqs. (3.1) to (3.19) are particularized to cover the shells treated here.

4. ELASTODYNAMIC EQUATIONS OF MOTION WITH FORCED ELECTRIC DRIVE

We first apply canonical Eq. (2.2) to the case of a vibrating shell having one electrical port and three mechanical ports. Let (u_1, u_2, w) be the displacement vector of a thin shell with components u_i, where $u_1 =$ axial, u_2 = tangential, and w = radial. Let π_{ij} be an elasticity differential operator defined by the elastodynamic equations of motion:

$$\sum_{j=1}^{3} \pi_{ij} u_j = \rho b \ddot{u}_i + P_i, \qquad i=1, 2, 3 \tag{4.1}$$

where

ρ = mass density of the shell

b = thickness of the shell

P_i = total external force per unit area acting on shell.

Explicit forms of π_{ij} are found in Flügge [7]. Defining k_{ij}, λ_{ij} to be "foundation" stiffness and damping dyadics respectively, we write the total forcing function in the theoretical form

$$P_i = p_i + \sum_{j=1}^{3} k_{ij} u_j + \sum_{j=1}^{3} \lambda_{ij} \dot{u}_j \tag{4.2}$$

where p_i is an individual external applied force per unit area (plus equivalent "electromechanical stresses") other than foundation stiffness and damping. Note that in general the latter forces couple all displacement components. The components of p_i are written with negative signs, in view of the definition of P_i:

$$p_i = (-p_1, -p_2, -p) \qquad \text{(dimensions, N/m}^2\text{)}. \tag{4.3}$$

Here the radial force p is positive outward (along the normal pointing away from the center of curvature of the shell).

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To solve the forced-vibration problem of a thin shell we adopt cylindrical coordinates r, θ, x and, suppressing r because the shell is thin, represent $u_i(\theta, x)$ by a spectral decomposition,

$$u_i = \begin{Bmatrix} u_1 \\ u_2 \\ w \end{Bmatrix} = \sum_{n=1}^{\infty} \begin{Bmatrix} U_{1n}(\theta, x) \\ U_{2n}(\theta, x) \\ W_n(\theta, x) \end{Bmatrix} q_n(t) = \sum_{n=1}^{\infty} U_{in}(\theta, x) q_n(t), \tag{4.4}$$

where U_{in} is the (dimensionless) displacement vector in the nth mode of free vibration, with natural frequency ω_n determined by the eigenvalue solutions for given boundary conditions, and q_n is an amplitude of displacement. It is assumed that these modes satisfy an orthogonality principle. By definition of free modes with no dissipation,

$$\sum_{j=1}^{3} \pi_{ij} U_{jn} = -\rho b \omega_n^2 U_{in}, \qquad i = 1, 2, 3. \tag{4.5}$$

Since it is important here to include structural damping, replace π_{ij} by $\overline{\pi}_{ij}$, which is defined to contain damping terms, and define a perturbed eigenvalue $\overline{\omega}_n$ to be the solution of the equation

$$\sum_{k=1}^{3} \overline{\pi}_{ik} U_{kn} = -\overline{\omega}_n^2 \left(\rho b U_{in} + j \rho b \sum_{m=1}^{3} \mathfrak{D}_{nm} U_{im} \right), \qquad i = 1,2,3\,, \tag{4.6}$$

in which $\mathfrak{D}_{nm}$ is the structural damping dyadic and $j = \sqrt{-1}$. We assume that this damping couples all modes, particularly in the higher orders. For the lower orders (which are of chief interest) we assume an internal friction of Solid Type II [8] and write the damping dyadic as an identity dyadic multiplied by a frequency dependent damping factor η_n, viz.,

$$\mathfrak{D}_{nm} = \delta_{mn} \eta_n . \tag{4.7}$$

From this we have

$$\overline{\omega}_n^2 = \omega_n^2 (1 + j\eta_n), \qquad |\eta_n| << 1 \tag{4.8}$$

$$\eta_n = \frac{\omega}{\omega_n} \left(\frac{R_n \omega_n}{K_n} \right) \tag{4.9}$$

in which (for purposes of defining the damping only) the modes are considered to be single-degree-of-freedom systems having mechanical resistance R_n, and stiffness K_n.

Substituting the proposed solution of Eq. (4.4) into Eq. (4.1) we arrive at

$$\sum_{n=1}^{\infty}\left\{\rho b U_{in}\ddot{q}_n(t) + \sum_{k=1}^{3}\lambda_{ik}U_{kn}\dot{q}_n(t) + \left[\rho b\bar{\omega}_n^{-2}U_{in} + \sum_{l=1}^{3}k_{il}U_{ln}q_n(t)\right]\right\} + p_i = 0, \quad i = 1, 2, 3. \tag{4.10}$$

To isolate individual modes we must use the orthogonality conditions of the modes of free vibration of a thin elastic shell. This is given by [6]

$$\sum_{i=1}^{3}\int U_{in}U_{im}\,dS(\theta, x) = \delta_{mn}N_n \tag{4.11}$$

where

$$N_n = \int\left(|U_{1n}|^2 + |U_{2n}|^2 + |W_n|^2\right)dS(\theta, x). \tag{4.12}$$

Thus we form a dot product of the spectral Eq. (4.10) with $U_{i(m)}$ and integrate over the area and add the three equations to obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\rho b&\left\{\ddot{q}_m(t)\sum_{i=1}^{3}\int U_{im}U_{in}dS + \dot{q}_m(t)\sum_{k=1}^{3}\sum_{i=1}^{3}\int\lambda_{ik}U_{km}U_{in}dS\right.\\ &+ q_m(t)\sum_{l=1}^{3}\sum_{i=1}^{3}\int k_{il}U_{lm}U_{in}dS\\ &\left.+ q_m(t)\sum_{i=1}^{3}\rho b\bar{\omega}_n^2\int U_{im}U_{in}dS\right\}\\ &+\sum_{i=1}^{3}\int p_iU_{in}dS = 0.\end{aligned} \tag{4.13}$$

It is seen that the foundation stiffness and damping couple all modes. Examination of Eq. (4.13) shows that mode isolation is possible only if

$$k_{il} = k_{ii} = k, \qquad i = 1, 2, 3 \tag{4.14}$$

$$\lambda_{ik} = \lambda_{ii} = \lambda, \qquad i = 1, 2, 3. \tag{4.15}$$

While such conditions are extremely restrictive, their use where justified makes Eq. (4.10) tractable. Adopting these simplifications and applying the orthogonality principle, we arrive at the statement that

$$\sum_{i=1}^{3} \int p_i U_{in} dS + N_n[(\rho b \overline{\omega}_n^2 + k) q_n(t) + \lambda \dot{q}_n(t) + \rho b \ddot{q}_n(t)] = 0 . \qquad (4.16)$$

Let the time dependence of the forcing functions be harmonic, viz, $\exp(j\omega t)$. Solve for $q_n(\omega)$:

$$q_n(\omega) = \frac{-\sum_{i=1}^{3} \int p_i U_{in} dS}{\overline{D}_n(\omega)} \quad \text{(dimensions, m)}, \qquad (4.17)$$

$$\overline{D}_n(\omega) = N_n \rho b \left(\overline{\omega}_n^2 - \omega^2 + j\omega \frac{\lambda}{\rho b} + \frac{k}{\rho b} \right) \quad \text{(dimensions, N/m)}. \qquad (4.18)$$

Thus the vector displacements are given by the product of known modal shapes and known amplitudes, namely,

$$u_i(\theta, x, \omega) = \sum_{n=1}^{\infty} U_{in}(\theta, x) q_n(\omega). \qquad (4.19)$$

Point and Band Influence Functions

We now specialize the forcing function in two ways: point and band. In the case of a point forcing function we assume that the forcing function is delta-distributed in θ, x, so that

$$p_i = f_i \delta(\xi_1 - \theta) \delta(\xi_2 - x) \qquad (4.20)$$

where

$$f_i = (-f_1, -f_2, -f_w) \quad \text{(dimensions, N)}. \qquad (4.21)$$

Thus,

$$-\sum_{i=1}^{3} \int p_i U_{in} dS = -\sum_{i=1}^{3} \int f_i U_{in} \delta(\xi_1 - \theta) \delta(\xi_2 - x) dS(\theta, x)$$

$$= f_1 U_{1n}(\xi_1, \xi_2) + f_2 U_{2n}(\xi_1, \xi_2) + f_w W_n(\xi_1, \xi_2) . \qquad (4.22)$$

The most general formula, valid for excitation in all orthogonal directions, is therefore given by

$$u_i(\theta, x | \xi_1, \xi_2) = \sum_{n=1}^{\infty} \sum_{j=1}^{3} \frac{(\gamma_{ijn})(\theta, x | \xi_1, \xi_2)}{\overline{D}_n(\omega)} , \qquad i = 1, 2, 3 \qquad (4.23)$$

where

$$\gamma_{ijn} = \begin{bmatrix} f_1 U_{1n}(\alpha)U_{1n}(\xi) & f_2 U_{1n}(\alpha)U_{2n}(\xi) & f_w U_{1n}(\alpha)W_n(\xi) \\ f_1 U_{2n}(\alpha)U_{1n}(\xi) & f_2 U_{2n}(\alpha)U_{2n}(\xi) & f_w U_{2n}(\alpha)W_n(\xi) \\ f_1 W_n(\alpha)U_{1n}(\xi) & f_2 W_n(\alpha)U_{2n}(\xi) & f_w W_m(\alpha)W_n(\xi) \end{bmatrix}. \tag{4.24}$$

Here α, ξ are surface location vectors with components $\alpha = (\alpha_1, \alpha_2)$; and $\xi = (\xi_1, \xi_2)$.

When the point forcing function is in one direction only (when only f_1, f_2, or f_w are operative), the following influence functions can be obtained:

$$\frac{u_1}{f_1} = g_1(\alpha|\xi) = \sum_{n=1}^{\infty} \frac{U_{1n}(\alpha)U_{1n}(\xi)}{\overline{D}_n(\omega)} \qquad \text{(dimensions, m/N)}$$

$$\frac{u_2}{f_2} = g_2(\alpha|\xi) = \sum_{n=1}^{\infty} \frac{U_{2n}(\alpha)U_{2n}(\xi)}{\overline{D}_n(\omega)} \qquad \text{(dimensions, m/N)}$$

$$\frac{w}{f_w} = g_w(\alpha|\xi) = \sum_{n=1} \frac{W_n(\alpha)W_n(\xi)}{\overline{D}_n(\omega)} \qquad \text{(dimensions, m/N)}\,. \tag{4.25}$$

All the above equations represent point-excited influence functions. In the case of band-excited influence functions we proceed as follows. Let $p_i(\theta, x)$ be a typical component of driving pressure and define it to be a rectangular function, that is, a function that is zero everywhere on the cylinder except across a band Δ_1 units wide in x and Δ_2 units wide in θ. Thus

$$p_i(x, \theta) = \overline{p}_i(\xi_{\Delta 1}, \xi_{\Delta 2}, \Delta_1, \Delta_2) f_1(\theta, x) \Pi(x - \xi_{\Delta 1}|\Delta_1)\, \Pi(\theta - \xi_{\Delta 2}|\Delta_2)\,. \tag{4.26}$$

Here $\overline{p}_i(\xi_{\Delta 1}, \xi_{\Delta 2}, \Delta_1, \Delta_2)$ is the average forcing function (N/m^2) in a band of width Δ_1 in x and Δ_2 in θ centered at the surface coordinates $x = \xi_{\Delta 1}$, $\theta = \xi_{\Delta 2}$; $f_i(x, \theta)$ is the nondimensional variation of $p_i(x, \theta)$ with x and θ; and $\Pi(a - b|c)$ is a displaced rectangle function of unit height and base c centered at $a = b$. For bands of width small enough we take $f(x, \theta)$ to be constant with value of unity. Returning to Eq. (4.17), we write

$$I = -\sum_{i=1}^{3} \int p_i U_{in} dS = \int p_1 U_{1n} dS + \int p_2 U_{2n} dS + \int p_3 W_n dS\,. \tag{4.27}$$

Selecting one term (say the 3rd) on the r.h.s. for explicit integration, we define an integral mode shape W_n^0 by integrating over a cylindrical surface of radius R:

$$\int p_3 W_n dS = \overline{p}_3(\xi_{\Delta 1}, \xi_{\Delta 2}, \Delta_1, \Delta_2) W_n^0(\xi_{\Delta 1}, \xi_{\Delta 2}, \Delta_1, \Delta_2) \tag{4.28}$$

where

$$W_n^0(\xi_{\Delta 1}, \xi_{\Delta 2}, \Delta_1, \Delta_2) = \oint f_3(x, \theta)\Pi(x - \xi_{\Delta 1}|\Delta_1)_3 \Pi(\theta - \xi_{\Delta 2}|\Delta_2)_3 W_n(x, \theta)Rd\theta dx . \tag{4.29}$$

Defining U_{1n}^0, U_{2n}^0 in the same way, we write

$$I = \overline{p}_1 U_{1n}^0 + \overline{p}_2 U_{2n}^0 + \overline{p} W_n^0 . \tag{4.30}$$

The component of displacement u_i due to $\overline{p}_1$, $\overline{p}_2$, $\overline{p}_3$ can now be written in the form

$$u_i(\theta, x) = g_{i1}(\theta, x|\xi_{\Delta 1}, \xi_{\Delta 2})\overline{p}_1(\xi_{\Delta 1}, \xi_{\Delta 2}) + g_{i2}(\theta, x|\xi_{\Delta 1}, \xi_{\Delta 2}) \times \overline{p}_2(\xi_{\Delta 1}, \xi_{\Delta 2}) + g_{i3}(\theta, x|\xi_{\Delta 1}, \xi_{\Delta 2})\overline{p}_3(\xi_{\Delta 1}, \xi_{\Delta 2}) \tag{4.31}$$

where

$$g_{ij}(\theta, x|\xi_{\Delta 1}, \xi_{\Delta 2}) = \sum_{n=1}^{\infty} \frac{U_{in}(\theta, x)U_{jn}^0(\xi_{\Delta 1}, \xi_{\Delta 2})}{\overline{D}_n(\omega)}. \tag{4.32}$$

The symbol g_{ij} (dimensions, m^3/N) represents the displacement in the ith port due to a unit forcing function acting in the jth port. They are therefore band-type influence functions. They will be used later in this analysis to solve the problem of an axisymmetric, force-driven, circular cylindrical shell. Note that in thin-shell theory the word "displacement" unless otherwise noted, means "displacement of the reference surface of the shell."

Electromagnetic and Acoustic Forcing Functions

We now consider the problem of forced vibration of a piezoactive shell submerged in water when the forcing function is restricted to a sum of a piezoactive excitation p_{EM} and an acoustic excitation p_A. Thus we write

$$p_i = p_{i(EM)} + p_{i(A)}. \tag{4.33}$$

The form of $p_{i(EM)}$ may be obtained from Eq. (2.2) by noting that forcing functions differ from response functions of similar form by a change in sign. For the case of magnetostrictive coupling, the appropriate operators in Eqs. (2.1) and (2.2) are

$$\Psi'_{ij}\{\} = \oint \psi'_{ij}(\ )dS_j$$

$$\Psi_{ij}\{\} = \psi_{ij}(\ )$$

$$\mathcal{Z}_{ji}^{(m)}\{\} = \oint \mathcal{Z}_{ji}^{(m)}(\ )dS_i .$$

Thus, in the steady state,

$$p_i(\theta, x) = -\sum_{k=1}^{3} \psi_{ik}(\theta, x)I_k(\theta, x) - j\omega \sum_{l(N)} \mathcal{G}_{il}(\theta, x|\xi_1, \xi_2)u_{l(N)}(\xi_1, \xi_2) \tag{4.34}$$

where $\mathcal{G}_{il}$ is the acoustic Green's function (i.e., acoustic pressure gradient per shell reference surface velocity) for the domain and boundaries to be specified, $u_{l(N)}$ is the normal component of displacement of the reference surface in the lth port.

The total (vector) displacement due to this type of forcing function may be obtained by using Eqs. (4.31) and (4.34) together:

$$u_i(\theta_\Delta, x_\Delta) = \sum_l \sum_{\xi_{\Delta 1}, \xi_{\Delta 2}} g_{il}(\theta_\Delta, x_\Delta | \xi_{\Delta 1}, \xi_{\Delta 2}) \sum_q (-)\psi_{lq}(\xi_{\Delta 1}, \xi_{\Delta 2}) I_q(\xi_{\Delta 1}, \xi_{\Delta 2})$$

$$+ \sum_r \sum_{\xi_{\Delta 1}, \xi_{\Delta 2}} g_{ir}(\theta_\Delta, x_\Delta | \xi_{\Delta 1}, \xi_{\Delta 2}) \sum_s \sum_{\zeta_{\Delta 1}, \zeta_{\Delta 2}}$$

$$\times \mathcal{G}_{rs}(\xi_{\Delta 1}, \xi_{\Delta 2} | \zeta_{\Delta 1}, \zeta_{\Delta 2}) u_s(\zeta_{\Delta 1}, \zeta_{\Delta 2}) \tag{4.35}$$

in which the subscripts l, i, r, q, s designate ports, and u_s in the second term of the r.h.s. is to be interpreted as a normal component of displacement. The summations over ξ_Δ, $\xi_{\Delta 2}$, $\zeta_{\Delta 1}$, $\zeta_{\Delta 2}$ denote finite-element areal integration.

Equation (4.35) is a finite-element approximation of an integral equation in the unknown velocities u_i, u_s. The principal factors in this equation are the entities g_{il}, ψ_{lq}, $\mathcal{G}_{rs}$. The influence coefficient g_{il} can be determined from a solution of the problem of forced vibration of a shell by a point or band forcing function. The transduction ψ_{lq} is a matrix of electromechanical coefficients which convert currents pertaining to the qth port to force per unit area at the lth port. The form of these coefficients depends on the choice of independent variables in the canonical set and on piezoactive material constants. The radiation Green's function $\mathcal{G}_{rs}$ is calculated from a solution of a delta-driven Helmholtz equation in the acoustic pressure, for the domain and boundary conditions chosen. If θ, x are discrete, Eq. (4.35) reduces to a set of simultaneous equations in the unknown displacements $u_i(\theta_{\Delta 1}, x_{\Delta 1})$.

Determination of g_{il}—It is seen from Eq. (4.32) that g_{il} is constructed from the modes of free vibration of a dissipationless shell (i.e., from U_{in}). To find these modes we consider the free vibration of a thin shell and choose to investigate free (time) periodic waves of arbitrary wavelength which make up a surface vibration pattern. To find their shape we let $\mathbf{M}$ be the vector displacement with components M_i, where $M_1 = U$, $M_2 = V$, $M_3 = W$, and assume that the equations of motion can be formulated in terms of a differential elastodynamic operator $\mathfrak{L}_{ij}$. The eigenvalue problem of free vibration is then formulated in abstract form by the following equations in a selected coordinate system:

$$\text{In the continuum domain:} \sum_{j=1}^{3} \mathfrak{L}_{ij} M_j = 0, \qquad i = 1, 2, 3 \tag{4.36}$$

$$\text{On the edges:} \quad \sum_{j=1}^{3} B_{ij}M_j = \sum_{j=1}^{3} \Lambda_i C_{ij} M_j, \qquad i = 1, 2, 3 \tag{4.37}$$

where B_{ij}, C_{ij} are differential operators and Λ_i is an eigenvalue of $\mathcal{K}_{ij}$ (see Eqs. (3.6) and (4.6)). The solution of this problem is, in general, very difficult to write for arbitrary coordinates. To reduce difficulties we select a system such that Eq. (4.36) is solvable by the method of separation of variables. Let θ, x be such coordinates. Then, for separable functions Θ, X we write

$$U, V, W \propto \Theta(\theta)X(x)e^{j\omega t}. \tag{4.38}$$

A shell of particular interest in this analysis is a circular cylinder of radius a and thickness b ($b \ll a$), whose length is finite. Thus we choose $\Theta(\theta)$ to be $\sin q\theta$ or $\cos q\theta$, and $X(x)$ to be exponential, i.e.,

$$U, V, W \sim \begin{Bmatrix} \sin q\theta \\ \cos q\theta \end{Bmatrix} \exp\left[\frac{\alpha_r x}{a} + j\omega t\right], \qquad r = 1, 2, \ldots, Q. \tag{4.39}$$

Here α_r is a separation constant, and a is a reference mean radius of the cylinder. For the most general case it is known [9] that $Q = 8$. Substitution of Eq. (4.38) into Eq. (4.36) changes the differential operator $\mathfrak{L}_{ij}$ to an algebraic operator L_{ij}. Thus the eigenvalue problem reads

$$\mathbf{LM} = 0 \tag{4.40}$$

or

$$\sum_{j=1}^{3} L_{ij}(\alpha_r, q, \omega)M_j = 0, \qquad \begin{aligned} i &= 1, 2, 3;, \\ q &= 1, 2, \ldots, \infty. \end{aligned} \tag{4.41}$$

To obtain nontrivial solutions we solve the characteristic equation

$$|L_{ij}(\alpha_r, q, \omega)| = 0. \tag{4.42}$$

This is an eighth-order equation whose roots α_r give the allowed values of the separation constant α which satisfy the equations of motion of the shell for selected values of frequency ω and q. The roots are expressed in terms of the original elastic moduli, wave numbers, etc., of the shell. When these roots are found one can substitute them sequentially in $\mathbf{L}$ and form a series of 3 X 3 matrixes:

$$L_r = \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}_r, \qquad r = 1, 2, \ldots, 8. \tag{4.43}$$

For each choice of r we then find the amplitude ratios by the cofactor method:

$$U:V:W \propto \text{cofactors of any row of } L_r. \tag{4.44}$$

Of the three displacements we next choose the radial displacement W to be our reference displacement, suppress the time dependence exp $(j\omega t)$, and write it as a sum of exponentials with arbitrary amplitudes:

$$W = \sum_{r=1}^{8} W_r, \qquad W_r = C_r \exp\left(\frac{\alpha_r x}{a}\right). \tag{4.45}$$

Using the amplitude ratios determined from Eq. (4.43), we can construct U and V as follows:

$$U = \sum_{r=1}^{8} \left(\frac{U}{W}\right)_r W_r \tag{4.46}$$

$$V = \sum_{r=1}^{8} \left(\frac{V}{W}\right)_r W_r . \tag{4.47}$$

We now fit the proposed solutions, Eqs. (4.45) through (4.47), to the boundary conditions. The amplitudes C_r constitute eight unknowns. For a finite cylindrical shell there are two edges with a possible maximum of four boundary conditions per edge. Thus we can write an 8×8 matrix equation in the form

$$[S(\beta, \omega)]\ [C] = 0 \tag{4.48}$$

in which parameter $\beta = \alpha_r l/a$. To obtain nontrivial solutions we set

$$|S(\beta, \omega)| = 0 , \tag{4.49}$$

which is a transcendental equation with an infinite number of roots

$$\beta_m = \frac{\alpha_r l_m}{a}, \qquad m = 1, 2, \ldots, \infty. \tag{4.50}$$

Hence for a given frequency ω_m one finds α_r, l_m which simultaneously satisfy Eqs. (4.49) and (4.42). The length l_m of the shell is the length required for the free mode of vibration W_m to occur at ω_m [9]. Alternatively, one can fix the length $l = l_f$ and seek a value of ω_m which simultaneously satisfies Eqs. (4.42) and (4.49). This method is required for the purposes of this analysis and will be adopted. Thus, for an infinity of choices ω_m, $m = 1, 2, \ldots, \infty$, we find an infinity of mode shapes W_m (for fixed length l_f) which can then be used to calculate Eq. (4.45), so that

$$W(x, q) = \sum_{r=1}^{8} W_r\,(x, q), \qquad q = 0, 1, \ldots, \infty \tag{4.51}$$

$$W_m(x, q) = \sum_{r=1}^{8} C_r^{(m)} \exp\left(\frac{\beta_m x}{l_f}\right)\begin{pmatrix}\sin q\theta \\ \cos q\theta\end{pmatrix}. \tag{4.52}$$

From this reference displacement one can then construct U, V by means of Eqs. (4.46) and (4.47) (to within a scale factor). When the modal shapes U_m, V_m, W_m (or U_{1m}, U_{2m}, W_m) are determined, one can construct the influence function

$$g_{il}(\theta, x \,|\, \xi_{\Delta 1}, \xi_{\Delta 2})$$

from Eq. (4.32).

5. FORCED VIBRATION OF A FREE-FLOODED MAGNETOSTRICTION RING

We consider in detail the axisymmetric case of an electrically driven free-flooded magnetostriction ring transducer of mean radius a and (transverse) rectangular cross-section submerged in water and radiating sound into an unbounded medium. There are two mechanical ports (for radial and axial velocity). The axial coordinate is x, and all axial variables are labeled subscript 1. The tangential coordinate is θ, and all tangential variables are labeled subscript 2. The radial direction is labeled subscript 3. The ring is wound toroidally with N turns of wire so that the current matrix has one electrical port I_1 and the resultant magnetic flux matrix is given by flux Φ_2. Dividing the ring surface of thickness b into J bands of equal area $2\pi a\Delta_1$ on the lateral surface and K annular bands of equal width on the top and bottom surface, deleting θ, then designating k, l as running index for bands, we reduce the components of Eq. (4.35) to the set of simultaneous equations in velocities $\dot{u}_1 = j\omega u_1$, $\dot{w} = j\omega w$, as follows:

$$\dot{u}_1(k) = -\sum_{l}^{J} j\omega g_{13}(k, l)\psi_{31}(l)I_1(l) + S_D^{(\dot{u})}$$

$$-\sum_{l}^{J}\sum_{r}^{J+K} j\omega g_{13}(k, l)\mathcal{G}(l, r)v_N(r), \qquad k = 1, 2, \ldots, K \tag{5.1}$$

$$\dot{w}(k) = -\sum_{i}^{J} j\omega g_{33}(k, l)\psi_{31}(l)I_1(l) + S_D^{(\dot{w})}$$

$$-\sum_{l}^{J}\sum_{r}^{J+K} j\omega g_{33}(k, l)\mathcal{G}(l, r)v_N(r), \qquad k = 1, 2, \ldots, J \tag{5.2}$$

where $v_N(r)$ is the normal velocity on the rth band and S_D is a structural damping, present only in water, due to "foundation forces" (method of support, etc). It is convenient to calculate S_D by defining a (real) damping constant R_s as follows:

$$S_D^{(\dot{u})} = -\sum_r^J j\omega g_{13}(k, r) R_s^{(\dot{u})} \dot{U}(r)$$

$$S_D^{(\dot{w})} = -\sum_r^J j\omega g_{33}(k, r) R_s^{(\dot{w})} \dot{W}(r).$$

The solution of this set of simultaneous equations yields all the surface velocities due to a constant current drive. When the current drive I_1 is replaced by a voltage drive E_1 we write

$$E_1 = E_1^{(g)} - I_1 Z_{11}^{(g)}, \tag{5.3}$$

in which $Z_{11}^{(g)}$ is a "generator" internal impedance. Substituting this into Eq. (2.1), we have (for $\mathfrak{M}$ mechanical ports),

$$E_1 = Z_{11}^{(b)} I_1 + \sum_{j=1}^{\mathfrak{M}} \oint \psi'_{ij} v_j dS_j. \tag{5.4}$$

We then solve for I_1 and find

$$I_1 = \frac{E_1}{Z_{11}^{(b)}} - \frac{\oint \psi'_{13} \dot{w} dS + \oint \psi'_{11} \dot{u}_1 dS}{Z_{11}^{(b)}} \tag{5.5}$$

Thus, for constant voltage drive the unknown reference surface velocities are found by solving the following set of simultaneous equations:

$$\dot{u}_1(k) + \sum_l^J \sum_r^{J+K} j\omega g_{13}(k, l) \mathcal{G}(l, r) v_N(r) + S_D^{(\dot{u})}$$

$$- \sum_l^J \sum_r^J 2\pi a \Delta_1(r) j\omega g_{13}(k, l) \psi_{31}(l) \psi'_{13}(r) \frac{\dot{w}(r)}{Z_{11}^{(b)}}$$

$$- \sum_{l}^J \sum_\rho^K j\omega g_{13}(k, l) \psi_{31}(l) \psi'_{11}(s) 2\pi a(s) \Delta_3(s) \frac{\dot{u}_1(s)}{Z_{11}^{(b)}} = -\sum_l^J j\omega g_{13}(k, l) \psi_{31}(l) \frac{E_1}{Z_{11}^{(b)}},$$

$$k = 1, 2, \ldots, K \tag{5.6}$$

$$\dot{w}(k) + \sum_{l}^{J} \sum_{r}^{J+K} j\omega g_{33}(k, l)\mathcal{G}(l, r)v_N(r) + S_D^{(\dot{w})}$$

$$-\sum_{l}^{J} \sum_{r}^{J} 2\pi a \Delta_1(r) j\omega g_{33}(k, l)\psi_{31}(l)\psi'_{13}(r) \frac{\dot{w}(r)}{Z_{11}^{(b)}}$$

$$-\sum_{l}^{J} \sum_{s}^{K} 2\pi a(s)\Delta_3(s) j\omega g_{33}(k, l)\psi_{31}(l)\psi'_{11}(s) \frac{\dot{u}_1(s)}{Z_{11}^{(b)}} = -\sum_{l}^{J} j\omega g_{33}(k, l) \frac{\psi_{31}(l)E_1}{Z_{11}^{(b)}},$$

$$k = 1, 2, \ldots, J. \tag{5.7}$$

The two sets of equations, (5.1)-(5.2) and (5.6)-(5.7), are the basic results of this analysis, since a knowledge of the surface velocities determines all dynamic behavior of the transducer. In the next section we specialize the parameters which enable explicit numerical solutions to be obtained.

Formulas for the Prediction of Performance of a Free-Flooded Magnetostrictive Cylindrical Shell

In the first approximation the electromechanical constitutive relations, referred to a three-dimensional coordinate system x_i, are given by the following linear set, which is valid in the low-frequency range:

$$T_i = C_{ij}^B S_j - h_{ik}^{(t)} B_k, \quad j = 1, 2, \ldots, 6; \quad k = 1, 2, 3$$

$$H_l = -h_{lm} S_m + (\mu^S)^{-1} B_l, \quad l = 1, 2, 3; \quad m = 1, 2, \ldots, 6 \tag{5.8}$$

where

T_i = mechanical stress (N/m^2)

C_{ij}^B = elastic strain/stress moduli at constant induction (N/m^2)

$h_{ik}^{(t)}$ = transposed piezomodulus relating stress to induction (N/Wb)

B_k = magnetic induction (Wb/m^2)

H_l = magnetic intensity ((A)/m)

h_{lm} = piezomodulus ((A)/m)

$(\mu^S)^{-1}$ = inverse permeability at constant strain [A/(Wb/m)].

For a toroidally wound coil,

$$\psi'_{13} = -\psi_{31} = \frac{h_{22}\mu^S bN\chi}{2\pi a^2}, \quad \left(\frac{\text{V}}{\text{vol. vel.}}\right) \tag{5.9}$$

where χ is a complex eddy-current loss factor ($\chi = \chi_R - j\chi_I$). The impedance $Z_{11}^{(b)}$ has explicit form

$$Z_{11}^{(b)} = \frac{j\omega N^2\mu^S\chi bl}{2\pi a\gamma} = R_{11}^{(b)} + j\omega L_{11}^{(b)}, \quad \left(\frac{\text{V}}{\text{A}}\right) \tag{5.10}$$

where l is the axial length of the shell and γ is a leakage coefficient. An explicit form for the leakage coefficient γ must be estimated from experiment (plus calculation, when advantageous). With the substitutions of Eqs. (5.9) and (5.10) into Eqs. (5.1)-(5.2) and (5.6)-(5.7), we complete the formulation of the acoustic-mechanical system in terms of an admittance-type influence function. There is another type of influence function, namely the impedance type, which plays a significant role in predicting the performance of the transducer. To obtain a formulation in terms of impedance, we first invert the influence matrix g_{ij} to form the impedance matrix $\mathcal{Z}_{ij}^{(m)}$, such that

$$\mathcal{Z}_{ij}^{(m)} = (j\omega g_{ij})^{-1}. \tag{5.11}$$

Then we multiply Eqs. (5.1) and (5.2) by this inverse* and sum the running integer k over J sidebands. Noting that

$$\sum_k^J \mathcal{Z}_{ij}^{(m)}(q,k)[Z_{ij}^{(m)}(k,l)]^{-1} = \delta(q,l) \tag{5.12}$$

and restoring the forced drive $p_j^{(g)}$ of Eq. (2.3) due to incident and diffracted acoustic waves, we see that Eq. (5.1), which deals with endbands, takes the form

$$\sum_k^J \mathcal{Z}_{11}^{(m)}(q,k)\dot{u}_1(k) + \sum_r^{J+K} \mathcal{G}(q,r)v_N(r) + \sum_s^J \mathcal{Z}_{13}^{(m)}(q,s)\dot{w}(s) = -\psi_{11}(q)I_1 + p_1^{(g)}(q),$$

$$q = 1, 2, \ldots, K. \tag{5.13}$$

Similarly Eq. (5.2), which deals with sidebands, takes the form

$$\sum_k^J \mathcal{Z}_{33}^{(m)}(q,k)\dot{w}(k) + \sum_r^{J+K} \mathcal{G}(q,r)v_N(r) + \sum_t^{J+K} Z_{31}(q,t)\dot{u}_1(t) = -\psi_{31}(q)I_1 + p_3^{(g)}(q),$$

$$q = 1, 2, \ldots, J. \tag{5.14}$$

When the drive is constant voltage the impedance formulation of Eqs. (5.6) and (5.7) becomes

*The matrix g_{ij} is invertible if and only if the number of modal functions used in obtaining g_{ij} is greater than or equal to the total number of reference surface bands used to describe the shell.

$$\sum_{k}^{J+K} \mathcal{Z}_{11}^{(m)}(q,k)\dot{u}_1(k) + \sum_{r}^{J+K} \mathcal{G}(q,r)v_N(r) + \sum_{s}^{J} Z_{13}(q,s)\dot{w}(s)$$

$$-\sum_{r}^{J} \psi_{11}(q)\psi'_{13}(r)\frac{\dot{w}(r)}{Z_{11}^{(b)}}\, 2\pi a\Delta_1(r)$$

$$-\sum_{t}^{K} \psi_{11}(q)\psi'_{11}(t)2\pi a(t)\Delta_3(t)\frac{\dot{u}_1(t)}{Z_{11}^{(b)}} = -\,\psi_{11}(q)\frac{E_1}{Z_{11}^{(b)}} + p_1^{(g)}(q),$$

$$q = 1, 2, \ldots, K \tag{5.15}$$

for sidebands, and

$$\sum_{k}^{J} \mathcal{Z}_{33}^{(m)}(q,k)\dot{w}(k) + \sum_{r}^{J+K} \mathcal{G}(q,r)v_N(r) + \sum_{k}^{J+K} Z_{31}(q,k)\dot{u}(k) \tag{5.16}$$

$$-\sum_{r}^{J} \psi_{31}(q)\psi'_{13}(r)2\pi a\Delta_1(r)\frac{\dot{w}(r)}{Z_{11}^{(b)}}$$

$$-\sum_{t}^{K} \psi_{31}(q)\psi'_{11}(t)2\pi a(t)\Delta_3(t)\frac{\dot{u}_1(t)}{Z_{11}^{(b)}} = -\,\psi_{31}(q)\frac{E_1}{Z_{11}^{(b)}} + p_3^{(g)}(q),$$

$$q = 1, 2, \ldots, J$$

for endbands (where we have again restored $p^{(g)}(q)$). The solution of the above-developed basic Eqs. (5.1), (5.2), (5.6), (5.7), (5.13), (5.14), (5.15), and (5.16) enables us to write the principal formulas needed for the prediction of the electroacoustic performance of a free-flooded magnetostriction ring vibrator. Assuming all velocities to have been obtained, we proceed to determine the following quantities.

Constant-Current and Constant-Voltage Specific Acoustic Impedances—Setting $\mathcal{G} = 0$ and neglecting endbands, we find from Eq. (5.14) that

$$\sum_{k}^{J} Z_{33}^{\text{o.c.}}(q,k)\dot{w}(k) = p_3^{(g)}(q) - \psi_{31}I_1(q) \tag{5.17}$$

where

$$Z_{33}^{\text{o.c.}}(q,k) = \mathcal{Z}_{33}^{(m)}(q,k)$$

$$= \text{constant-current specific acoustic impedance.} \tag{5.18}$$

Setting $\mathcal{G} = 0$ in Eq. (5.16) and neglecting endbands, we find that

$$\sum_{k}^{J} Z_{33}^{\text{s.c.}}(q,k)\dot{w}(k) = p_3^{(g)}(q) - \frac{\psi_{31}(q)E_1(q)}{Z_{11}^{(b)}} \tag{5.19}$$

where

$$Z_{33}^{\text{s.c.}}(q,k) = Z_{33}^{\text{o.c.}}(q,k) - \frac{\psi_{31}(q)\psi'_{13}(r)}{Z_{11}^{(b)}}$$

$$= \text{constant-voltage specific acoustic impedance.} \tag{5.20}$$

Pressure Field and Source Level—Letting $p^{(g)} = 0$ we solve Eqs. (5.13) and (5.14) for $\dot{w}^{(I)}(k)$ and $\dot{u}_1^{(I)}(k)$, i.e., for given $v_N^{(I)}(k)$, and find the true surface velocities $\mathcal{V}_N$ by using thickness correction factors (see Sec. 9). Then the acoustic pressure at field point $\mathbf{x}$ is seen from Eq. (2.16) to be

$$p_I(\mathbf{x}) = \sum_{k}^{2(J+K)} G(\mathbf{x},k)\mathcal{V}_N^I(k) \tag{5.21}$$

where G is the true surface radiation Green's function (see Sec. 9). The acoustic pressure at $\mathbf{x}$ due to constant voltage (p_E) is similarly obtained by solving Eq. (5.6) for v_N^E with $p^{(g)} = 0$, then applying thickness correction factors (Sec. 9) to find $\mathcal{V}_N$, and finally writing

$$p_E(\mathbf{x}) = \sum_{k}^{2(J+K)} G(\mathbf{x},k)\mathcal{V}_N^E(k). \tag{5.22}$$

The transmitting current response ($(\text{N/m})\text{A}^{-1}$) at a far-field point $|x| \to \infty$ is given by

$$\text{Transmitting current response} = \lim_{|x|\to\infty} |\mathbf{x}| \sum_{k=1}^{2(J+K)} G(|\mathbf{x}|,k)\frac{\mathcal{V}_N^I}{I_1}(k) \tag{5.23}$$

where $\mathcal{V}_N(k)$ is the surface velocity of the Nth band (both radial and axial) due to unit current excitation, and G is the actual (not reference) surface radiation Green's function. For voltage excitation $\mathcal{V}_N^I(k)/I$ is replaced by $\mathcal{V}_N^E(k)/E$. We note again that true surface velocity $\mathcal{V}_N$ differs from reference surface velocity v_N. A discussion of this is found in Sec. 9.

Electrical Input Impedance—From Eq. (2.18) we write the ratio of the voltage E_i due to the current I_1 in the form $Z_{il}^{(e)}$. Since we consider only radial (subscript 3) and axial (subscript 1) volume velocities, we have

$$Z_{i1}^{(e)} = Z_{i1}^{(b)} - \psi_{i1}\hat{Z}_{13}^{-1}\psi'_{31} - \psi'_{i1}\hat{Z}_{11}^{-1}\psi_{11} - \psi'_{i3}\hat{Z}_{33}^{-1}\psi_{31} - \psi'_{i3}\hat{Z}_{31}^{-1}\psi_{11}.$$

Assuming that $i = 1$, i.e., that the only voltage is E_1, we write further, for l bands,

$$Z_{11}^{(e)} = Z_{11}^{(b)} + \sum_{l}^{K} \frac{2\pi a\Delta(l)\psi'_{11}(l)\dot{u}_1(l)}{I_1(l)} + \sum_{l}^{J} \frac{\psi'_{13}(l)\dot{W}(l)2\pi a\Delta(l)}{I_1(l)} \tag{5.24}$$

in which $\dot{u}_1$ and $\dot{W}$ are determined from Eqs. (5.1) and (5.2).

Receiving Response—We return to Eqs. (5.17) and (5.18) and solve for the velocity $\dot{w}^{(p^{(0)})}(k)$ when the current $I_1(q)$ vanishes, and when the rigid-body diffraction pressure $(p^{(D)})$ also is negligible. Then, assuming the receiver transducer is electrically terminated in an electrical impedance $Z_{11}^{(g)}$, we write $I_1 = - E_1/Z_{11}^{(g)}$ in Eq. (2.1) and solve for E_1. Thus the receiving response is given by

$$\frac{E_1}{p^0} = \frac{Z_{11}^{(g)}}{Z_{11}^{(g)} + Z_{11}^{(b)}} \left[\sum_{l}^{J} \frac{\psi'_{13}(l)\dot{W}(l)2\pi a\Delta(l)}{p^0} + \sum_{k}^{K} \frac{\psi'_{11}(k)\dot{u}(k)2\pi a(k)\Delta(k)}{p^0} \right]. \tag{5.25}$$

The open-circuit receiving response is then given by the condition $Z_{11}^{(g)} \to \infty$.

Electrical Admittance—When $E_i = E_1$ and $I_i = I_1$, then from Eq. (2.1),

$$E_1 = Z_{11}^{(b)} I_1 + \sum_{j=1}^{2} \Psi'_{1j}\{v_j\}, \qquad \Psi'_{1j}\{\} = \oint \psi'_{ij}(\ )dS$$

where v_j is the velocity at the mechanical port $\mathfrak{M}_j$. Dividing by E_1 and forming the ratio $I_1/E_1 = Y_{11}^{(e)}$, we write

$$Y_{11}^{(e)} = \frac{1}{Z_{11}^{(b)}} \left[1 - \sum_{l=1}^{J} \frac{\psi'_{13}(l)\dot{w}^{(E)}(l)2\pi a\Delta(l)}{E_1} - \sum_{k=1}^{K} \frac{\psi'_{11}\dot{u}(k)2\pi a(k)\Delta(k)}{E_1} \right]. \tag{5.26}$$

Steady State Radiated Power—When the peak normal surface velocities $\mathcal{V}_N$ are determined for any electrical drive into a source-free medium, we can apply Eq. (2.17) to find the rms radiated power $\mathcal{W}$, i.e.,

$$\mathcal{W} = \frac{1}{2} \mathrm{Re} \sum_{k}^{2(J+K)} \sum_{l}^{2(J+K)} \mathcal{V}_N^*(k) S(k) G(k, l) \mathcal{V}_N(l) \tag{5.27}$$

where $S(k)$ is the area of the kth band, G is the true surface-radiation Green's function, and * signifies complex conjugate.

Steady State Mechanical Power Dissipated As Internal Loss—During in-vacuo vibration at constant current the real part of the mechanical impedance is assumed to consist of $R' + R''$, where R' is the mechanical resistance resulting from coupling of the magnetostrictive field to the elastic field and R'' is a purely internal (friction-type) mechanical resistance.

In the more general case, the mechanical resistance is a matrix $R[k, l]$, which represents the equivalent mechanical force (associated with internal resistance) at band k due to a reference surface velocity at band l. When the normal (peak) reference surface velocities $v_N^{(I)}$ are found (with $\mathcal{G} \equiv 0$), the steady state mechanical power $\mathfrak{W}_D$ dissipated as internal loss by the formula

$$\mathfrak{W}_D{}^I = \frac{1}{2} \mathrm{Re} \sum_{k,l}^{J+K} v_N^{*I}(k) R^I(k, l) v_N{}^I(l) . \tag{5.28}$$

In the particular case where $R[k, l]$ is a diagonal matrix with only one component which is the same for all velocity distributions, we have

$$\mathfrak{W}_D^I = \frac{1}{2} \mathrm{Re} \sum_{k=1}^{J+K} |v_N{}^I|^2 R^I, \qquad R^I = R' + R''. \tag{5.29}$$

At constant voltage drive under analogous conditions, in terms of impedance $\mathfrak{Z}$ (see Sec. 6),

$$\mathfrak{W}_D^E = \frac{1}{2} \mathrm{Re} \sum_{k,l}^{J+K} v_N^{*E}(k) R^E(k, l) v_N{}^E(l) \tag{5.30}$$

$$\mathfrak{W}_D^E = \frac{1}{2} \mathrm{Re} \sum_{l}^{J+K} |v_N{}^E|^2 R^E, \qquad R^E = R'' + \mathrm{Re}\, \mathfrak{Z} . \tag{5.31}$$

6. MODAL MASS AND COUPLING

When the vibratory motion of the shell is analyzed into a superposition of an infinity of modes, the electroacoustic performance may be said to mirror loosely these modes by showing an infinite sequence of peaks and valleys of response to transient and steady state drivers. It is convenient to consider each mode as a single-degree-of-freedom system of special type, possessing a modal mass, modal resonant frequency, modal stiffness, modal resistance, and a modal coefficient of electromechanical coupling. It is also useful to isolate a single selected peak and its associated valleys from the total electroacoustic response, and to find for it an effective mass, effective stiffness, etc., which take into account contributions of an infinity of modes at a specified frequency. Both of these approaches are analyzed in this section.

Modal Mass (First Formulation)

The concept of modal mass can be defined in several ways. Different choices, used consistently, yield the same final results. However, the analytic formulation of modal resistance (modal mechanical Q) places restrictions on the possible definitions of modal mass. When the concept of mechanical Q is explicitly used in the analysis the modal mass must be a real positive number. We consider now a first formulation of modal mass.

The generalized coordinate $q_n(\omega)$ (Eq. (4.17)) has the form

$$q_n(\omega) = \frac{-\int \mathbf{p} \cdot \mathbf{U}_n dS}{\rho b N_n \overline{F}(\omega)} \tag{6.1}$$

where

$$\overline{F}(\omega) = \overline{\omega}_n^{-2} - \omega^2 + \text{loss terms.} \tag{6.2}$$

For a description of the loss terms see Secs. 7 and 8.

Now we write the normalization N_n in terms of three entities A_{1n}, A_{2n}, A_{3n}, as follows:

$$N_n = (A_{1n}^2 + A_{2n}^2 + A_{3n}^2)2\pi al \tag{6.3}$$

where

$$A_{1n}^2 = \frac{1}{2\pi al}\int U_{1n}^2 dS$$

$$A_{2n}^2 = \frac{1}{2\pi al}\int U_{2n}^2 dS$$

$$A_{3n}^2 = \frac{1}{2\pi al}\int W_n^2 dS.$$

In general there is an A_{in}^2 which has the largest magnitude. For ease in writing assume this to be A_{3n}^2. Then

$$\frac{N_n}{A_{3n}^2} = \left[1 + \left(\frac{A_{1n}}{A_{3n}}\right)^2 + \left(\frac{A_{2n}}{A_{3n}}\right)^2\right] 2\pi al. \tag{6.4}$$

By simple manipulations we obtain

$$q_n(\omega) = \frac{-\int\left(\mathbf{p} \cdot \frac{\mathbf{U}_n}{A_{3n}^2}\right) dS}{M_n \overline{F}(\omega)} \tag{6.5}$$

where

$$M_n = M_s \left[1 + \left(\frac{A_{1n}}{A_{3n}} \right)^2 + \left(\frac{A_{2n}}{A_{3n}} \right)^2 \right] \tag{6.6}$$

with

$$M_s = \rho b 2\pi a l = \text{static mass of shell.} \tag{6.7}$$

The quantity M_n is the nth modal mass (i.e., generalized mass) normalized to the mode shape of maximum amplitude. The nth modal stiffness is given by

$$K_n = \Omega_n^2 M_s \left[1 + \left(\frac{A_{1n}}{A_{3n}} \right)^2 + \left(\frac{A_{2n}}{A_{3n}} \right)^2 \right], \tag{6.8}$$

where Ω_n^2 is the in-vacuo resonant-frequency at the nth mode.

Form of D_n in the Constant-Voltage Case

We begin with the constant-current case (Eq. (4.18)) and write $\overline{D}_n$, including coupled core loss (R'), such that

$$\overline{D}_n = N_n \rho b \left(\omega_n^2 - \omega^2 + \frac{j\omega R_n''}{M_n} + \frac{j\omega R_n'}{M_n} \right) \tag{6.9}$$

where

$$\omega_n^2 = \Omega_n^2 (1 - k_n^2 \chi_{R_n}), \qquad \Omega_n^2 = \frac{K_n}{M_n} \tag{6.10}$$

$$R_n' = \frac{K_n k_n^2 \chi_{I_n}}{\omega} \tag{6.11}$$

$$\chi_n = \chi_{R_n} - j\chi_{I_n} . \tag{6.12}$$

Here Ω_n is the in-vacuo nth modal velocity resonance of the elastic system, independent of electromechanical coupling, χ_n is the modal hysteresis and eddy-current factor and K_n, M_n are the nth modal stiffness and mass, respectively. It is noted again that R_n' is the mechanical loss resistance associated with the magnetic properties of the core, while R_n'' is the mechanical loss resistance associated with internal friction. Regrouping terms, we cast $\overline{D}_n$ into the form

$$\overline{D}_n = N_n \rho b \frac{j\omega}{M_n} \left(\frac{K_n}{j\omega} - \frac{K_n k_n^2 \chi_{R_n}}{j\omega} + j\omega M_n + R_n'' + R_n' \right). \tag{6.13}$$

The terms in parentheses are mechanical impedances (dimensions, N-sec/m).

In the constant-voltage case we are required to add a mechanical impedance of the form [5, chap. 4],

$$\left[\frac{-\psi\psi'}{(Z^c + Z^l)}(2\pi a l)^2\right]_n \tag{6.14}$$

where Z^l is the leakage impedance. We assume that for all modes

$$\left[(2\pi a l)^2\left(\frac{-\psi\psi'}{Z^c}\right)\right]_n - \frac{K_n k_n^2 \chi_{R_n}}{j\omega} + \frac{K_n}{\omega} k_n^2 \chi_{I_n} = 0 . \tag{6.15}$$

Since this means that the frequency of velocity resonance rises due to cancellations in Eq. 6.13 of terms in k^2, we can, for the constant-voltage case, eliminate all k^2 from $\bar{D}_n$ and retain an added residual mechanical impedance

$$\mathfrak{A}_n = \left[\frac{\psi\psi' Z^l}{Z^c(Z^c + Z^l)}(2\pi a l)^2\right]_n . \tag{6.16}$$

Using the significant grouping

$$\frac{\mathfrak{A}_n}{\Omega_n M_n} = \left[\frac{(\psi\psi' Z^l)(2\pi a l)^2}{Z^c(Z^c + Z^l)}\right]_n \left(\frac{1}{\Omega_n M_n}\right) \tag{6.17}$$

we may formulate the losses in terms of a measurable Q_n^E as follows:

$$\frac{R_n''}{\Omega_n M_n} + \frac{\mathfrak{A}_n}{\Omega_n M_n} = \frac{1}{\Omega_n M_n}(R_n'' + \mathrm{Re}\,\mathfrak{A}_n) + j\frac{\mathrm{Im}\,(\mathfrak{A}_n)}{\Omega_n M_n}$$

$$= \frac{1}{Q_n^E} + j\frac{\mathrm{Im}\,(\mathfrak{A}_n)}{\Omega_n M_n} . \tag{6.18}$$

Thus, writing $\hat{D}_n$ to be the effective $\overline{D}_n$ for the constant-voltage case we have

$$\hat{D}_n = N_n \rho b \frac{j\omega}{M_n}\left[\frac{K_n}{j\omega} + j\omega M_n + R_n'' + \left(\frac{\psi\psi' Z^l}{Z^c(Z^c + Z^l)}(2\pi a l)^2\right)_n\right] \tag{6.19}$$

or

$$\hat{D}_n = N_n \rho b\left[\Omega_n^2 - \omega^2 - \omega\frac{\mathrm{Im}\,(\mathfrak{A}_n)}{M_n} + j\omega\frac{\Omega_n}{Q_n^E}\right] \tag{6.20}$$

Thus, in the constant-voltage case we shall use this form D_n to calculate the generalized coordinate $q_n(\omega)$ in Eq. (6.5).

Consistent theoretical formulas for Q_n^I and Q_n^E are found from the previously given definition of $\hat{D}_n$. In terms of the auxiliary quantities $\mathcal{Q}_n^I$, $\mathcal{Q}_n^E$ they are,

$$\mathcal{Q}_n^I(\omega) = \frac{\omega_n M_n}{R_n''(\omega_n) + \dfrac{K_n k_n^2 \chi_{I_n}(\omega)}{\omega}}, \tag{6.21}$$

$$\mathcal{Q}_n^E(\omega) = \frac{\Omega_n M_n}{R_n''(\omega_n) + \mathrm{Re}\ \mathcal{A}_n(\omega)}, \tag{6.22}$$

$$Q_n^I \equiv \mathcal{Q}_n^I(\omega_n), \qquad Q_n^E \equiv \mathcal{Q}_n^E(\Omega_n) \tag{6.23}$$

Here $R_n''(\omega_n)$ means resistance calculated at ω_n.

There are two options for calculating the effective Q_n at constant current and constant voltage, Q_n^I and Q_n^E: (a) Q_1^I is found from experiment, and Q_1^E is then determined by theory. Then all Q_n^I, Q_n^E are determined from Q_1^I and Q_1^E by assumption; (b) Q_n^I is determined by experiment for each mode, and Q_n^E is then found from Q_n^I by theory. Note that in determining Q_1^E, we first assume the ring mass to be M_1 and the ring stiffness to be K_1 as determined from the Butterworth and Smith model [10] (i.e., from static values). Actually the modal M_1 may differ from the static mass if there is a variation of radial displacement with axial distance. This difference is first assumed to be negligible. If it is not negligible, an iteration scheme may be used in which the variation of radial displacement with axial distance determined by assuming M_1 to be the static mass is itself used to determine M_1. This is the first iteration. It results in a new distribution of radial displacement and a new M_1. Upon further iterations, successive radial displacement curves are assumed to stabilize. When stability is achieved a final M_1 is calculated.

Alternatively, if we begin with an experimental determination of Q_1^E the above procedures can be applied again by interchanging Q_n^E by Q_n^I wherever they appear.

The definition of modal mass is intimately associated with the scale that can be given to the (dimensionless) mode shapes when calculating displacements. For example, if $\rho b N_q = M_q$ is defined to be the modal mass, and if radial mode shape is selected to serve as reference shape, then the scale of W_q is $\sqrt{J_q}$, where

$$J_q = (W_q, W_q) = \frac{M_q}{\rho b\left[1 + \dfrac{(U_q, U_q)}{(W_q, W_q)} + \dfrac{(V_q, V_q)}{(W_q, W_q)}\right]}.$$

The M_q is a selected real positive number. Using this scale we change the dimensionless quantities U_q, V_q, and W_q into quantities with dimensions of reciprocal length by writing

$$\frac{U_q}{\sqrt{J_q}}, \frac{V_q}{\sqrt{J_q}}, \frac{W_q}{\sqrt{J_q}}.$$

When scaled in this way these quantities have absolute values and can be used in the analysis freely as such. However, as noted, such scaling is one of several possible choices.

Modal Mass (Second Formulation)

We consider the average radial velocity due to unit radial and end applied pressures. The average radial velocity is then given by

$$\frac{\dot{W}_{\text{av}}}{P_{\text{unit}}} = j\omega \sum_{n=1}^{\infty} \frac{\frac{1}{l}\int W_n(x)dx \int W_n(x) 2\pi a dx + \frac{1}{l}\left[\int W_n(x)dx \int 4\pi r U_n(r)dr\right]}{\rho b N_n \left(\omega_n^2 - \omega^2 + j\omega \frac{R''}{M_n} + \frac{j\omega R'}{M_n}\right)} \tag{6.24}$$

where

$$N_n = \oint (U_{1n}^2 + W_n^2)dS \tag{6.25}$$

and resistances R'', R' are implicit functions of frequency. We now define a modal mass in the following way (second formulation):

$$M_n = M_s \left\{ \frac{\oint (U_{1n}^2 + W_n^2)dS}{\frac{2\pi a}{l}\left[\int W_n(x)dx\right]^2 + 4\pi \frac{1}{l}\int U_n(r)rdr \int W_n(x)dx} \right\} \tag{6.26}$$

and

$$M_s = 2\pi a l \rho_s b = \text{total static mass of shell.} \tag{6.27}$$

Since the factor in braces is dimensionless we can interpret U_{1n}, W_n as (a) having no dimension, in accord with this entire analysis, or (b) having the dimension of displacement, which may be convenient in applications. The above definition of modal mass may be justified by noting that $M_n\omega$ corresponds to a mechanical impedance due to integration over area. It is also to be noted that when $\int W_n(x)dx$ vanishes, the modal mass in the nth mode is infinite, i.e., motion in the nth mode is not possible. When many modes participate in the total dipslacement, the concept of modal mass is not directly applicable. We can, however, define a complex dynamic quantity Z_n, which is a specific acoustic impedance evaluated at each frequency where the total displacement becomes a maximum. Considering only radial motion we take the radial velocity due to both radial and axial surface pressures to be

$$\dot{w}(x) = g_{33}p_3 + g_{31}p_1 . \tag{6.28}$$

By definition of g_{il} the radial displacement at constant voltage due to unit axial and radial pressures is

$$w(x) = \overline{p}_{3\,\mathrm{unit}} \sum_{q=1}^{\infty} \frac{W_q(x)\left[W_q^{\circ}(\xi_\Delta) + 2U_q^{\circ}(\xi_\Delta)\right]}{D_q^E(\omega)} . \tag{6.29}$$

The ratio of average surface pressure ($=\overline{p}_{3\,\mathrm{unit}}$) to average surface velocity ($=\dot{w}_{\mathrm{av}}$) is defined as the effective complex modal impedance Z_n at the complex frequency $[\Omega]_n^{1/2}$, i.e.

$$Z_n = \frac{j[\Omega]_n^{1/2}}{-2\pi al} \mathfrak{M}_n^E$$

$$\mathfrak{M}_n^E = \frac{M_s}{[\Omega]_n \sum_{q=1}^{\infty} \frac{\mathcal{G}_q}{N_q[\Omega]_q}} \tag{6.30}$$

$$M_s = 2\pi al\rho b \tag{6.31}$$

$$\mathcal{G}_q = \frac{1}{2\pi al} \int W_q(x) 2\pi a dx \left\{ \int W_q(x) 2\pi a dx + 4\pi \int U_q(r) r dr \right\} \tag{6.32}$$

$$[\Omega]_q = \Omega_q^2 - \omega^2 + j\omega \frac{(R'' + \mathfrak{A}_n)}{M_q} \tag{6.33}$$

$$\mathfrak{A}_n = \frac{\psi_{31}\psi_{13}' Z^l (2\pi al)^2}{Z^c(Z^c + Z^l)} \tag{6.34}$$

$$M_q = \frac{M_s N_q}{\mathcal{G}_q} . \tag{6.35}$$

The symbol $\mathfrak{M}_n^E$ suggest the possibility of "complex mass." The concepts embodied in Z_n and $\mathfrak{M}_n^E$ have theoretical implications. These, however, are not explored further in this analysis except to note that when single modes only are considered the symbol $\mathfrak{M}_n^E$ reduces to the modal mass previously defined in Eq. (6.26), i.e.,

$$M_n = \frac{2\pi al\rho_s bN_n}{\mathcal{G}_n} . \tag{6.36}$$

We return now to the concept of modal mass formluated in Eq. (6.35). This definition of modal mass will aid in defining a complex multimode coupling factor. Note that, for a single-degree-of-freedom system consisting of a radially vibrating ring, the effective Young's modulus is

$$Y_1 = M_1 \Omega_1^2 \frac{a}{(2\pi l b)}, \tag{6.37}$$

where U_1 is negligible and $N_1/\mathcal{J}_1 = 1$. We define a complex dynamic coupling factor as

$$k_n^2 = \frac{h_{22}^2 \mu^S}{Y_n^*}, \tag{6.38}$$

where the dynamic complex equivalent Young's modulus in the nth mode is

$$Y_n^* = \left(\frac{a}{2\pi l b}\right) M_n \Omega_n^2 . \tag{6.39}$$

Modal Coupling, Modal Mass (Third Formulation)

In the linear approximation, the two constitutive relations for magnetostrictive systems at very low frequency can be solved for mechanical stress to yield the known formula

$$T_i = \sum_{j=1}^{6} C_{ij}^B (1 - k^2) S_j - h_{iq}^{(t)} H_q \mu^S \tag{6.40}$$

$$k^2 = \sum_{i}^{6} \sum_{j}^{6} \sum_{q}^{3} \left(C_{ji}^B\right)^{-1} h_{iq}^t h_{qj} \mu^S , \tag{6.41}$$

where T_i is the stress 6-vector, C_{ij}^B is the stress-strain modulus at constant induction, S_j is the strain 6-vector, h_{iq} is the piezoactive modulus, H_q is the magnetic intensity 3-vector, and μ^S is the permeability at constant strain (clamped permeability). The coupling factor k^2 is properly the material coefficient of electromechanical coupling defined for the lowest mode in the frequency range where elastic stiffness dominates the response of the continuum to applied forces. When the frequency of forced drive is increased, the concept of the electromechanical coupling factor must be enlarged to include strain deformation, which is expressible as a superposition of modes of shell vibration. Two such modal coupling factors of extended sense can be constructed, one based on strains accompanying modes in predominantly radial motion and one for predominantly axial motion. It is important to note that coupled radial and axial motion are understood to belong to shell vibrations in elastically coupled modes. From the canonical set (Eqs. (2.1) and (2.2)) it is seen that the total voltage E_1 corresponding to the total current I_1 and radial velocity $\dot{w}(x)$ is given by

$$E_1 = I_1 Z^b + \int \psi_{13}' \dot{w}(x) dS + \int \psi_{11}' \dot{u}(x) dS , \tag{6.42}$$

where $Z^b = Z^c + Z^l$, and ψ_{ij}' is the transduction coefficient (dimensions of volts per volume velocity). In terms of the displacement influence functions g_{33} and g_{13} and an equivalent (electromechanical/mechanical) driving pressure $\mathcal{P}$, the radial and axial displacements in coupled motion are

$$w(\omega, x) = \mathcal{P} g_{33}$$

$$u(\omega, x) = \mathcal{P} g_{13}. \tag{6.43}$$

For a constant current drive,

$$\mathcal{P} = -\psi_{31} I_1 \tag{6.44}$$

$$g_{33}(x, \xi_\Delta, \omega) = \sum_{n=1}^{\infty} \frac{W_n(x) W_n^\circ(\xi_\Delta)}{D_n^I(\omega)} \tag{6.45}$$

$$g_{13}(x, \xi_\Delta, \omega) = \sum_{n=1}^{\infty} \frac{U_n(x) W_n^\circ(\xi_\Delta)}{D_n^I(\omega)} \tag{6.46}$$

where

$$W_n^\circ = \int W_n(x) 2\pi a dx . \tag{6.47}$$

Here $W_n(x)$ is dimensionless, $W_n^\circ(\xi_\Delta)$ has the dimensions of m^2, and $D_n^I(\omega)$ the dimensions of N/m, so that g_{33} is interpreted as displacement per unit pressure. For convenience we introduce a complex number γ to account for leakage and define it by the relation

$$Z^b = \frac{Z^c}{\gamma} = \frac{j\omega N^2 \mu^S bl\chi}{2\pi a \gamma} \tag{6.48}$$

(see Eq. (5.10), which is repeated here for convenience). The symbol χ represents the complex eddy-current and hysteresis factor, and the other symbols are defined in the glossary. Inserting Eqs. (6.43) and (6.48) into Eq. (6.42) and factoring out $I_1 Z^b$, we reduce Eq. (6.42) to the form

$$E_1 = I_1 Z^b \left[1 + \sum_{n=1}^{\infty} \left(C_{EFF}^I \right)_n^2 \right] \tag{6.49}$$

where

$$\left(C_{EFF}^I \right)_n^2 = \frac{1}{Z^b} \left(-\psi_{13}' \psi_{31} \right) j\omega \; \frac{\left[\int W_n(x) 2\pi a dx \right] W_n^\circ(\xi_\Delta)}{D_n^I(\omega)}$$

$$+ \frac{1}{Z^b} \left(-\psi_{13}' \psi_{31} \right) j\omega \; \frac{\left[\int W_n(r) 2\pi r dr \;\; W_n^\circ(\xi_\Delta) \right]}{D_n^I(\omega)}$$

$$+ \frac{1}{Z^b} \left(-\psi_{11}' \psi_{31} \right) j\omega \; \frac{\left[\int U_n(x) 2\pi a dx \right] W_n^\circ(\xi_\Delta)}{D_n^I(\omega)} \tag{6.50}$$

Continued

$$+\frac{1}{Z^b}\left(-\psi'_{11}\psi_{31}\right)j\omega\frac{\left[\int U_n(r;\,x=l/2)2\pi r dr\right]W_n^\circ(\xi_\Delta)}{D_n^I(\omega)}. \tag{6.50}$$

The factor $D_n^I(\omega)$ will be discussed below.

Equation (6.50) is considered to be an effective squared coefficient of electromechanical coupling in the nth mode. The form shown can be reduced as follows. The second term and third term on the r.h.s. both vanish since the velocities in question lie in the surface, and thus do not constitute volume velocity as required by the canonical Eq. (2.1). Thus, Eq. (6.50) is reducible to two terms. Its numerical value depends on the values to be assigned to the transduction coefficients. From the geometrical configuration of the toroidal winding it is known [10] that

$$\psi'_{13} = \frac{h_{22}\mu^S bN\chi}{2\pi a^2} \quad \text{(dimensions, (V-sec/m}^3\text{) or (N/m}^2\text{A))}. \tag{6.51}$$

We can assume that application of positive I_1 causes the ring-shell (say) to expand in the radial direction. Actual numerical calculation shows that the shell contracts in the axial direction. We can conjecture, then, that

$$(\psi'_{11})(\psi'_{13}) > 0. \tag{6.52}$$

Equation (6.52) states that the product of the voltage per unit volume velocity induced in the coil due to constriction in the axial direction and the voltage per unit volume velocity induced by expansion in the tangential direction is positive. However, the mechanical transduction factor ψ_{31} (the equivalent pressure on the radial reference surface due to a positive current I_1) must be opposite in sign to ψ'_{13}, due to the antisymmetrical nature of piezomagnetic coupling. Hence we write

$$\psi_{31} = -\psi'_{13}. \tag{6.53}$$

Inserting Eqs. (6.51) and (6.52) into Eq. (6.50) yields the formula

$$\left(C_{EFF}^I\right)_n^2 = h_{22}^2\mu^S\chi\left(\frac{b}{2\pi a^3 l}\right)\gamma\frac{W_n^\circ(\xi_\Delta)}{D_n^I(\omega)}\left[\int W_n(x)2\pi a dx + 2\left(\frac{\psi'_{11}}{\psi'_{13}}\right)\int U_n(r;\,x=l/2)2\pi r dr\right]. \tag{6.54}$$

We designate this as the modal coefficient of electromechanical coupling. It contains a factor

$$D_n^I(\omega) = \rho_S bN_n[\omega]_n \tag{6.55}$$

where ρ_S is the static density, b the shell thickness, and N_n the mode shape normalization. To sufficient approximation,

$$N_n = \int\left(U_n^2 + W_n^2\right)2\pi a dx \quad \text{(dimensions, m}^2\text{)}. \tag{6.56}$$

The symbol $[\omega]_n$ (dimensions, $\sec^{-2}$) requires special discussion. To define it we first return to Eqs. (6.25) and (6.26) and assemble the following quantity:

$$M_n = \frac{\rho_S b N_n (2\pi a l)^2}{\left[\int W_n(x)\, 2\pi a dx + 2 \int U_n(r; x = l/2)\, 2\pi r dr\right] W_n^{\circ}(\xi_\Delta)}$$

$$M_n = \zeta_n M_S \tag{6.57}$$

$$\zeta_n = \frac{2\pi a l \oint \left(U_n^2 + W_n^2\right) dS}{\left[\int W_n(x)\, 2\pi a dx + 2 \int U_n(r; x = l/2)\, 2\pi r dr\right] W_n^{\circ}(\xi_\Delta)} \tag{6.58}$$

$$M_s = 2\pi a l \rho_S b\,. \tag{6.59}$$

We designate M_n as the modal mass (kilograms), while M_S is the static mass. The modal density is

$$\rho_n = M_n / 2\pi a l b\,. \tag{6.60}$$

Returning to Eq. (6.55), we now write $[\omega]_n$ explicitly in terms of modal frequency ω_n:

$$[\omega]_n = \omega_n^2 - \omega^2 + \frac{j\omega(R' + R'')}{M_n}, \tag{6.61}$$

in which R' (as noted earlier, in Eq. (6.9), etc.) is a mechanical resistance (Newton-seconds per meter) directly due to electromechanical coupling, R'' is a mechanical resistance independent of such coupling (internal function, etc.), and ω_n is the nth-order modal (resonant) frequency experimentally obtained by driving the ring shell through a range of many frequencies with a constant amplitude of current. These loss resistances can be assumed to be independent of mode number and frequency. Inclusion of such dependence can be made without changing the form of Eq. (6.61).

There is another modal frequency Ω_n, which occurs at a forced drive of constant voltage. As explained in Eq. (6.33), it appears in the form (repeated here for convenience) of

$$[\Omega]_n = \Omega_n^2 - \omega^2 + \frac{j\omega(R'' + \mathfrak{A})}{M_n} \tag{6.62}$$

$$\mathfrak{A} = \frac{\psi_{31}\psi_{13}' Z^l (2\pi a l)^2}{Z^c(Z^l + Z^c)} \quad \text{(dimensions, N-sec/m).} \tag{6.63}$$

From classical theory the relation between Ω_1 and ω_1 is established by the coefficient of electromechanical coupling k^2. We now broaden the concept involved in this relation and define a complex coefficient of electromechanical couping $\mathcal{K}_n^2$ such that

$$[\omega]_n = [\Omega]_n\left(1 - \mathcal{K}_n^2 \chi \gamma\right) \tag{6.64}$$

where

$$\mathcal{K}_n^2 \equiv \frac{h_{22}^2 \mu^S}{Y_n} \tag{6.65}$$

$$Y_n = \rho_n a^2 [\Omega]_n . \tag{6.66}$$

Substitution of appropriate parts of Eqs. (6.55) through (6.60) into Eq. (6.50) leads to two forms for the effective modal coupling:

$$\left(C_{EFF}^I\right)_n^2 = \frac{h_{22}^2 \mu^S \chi\gamma}{a^2 \rho_n [\omega]_n} \tag{6.67}$$

and

$$\left(C_{EFF}^I\right)_n^2 = \frac{\mathcal{K}_n^2 \chi\gamma}{1 - \mathcal{K}_n^2 \chi\gamma} . \tag{6.68}$$

The definition of $\left(C_{EFF}^I\right)_n$ contained in these formulas shows that it is a complex number and a function of frequency. If for any mode the drive frequency ω is chosen such that $\omega << \omega_n$(or Ω_n) and if losses (both electrical and mechanical) are neglected, then Eq. (6.68) reduces to

$$(k_{EFF})_n^2 = \frac{k_n^2 \gamma}{1 - k_n^2 \gamma} \tag{6.69}$$

$$k_n^2 = \frac{h_{22}^2 \mu^S}{\rho_n a^2 \Omega_n^2} . \tag{6.70}$$

Thus Eq. (6.70), by construction, is the modal analog of the conventional low-frequency coefficient of electromechanical coupling.

The coupling coefficients defined in Eqs. (6.69) and (6.70) depend on the modal number n only through the modal density and the modal frequency. Since on physical grounds the infinite sum called for by Eq. (6.49) must converge, ultimately $\left(C_{EFF}^I\right)_n$ must vanish as n increases. Furthermore, since the real part of the ratio E_1/I_1 in Eq. (6.49) can never be negative, it is required that

$$R^b [1 + \mathrm{Re}\{\mathcal{C}\}] - jX^b \,\mathrm{Im}\{\mathcal{C}\} > 0 \tag{6.71}$$

where

$$Z^b = R^b + jX^b \tag{6.72}$$

$$\mathcal{C} = \mathrm{Re}\{\mathcal{C}\} + j\,\mathrm{Im}\{\mathcal{C}\} \tag{6.73}$$

$$\mathcal{C} = \sum_{n=1}^{\infty} \left(C_{EFF}^{I}\right)_n^2 .$$

These requirements are automatically satisfied in any real embodiment of the transducer of this article under stable operating conditions.

7. HYSTERESIS AND EDDY-CURRENT LOSSES

A magnetostrictive ring excited by a toroidal coil constitutes a magnetic circuit without gap whose operation characteristics can be described using a flux Φ vs magnetomotive force $\mathcal{F}$ diagram (alternatively on the *B-H* plane). An applied direct current bias $\mathcal{F}_{dc}$ fixes the dc flux Φ_{dc} at the quiescent operating point of the ring. An alternating signal $\mathcal{F}_{ac}\, e^{j\omega t}$ may then be applied to the circuit so that the total flux at any moment is $\Phi_{dc} + \Phi_{ac} \exp(j\omega t)$. Considering only ac quantities, we write the dependence of flux upon drive in the form

$$\Phi_{ac}^{(i)} = \Phi_{ac}^{(i)}\left(\mathcal{F}_{ac}^{(i)}, \mathcal{F}_{dc}\right)$$

$$\Phi_{ac}^{(d)} = \Phi_{ac}^{(d)}\left(\mathcal{F}_{ac}^{(d)}, \mathcal{F}_{dc}\right),$$

in which the superscripts *i*, *d* represent increasing and decreasing parts, respectively, of the Φ_{ac}-vs-$\mathcal{F}_{ac}$ diagram. This description means that an applied alternating magnetomotive force induces an ac flux which has one of two values for any $\mathcal{F}_{ac}$, one appearing when $\mathcal{F}_{ac}$ lies on an increasing part of a cycle and the other when the same $\mathcal{F}_{ac}$ lies on a decreasing part of a cycle. The area enclosed by the loop on the $\Phi - \mathcal{F}$ plot represents the work done by $\mathcal{F}_{ac}$ in driving the flux Φ_{ac} around the magnetic circuit. The applied $\mathcal{F}_{ac}$ also does work against the elastic impedance of the ring due to magnetostriction.

The conventional model for the minor or dynamic hysteresis loop is a narrow ellipse drawn on a *B-H* plane and centered at the operating point of the major (or static) hysteresis loop. The large axis of this ellipse has a slope $\mu_i\mu_0$ (= incremental permeability) which because of the narrowness of the ellipse can represent the value of the incremental permeability during the entire cycle of applied magnetomotive force. The value of μ_i is customarily given as a real number. However, since energy is consumed over a cycle of hysteresis, it is required to make μ_i complex. Thus, one writes

$$\frac{\Delta B}{\mu_0 \Delta H} = \mu_i \chi, \qquad \chi = \chi_h e^{-j\eta}. \tag{7.1}$$

The negative sign indicates that ΔB lags $\mu_0 \Delta H$ by the hysteresis angle η. The core impedance of a magnetostrictive ring may thus be written (in the pure hysteretic version of the Butterworth and Smith model [10]) as

$$Z_c = \frac{E}{I_c} = j\omega L_c e^{-j\eta}, \qquad L_c = \frac{N^2 |\mu^S| bl}{2\pi a} \tag{7.2}$$

$$|\mu^S| = |\mu_i| \mu_0 \chi_h . \tag{7.3}$$

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The real part of Z_c contributes an electrical resistance, which accounts for cyclical loss. When the frequency of operation is high enough, the core loss increases due to eddy currents. Considering eddy currents alone, we again write

$$Z_c = \frac{E}{I_c} = j\omega L_c e^{-j\zeta}, \qquad L_c = \frac{N^2|\mu^S|bl}{2\pi a} \tag{7.4}$$

$$|\mu^S| = |\mu_i|\mu_0\chi_e. \tag{7.5}$$

Combining both hysteretic and eddy-current losses in one formula, we have

$$Z_c = j\omega L_c e^{-j(\eta+\zeta)}, \qquad L_c = \frac{N^2|\mu^S|bl}{2\pi a} \tag{7.6}$$

$$|\mu^S| = |\mu_i|\mu_0\chi_0 \quad , \quad \chi_0 = \chi_h\chi_e. \tag{7.7}$$

The numerical evaluation of the eddy-current angle is based on dimensional analysis. One first defines a critical frequency for elimination of thickness t, below which the above-noted simplified representation of the eddy-current loss is valid. This f_c is determined by [11, 12]

$$f_c = \frac{2\rho_e}{\pi|\mu_i|\mu_0 t^2}, \tag{7.8}$$

where

ρ_e = resistivity (dimensions, Ωm)

t = lamination thickness (dimensions, m).

For frequencies $f < f_c$ the loss factor χ due to eddy currents is conventionally given by the formula,

$$\chi = \left(\frac{\tanh\sqrt{\frac{jf}{f_c}}}{\sqrt{\frac{jf}{f_c}}}\right)$$

At low frequency the angle ζ of χ is

$$\zeta = \tan^{-1}\left[\frac{\frac{1}{3}\left(\frac{f}{f_c}\right) - \frac{17}{315}\left(\frac{f}{f_c}\right)^3}{1 - \frac{2}{15}\left(\frac{f}{f_c}\right)^2 + \frac{62}{2835}\left(\frac{f}{f_c}\right)^4}\right]. \tag{7.9}$$

The angle associated with hysteresis appears to be not directly predictable by formulas based on material parameters. However, by measurement one can find the angle $\eta = \angle\mu^S$ at very low frequency, i.e., at a frequency for which the eddy currents are negligible. This estimate of η is often used in practice.

The transduction force factor Z_{em} (i.e., the ratio E/v) is known from the Butterworth and Smith model to be

$$Z_{em} = \frac{h_{\theta\theta}\mu^S blN}{a}. \tag{7.10}$$

Introducing hysteresis and eddy-current angles, we write

$$Z_{em} = \frac{h_{\theta\theta}|\mu_i|\mu_0\chi_e\chi_h blN}{a} e^{-j(\eta+\zeta)}. \tag{7.11}$$

Since the motional impedance Z_{Mot} in vacuo is

$$Z_{\text{Mot}} = \frac{Z_{em}^2}{Z_m}, \quad Z_m = \text{mechanical impedance} \tag{7.12}$$

it is seen that at mechanical resonance (at Z_m = real number) the phasor

$$(Z_{\text{Mot}})_{\text{Res}} = \frac{|Z_{em}^2|}{\text{Re}\, Z_m} e^{-2j\beta}, \quad \beta = \eta + \zeta \tag{7.13}$$

$$|Z_{em}|^2 = \frac{h_{\theta\theta}^2(\mu_i\mu_0)^2(\chi_e\chi_h)^2 b^2 l^2 N^2}{a^2}. \tag{7.14}$$

Thus, $(Z_{\text{Mot}})_{\text{Res}}$ makes an angle of -2β with the horizontal on a motional impedance diagram.

Geometrical Aspects of the Motional Circle

The motional impedance Z_{Mot} $\left(= Z_{em}^2/Z_m\right)$, when plotted on a R_{Mot} vs X_{Mot} plot, has geometrical properties strongly determined by the function $1/Z_m$ (i.e., by an inversion). The properties of inversions are as follows. If $z = x + iy$, $w = u + iv$, then the transformation $w = 1/z$ carries circles into circles (considering a line as the limiting case of a circle). Thus if we are given the equation of a circle (or line)

$$a(x^2 + y^2) + bx + cy + d = 0, \tag{7.15}$$

the transformation $w = 1/z$ performed on this equation leads to the transformed circle (or line)

$$d(u^2 + v^2) + bu - cv + a = 0. \tag{7.16}$$

In particular the line $x = C_1$ transforms into the circle

$$u^2 + v^2 - \frac{u}{c_1} = 0, \tag{7.17}$$

which is tangent to the v axis at the origin. Now, near mechanical resonance the mechanical impedance Z_m is given by

$$Z_m \approx r_m + j\omega m + \frac{K}{j\omega} = r_m + jx_m . \tag{7.18}$$

On a r_m vs x_m plot, it is seen that Z_m is the straight line r_m = const. (say $r_m = C_1$ where C_1 is the resistance at zero reactance). Assuming Z_{em} is real and changes little for all frequencies in the vicinity of resonance, it is seen in the first approximation that the line $r_m = C_1$ transforms into a motional circle tangent to the line $R_{\text{Mot}} = 0$ with diameter $|Z_{em}^2|/C_1$ and center $|Z_{em}^2|/2C_1$. The circle passes through the origin $X_{\text{Mot}} = 0 = R_{\text{Mot}}$. When Z_{em}^2 is complex, with angle $-\ 2\beta$ $(\beta = \eta + \zeta)$, the transformed line is a circle with center at

$$R_{\text{Mot}} = \frac{|Z_{em}|^2}{2C_1} \cos 2\beta \tag{7.19}$$

$$X_{\text{Mot}} = -\frac{|Z_{em}|^2}{2C_1} \sin 2\beta \tag{7.20}$$

and diameter

$$|Z_{\text{Mot}}|_{\text{Res}} = \frac{|Z_{em}|^2}{C_1} . \tag{7.21}$$

This diameter terminates at the point $X_{\text{Mot}} = 0 = R_{\text{Mot}}$.

Eddy-Current and Hysteresis Plots

A sheet height l and thickness x, where $-a < x < +a$, magnetized perpendicular to the section $2al$ with magnetic flux density $B(x)$, satisfies a boundary value $B(a)$ according to the formula [12]

$$B(x) = B(a)\,\frac{\cosh mx}{\cosh ma}, \quad (\text{Wb/m}^2) \tag{7.22}$$

$$m = \pm\frac{j\mu_i(a)\mu_0\omega}{\sigma} \quad (\text{m}^{-1}) \tag{7.23}$$

where

$\mu_i(a)$ = permeability (dimensionless) at the surface at very low frequency (i.e., at zero eddy-current)

$\mu_0 = 4\pi \times 10^{-7}$ (Wb/m-A)

ω = radian frequency (sec^{-1})

σ = resistivity (V-m/A).

The mean value of B over the thickness is found by integration to be

$$B_{\text{mean}} = B(a) \frac{\tanh ma}{ma}. \tag{7.24}$$

In terms of a mean magnetization M_{mean} and mean magnetic intensity, we can also define

$$B_{\text{mean}} = \mu_0 H_{\text{mean}} + M_{\text{mean}}. \tag{7.25}$$

Writing the magnetization at the surface as

$$M(a) = \mu_i(a)\mu_0 H(a) \tag{7.26}$$

and dividing by $H(a)$, we have

$$\frac{B_{\text{mean}}}{H(a)} = \mu_0 \frac{H_{\text{mean}}}{H(a)} + \frac{M_{\text{mean}}}{H(a)}. \tag{7.27}$$

Since for a ferromagnetic substance

$$\mu_0 \frac{H_{\text{mean}}}{H(a)} << \frac{M_{\text{mean}}}{H(a)}, \tag{7.28}$$

one has approximately

$$M_{\text{mean}} = B_{\text{mean}} = B(a) \frac{\tanh ma}{ma} \tag{7.29}$$

$$= \mu_i(a)\mu_0 H(a) \frac{\tanh ma}{ma}$$

or

$$\frac{M_{\text{mean}}}{H(a)} = \mu_i(a)\mu_0 \frac{\tanh ma}{ma}. \tag{7.30}$$

One next defines the mean permeability for ferromagnetic materials ($\bar{\mu}$) by noting that

$$M_{\text{mean}} = \bar{\mu}\mu_0 H(a), \tag{7.31}$$

so that

$$\bar{\mu} = \frac{\tanh ma}{ma} \mu_i(a)\chi = |\mu_i(a)\chi| e^{-j\beta}. \tag{7.32}$$

Now since $\bar{\mu}$, $\mu_i(a)$ are complex numbers,

$$\beta = \angle\mu = \angle\mu_i(a) + \angle\chi \tag{7.33}$$

$$= \text{hysteresis angle} + \text{eddy-current angle}.$$

It is convenient to plot $|\chi|$ vs ma by selecting as parameter $\mu_i(a) = |\mu_i(a)|e^{-j\eta} = \mu_i(a)$ $(\cos\eta - j\sin\eta)$ for the particular material (a procedure equivalent to choosing the permeability for the condition of zero eddy current as a real number together with a choice of hysteresis angle η). In this calculation the angle β is omitted. Thus, we plot

$$|\chi| = \left|\frac{\tanh ma}{ma}\right| \text{ vs } ma, \tag{7.34}$$

and note that $0 < |\chi| < 1$ (which means that the average permeability is less than or equal to the surface permeability). When the eddy-current angle ζ ($\angle\chi$) is taken into account, we plot β vs ma, where (as before) $\beta = \eta + \zeta$. The joint plot of $|\chi|$ and β vs ma is called a Fukushima plot.

Fukushima plots may be used to obtain $\mu_h(\omega)$. At mechanical resonance ($\omega = \omega_R$) we measure the total angle β. By successive trials using the Fukushima plot we obtain $\mu_h(\omega_R)$ and $\angle\chi(\omega_R)$. The values of μ_h and χ at any other frequency are then obtained by formulas noted above.

8. LEAKAGE FLUX

The introduction of leakage flux into the magnetic circuit of the toroidally wound magnetostrictive core requires careful handling. In this section the problem is reviewed and given a formulation useful for numerical calculation.

When the leakage flux Φ^l is not negligible relative to the flux due to the magnetomotive force, the total impedance of the windings and the core is given by

$$Z^b = R^l + j\omega L^l + j\omega(L_1 - jL_2) = Z^l + Z^c \tag{8.1}$$

where ωL_2 is the core loss resistance due to eddy currents and hysteresis, R^l is the coil resistance, and L^l is the leakage inductance. Thus,

$$Z^b = R^b + jX^b; \quad R^b = R^l + \omega L_2; \quad X^b = \omega(L_1 + L^l). \tag{8.2}$$

The electric equation which includes leakage impedance and eddy-current plus hysteresis losses is

$$\begin{aligned} E_1 &= \frac{j\omega\chi_R\mu^S blN^2}{2\pi a} I_1 \\ &+ \frac{\omega\chi_I\mu^S blN^2}{2\pi a} I_1 \\ &+ \oint \psi'_{13}\dot{w}(x)dS + \oint \psi'_{11}\dot{u}(x)dS \\ &+ Z^l I_1 . \end{aligned} \tag{8.3}$$

Here,

$$\mu^S = \mu_i \mu_0 \tag{8.4}$$

where

μ_i = incremental permeability.

The leakage impedance can be calculated from the formula

$$Z^l = R^l + j\omega \frac{N^2}{2\pi a} \mu_0 (A_{\text{coil}} - A_{\text{core}}). \tag{8.5}$$

From the above formulas it is seen that

E = voltage drop due to leakage

+ voltage induced by magnetization of the core

or

$$E = IZ^l + j\omega N \Phi_c' \tag{8.6}$$

where the core flux Φ_c' is defined by

$$j\omega N \Phi_c' = \frac{j\omega\mu^S b l N^2 \chi}{2\pi a} I + \oint \psi_{13}' \dot{w} dS + \oint \psi_{11}' \dot{u}_1 dS. \tag{8.7}$$

Otherwise expressed, the magnetomotive force in the magnetic core circuit is

$$NI = (j\omega N \Phi_c') \frac{2\pi a}{j\omega\mu^S b l N \chi} - \frac{\oint \psi_{13}' \dot{w} dS + \oint \psi_{11}' \dot{u} dS}{j\omega\mu^S b l N \chi}. \tag{8.8}$$

Let Φ_l be the magnetic flux leakage. The magnetomotive force which supports this leakage is

$$NI = j\omega N \Phi_l. \tag{8.9}$$

The total magnetomotive force required to generate flux Φ_c' in the presence of motion $\dot{w}$, $\dot{u}$ and flux leakage Φ_l is the sum of Eqs. (8.8) and (8.9).

9. ACOUSTIC LOADING

The acoustic loading of the free-flooded ring-shell presents the following problem. Ring-shells can be described as having either one surface (reference surface) or two surfaces (inner and outer). According to thin-shell theory the calculation of elastic deformation is best done on the reference surface, the shell thickness being integrated out. Acoustic radiation theory, however, requires the two-surface description since acoustic pressures depend on true normal surface velocities. In the analysis presented above, acoustic loading is accounted for by the factor $\mathcal{G}(m, n)$, which was defined as the pressure on reference surface band m due to the normal velocity of reference surface band n. To make the elastic and acoustic fields compatible we define four surface-radiation Green's functions, $G(m_A, n_A)$, $G(m_A, n_B)$, $G(m_B, n_A)$, $G(m_B, n_B)$, related to but not identical with $\mathcal{G}$, in which A, B refer respectively to the inner and outer surfaces of the ring-shell. Thus $G(m_B, n_A)$, for example, signifies the pressure on the exterior surface band m due to the velocity on the interior surface band n. We also define two thickness correction factors $f(A)$ and $f(B)$, which convert the magnitudes of inner and outer surface velocities respectively to reference surface velocities, and attach to each factor one of four sign symbols α, β, γ, δ. These symbols, though differently defined, are either + or −, depending on certain sign conventions associated with force and velocity.

Using these newly defined quantities, we expand the reference surface radiation Green's function $\mathcal{G}$ into a sum of four, inner and outer shell, surface radiation Green's functions G as follows:

$$\begin{aligned}\mathcal{G}(m, n) = {} & G(m_A, n_A) f(A)\alpha + G(m_A, n_B) f(B)\beta + G(m_B, n_A) f(A)\gamma \\ & + G(m_B, n_B) f(B)\delta . \end{aligned} \tag{9.1}$$

In this constriction the reference band m, which is a single surface, is replaced by two surfaces m_A, m_B, representing the inner and outer surface on a thickness b of the shell. This replacement, necessary as it is for the acoustic loading to be calculated, must be corrected for by the correction factor $f(A)$, or $f(B)$, as already noted, since all elastic calculations are made with regard to the reference surface only. Further examination of Eq. (9.1) shows that it is the algebraic sum of all pressure loads identified with reference band m due to all velocities identified with reference band n. The functions $G(m_A, n_A)$, etc., are complex numerics obtained by a special computer program (SHIP [13]) whose derivation and significance are discussed in this section (page 48).

Calculation of Thickness Correction Factors

As noted, the solution of the acoustic-radiation problem requires a three-dimensional dynamic elasticity analysis. Since such analyses are either intractable or exceptionally tedious, we have used an approximate analysis, judged here to be of sufficient validity for our work. To this end we apply to the dynamic motion of the finite-length shallow magnetostrictive ring-shell with moderately thick walls, the theory of the axially symmetric extensional vibrations of a free circular disk worked out by Gustafsson and Kane [14]. In these authors' theory a thick disk of axial height $2h$, inner radius a^*, and outer radius b^*, is assumed to have displacements of the form

$$\begin{aligned} u_r &= q_r(r, t) \\ u_\theta &= 0 \\ u_z &= \frac{Z}{h}\, q_z(r, t), \end{aligned} \tag{9.2}$$

in which the radial q_r and axial q_z are to be determined by the use of two potentials ϕ_1, ϕ_2:

$$\left(\nabla^2 + \delta_1^2\right)\phi_1 = 0; \qquad \left(\nabla^2 + \delta_2^2\right)\phi_2 = 0$$

$$q_r = \frac{\partial \phi_1}{\partial r} + \frac{\partial \phi_2}{\partial r}; \qquad q_z = \sigma_1 \phi_1 + \sigma_2 \phi_2 \tag{9.3}$$

where δ_1, δ_2, σ_1, σ_2 are frequency-dependent parameters (see Appendix A page). The complementary solutions of the differential equations are known to be of the form

$$\phi_1 = A_1 J_0(\delta_1 r) + B_1 Y_0(\delta_1 r)$$

$$\phi_2 = A_2 J_0(\delta_2 r) + B_2 Y_0(\delta_2 r), \tag{9.4}$$

in which J_0, Y_0 are Bessel functions of the first and second kind, and A_1, A_2, B_1, B_2 are to be determined from the boundary conditions. Choosing the particular case of traction-free vibrations and substituting these expressions for ϕ_1, ϕ_2 into the boundary conditions (i.e., zero normal stress and zero sheer at each edge) leads to a matrix in the four quantities A_1/B_2, A_2/B_2, B_1/B_2, B_2 in terms of the unknown B_2 (see Ref. 14 for details). To fix B_2, we assume that the radial displacement q_r has a prescribed value at the particular radius ($r = a$) corresponding to the mean radius of a ring-shell viewed as an elastically thin shell. Thus ϕ_1, ϕ_2 are determined and with them q_r, q_z and, finally, u_r, u_z. The thickness correction factors $f(A)$, $f(B)$ are then

$$f(A) = \frac{q_r(a, t)}{q_r(a^*, t)}; \qquad f(B) = \frac{q_r(a, t)}{q_r(b^*, t)}. \tag{9.5}$$

A similar set of end correction factors can also be defined:

$$f_{\mathrm{end}}(A) = \frac{q_z(a, t)}{q_z(a^*, t)}; \qquad f_{\mathrm{end}}(B) = \frac{q_z(a, t)}{q_z(b^*, t)}. \tag{9.6}$$

The signs to be associated with these factors are discussed in the following section.

Sign Conventions

The application of Eq. (9.1) requires a determination of the sign symbols α, β, γ, δ. Each of these symbols is a ratio of the sign attributed to the force on band m and that of the velocity on band n. The pair of numbers m, n are as found in the symbol $G(m, n)$. These are the conventions for determining force and velocity signs:

1. A normal surface velocity pointing into the ring material is negative. Thus, a normal surface velocity pointing into the medium is positive.

2. A force per unit area on the inner surface is negative if the ring expands under the force and positive if it contracts. A force per unit area on the outer surface is positive if the ring expands and negative if it contracts.

To illustrate the application of these conventions, consider the following example. Let there be a reference surface having three bands a, b, c and one endband d (Fig. 1a). The actual surfaces are labeled 1 through 8, as shown in Fig. 1b. We first calculate $\mathcal{G}(c, a)$. From Eq. (9.1), this is written

$$\mathcal{G}(c, a) = G(3,1)f(1)\alpha + G(3,8)f(8)\beta + G(6,1)f(1)\gamma + G(6,8)f(8)\delta. \quad (9.7)$$

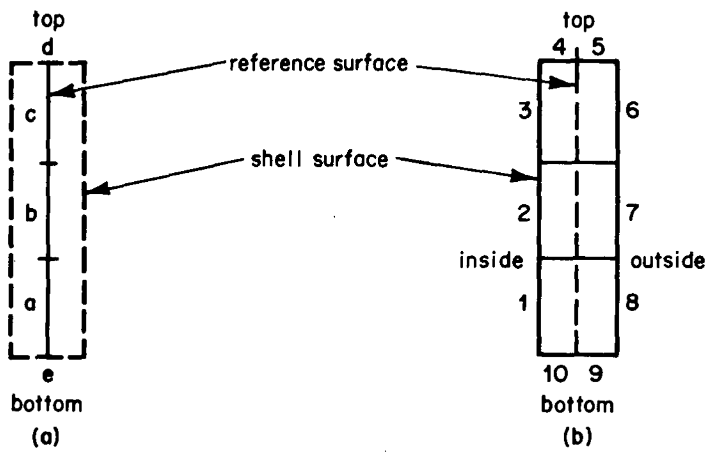

Fig. 1—Subdivision of a cylindrical shell into three sidebands and two endbands: (a) Reference surface cross section and (b) Actual shell surface cross section

To determine α, β, γ, δ we assume that the ring is expanding both radially and axially. This choice is arbitrary. Then the inside surface velocity is minus, the outside surface velocity is plus, and the end velocities are both plus. The inside (medium-induced) force is minus, the outside force is plus, and the end forces are both plus. Thus, for example,

$$\alpha = \frac{\text{force sign on band 3}}{\text{velocity sign on band 1}} = \frac{-}{-} = +. \quad (9.8)$$

Similarly,

$$\beta = \frac{\text{force sign on band 3}}{\text{velocity sign on band 8}} = \frac{-}{+} = - \quad (9.9)$$

$$\gamma = - \quad (9.10)$$

$$\delta = +. \quad (9.11)$$

When we require $\mathcal{G}(c, d)$, then, following the above rules, we find

$$\mathcal{G}(c, d) = -G(3,4)f(4) - G(3,5)f(5) + G(6,4)f(4) + G(6,5)f(5). \quad (9.12)$$

The above sign conventions have been found to agree with signs established by more complex displacement-force considerations.

When the shell thickness is not vanishingly thin, an area correction must be added to each term in Eq. (9.1). Let

$S(A)$ = shell area of inner surface

$S(B)$ = shell area of outer surface

$S(R)$ = shell area of reference surface.

Then the addition of area correction factors modifies Eq. (9.1) to read

$$\begin{aligned}\mathcal{G}(m,n) = {} & G(m_A, n_A) f(A)[S(A)/S(R)]\,\alpha \\ & + G(m_A, n_B) f(B)[S(A)/S(R)]\,\beta \\ & + G(m_B, n_A) f(A)[S(B)/S(R)]\,\gamma \\ & + G(m_B, n_B) f(B)[S(B)/S(R)]\,\delta\,. \end{aligned} \tag{9.13}$$

Calculation of Acoustic Loading From Known Surface Velocities

The acoustic loading on the vibrating shell is determined in this report by use of the surface Helmholtz integral formulation, a description of which follows. Let time be given by the real part of exp $(j\omega t)$. The complex amplitude of acoustic pressure in a fluid of density ρ_f at a frequency ω is then given by $p = j\omega\rho_f\Phi$, where Φ is the acoustic velocity potential. The acoustic particle velocity $\mathbf{v} = -\nabla\Phi$, so that one can write $\nabla p = -j\omega\rho_f\mathbf{v}$. At the surface of the ring where the displacement is $\mathbf{u}$, the component of this equation in the direction of the normal $\mathbf{n}$ is

$$\left(\frac{\partial p}{\partial n}\right)_{\text{surface}} = -j\omega\rho_f(j\omega\mathbf{u}\cdot\mathbf{n}) = \omega^2\rho_f\mathbf{u}\cdot\mathbf{n} = \omega^2\rho_f v\,. \tag{9.14}$$

To find the acoustic pressure loading $p(\mathbf{x})$ at frequency ω on any point $\mathbf{x}$ of the surface $S(\mathbf{x})$, we use the Helmholtz integral equation:

$$2\pi p(\mathbf{x}) = \oint_S p(\mathbf{x}_0)\,\frac{\partial}{\partial n}\,G_\omega(\mathbf{x}|\mathbf{x}_0|)\,dS(\mathbf{x}_0) + j\omega\rho_f\oint \frac{\partial p(\mathbf{x}_0)}{\partial n}\,G_\omega(\mathbf{x}|\mathbf{x}_0)\,dS(\mathbf{x}_0). \tag{9.15}$$

In an unbounded medium,

$$G_\omega(\mathbf{x}|\mathbf{x}_0) = \frac{\exp\left(-jk|\mathbf{x}-\mathbf{x}_0|\right)}{|\mathbf{x}-\mathbf{x}_0|}\,. \tag{9.16}$$

The complexity of Eq. (9.15) is so great that one usually resorts to numerical means to accomplish the required integration. Thus we divide the surface of the ring-shell into M subsegments by transversely segmenting the axial length and radially segmenting the ends (Fig. 2). We label the bands as shown, with the requirement that the number of bands on the inner and outer surfaces be equal and of the same width and that the number of bands on each end also be equal and of the same width. The Helmholtz integral is then written as

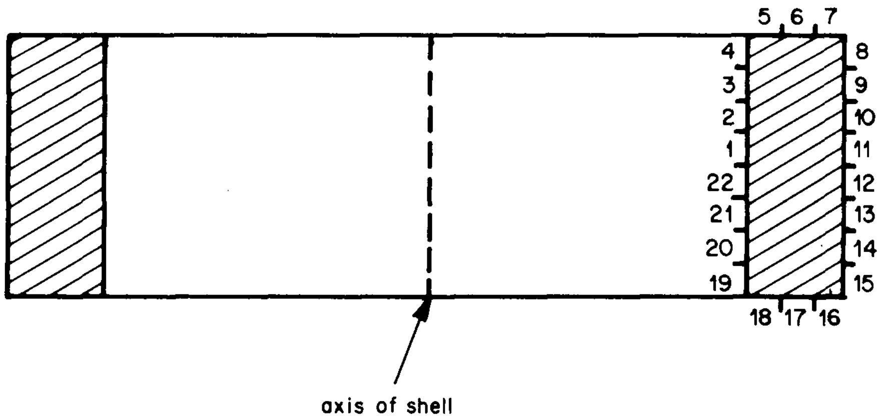

Fig. 2—Numbering of the bands on a cylindrical shell for eight sidebands and three top (and bottom) endbands

$$2\pi p(s_i) = \oint p(s_j) \frac{\partial}{\partial n_j} \left[\frac{e^{-jkR(s_i,s_j)}}{R(s_i, s_j)} \right] dS(s_j)$$

$$+j\omega\rho_f \oint v(s_j) \frac{e^{-jkR(s_i,s_j)}}{R(s_i, s_j)} dS(s_j), \qquad i = 1, 2, \ldots, M. \tag{9.17}$$

Since the problem is axisymmetric, the pressure and the normal velocity must be independent of azimuth angle. To facilitate numerical work we choose the subsegments so small as to make $p(s_i)$ and $v(s_j)$ substantially constant in the integral over the sth band, we rewrite the above equations in the following way:

$$p(s_l) - \sum_{i=1}^{M} p(s_i)P_{il} = \sum_{i=1}^{M} v(s_i)Q_{il}, \qquad l = 1, 2, \ldots, M \tag{9.18}$$

where

$$P_{il} = \frac{1}{2\pi} \int \frac{\partial}{\partial n_i} \frac{e^{-jkR(s_i,s_j)}}{R(s_i, s_j)} dS(s_i) \tag{9.19}$$

$$Q_{il} = \frac{j\omega\rho_f}{2\pi} \int \frac{e^{-jkR(s_i,s_j)}}{R(s_i, s_j)} dS(s_i). \tag{9.20}$$

In operator notation, the set of equations may be written as

$$\sum_{i=1}^{M} P_{li}^{0} p(s_i) = \sum_{i=1}^{M} Q_{li} v(s_i), \qquad l = 1, 2, \ldots, M \tag{9.21}$$

where

$$P_{li}^{(0)} = \delta_{li} - P_{li}. \tag{9.22}$$

Since this is a square matrix of simultaneous algebraic equations the solution is found by direct inversion:

$$p(s_l) = \sum_{i=1}^{M} G(s_l, s_i) v(s_i). \tag{9.23}$$

Thus, for every known surface velocity at point s_i one can calculate a surface pressure at point s_l. The calculation of the matrix of coefficients P_{il} and Q_{il} is accomplished by the high-speed algorithm (SHIP), which overcomes tedious numerical difficulties involved by directly evaluating Eqs. (9.18), (9.19), and (9.20). The free-space Green's function is expanded in cylindrical coordinates, so that

$$\frac{e^{-ikR}}{R} = \frac{1}{2} \sum_{m} \epsilon_m \cos(m\theta) \int_0^\infty \frac{J_m(l, r) J_m(lr_0)}{(l^2 - k^2)^{1/2}} \, l \exp\left[(l^2 - k^2)^{1/2} |Z - Z_0|\right] dl. \tag{9.24}$$

In this form the spatial parts of the integrations in Eqs. (9.19) and (9.20) can be done analytically, leaving each matrix element as an infinite integral over a single variable l. These integrals are evaluated by numerical integration up to a value of l which is sufficiently large to enable the Bessel functions to be replaced by their asymptotic forms. Once this point has been reached in all cases the remainder of the integration can be evaluated analytically. An effort is made to avoid duplicating calculations (e.g., matrix elements which are known by symmetry to be equal are evaluated only once; all Bessel functions used in the program are calculated once and stored in an array; trigonometric functions, where possible, are evaluated by recursion, etc.).

Characteristic frequencies (frequencies for which the matrix P_{li}^{0} is ill conditioned) are usually not an important consideration in problems concerning rings. The lowest characteristic wave number is always greater than π/b, so that there can be no characteristic wave numbers unless the ring is more than 1/2 wavelength thick. In practice, rings are almost never driven at such a high frequency.

CONCLUSION TO PART 1

In this report the modal analysis of elastic continuum has been joined to the analysis of multilooped electric circuits and multipole acoustic radiation. Since the acoustic reaction of the medium depends on normal surface velocities, the equations of motion have been formulated as integral equations in the unknown velocities (including non-normal components). Piezoactivity has been accounted for by electromechanical transduction factors and coupling coefficients. The important concepts of modal mass and modal coupling have been defined and discussed. Although the composite theoretical structure of all fields is complete, its limitations must be borne in mind. They are as follows: (a) The elastic, electromagnetic, and acoustic fields are treated as linear, small-amplitude fields (i.e., high power effects are not considered). (b) Whereas the acoustic loading may be high, the internal dissipation losses due to friction, hysteresis eddy currents, etc., are considered modest. (c) The magnetostrictive shell is elastically considered to be monolayer rather than multilayer, meaning that the scroll nature of a true ring-shell transducer is not considered when the equations of motion are written, although the scroll nature is considered when hysteresis and eddy currents are analyzed. (d) Thin-shell theory is used throughout, except in the problem of acoustic radiation which is advantageously treated by using a finite thickness of shell. Plausible thickness correction factors have been deduced to satisfy the radiation requirement.

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Appendix A

CONSTANT VOLTAGE AND CONSTANT CURRENT DRIVE

If for a given frequency we elect to calculate electrical impedance Z_{ij}^E, then we use the $E - p$ canonical set. In matrix form this set is

$$E_i = \sum_{j=1}^{\mathfrak{N}} Z_{ij}^b I_j + \sum_{k=1}^{\mathfrak{M}} \oint \psi'_{ik} v_k dS_k , \quad i = 1, 2, \ldots \mathfrak{N} \tag{A1}$$

$$p_q = \sum_{r=1}^{\mathfrak{N}} \psi_{qr} I_r + \sum_{s=1}^{\mathfrak{M}} \oint Z_{qs}^I v_s dS_s , \quad q = 1, 2, \ldots \mathfrak{M} . \tag{A2}$$

The electrical impedance then becomes

$$Z_{ij}^E = Z_{ij}^b + \sum_{k=1}^{\mathfrak{M}} \left(\oint \psi'_{ik} v_k dS_k \right) I_j^{-1} , \quad i, j = 1, 2 \ldots \mathfrak{N} . \tag{A3}$$

Setting the external mechanical pressure $p_q \equiv 0$ reduces the mechanical equation to

$$\sum_{s}^{\mathfrak{M}} \oint Z_{qs}^I v_s dS_s = -\sum_{r}^{\mathfrak{N}} \psi_{qr} I_r , \quad q = 1, 2, \ldots \mathfrak{M} . \tag{A4}$$

This is an integral equation of the first kind in the unknown velocities v_s. Here Z_{qs}^I is the electrically coupled acoustic impedance matrix of the elastic body with dimensions of mechanical stress per volume velocity.

If we elect at the same frequency to calculate electrical admittance Y_{qr}^E, then we use the $I - p$ canonical set by solving Eq. (A1) for I_j and substituting in Eq. (A2). The electrical admittance then has the form

$$Y_{qr}^E = \left(Z_{qr}^b \right)^{-1} - \sum_{t}^{\mathfrak{N}} \sum_{k}^{\mathfrak{M}} \left(Z_{qt}^b \right)^{-1} (\oint \psi'_{tk} v_k dS_k) E_r^{-1} , \quad q, r = 1, 2, \ldots \mathfrak{N} . \tag{A5}$$

By construction Y_{qr}^E is the inverse of Z_{ij}^E, provided E and I satisfy the canonical set. Setting $p_q = 0$ again, we arrive at the mechanical equation

$$\sum_{i=1}^{\mathfrak{N}} \sum_{r=1}^{\mathfrak{N}} \sum_{s=1}^{\mathfrak{M}} \oint v_s \left[Z_{qs}^I - \psi_{qr} \left(Z_{ri}^b \right)^{-1} \psi'_{is} \right] dS_s = -\sum_{i=1}^{\mathfrak{N}} \sum_{r=1}^{\mathfrak{N}} \psi_{qr} \left(Z_{ri}^b \right)^{-1} E_i . \tag{A6}$$

This again is an integral equation of the first type in the unknown velocities v_s. The acoustic impedance

$$Z_{qs}^{E} = Z_{qs}^{I} - \sum_{i=1}^{\mathfrak{N}} \sum_{r=1}^{\mathfrak{N}} \psi_{qr} \left(Z_{ri}^{b} \right)^{-1} \psi_{is}' \tag{A7}$$

is no longer the electrically coupled impedance as in the case of the $E - p$ system, but is increased due to cancellation of the negative stiffness effect. It is (to close approximation) the purely elastic impedance of the electromechanical converter and as such is regarded as known. Thus Z_{qs}^{I} can be calculated from Eq. (A7). The velocities calculated by use of Eq. (A4) are identical with the velocities calculated by Eq. (A6).

If the frequency of forced drive is changed we can (a) maintain $|I|$ constant or (b) maintain $|E|$ constant. Equations (A4) and (A6) report the same velocities when either (a) or (b) is adopted. However, if $|I|$ is maintained constant as ω is changed, the velocities reported by Eq. (A6) are at different voltages. Similarly, if $|E|$ is maintained constant the velocities reported by Eq. (A4) are at different currents.

Appendix B

RADIATION MODAL IMPEDANCES

In Eq. (3.1) the forcing function P_i is deemed to have an acoustic component (i.e., a component dependent upon normal velocity). To illustrate, we take the case of cylindrical shells and consider the normal radial velocity. Let $\mathcal{A}_{ij}(\omega)\delta_{j3}$ be the acoustic operator having only a normal (33) component. Then the inner product of the acoustic reaction with a normal mode has the form

$$\left(\sum_j \mathcal{A}_{ij}\delta_{j3}\{U_i\},\ Y_i\right). \tag{B1}$$

The acoustic operator $\mathcal{A}$ has an adjoint $\mathcal{A}^*$ such that

$$(\mathcal{A}_{33}\{U_3\},\ Y_3) = (U_3,\ \mathcal{A}_{33}^* Y_3). \tag{B2}$$

Now in the case where normal modes can be described as spherical, spheroidal, or circular cylindrical harmonics, we may define the following adjoint operation in terms of modal resistances $R(q)$ and modal reactances $M(q)$, so that

$$\mathcal{A}^*(\omega)[Y_i(q;x_1,x_2)] = [j\omega R(q;\omega) - \omega^2 M(q;\omega)]\, Y_3(q;x_1,x_2). \tag{B3}$$

Thus,

$$\begin{aligned}(\mathcal{A}_{33}\{U_3(x_1,x_2)\},\ Y_i(q;x_1,x_2))\\ = ([j\omega R(q;\omega) - \omega^2 M(q;\omega)]\, U_3(x_1,x_2),\ Y_3(q;x_1,x_2)).\end{aligned} \tag{B4}$$

The amplitude factor for the acoustic reaction therefore involves only the inner product of the normal components (3-direction) of displacement. We compare this with the amplitude factor for the general forcing function (Eq. (3.10)) and note that the latter involves the inner product of all components of displacement with the modal functions. To group both acoustic and nonacoustic amplitude factors together one defines a shape factor $S(q)$ such that

$$(U_3(x_1,x_2),\ Y_3(q;x_1,x_2)) = \frac{(U_i(x_1,x_2),\ Y_i(q;x_1,x_2))}{S(q)} \tag{B5}$$

where

$$S(q) = 1 + \frac{(U_1(x_1, x_2),\ Y_1(q; x_1, x_2))}{(U_3(x_1, x_2),\ Y_3(q; x_1, x_2))} + \frac{(U_2(x_1, x_2),\ Y_2(q; x_1, x_2))}{(U_3(x_1, x_2),\ Y_3(q; x_1, x_2))}. \tag{B6}$$

Since $U_i(x_1, x_2)$ is expandable in orthogonal modes $Y_i(q; x_1, x_2)$, $S(q)$ can be further reduced to the form

$$S(q) = 1 + \frac{(Y_1(q; x_1, x_2),\ Y_1(q; x_1, x_2))}{(Y_3(q; x_1, x_2),\ Y_3(q; x_1, x_2))} + \frac{(Y_2(q; x_1, x_2),\ Y_2(q; x_1, x_2))}{(Y_3(q; x_1, x_2),\ Y_3(q; x_1, x_2))}. \tag{B7}$$

When the shell vibration has other normal velocity components (e.g., in the axial direction at each edge) we may construct similar shape factors. However, in all cases treated here the axial component of acoustic radiation is neglected. We now return to Eq. (3.1) and group all terms in the steady state in a power series of ω. Thus,

$$\omega^2 \left\{ (U_i(x_1, x_2),\ m_E Y_i(q; x_1, x_2)) + (U_i(x_1, x_2),\ Y_i(q; x_1, x_2)) \times \left[\frac{M(q)}{S(q)} + M_f(q) \right] \right\} - j\omega \left\{ (U_i(x_1, x_2),\ Y_i(q; x_1, x_2)) \left[\frac{R(q)}{S(q)} + R_f(q) \right] \right\} \tag{B8}$$

$$-(U_i(x_1, x_2),\ Y_i(q; x_1, x_2)) K_f(q) - \Lambda^2(q)(U_i(x_1, x_2),\ m_E Y_i(q; x_1, x_2)) = (Q_i(x_1, x_2),\ Y_i(q; x_1, x_2)). \tag{B9}$$

The quantity Q_i represents all terms of P_i excluding foundation and acoustic response. Here we have added a foundation forcing operator $F(\omega)$ which provides an amplitude factor of the form

$$(F(\omega)\ \{U_i(x_1, x_2)\},\ Y_i(q; x_1, x_2)) = (K_f(q) + j\omega R_f(q) - \omega^2 M_f(q))(U_i(x_1, x_2),\ Y_i(q; x_1, x_2)), \tag{B10}$$

that is, the foundation supplies stiffness K_f, resistance R_f, and mass M_f, which, though shown here as lumped quantities, may be functions of coordinates.

Solving Eq. (B9), we obtain

$$(U_i(x_1, x_2),\ Y_i(q; x_1, x_2)) = \frac{-(Q_i(x_1, x_2),\ Y_i(q; x_1, x_2))}{m_E D(\omega)} \tag{B11}$$

where

$$D(q;\omega) = \Lambda^2(q) - \omega^2 \left[1 + \frac{M(q)}{m_E S(q)} + \frac{M_f(q)}{m_E}\right]$$

$$+ j\omega \left[\frac{R(q)}{m_E S(q)} + \frac{R_f(q)}{m_E}\right] + \frac{K_f(q)}{m_E} . \tag{B12}$$

Using Eq. (B10) in Eq. (3.4) and considering that the othorgonality weighting function ζ is the constant $m_E = \rho b$ for a thin shell, we write the thin-shell expansion constants as

$$A(q;\omega) = \frac{-\left(\sum_i Q_i(x_1, x_2; \omega),\ Y_i(q; x_1, x_2)\right)}{m_E D(q;\omega) \sum_j (Y_j(q; x_1, x_2),\ Y_j(q; x_1, x_2))} . \tag{B13}$$

In general there is a largest (Y_i, Y_i) (call it (Y_3, Y_3)) in the sum $\sum_j (Y_j, Y_j)$. Dividing by this (Y_i, Y_i) we rewrite A as

$$A(q, r; \omega) = \frac{-(Y_3, Y_3)^{-1} \sum_t (Q_t, Y_t)\mathcal{S}}{M_E(q) D(q;\omega)} \tag{B14}$$

where $\mathcal{S}$ = reference area of thin shell. The symbol $M_E(q, r)$ designates the modal mass in the q mode and is defined as

$$M_E(q) = m_E \mathcal{S}\, S(q) \tag{B15}$$

where

$$m_E \mathcal{S} = \text{static mass of unloaded shell.} \tag{B16}$$

Substituting Eq. (B14) into Eq. (3.4) we find the steady state displacement in a forced harmonic drive of a thin shell to be

$$U_j(x_1, x_2; \omega) = -\sum_q \left\{ \frac{(Y_3(q; x_1, x_2),\ Y_3(q; x_1, x_2))^{-1} \mathcal{S}}{M_E(q;\omega) D(q;\omega)} \right.$$

$$\left. \times \left[\sum_t ((Q_t(x_1, x_2; \omega),\ Y_t(q; x_1, x_2))\right] Y_j(q; x_1, x_2) \right\} \tag{B17}$$

$$j = 1, 2, 3$$

$$q = 1, 2, \ldots, \infty$$

$$t = 1, 2, 3.$$

We note in Eq. (B17) that the amplitude of motion in any mode (i.e., in any Y_j) is independent of coordinates x_1, x_2, but does depend on the modal numbers q and on the frequency ω. This amplitude is maximized when $D(q; \omega)$ is minimized, i.e., at the frequencies of amplitude resonance $(\omega(q))$,

$$|D_n(\omega(q))| = \text{minimum}. \tag{B18}$$

Appendix C

MATHEMATICAL MODEL OF A FREE-FLOODED MAGNETOSTRICTIVE RING TRANSDUCER

The analysis carried out in the main body of this report deals primarily with the vibration of an elastic continuum describable by mode superposition. This extensive development in modes is superfluous when the shell axial length is much less than a wavelength of the lowest mode of free axial vibration. We present here a formulation of the mathematical model of a free-flooded magnetostrictive shell of such short axial length that a steady state solution by Fourier methods is directly available. We restrict the analysis to axisymmetric motion. While portions of this appendix are a restatement of classical theory, other parts contain new information. The chief objective in presenting this theory in one complete form is to provide a reservoir of defined parameters, concepts, circuits, etc., from which needed parameters appearing in the main body of this report can be drawn or to which they can be compared.

THEORY OF THE MAGNETOSTRICTIVE RING SOURCE: SPECIFIC EQUATIONS

Mechanical Shell Equations

We consider a thin shell of revolution (Fig. C1) with an axisymmetric reference surface, whose points may be described by the meridian coordinate s, the circumferential coordinate angle θ, the normal coordinate ζ, and the angle ϕ which the normal makes with the "axis of revolution." Since ϕ and s are not independent, we shall refer to either, as convenient. According to thin-shell theory, for shell thickness b and shell radius of curvature R, we shall assume $b^2/R^2 << 1$. Then the displacement of any point on the reference surface will be given by

$$\mathbf{U}(\phi, \zeta, t) = u_\phi \mathbf{t}_\phi + w\mathbf{t}_\zeta + \zeta\beta_\phi \mathbf{t}_\phi ,$$

where u_ϕ, w are displacements in the coordinate directions ϕ and ζ respectively, and β_ϕ is the rotation of the reference normal in the plane containing the shell axis (Fig. C2). When the shell is deformed, we assume infinitesimal strain theory to hold and, indicating differentiation by a comma, we write the strains in the form

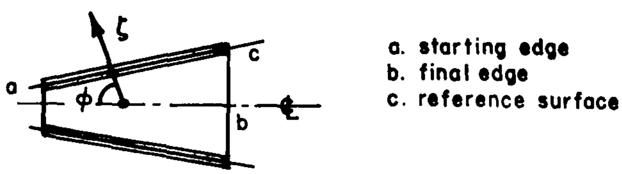

Fig. C1—Coordinates for a rotationally symmetric thin shell

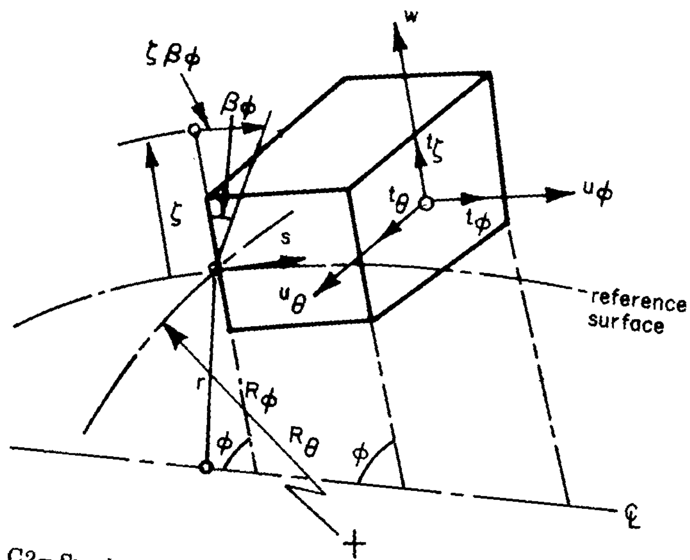

Fig. C2—Symbols for stresses, displacement, and angles associated with a typical volume element on a rotationally symmetric thin shell

$$S_\phi = \frac{(\epsilon_\phi + \zeta k_\phi)}{\left(1 + \dfrac{\zeta}{R_\phi}\right)}$$

$$S_\theta = \frac{(\epsilon_\theta + \zeta k_\theta)}{\left(1 + \dfrac{\zeta}{R_\theta}\right)}$$

$$2S_{\phi\theta} = 0; \quad S_\zeta = 0; \quad 2S_{\theta\zeta} = 0;$$

$$2S_{\phi\zeta} = \frac{\gamma_{\phi\zeta}}{\left(1 + \dfrac{\zeta}{R_\phi}\right)}$$

where

$$\epsilon_\phi = u_{\phi,s} + w/R_\phi;$$

$$\epsilon_\theta = u_\phi \cot\phi/R_\theta + w/R_\theta$$

$$k_\phi = \beta_{\phi,s}$$

$$k_\theta = \beta_\phi \cot\phi/R_\theta$$

$$\gamma_{\phi\zeta} = w_{,s} - u_\phi/R_\phi + \beta_\phi,$$

where R_ϕ, R_θ are the principal radii of curvature of the reference surface and r is the distance of the reference surface from the axis. From geometrical considerations, it is seen that $r = R_\theta \sin\phi$, and $ds = R_\phi d\phi$. The stresses accompanying these strains are defined by Fig. C3. By use of the stresses T_ϕ, T_θ, $T_{\phi\zeta}$ we define the stress resultants N_ϕ, Q_ϕ, M_ϕ, N_θ, M_θ in the usual way,*

$$\begin{Bmatrix} N_\phi \\ Q_\phi \\ M_\phi \end{Bmatrix} = \int_\zeta \begin{Bmatrix} T_\phi \\ T_{\phi\zeta} \\ \zeta T_\phi \end{Bmatrix} \left(1 + \frac{\zeta}{R_\theta}\right) d\zeta$$

$$\begin{Bmatrix} N_\theta \\ M_\theta \end{Bmatrix} = \int_\zeta \begin{Bmatrix} 1 \\ \zeta \end{Bmatrix} T_\theta \left(1 + \frac{\zeta}{R_\phi}\right) d\zeta .$$

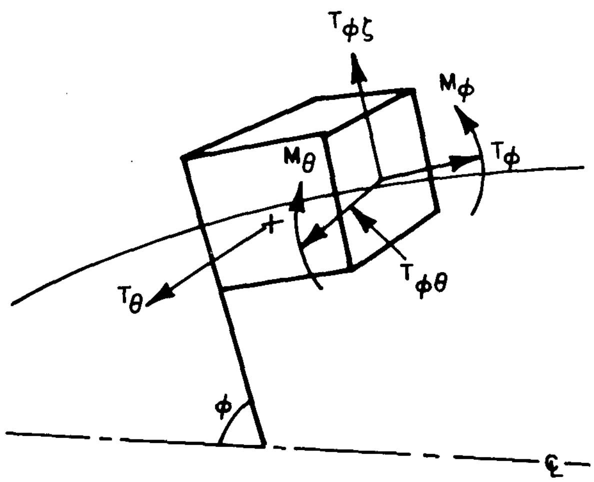

Fig. C3—Bending moment and shear stresses of a typical element on the reference surface

The equations of motion are standard,*

$$(rN_\phi)_{,s} - N_\theta \cos\phi + \frac{rQ_\phi}{R_\phi} + rp_\phi = r(b_1 u_{\phi,tt} + b_2 \beta_{\phi,tt})$$

$$(rQ_\phi)_{,s} - r\left(\frac{N_\phi}{R_\phi} + \frac{N_\theta}{R_\theta}\right) + rp_\zeta = rb_1 w_{,tt}$$

$$(rM_\phi)_{,s} - M_\theta \cos\phi - rQ_\phi + rm_\phi = r(b_2 u_{\phi,tt} + b_3 \beta_{\phi,tt})$$

*H. Kraus, *Thin Elastic Shells*, 2nd ed., Wiley & Sons, New York, 1973.

where

$$b_i = \int_{\zeta_1}^{\zeta_2} \rho\left(1 + \frac{\zeta}{R_\phi}\right)\left(1 + \frac{\zeta}{R_\theta}\right) \zeta^{(i-1)} d\zeta, \qquad i = 1, 2$$

$\mathbf{p}$ = vector mechanical surface load = $p_\phi \mathbf{t}_\phi + p_\zeta \mathbf{t}_\zeta$

$\mathbf{m}$ = vector surface couple = $\mathbf{m}_\phi \mathbf{t}_\theta$

ρ = shell density.

The types of deformation described by these equations is shown in Fig. C4. At the starting edge and final edge, the boundary conditions to be prescribed are

either w or Q_ϕ

either u_ϕ or N_ϕ

either β_ϕ or M_ϕ.

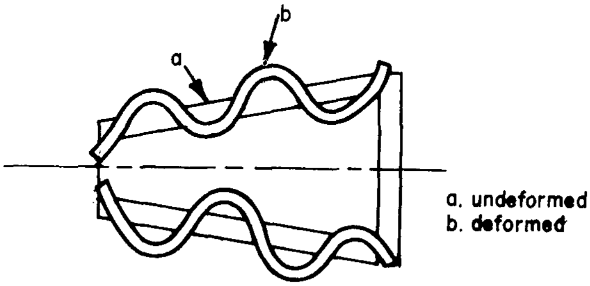

Fig. C4—Displacements described by standard equations

MAGNETIC FIELD EQUATIONS

At each point in the interior of the shell we require that

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{H} = 0.$$

On the surface of the shell

$$\mathbf{B} \cdot \mathbf{t}_\zeta = 0,$$

viz, $\mathbf{B}$ must be in the plane defined by $\mathbf{t}_\phi$ and $\mathbf{t}_\theta$; and

$$\mathbf{t}_\zeta \times \mathbf{H} = \mathbf{R} = \text{vector surface current density.}$$

Thus, from $\nabla \times \mathbf{H} = 0$, the differential equations governing the components of $\mathbf{H}$ in the $\mathbf{t}_\phi$, $\mathbf{t}_\theta$, $\mathbf{t}_\zeta$ directions are

$$\frac{\partial}{\partial \zeta}\left(1 + \frac{\zeta}{R_\theta}\right)H_\theta = 0; \quad \frac{\partial}{\partial \phi} H_\zeta - R_\phi \frac{\partial}{\partial \zeta}\left(1 + \frac{\zeta}{R_\phi}\right)H_\phi = 0; \quad \frac{\partial}{\partial \phi} r\left(1 + \frac{\zeta}{R_\theta}\right) H_\theta = 0.$$

If there are N windings carrying current $Ie^{j\omega t}$ over the shell surface, then*

$$\mathbf{K} = \frac{NIe^{j\omega t}}{2\pi r\left(1 + \frac{\zeta}{R_\theta}\right)} \mathbf{t}_\phi .$$

Thus, if one sets

$$H_\theta = \frac{NIe^{j\omega t}}{2\pi r\left(1 + \frac{\zeta}{R_\theta}\right)}; \quad H_\zeta = H_\phi = 0; \quad B_\zeta = B_\phi = 0,$$

it will be seen that magnetic field equations and surface conditions are satisfied.

ELECTROMECHANICAL CONSTITUTIVE RELATIONS

In the first approximation the electromechanical constitutive relations, when referred to a three-dimensional coordinate system x_i, are given by the linear set of equations

$$T_i = C_{ij}^B\left(S_j - \alpha_j^S\Theta\right) - h_{ik}^t B_k \quad , \quad j = 1, 2, \ldots, 6; \quad k = 1, 2, 3$$

$$H_\ell = -h_{\ell m}\left(S_m - \alpha_m{}^S\Theta\right) + \gamma^S B_\ell , \quad \ell = 1, 2, 3; \quad m = 1, 2, \ldots, 6$$

where α_j is the coefficient of thermal expansion and Θ the temperature increment. Since there are only three component stresses T_ϕ, T_θ, $T_{\phi\zeta}$ and one component of magnetic intensity H_θ, we write the explicit forms of the above set in the following way:

$$T_\phi = C_{\phi\phi}^B S_\phi + C_{\phi\theta}^B S_\theta - \left\{C_{\phi\phi}^B \alpha_\phi + C_{\phi\theta}^B \alpha_\theta + C_{\phi\zeta}\alpha_\zeta\right\}\Theta - h_{\phi\theta}^t \beta_\theta$$

$$T_\theta = C_{\theta\phi}^B S_\phi + C_{\theta\theta}^B S_\theta - \left\{C_{\theta\phi}\alpha_\phi + C_{\theta\theta}\alpha_\theta + C_{\phi\zeta}\alpha_\zeta\right\}\Theta - h_{\theta\theta}^t B_\theta$$

$$T_6 = T_{\phi\zeta} = C_{\phi\zeta\phi\zeta}^B S_{\phi\zeta} = C_{66}^B S_6$$

$$H_\theta = -h_{\theta\theta}\left(S_\theta - \alpha_\theta{}^S\Theta\right) - h_{\theta\phi}\left(S_\phi - \alpha_\phi{}^S\Theta\right) + \gamma^S B_\theta .$$

*See ref. [2].

Applying basic definitions and integrating across the thickness, we arrive at the following formulas for the stress resultants and stress couples:

$$N_\phi = C_{11}^B \epsilon_\phi + C_{13}^B \epsilon_\theta + d_{11}^B k_\phi + d_{13}^B k_\theta - N_{th\phi} - B_\alpha$$

$$N_\theta = C_{31}^B \epsilon_\phi + C_{33}^B \epsilon_\theta - N_{th\theta} - B_\beta$$

$$N_{\phi\zeta} = Q_\phi = C_{66}^B \gamma_{\phi\zeta}$$

$$M_\phi = d_{11}^B \epsilon_\phi + d_{13}^B \epsilon_\theta + k_{11}^b k_\phi + k_{13}^B k_\theta - M_{th\phi} - B_\gamma$$

$$M_\theta = d_{31}^B \epsilon_\phi + d_{33}^B \epsilon_\theta + k_{31}^B k_\phi + k_{33}^B k_\theta - M_{th\theta} - B_\delta$$

$$\mathcal{H}_\theta = -h_{31} \epsilon_\phi - h_{33} \epsilon_\theta - f_{31} k_\phi - f_{33} k_\theta - H_{th\theta} + B_\epsilon$$

where the subscripts 1, 2, 3 correspond to coordinates ϕ, ζ, θ, respectively, and

$$\begin{Bmatrix} C_{11} \\ d_{11} \\ k_{11} \end{Bmatrix}^B = \int C_{\phi\phi}^B \begin{Bmatrix} 1 \\ \zeta \\ \zeta^2 \end{Bmatrix} \frac{P}{Q}\, d\zeta, \qquad P = 1 + \frac{\zeta}{R_\theta}\,; \qquad Q = 1 + \frac{\zeta}{R_\phi}$$

$$\begin{Bmatrix} C_{13}, C_{31} \\ d_{13}, d_{31} \\ k_{13}, k_{31} \end{Bmatrix}^B = \int \left(C_{\phi\theta}^B, C_{\theta\phi}^B\right) \begin{Bmatrix} 1 \\ \zeta \\ \zeta^2 \end{Bmatrix} d\zeta$$

$$\begin{Bmatrix} C_{33} \\ d_{33} \\ k_{33} \end{Bmatrix}^B = \int C_{\theta\theta}^B \begin{Bmatrix} 1 \\ \zeta \\ \zeta^2 \end{Bmatrix} \frac{Q}{P}\, d\zeta$$

$$\begin{Bmatrix} h_{31} \\ f_{31} \end{Bmatrix}^S = \int h_{\theta\phi}^S \begin{Bmatrix} 1 \\ \zeta \end{Bmatrix} d\zeta$$

$$\begin{Bmatrix} h_{33} \\ f_{33} \end{Bmatrix}^S = \int h_{\theta\theta}^S \begin{Bmatrix} 1 \\ \zeta \end{Bmatrix} \frac{P}{Q}\, dS$$

$$c_{66}^{B} = \int C_{\phi\zeta\phi\zeta} \frac{P}{Q}\, d\zeta$$

$$\begin{Bmatrix} B_\alpha \\ B_\gamma \end{Bmatrix} = \int h_{\phi\theta}^{t} B_\theta P \begin{Bmatrix} 1 \\ \zeta \end{Bmatrix} d\zeta\,; \quad \begin{Bmatrix} B_\beta \\ B_\delta \end{Bmatrix} = \int h_{\theta\theta}^{t} B_\theta Q \begin{Bmatrix} 1 \\ \zeta \end{Bmatrix} d\zeta\,;$$

$$B_\epsilon = \int \gamma^{S} B_\theta Q\, d\zeta\,; \quad \mathcal{H}_\theta = \int_\zeta H_\theta \left(1 + \frac{\zeta}{R_\phi}\right) d\zeta .$$

(Note: $T_4 = T_{23} = T_{\zeta\theta}$; $T_5 = T_{13} = T_{\phi\theta}$; $T_6 = T_{12} = T_{\phi\zeta}$). We shall define $N_{th\phi}$, $N_{th\theta}$, $M_{th\phi}$, $M_{th\theta}$ in the following section.

CONSTITUTIVE EQUATIONS OF THE MAGNETOSTRICTIVE RING

We now apply the above general equations to the specific case of a right circular cylinder of short axial length (a ring) made of magnetostrictive metal. It will be assumed that the material is transversely isotropic in its elastic properties and is wound toroidally with electric current-carrying wire. The elastic constants, therefore, take on the forms

$$C_{11}^{B} = E_\phi{}^{B}; \quad C_{33}^{B} = E_\theta{}^{B}; \quad C_{13}^{B} = E_\theta{}^{B} \nu_{\theta\phi}; \quad C_{31}^{B} = E_\phi{}^{B} \nu_{\phi\theta}$$

$$E_i^{B} = Y_i^{B} \; (1 - \nu_{ij}\nu_{ji}),$$

where Y_i^B = Young's modulus in the i direction,

$$C_{66}^{B} = 2\left(C_{11}^{B} - C_{13}^{B}\right) = G_{13} .$$

Most magnetostrictive materials exhibit no isotropy in the modulus h. However, various authorities* assume $h_{\theta\phi}^{S} = -1/2\, h_{\theta\theta}^{S}$. The basic constitutive relations in stress resultant form now reduce to the set

$$N_\phi = K_{11}\epsilon_\phi + K_{13}\epsilon_\theta + D_{11}k_\phi + D_{13}k_\theta - N_{th\phi} - B_\alpha$$

$$N_\theta = K_{31}\epsilon_\phi + K_{33}\epsilon_\theta + D_{31}k_\phi + D_{33}k_\theta - N_{th\theta} - B_\beta$$

$$Q_\phi = L_{66}\gamma_{\phi\zeta}$$

$$M_\phi = D_{11}\epsilon_\phi + D_{13}\epsilon_\theta + E_{11}k_\phi + E_{13}k_\theta - M_{th\phi} - B_\gamma$$

$$M_\theta = D_{31}\epsilon_\phi + D_{33}\epsilon_\theta + E_{31}k_\phi + E_{33}k_\theta - M_{th\theta} - B_\delta$$

$$\mathcal{H}_\theta = -h_{31}\epsilon_\phi - h_{33}\epsilon_\theta - f_{31}k_\phi - f_{33}k_\theta - H_{th\theta} - B_\epsilon .$$

*Y. Masiyama, "On the Magnetostriction of Iron-Nickel Alloys," The Science Reports of the Tohoku Imperial University, First Series 20, 574-593 (1931).

Assuming that ζ/R_i is small compared to unity, we write the definitions

$$K_{11} = \int E_\phi^B d\zeta; \quad K_{13} = \int E_\theta \nu_{\theta\phi} d\zeta$$

$$D_{11} = \int E_\phi{}^B \zeta d\zeta; \quad D_{13} = \int E_\theta \nu_{\theta\phi} \zeta d\zeta$$

$$K_{31} = \int E_\phi{}^B \nu_{\phi\theta} d\zeta; \quad K_{33} = \int E_\theta{}^B d\zeta; \quad D_{31} = \int E_\phi{}^B \nu_{\phi\theta} \zeta d\zeta$$

$$D_{33} = \int E_\theta{}^B \zeta d\zeta; \quad E_{11} = \int E_\phi{}^B \zeta^2 d\zeta; \quad E_{13} = \int E_\theta \nu_{\theta\phi} \zeta^2 d\zeta$$

$$L_{66} = \int G_{13} d\zeta; \quad E_{31} = \int E_\phi{}^B \nu_{\phi\theta} \zeta^2 d\zeta; \quad E_{33} = \int E_\theta \zeta^2 d\zeta$$

$$\begin{Bmatrix} N_{th\phi} \\ M_{th\phi} \end{Bmatrix} = \int (E_\phi \alpha_{t\phi} + \nu_{\phi\theta} E_\theta \alpha_{t\theta}) \begin{Bmatrix} 1 \\ \zeta \end{Bmatrix} \Theta d\zeta$$

$$\begin{Bmatrix} N_{th\theta} \\ M_{th\theta} \end{Bmatrix} = \int (E_\theta \alpha_{t\theta} + \nu_{\theta\phi} E_\phi \alpha_{t\phi}) \begin{Bmatrix} 1 \\ \zeta \end{Bmatrix} \Theta d\zeta$$

$$\begin{Bmatrix} B_\alpha \\ B_\gamma \end{Bmatrix} = \int h_{55} B_\theta \begin{Bmatrix} 1 \\ \zeta \end{Bmatrix} d\zeta; \quad \begin{Bmatrix} B_\beta \\ B_\delta \end{Bmatrix} = h_{33} B_\theta \begin{Bmatrix} 1 \\ \zeta \end{Bmatrix} d\zeta.$$

The algebraic signs to be associated with the stress resultants and couples are chosen for convenience (Fig. C5). On the stress plane s = const., and the stress resultant N_ϕ is positive (i.e., tensile) if its vector representation points in the direction of ϕ_+ (as shown). The stress N_θ is positive if it turns as a right-hand screw in the direction of ϕ_+. The terms $D_{11}k_\phi$, $D_{13}k_\theta$, $D_{31}k_\phi$, $D_{33}k_\theta$ are positive in the directions shown in Fig. C6. The flux-density vector B_θ is positive if it points in the direction of positive N_θ. The shear-stress vector Q_ϕ is positive if it is directed along the outward normal to the reference surface.

DYNAMIC EQUATION OF MOTION OF MAGNETOSTRICTIVE RING

For a right circular cylinder the radii of curvature reduce to $R_\phi \to \infty$ and $R_\theta = a$ = radius of cylinder. We restrict our attention here to radial motions only, and in the first approximation assume (a) the transverse shear Q_ϕ is negligible; (b) stresses due to rotary motion (i.e., $D_{31}k_\phi$, $D_{33}k_\theta$) are negligible; and (c) the low-frequency linear piezomagnetic constitutive equations are valid.

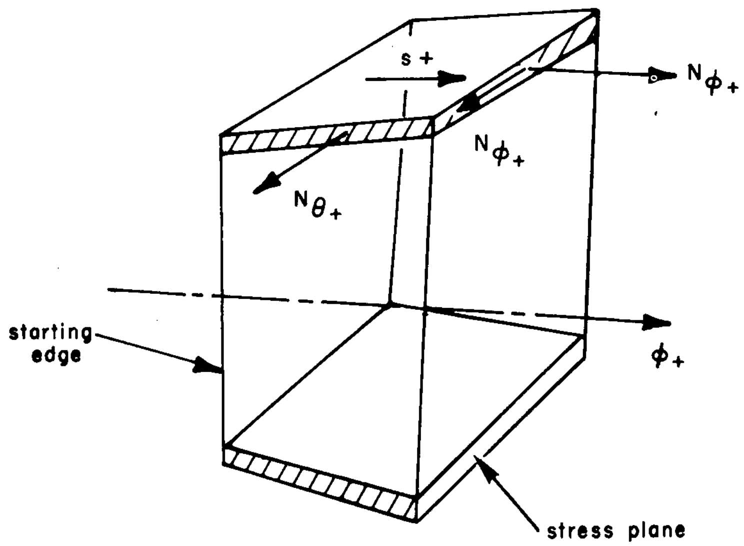

Fig. C5—Convention of positive resultant stresses and angles used in this report

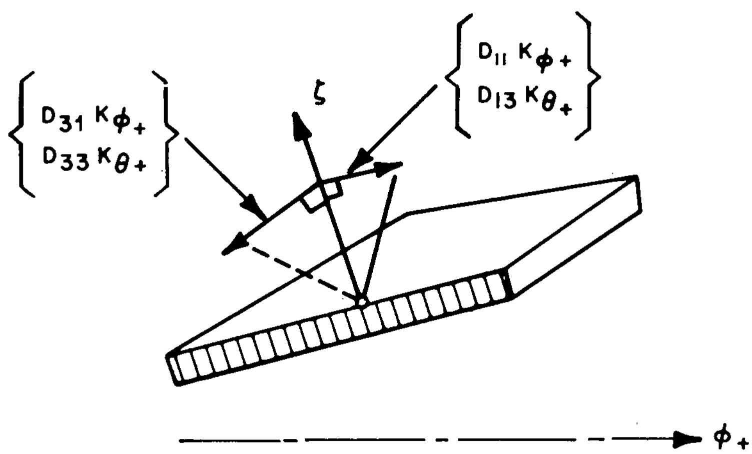

Fig. C6—Convention of positive directions for $D_{31}K_{\phi+}, D_{33}K_{\theta+}$ and $D_{11}K_{\phi+}, D_{13}K_{\theta+}$

Noting that $\phi = 90^\circ$ and ζ/R_ϕ is negligible, we write

$$N_\theta = \int \left(E_\theta{}^B\, d\zeta\right)\left(\frac{w}{a}\right) - \int \left(E_\theta{}^B \alpha_{t_\theta} + \nu_{\theta\phi} E_\phi{}^B \alpha_{t_\phi}\right) \Theta\, d\zeta - \int h_{\theta\theta} B_\theta\, d\zeta .$$

We next assume that all parameters are independent of thickness b of the shell. Thus,

$$N_\theta = \frac{Y_\theta{}^B b w}{(1-\nu^2)a} - \frac{\left(Y_\theta{}^B \alpha_{t\theta} + \nu Y_\phi{}^B \alpha_{t\phi}\right) b\,\Theta}{(1-\nu^2)} - h^t_{\theta\theta} B_\theta\, b .$$

To find B_θ in terms of $\mathcal{H}_\theta$ we turn to the formula for $\mathcal{H}_\theta$ and write

$$\mathcal{H}_\theta = -h_{31}\epsilon_\phi - h_{33}\epsilon_\theta + B_\epsilon,$$

neglecting the thermal effect $H_{th\theta}$. Since $B_\epsilon = \gamma^S B_\theta b$,

$$B_\theta = h_{\theta\theta}^S \mu^S \left(\frac{w}{a}\right) + \mu^S H_\theta, \qquad \epsilon_\phi = 0$$

where μ^S = incremental permeability at the induction B_0 (polarizing induction). Thus,

$$N_\theta = \frac{Y_\theta{}^B bw}{(1-\nu^2)a} - \frac{\left(Y_\theta{}^B \alpha_{t\theta} + \nu Y_\phi{}^B \alpha_{t\phi}\right) b\Theta}{(1-\nu^2)} - \left(h_{\theta\theta}^S\right)^2 \mu^S b \, \frac{w}{a} - h_{\theta\theta} b \mu^S H_\theta .$$

Now

$$\frac{w\left(h_{\theta\theta}^S\right)^2 \mu^S Y_\theta{}^B (1-\nu^2)}{aY_\theta{}^B (1-\nu^2)} = \frac{Y_\theta{}^B bw}{(1-\nu^2)a} \, k_c^2; \qquad k_c^2 = \frac{\left(h_{\theta\theta}^S\right)^2 \mu^S (1-\nu^2)}{Y_\theta{}^B}$$

where k_c is the coefficient of electromechanical coupling. Also we desire to consider certain transduction losses due to eddy currents and hysteresis which can be accounted for by replacing μ^S by the product $\mu^S \chi$, where $\chi = \chi_0 \exp\,[-j(\eta + \sigma)]$. The stress resultant now appears in the form

$$N_\theta = \frac{Y_\theta{}^B bw}{(1-\nu^2)a}\left(1 - k_c^2 \chi\right) - \frac{\left(Y_\theta{}^B \alpha_{t\theta} + \nu Y_\phi{}^B \alpha_{t\phi}\right) b\Theta}{1 - \nu^2} - h_{\theta\theta} b \mu^S H_\theta .$$

Substituting this into the equation of radial motion, we obtain

$$ab_1 \frac{\partial^2 w}{\partial t^2} + Y_\theta{}^B \frac{bw}{(1-\nu^2)a}\left(1 - k_c^2 \chi\right) = ap_\zeta + \frac{\left(Y_\theta{}^B \alpha_{t\theta} + \nu Y_\phi{}^B \alpha_{t\phi}\right) b\Theta}{1 - \nu^2}$$

$$+ h_{\theta\theta} b \mu^S H_\theta \chi .$$

The dimensions of this equation are Newton's per meter (force per unit axial length). The l.h.s. of this equation can be interpreted as the inertial and stiffness reaction forces of the shell per unit axial length corresponding to the applied forces (pressure, thermal, and magnetostrictive) on the r.h.s. Since there is a purely mechanical loss due to internal friction accompanying shell motion, we shall add to the l.h.s. a resistance force proportional to reference surface velocity, writing it in the form $\mathcal{R}_m \, \partial w/\partial t$, where the resistance constant $\mathcal{R}_m$ has the dimensions N-sec/m^2. In addition we shall consider the external pressure p_ζ to consist of a normally applied (tensile) force $\mathcal{F}$ per unit area and an acoustic load p_L. In symbols

$$p_\zeta = \mathcal{F} - p_L .$$

Substituting the explicit form for H_θ found earlier, we now write the equation of motion as follows:

$$ab_1 \frac{\partial^2 w}{\partial t^2} + \mathcal{R}_m \frac{\partial w}{\partial t}$$

$$+Y_\theta^B \frac{bw}{(1-\nu^2)a}\left(1 - k_c^2\chi\right) + ap_L = a\mathcal{F} + \frac{\left(Y_\theta^B \alpha_{t\theta} + \nu Y_\phi^B \alpha_{t\phi}\right) b\Theta}{1 - \nu^2} + h_{\theta\theta} b \mu^S \frac{NI\chi}{2\pi a}.$$

We next assume that the acoustic load can be written in terms of a resistance component $r_L(\omega)$, an inertial component $m_L(\omega)$ and a stiffness component $k_L(\omega)$, such that

$$p_L = r_L \frac{\partial w}{\partial t} + m_L \frac{\partial \dot{w}}{\partial t} + k_L \int \dot{w}\,dt$$

$$\left(\text{dimensions of } r_L,\ \frac{\text{N-sec}}{\text{m}^3};\ \text{dimensions of } m_L,\ \frac{\text{N-sec}^2}{\text{m}^3};\ \text{dimensional of } k_L,\ \frac{\text{N}}{\text{m}^3}\right).$$

Treating the applied quantities ($\mathcal{F}$, Θ, I) as sinusoidal ($e^{j\omega t}$) and writing

$$\chi = \chi_R - j\chi_I, \quad \mathcal{R}_m = aR_m, \quad \left(\text{dimensions of } R_m,\ \frac{\text{N-sec}}{\text{m}^3}\right),$$

The equation of motion, in terms of velocity $v = j\omega w$, results in

$$\left[j\omega(b_1 + m_L) + \left(R_m + r_L + \frac{Y_\theta^B b k_c^2 \chi_I}{(1-\nu^2)a^2\omega}\right) + \frac{k_L}{j\omega} + \frac{Y_\theta^B b\left(1 - k_c^2\chi_R\right)}{(1-\nu^2)a^2 j\omega}\right] v = \mathcal{F} + \frac{\left(Y_\theta^B \alpha_{t\theta} + \nu Y_\phi^B \alpha_{t\phi}\right) b\Theta}{(1-\nu^2)a} + \frac{h_{\theta\theta} b \mu^S NI\chi}{2\pi a^2}.$$

All terms in this equation can be interpreted as mechanical pressure (or stress) with dimensions N/m^2. Multiplying through by the reference area of the shell ($2\pi al$, l = axial length of shell) we can write

$$F = \frac{-h_{\theta\theta}\mu^S \chi b l N I}{a} + \left[j\omega(\rho V + M_L) + \left(R_m'' + R_L + \frac{Y_\theta^B b l k_c^2 2\pi \chi_I}{(1-\nu^2)a\omega}\right) - j\frac{K_L}{\omega} - j\frac{Y_\theta^B b l 2\pi\left(1 - k_c^2\chi_R\right)}{(1-\nu^2)a\omega}\right] v - \frac{\left(Y_\theta^B \alpha_{t\theta} + \nu Y_\phi^B \alpha_{t\phi}\right) b l 2\pi\Theta}{(1-\nu^2)}$$

where

F = mechanical force = $\mathfrak{F}(2\pi al)$

ρ = density of shell

V = volume of shell

$$M_L = m_L(2\pi al),\ K_L = k_L(2\pi al)$$

$$R'_m = R_m(2\pi al)$$

$$R_L = r_L(2\pi al).$$

For purposes of convenience in later discussions, we rewrite the equation for F in the form

$$F = -Z_{em}I + Z_m v$$

or

$$v = \frac{F + Z_{em}I}{Z_m},$$

where

$$Z_{em} = h_{\theta\theta}\mu^S \chi blN/a; \qquad Z_m = (R'_m + R''_m + R_L) + j(X'_m + X''_m + X_L)$$

with

R''_m = mechanical loss coefficient which is independent of mechanical coupling and acoustic load

$$R'_m = Y_\theta{}^B blk_c{}^2 2\pi\chi_I/(1 - \nu^2)(a\omega)$$

R_L = real part of acoustic load

$$X'_m = Y_\theta{}^B bl2\pi\chi_R k_c{}^2/(1 - \nu^2)(a\omega)$$

$$X''_m = \omega\rho V - Y_\theta{}^B bl2\pi/(1 - \nu^2)(a\omega)$$

X_L = imaginary part of acoustic load = ωM_L.

We also adopt the following notation for later use:

$$Z_m{}^0 = R'_m + R''_m + j(X'_m + X''_m)$$

$$\mathrm{Re}\{Z_m{}^0\} = R_d = R'_m + R''_m$$

$$\mathrm{Re}\ \{Z_m\} = R_d + R_L = R_{m(\mathrm{TOT})}.$$

ELECTRICAL IMPEDANCE OF THE MAGNETOSTRICTIVE RING

It has been noted above that the magnetic flux density B_θ, under the simplifying assumptions of thin-shell theory and negligible thermal effects, has the form

$$B_\theta = h_{\theta\theta}\mu^S\left(\frac{w}{a}\right)\chi + \mu^S\chi H_\theta .$$

The sinusoidal magnetic flux Φ threading the cross-sectional area bl may therefore be expressed in terms of velocity and current as

$$\Phi = B_\theta bl = \frac{h_{\theta\theta}\mu^S\chi bl}{a(j\omega)}v + \frac{NI\mu^S\chi bl}{2\pi a} .$$

The applied (external) voltage E is therefore given by

$$E = j\omega N\Phi$$

or

$$E = j\omega\frac{N^2I\mu^S\chi bl}{2\pi a} + \frac{h_{\theta\theta}\mu^S\chi blN}{a}v .$$

Due to flux leakage, there is an additional purely electrical voltage drop in the exciting circuit which we assume to be proportional to current. Inserting this additional drop in the above formula (which does not contain it) and using appropriate definitions, we may write

$$E = (Z^l + Z^c)I + Z_{em}v$$

where

$Z^b = Z^l + Z^c =$ total blocked electrical impedance

$Z^l =$ electrical impedance due to flux leakage $\equiv R_l + j\omega L_l$

$Z^c = j\omega N^2\mu^S\chi bl/(2\pi a) = R_c + j\omega L_c$

$R_c = \omega N^2\mu^S\chi_I bl/(2\pi a)$

$L_c = N^2\mu^S\chi_R bl/(2\pi a)$

$R_l =$ resistance of wiring

$L_l =$ inductance due to flux leakage.

Substituting for velocity by using the formulas developed in the previous section, we obtain the following set of canonical equations:

$$E = \left(Z^l + Z^c + \frac{Z_{em}^2}{Z_m}\right) I + \left(\frac{Z_{em}}{Z_m}\right) F$$

$$v = \left(\frac{Z_{em}}{Z_m}\right) I + \left(\frac{F}{Z_m}\right).$$

We wish to obtain the electrical driving-point impedance, defined by the relation

$$Z_{ee} = \left(\frac{E}{I}\right)_{F=0} = Z^l + Z^c + \left(\frac{Z_{em}^2}{Z_m}\right) = R_{ee} + jX_{ee}$$

$$= Z^l + Z^c + \left(\frac{Z_m}{Z_{em}^2}\right)^{-1}.$$

In view of the definitions provided above, we can write this explicitly in the form

$$Z_{ee} = Z_l + Z_c + \left\{\left(\frac{Z_{em}^2}{j\omega\rho V}\right)^{-1} + \left(\frac{Z_{em}^2}{j\omega M_L}\right)^{-1} + \left(\frac{Z_{em}^2}{R_m''}\right)^{-1} + \left(\frac{Z_{em}^2}{R_L}\right)^{-1}\right.$$

$$\left. + \left[\frac{Z_{em}^2}{\left(\dfrac{Y_\theta{}^B bl k_c{}^2 2\pi\chi_I}{(1-\nu^2)a\omega}\right)}\right]^{-1} + \left[\frac{Z_{em}^2 j\omega}{\dfrac{Y_\theta{}^B bl 2\pi\left(1 - k_c{}^2\chi_R\right)}{(1-\nu^2)a}}\right]^{-1}\right\}^{-1}$$

where

$$Z_{em}^2 = \left[h_{\theta\theta}^2(\mu^S)^2(bl)^2N^2\left(\chi_R{}^2 - \chi_I{}^2 - j2\chi_R\chi_I\right)\right]\Big/a^2.$$

The term in braces is in the form of the reciprocal of six reciprocal quantities. It can therefore be interpreted as the impedance of six elements connected in parallel. Figure C7 is an equivalent circuit that describes the electrical driving-point impedance. In the figure,

$$R_l = R_{\text{leakage}}$$

$$L_l = L_{\text{leakage}}$$

$$R_c = \omega N^2\mu^S\chi_I bl/2\pi a$$

$$L_c = N^2\mu^S\chi_R bl/2\pi a$$

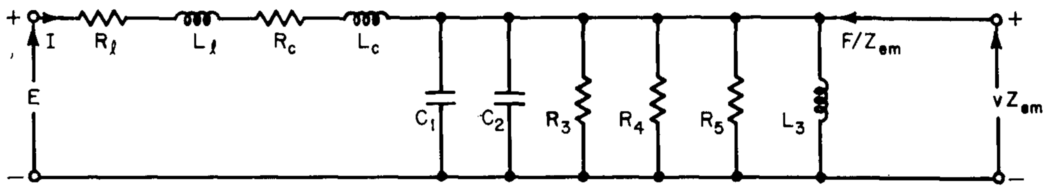

Fig. C7—Equivalent circuit of a magnetostrictive ring transducer vibrating as a single-degree-of-freedom system

$$C_1 = \rho V/Z_{em}^2$$

$$C_2 = M_L/Z_{em}^2$$

$$R_3 = Z_{em}^2/R_m''$$

$$R_4 = Z_{em}^2/R_L$$

$$R_5 = Z_{em}^2\Big/\left[Y_\theta{}^B blk_c{}^2 2\pi\chi_I/(1-\nu^2)a\omega\right]$$

$$L_3 = Z_{em}^2(1-\nu^2)a\Big/\left[Y_\theta{}^B bl2\pi(1-k_c{}^2\chi_R)\right].$$

It will be noted that Z_{em}^2 is a complex number, so that the parameters of the equivalent circuit are themselves complex numbers.

ELECTROACOUSTIC EFFICIENCY

If $v\exp(j\omega t)$ is the radial velocity of the reference surface and R_L the effective radiation resistance of the medium, then the radiated acoustic power is $R_L|v|^2$. Similarly, if the driving-point current is $I\exp(j\omega t)$ and the real part of the driving-point electrical impedance is $\mathrm{Re}\{Z_{ee}\} = R_{ee}$, then the dissipated power seen from the electrical terminals is $|I|^2R_{ee}$. Thus the electroacoustic efficiency η is the ratio $(R_L/R_{ee})|v/I|^2$. Now when $F = 0$ the ratio $(v/I) = Z_{em}/Z_m$. Thus, the efficiency is

$$\eta = \frac{R_L}{R_{ee}}\frac{|Z_{em}|^2}{|Z_m|^2}$$

where

$$|Z_{em}|^2 = h_{\theta\theta}^2(\mu^S)^2(\chi_R{}^2 + \chi_I{}^2)b^2l^2N^2\big/a^2$$

$$|Z_m|^2 = \left[R_m'' + R_L + \frac{Y_\theta{}^B blk_c{}^2 2\pi\chi_I}{(1-\nu^2)a\omega}\right]^2 + \left[\omega(\rho V + M_L) - \frac{Y_\theta{}^B bl2\pi(1-k_c{}^2\chi_R)}{(1-\nu^2)a\omega}\right]^2$$

$$R_{ee} = R_l + R_c + \frac{1}{|Z_m|^2}\left(\left[\frac{h_{\theta\theta}^2(\mu^S)^2 b^2 l^2 N^2}{a^2}\left(\chi_R{}^2 - \chi_I{}^2\right)\right]\left[R_m'' + R_L\right.\right.$$

$$\left. + \frac{Y_\theta{}^B blk_c{}^2 2\pi\chi_I}{(1-\nu^2)a\omega}\right] + \left\{-2\chi_R\chi_I\left[h_{\theta\theta}^2(\mu^S)^2 b^2 l^2 N^2/a^2\right]\right\}\left[\omega(\rho V + M_L)\right.$$

$$\left.\left. - \frac{Y_\theta{}^B bl2\pi\left(1 - k_c{}^2\chi_R\right)}{(1-\nu^2)a\omega}\right]\right).$$

COMBINED ELASTIC-MAGNETOSTRICTION-ACOUSTIC FIELDS

The equations of motion of forced vibration of a magnetostrictive ring in a liquid medium, in which the magnetostriction effect is mathematically accounted for by analog, to the thermal effect, may be written as follows in operator notation:

$$M_i\{u_\phi, w\} = \mathfrak{F}_i^{(p)}\{p\} + \mathfrak{F}_i^{(H)}\{H\} + \mathfrak{F}_i^{(\Theta)}\{\Theta\}$$

where M_i, $\mathfrak{F}_i$ are differential (or integral) operators; u_ϕ, w are displacements; and p, H, θ are forcing parameters (pressure, magnetic intensity, and heat). The particular solutions of this equation (in the steady state) are directly obtainable by symbolic inversion of the operator:

$$\{v_\phi, v_w\} = j\omega M_i^{-1}\mathfrak{F}_i^{(p)}\{p\} + j\omega M_i^{-1}\mathfrak{F}_i^{(H)}\{H\} + j\omega M_i^{-1}\mathfrak{F}_i^{(\Theta)}\{\Theta\}$$

where $v_\phi = j\omega u_\phi$, and $v_w = j\omega w$. We now define the products $j\omega M_i^{-1}\mathfrak{F}$ on the r.h.s. as influence coefficients, i.e., as resultant surface velocities per unit external pressure, magnetic intensity, and heat. For convenience we adopt the following symbols:

$j\omega M_i^{-1}\mathfrak{F}_i^{(b)} = {}_{(p)}\nu_{jl}^{(w)}$ = radial surface velocity of the jth band due to a unit applied external pressure on the lth band at zero applied magnetic intensity and heat.

$j\omega M_i^{-1}\mathfrak{F}^{(H)} = {}_{(H)}\nu_{jl}^{(w)}$ = radial surface velocity of the jth band due to a unit applied magnetic intensity on the lth band at zero applied external pressure and heat.

$j\omega M_i^{-1}\mathfrak{F}^{(\Theta)} = {}_{(\Theta)}\nu_{jl}^{(w)}$ = radial surface velocity of the jth band due to a unit applied thermal excitation on the lth band at zero pressure and magnetic intensity.

When axial displacement u rather than radial displacement w is to be calculated, we replace the superscripts w by u. The three influence coefficients noted above are assumed to be determinable by calculation using the shell theory outlined above. With these coefficients on hand we may formulate the problem of finding (say) all radial surface velocities w_j, $j = 1, 2, \ldots, M$, through solving the set of linear simultaneous equations,

$$\sum_{l=1}^{M} \sum_{i=1}^{M} {}_{(p)}\nu_{jl}^{(w)} \mathcal{G}_{li} w_i + \sum_{l=1}^{M} {}_{(H)}\nu_{jl}^{(w)} H_l = w_j, \quad j = 1, 2, \ldots, M.$$

We note in particular that w_j is a function of radius r, so that the thickness effect is not negligible in these equations.

EVALUATION OF PARAMETERS OF MATERIAL AND PERFORMANCE

When the external applied mechanical force F vanishes, it is seen from the first canonical set that the driving-point electrical impedance Z_{ee} is given by

$$Z_{ee} = Z^l + Z^c + \left(Z_{em}^2/Z_m\right).$$

We write (as before) the eddy-current factor as a complex number,

$$\chi = \chi_R - j\chi = |\chi| e^{-j\beta}; \quad \beta = \tan^{-1}\left(\frac{\chi_I}{\chi_R}\right), \quad |\chi|^2 = \chi_R{}^2 + \chi_I{}^2.$$

The transduction force factor Z_{em} is thus also written as

$$Z_{em} = \frac{h_{\theta\theta} \mu^S b l N |\chi| e^{-j\beta}}{a}.$$

Similarly, the core impedance may be represented by

$$Z_c = R_c + j\omega L_c = |Z_c| e^{j\Phi_c}$$

where

$$R_c = \omega N^2 \mu^S \chi_I b l/2\pi a$$

$$L_c = N^2 \mu^S \chi_R b l/2\pi a$$

$$\tan \Phi_c = \omega L_c/R_c = \chi_R/\chi_I.$$

Since $\tan \beta = \cot \Phi_c$, $(\beta + \Phi_c) = \pi/2$. Hence,

$$Z_{em}^2 = \frac{h_{\theta\theta}^2 (\mu^S)^2 b^2 l^2 N^2 |\chi|^2 e^{-j\pi} e^{j2\Phi_c}}{a^2} = -|Z_{em}^2| e^{j2\Phi_c}.$$

Now

$$Z_{Mot} = \frac{Z_{em}^2}{Z_m} = \frac{|Z_{em}^2| e^{-j2\beta}}{Z_m} = R_{Mot} + jX_{Mot}.$$

We consider first that $\beta \equiv 0$ and $|Z_{em}^2| = \text{const.} = A$. Then

$$Z_{Mot} = (R'_m + jX'_m)^{-1} = R'_m |Y'_m|^2 - jX'_m |Y'_m|^2 = Y'_m$$

where

$$R'_m = \frac{R_m}{A}$$

$$X'_m = \frac{X_m}{A}.$$

We then define an angle θ by the relation

$$\cos\theta = \frac{R'_m |Y'_m|^2}{|Y'_m|} = \frac{R_{Mot}}{|Z_{Mot}|}.$$

Thus,

$$|Y'_m| = \frac{1}{R'_m} \cos\theta = \frac{1}{R'_m} \frac{R_{Mot}}{|Z_{Mot}|}.$$

This is seen to be

$$R_{Mot}^2 + X_{Mot}^2 = \frac{1}{R'_m} R_{Mot}$$

or

$$\left(R_{Mot} - \frac{A}{2R_m}\right)^2 + X_{Mot}^2 = \frac{1}{4} \frac{A^2}{R_m{}^2}.$$

This is a circle with center at $R_{Mot} = |Z_{em}^2|/2R_m$; $X_{Mot} = 0$ and radius of $|Z_{em}^2|/2R_m$. We consider next that β is finite. Then the circle is rotated clockwise by angle 2β. The new coordinates R'_{Mot}, X'_{Mot}, are related to the old by the formula

$$R_{Mot} = R'_{Mot} \cos 2\beta + X'_{Mot} \sin 2\beta$$

$$X_{Mot} = -R'_{Mot} \sin 2\beta + X'_{Mot} \cos 2\beta.$$

Thus, the rotated circle has the mathematical form

$$\left(R'_{Mot} - \frac{A}{2R_m}\cos 2\beta\right)^2 + \left(X'_{Mot} - \frac{A}{2R_m}\sin 2\beta\right)^2 = \frac{A^2}{4{R_m}^2}\,.$$

The center of this circle is at

$$R'_{Mot} = \frac{A}{2R_m}\cos 2\beta\,; \qquad X'_{Mot} = \frac{A}{2R_m}\sin 2\beta.$$

When $R'_{Mot} \equiv 0$, the circle intercepts the ordinate at two points:

$$X'_{Mot} = -\frac{A}{2R_m}\left[\sin|2\beta| \pm (1-\cos^2|2\beta|)^{1/2}\right].$$

The tip of the diameter of the rotated circle, originating at the origin, has the coordinates

$$\left.\begin{aligned} R'_{Mot} &= R_{Mot}\cos|2\beta| \\ X'_{Mot} &= -R_{Mot}\sin|2\beta| \end{aligned}\right\} \quad \text{at} \quad X_{Mot} = 0.$$

Now the Im $\{Z_m\} = \omega M^* - S^*/\omega$ where M^*, S^* are the general expressions for mass and stiffness of the ring. If we set $\omega = \omega_R + \Delta\omega = \omega_R(1 + \Delta\omega/\omega_R)$, where ω_R is the resonant frequency as defined above, we see that for $\Delta\omega/\omega << 1$,

$$\text{Im}\,\{Z_m\} \approx (\omega_R + \Delta\omega)M^* - S^*\left(1 - \frac{\Delta\omega}{\omega_R}\right)\Big/\omega_R\,.$$

Since ${\omega_R}^2 \equiv S^*/M^*$, we reduce the above to Im $\{Z_m\} \approx 2\Delta\omega M^*$. Hence,

$$\tan\Phi_m = -\frac{2(\omega-\omega_R)M^*}{\text{Re}\,\{Z_m\}}\,, \qquad \Delta\omega/\omega_R << 1.$$

Now let $\omega'' = \omega_R + \Delta\omega''$, $\omega' = \omega_R - \Delta\omega'$, then

$$\frac{1}{\omega''} + \frac{1}{\omega'} = \frac{1}{\omega_R}\left(\frac{1}{1+\frac{\Delta\omega''}{\omega_R}} + \frac{1}{1-\frac{\Delta\omega'}{\omega_R}}\right)$$

$$= \frac{1}{\omega_R}\left(\frac{2 - \frac{\Delta\omega'}{\omega_R} + \frac{\Delta\omega''}{\omega_R}}{1 + \frac{\Delta\omega''}{\omega_R} - \frac{\Delta\omega'}{\omega_R} - \frac{\Delta\omega''\Delta\omega'}{{\omega_R}^2}}\right)$$

If

$$\frac{\Delta\omega'}{\omega_R} << 1, \quad \frac{\Delta\omega''}{\omega_R} << 1,$$

then

$$\frac{1}{\omega''} + \frac{1}{\omega'} = \frac{2}{\omega_R}.$$

Now

$$\mathrm{Re}\,\{Z_m\} = R_m'' + R_L + \frac{Y_\theta{}^B blk_c{}^2 2\pi\chi_I}{(1-\nu^2)a\omega} = D - \frac{G_2}{\omega g}$$

where

$$\frac{G_2}{g} \equiv -\frac{Y_\theta{}^B blk_c{}^2 2\pi\chi_I}{(1-\nu^2)a}$$

$$D \equiv R_m'' + R_L.$$

From the previous derivations it was seen that

$$\tan\Phi = -\frac{2(\omega-\omega_R)M^*}{D - \dfrac{G_2}{\omega g}}.$$

At the quadrantal frequencies ω_2, ω_3, by definition,

$$D - \frac{G_2}{\omega_2 g} = \frac{-2(\omega_2-\omega_R)M^*}{(\tan\Phi)_{\Phi=\pi/4}}$$

$$D - \frac{G_2}{\omega_3 g} = \frac{-2(\omega_3-\omega_R)M^*}{(\tan\Phi)_{\Phi=-\pi/4}}$$

By adding these two equations, we obtain

$$2D - \frac{G_2}{g}\left(\frac{1}{\omega_3} + \frac{1}{\omega_2}\right) = 2(\omega_3-\omega_2)M^*.$$

Assuming that

$$\omega_3^{-1} + \omega_2^{-1} \approx \frac{2}{\omega_R},$$

we get

$$\mathrm{Re}\,\{Z_m\} = D - \frac{G_2}{\omega_R g} = (\omega_3-\omega_2)M^*.$$

The mechanical quality factor is then defined by

$$Q_M \equiv \frac{\omega_R}{(\omega_3 - \omega_2)} .$$

With these factors it is now possible to evaluate important performance parameters of the ring.

EVALUATION OF $h_{\theta\theta}$

We first note that from the above development we have two formulas for Re $\{Z_m\}$, viz,

$$\text{Re}\,\{Z_m\} = \frac{|Z_{em}|^2}{D_m}; \quad \text{Re}\,\{Z_m\} = (\omega_3 - \omega_2)M^* .$$

Hence

$$|Z_{em}| = \sqrt{(\omega_3 - \omega_2)M^* D_m}$$

and

$$h_{\theta\theta} = \frac{a\sqrt{(\omega_3 - \omega_2)M^* D_m}}{\mu^S b l N |\chi|} .$$

Thus, to find the magnetostriction constant h we must measure the quadrantal frequencies ω_3, ω_2, the effective mass M^*, the diameter of the motional impedance circle D_m, and the absolute value of the eddy current "shielding" $|\chi|$. The factor μ^S is the reversible permeability at zero frequency, which can be calculated from the core inductance and the constants of the coil winding. The factors a, b, l are the mean radius of the ring, its thickness, and its axial length, respectively.

EVALUATION OF VELOCITY, DISPLACEMENT, AND LOSSES

At resonance, defined by $\omega_R \equiv (S^*/M^*)^{1/2}$, the mechanical impedance in the presence of an acoustic load is $R_d + R_L$. The corresponding electrical impedance is $Z_{em}^2/(R_d + R_L)$. From the canonical set (in the mobility analogy) the velocity is given by

$$v = \frac{Z_{em}}{Z_m} I + \frac{F}{Z_m} .$$

Note that v is the "velocity of the ring" in a one-dimensional description. If we set $F = 0$ and consider the condition of mechanical resonance, we see that

$$|v|_{\omega=\omega_R} = \frac{|Z_{em}|}{(R_d + R_L)} |I| .$$

Similarly, the displacement amplitude $|x|$ in the presence of acoustic load is

$$|x| = \frac{|v|}{\omega_R}\,.$$

The internal mechanical power loss at mechanical resonance due to motion in the presence of an acoustic load is given by

$$(\text{Mech. Power Loss})_{\omega=\omega_R} = R_d |v_{\text{rms}}|^2 = \frac{|Z_{em}|^2}{(R_d + R_L)^2} R_d |I_{\text{rms}}|^2 .$$

In the presence of radiation the ratio of mechanical to macro-eddy-current loss, represented by R_c is

$$\frac{\text{mechanical loss}}{\text{macro-eddy-current loss}} = \frac{|Z_{em}|^2 R_d}{(R_L + R_d)^2 R_c} = \frac{Z_{em}^2}{R_L + R_d}\left(\frac{R_L}{R_L + R_d}\right)\left(\frac{R_d}{R_L}\right)\left(\frac{1}{R_c}\right)$$

$$= \frac{D_L{}^2}{D_v R_c}$$

where D_L, D_v are diameters of the loaded and unloaded circles, respectively. The internal mechanical loss resistance R_m'' depends on micro-eddy-currents, magnetic hysteresis, thermoelastic effects, and internal friction. It can be estimated by subtracting $(Y_\theta{}^B hlk_c{}^2 2\pi\chi_I)/(1 - \nu^2)a\omega$ from Re $\{Z_m{}^0\}$.

ELECTROSTRICTIVE TRANSDUCTION

When the transduction principle is electrostrictive rather than magnetostrictive, the preceding analysis is directly applicable with a change of parameters. We first select a set of constitutive equations of the piezoelectric type, adapted to the low-frequency linear behavior of an electrostrictive ceramic:

$$T = C^D S - h_t D$$

$$\mathcal{E} = -hS + \beta^S D$$

in which all parameters are matrixes with the following meanings:

T = stress

h = electrostriction constant (subscript t means transposed matrix)

S = strain

D = dielectric displacement

$\mathcal{E}$ = voltage gradient

β^S = inverse permittivity.

Note that the units of h are volts per meter or Newton's per Coulomb. We consider two cases, depending on the axis of polarization.

1. Polarized across the radial thickness or 3-direction; i.e., the ceramic cylinder is electroded on the inside and outside radius

2. Polarized in the tangential or 1-direction; i.e., the ceramic cylinder is constructed of ("barrel") staves electroded on the certical faces which abut each other.

A third case, in which the electrostrictive ring shell is polarized in the axial or 2-direction, i.e., the cylinder is electroded on the ends, is not considered here.

Case 1. Radial Polarization

The appropriate constitutive relations when the polarization is across the thickness of the cylinder (in the 3-direction) are

$$T_1 = C_{11}^D S_1 - h_{31} D_3$$

$$\mathcal{E}_3 = -h_{31} S_1 + \beta_{33}^S D_3 .$$

Since these are low-frequency equations in which the only mechanical impedance is the stiffness of the shell, we consider the dynamic vibration of the transducer over a frequency range of its first radial mode. The analysis is then one-dimensional in the radial displacement ξ driven by the radial force F_r. In linear terms,

$$F_r = F_1 \delta\theta = T_1 bl\delta\theta$$

where F_1 = tangential force, b = radial thickness, l = axial length of the ring, and $\delta\theta$ = the incremental tangential angle. The dynamic equation is then

$$\frac{\partial^2 \xi}{\partial t^2} (\rho a \delta\theta) bl = -T_1 bl\delta\theta$$

or

$$\rho a \frac{\partial^2 \xi}{\partial t^2} = -T_1$$

where a = mean radius of the shell and ρ = density of shell. Substituting for T_1 from the constitutive relations reduces this to

$$\rho a \frac{\partial^2 \xi}{\partial t^2} + \frac{C_{11}^D \xi}{a} = h_{31} D_3 .$$

In terms of radial velocity $v_r = \xi / j\omega$, this is

$$j\omega\rho a v + \frac{C_{11}^D v}{j\omega a} = \frac{h_{31} I}{j\omega 2\pi a l}$$

in which we have set the electric current

$$I \equiv (j\omega D_3)(2\pi a l).$$

Multiplying through by $2\pi lb$ leads to the relation

$$\left(j\omega\rho V + \frac{C_{11}^D 2\pi lb}{j\omega a}\right) v = \left(\frac{h_{31} b}{j\omega a}\right) I,$$

or

$$z_m v = \psi I$$

$$\psi = \frac{h_{31} b}{j\omega a}$$

$$z_m = j\omega M_{eq} + \frac{K_{eq}}{j\omega} + R_m$$

$$M_{eq} = \rho V$$

$$K_{eq} = \frac{2\pi lb C_{11}^D}{a}$$

$$V = 2\pi a l b.$$

Here we have included an internal resistance R_m in the definition of mechanical impedance z_m. When there is an acoustic load $F_L = -vZ_L$, the total mechanical impedance is written as Z_M, where

$$Z_M v = \psi I.$$

The electrical impedance is now determined from the second constitutive relation in terms of the applied voltage $E_3 = b\,\mathcal{E}_3$:

$$E_3 = -h_{31} b \frac{\xi}{a} + \beta_{33}^S b D_3$$

or

$$E_3 = \frac{-h_{31} b v}{j\omega a} + \frac{\beta_{33}^S b I}{j\omega 2\pi a l}.$$

It is thus seen that the transduction coefficient $\mathcal{T}_{em}$ (or T_{me}), as defined by the canonical equations

$$E = Z_b I + T_{em} v$$

$$F = T_{me} I + Z_M v,$$

is given by

$$T_{em} = \frac{-h_{31} b}{j\omega a} = -\psi,$$

and the blocked electrical impedance is

$$Z_b = \frac{\beta_{33}^S b}{j\omega 2\pi a l}.$$

Since

$$v = \psi I / Z_M,$$

we see that

$$E_3 = Z_b I + \left(\frac{-\psi\psi}{Z_M}\right) I,$$

so that the total electrical impedance Z_{ee} is

$$Z_{ee} = Z_b + \left(\frac{-\psi\psi}{Z_M}\right).$$

Here we define the motional electrical impedance Z_{Mot} by

$$Z_{Mot} = \frac{-\psi\psi}{Z_M}$$

or

$$Z_{Mot} = \frac{h_{31}^2 b^2}{\omega^2 a^2 (z_m + Z_L)}.$$

Case 2. Tangential Polarization

We select the constitutive relations

$$S = s^E T + d_t \mathfrak{E}$$

and

$$D = dT + \epsilon^T \mathfrak{E}$$

where d = electrostriction constant (subscript t means transposed matrix). For tangential polarization in a cylindrical ring having N "barrel" staves these relations reduce to

$$S_3 = s_{33}^E T_3 + d_{33} \mathcal{E}_3$$

$$D_3 = d_{33} T_3 + \epsilon_{33}^T \mathcal{E}_3 .$$

Replacing T_1 of Case 1. by T_3, we see that the dynamic equation is given by

$$\rho a \frac{\partial^2 \xi}{\partial t^2} = -T_3$$

$$\rho a \frac{\partial^2 \xi}{\partial t^2} = -\frac{S_3}{s_{33}^E} - \frac{d_{33}}{s_{33}^E} \mathcal{E}_3$$

or

$$\rho a \frac{\partial^2 \xi}{\partial t^2} + \frac{\xi}{a s_{33}^E} = \frac{-d_{33}}{s_{33}^E} \mathcal{E}_3 .$$

In steady state

$$\rho a j\omega v + \frac{v}{j\omega a s_{33}^E} = \frac{-d_{33}}{s_{33}^E \left(\frac{2\pi a}{N}\right)} E_3 .$$

Multiplying the dynamic equation through by $2\pi lb$, we obtain

$$v = \frac{\phi}{z_m} E_3 , \qquad \phi = \frac{-d_{33} lbN}{s_{33}^E a} .$$

Let

$$z_m = (\rho V) j\omega + \left(\frac{2\pi lb}{j\omega a s_{33}^E}\right)$$

or

$$z_m = j\omega M_{eq} + \frac{K_{eq}}{j\omega} + R_m$$

where

$$M_{eq} = \rho V$$

$$K_{eq} = \frac{2\pi lb}{a s_{33}^E}$$

R_m = internal resistance.

When there is an acoustic load $F_L = -vZ_L$, we replace z_m by Z_M where

$$Z_M = z_m + Z_L .$$

Thus, for a water load,

$$v = \frac{\phi E_3}{Z_M} .$$

The stress T_3 in water is therefore

$$T_3 = \frac{S_3}{s_{33}^E} - \frac{d_{33}}{s_{33}^E} \mathcal{E}_3$$

$$T_3 = \frac{\phi E_3}{Z_M j\omega s_{33}^E a} - \frac{d_{33} E_3 N}{s_{33}^E 2\pi a} .$$

Substituting this formula for T_3 into the canonical equation for D_3, and noting that

$$D_3 = \frac{I_3}{j\omega l b N},$$

we reduce the equation for the electrical admittance to the form

$$\frac{I_3}{E_3} = \frac{\phi^2}{Z_M} + \frac{j\omega_{33}^T N^2 lb}{2\pi a} \left(1 - k_{33}^2\right)$$

$$k_{33}^2 = \frac{d_{33}^{E2}}{s_{33}^E \epsilon_{33}^T} .$$

GLOSSARY

a = mean radius of reference surface (m)

$a(k)$ = radius of cylinder at kth band (m)

B = magnetic induction (Wb/m^2)

b = shell thickness (m)

C_{ij}^{B} = stress-strain modulus at constant induction (N/m^2)

$D_n(\omega)$ = generalized stiffness (N/m)

E = voltage (V)

G = true surface radiation Green's function (N-sec/m^3)

$\mathcal{G}(l, r)$ = reference surface radiation Green's function (N-sec/m^3)

g_{ij} = influence coefficient (m^3/N-sec)

H_k = magnetic intensity (A/m)

$h_{ik}^{(t)}$ = transposed piezomodulus (N/Wb or A/m)

I = current (A)

k^2 = low-frequency coefficient of electromechanical coupling

l = shell axial length (m)

M_n = modal mass (N-sec^2/m)

M_s = static mass (N-sec^2/m)

N = number of turns of wire

N_n = mode shape normalization factor (dimensionless)

$\mathcal{P}$ = equivalent driving pressure (N/m^2)

p = acoustic pressure (N/m^2)

$\bar{p}$ = average pressure (N/m^2)

R', R'' = mechanical resistance (N-sec/m)

S = surface area (m^2)

S_j = strain 6-vector (dimensionless)

T_i = stress 6-vector (N/m^2)

$\dot{u}$ = velocity in axial direction (m/sec)

v_N = normal component of velocity (m/sec)

W_n = modal radial displacement (dimensionless)

W_n^0 = surface integral of W_n (m^2)

$\dot{w}$ = velocity in radial direction (m/sec)

Y_n = modal Young's modulus at constant magnetic induction (N/m^2)

Z^b = total blocked electrical impedance (V/A)

Z^c = electrical core impedance (V/A)

Z^l = electrical leakage flux impedance (V/A)

$\mathcal{Z}_{ij}^{(m)}$ = mechanical influence function under open-circuit conditions (N-sec/m^5)

$\hat{Z}_{ij}$ = specific acoustic impedance (N-sec/m^3)

$\hat{Z}_Y$ = negative stiffness specific acoustic impedance (N-sec/m^3)

γ = complex number accounting for leakage (dimensionless); sign symbol

$\Delta_i(k)$ = finite increment of coordinate (m)

μ^S = permeability at constant strain (Wb/A-m or H/m)

ν = Poisson's ratio

ρ_n = modal density (N-sec^2/m^4)

ρ_s = static density (N-sec^2/m^4)

Φ_i = magnetic flux vector (Wb)

χ = eddy current and hysteresis factor (dimensionless)

ψ_i = transduction coefficient (N/A-m^2)

ψ_i' = transduction coefficient (V-sec/m^3)

ω = angular frequency (1/sec)

V = volt

ϵ_m = Neumann factor ($\epsilon_0 = 1$; $\epsilon_m = 2, m \neq 0$)