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A Categorization of Two-Person Zero-Sum Differential Games

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The upper and lower game values are used to distinguish among three distinct types of games. The difference between these two values serves as a criterion for a player's best playing order—first, second, or in any order. Examples are given of each type. Furthermore, saddlepoint strategies are given for a game of the play-first type.

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CONTENTS

A CATEGORIZATION OF TWO-PERSON ZERO-SUM DIFFERENTIAL GAMES

INTRODUCTION

In a two-person zero-sum differential game in which all strategies of one player are playable with all strategies of the other player, several relations are known to hold [1-13]. One relationship is that the upper game value (i.e., inf. sup. value) is never less than the lower game value (i.e., sup. inf. value). Another is that if a saddlepoint strategy exists then the saddlepoint value is unique and is equal to the upper and lower game values. A third is that a player's saddlepoint strategy assures him, regardless of his opponent's strategy, a cost that is at least as favorable as the saddlepoint game value. In a recent paper [14], it is shown by example that these relations can fail to hold in differential games in which open-loop strategy pairs are subject to terminating conditions. By an example, the same fate is demonstrated in this report for closed-loop strategies.

The choice of a particular strategy by one player can limit the choices of strategies of the other player. For example, some strategies of the other player may violate state constraints or never bring the game to termination. In general, therefore, it is not valid to assume that all strategies of one player are playable with all strategies of the other player. Because of this, it is not necessarily in a player's best interest to observe his opponent's strategy before choosing his own. He may well want to disclose his strategy first, provided that a more favorable outcome could result. Whether it is best to play first or second is determined by the difference between the upper and lower game values. If this difference is positive, each player will want his opponent to play first. If it is negative, each player will want to be first in selecting a strategy. There is little concern if this difference is zero. Consequently, the difference between the upper and lower game values permits a categorization of two-person zero-sum differential games. These three types are referred to as play-second, play-first, and play-anytime. Examples of each are discussed in the third section, and a saddlepoint strategy pair of a play-first game is presented in the fifth section. Before describing mathematically the relations between the various game values, we give a rigorous development of the two-person zero-sum differential game.

STATEMENT OF A DIFFERENTIAL GAME

We consider a two-person zero-sum differential game with state equations

$$\dot{x} = f(x, u, v) \tag{1}$$

Note: Manuscript submitted March 6, 1974.

where

x is contained in E^n
u is contained in E^r
v is contained in E^s
f is a Borel measurable function with domain $E^n \times E^r \times E^s$ and range in E^n.

The playing space denoted by X is a Lebesgue measurable subset of E^n. The target Θ is a closed set contained in the closure of X.

The two players are denoted by P and E. They choose the values of u and v, respectively, for all points x of the playing space X. Let U and V be compact subsets of E^r and E^s, respectively. Let $\mathcal{P}$ denote the set of all strategies $p : X \to U$ such that there is some Borel measurable function $p_1 : E^n \to U$ with $p = p_1|X$. The notation $p_1|X = p$ means that $p_1(x) = p(x)$ for all $x \in X$, i.e., p is the restriction of p_1 to X. Let $\mathcal{E}$ denote the set of all strategies $e : X \to V$ such that $e = e_1|X$ for some Borel measurable function $e_1 : E^n \to V$. The sets $\mathcal{P}$ and $\mathcal{E}$ constitute the sets of admissible strategies for players P and E, respectively.

Let $x_0 \in X$. For $p \in \mathcal{P}$ and $e \in \mathcal{E}$, the pair (p, e) is said to be a playable strategy pair at the point $x_0 \in X$ if it generates at least one terminating trajectory φ satisfying the equation

$$\varphi(t) = \varphi(t_0) + \int_{t_0}^{t} f(\varphi(\tau), p(\varphi(\tau)), e(\varphi(\tau)))d\tau \tag{2}$$

for all $t \in [t_0, t_f]$, where $\varphi(t_0) = x_0$, $\varphi(t) \in X$ for all $t \in [t_0, t_f)$, and t_f is the first time for which $\varphi(t_f) \in \Theta$. Let $\mathcal{T}(x_0)$ denote the set of all playable strategy pairs at the point x_0. Define

$$\mathcal{T}(X) = \bigcap_{x_0 \in X} \mathcal{T}(x_0).$$

We assume that $\mathcal{T}(X)$ is nonempty.

For $x_0 \in X$, define $\mathcal{P}(x_0)$ as the set of $p \in \mathcal{P}$ such that there is some member e of $\mathcal{E}$, with (p, e) playable. The set $\mathcal{E}(x_0)$ is defined analogously.

Let $x_0 \in X$. For each $e \in \mathcal{E}(x_0)$, we define $\mathcal{P}(e, x_0)$ to be the set of all strategies $p \in \mathcal{P}(x_0)$, such that (p, e) is a playable strategy pair at x_0. The set $\mathcal{E}(p, x_0)$ is defined analogously for each $p \in \mathcal{P}(x_0)$. Note that the following equalities hold:

$$\mathcal{P}(x_0) = \bigcup_{e \in \mathcal{E}(x_0)} \mathcal{P}(e, x_0)$$

$$\mathcal{E}(x_0) = \bigcup_{p \in \mathcal{P}(x_0)} \mathcal{E}(p, x_0) .$$

Observe that if the initial state of the game is x_0 and if player E has not yet announced some strategy from $\mathcal{E}(x_0)$, then player P may choose any strategy from $\mathcal{P}(x_0)$. However, if E announces $e \in \mathcal{E}(x_0)$ as his choice, then P may choose only from $\mathcal{P}(e, x_0)$ which is, in general, a much smaller class of strategies. In a game situation it is apparent then that a player must decide whether it is more important to limit his opponent's playable strategies or to let his opponent disclose the strategy he intends to play.

Let $x_0 \in X$ and $(p, e) \in \mathcal{T}(x_0)$. We define $T(x_0; p, e)$ to be the set of all solutions φ of Eq. (2) emanating from x_0 and associated with (p, e). Let $\varphi \in T(x_0; p, e)$. Associated with the quadruple (x_0, p, e, φ) there is a real number $V(x_0; p, e, \varphi)$, termed the cost, and defined by

$$V(x_0; p, e, \varphi) = \int_{t_0}^{t_f} f_0 \left(\varphi(\tau),\ p[\varphi(\tau)], \dot{e}[\varphi(\tau)]\right) d\tau \tag{3}$$

where f_0 is a real-valued, bounded Borel measurable function with domain $E^n \times E^r \times E^s$. For each $x_0 \in X$, the player P desires to minimize $V(x_0; p, e, \varphi)$, whereas player E wants to maximize it.

Let $x_0 \in X$, $(p^*, e^*) \in \mathcal{T}(x_0)$ and $\varphi^* \in T(x_0; p^*e^*)$. The pair (p^*, e^*) constitutes a saddlepoint strategy pair at the point x_0 if and only if

$$V(x_0; p^*, e^*, \varphi^*) \leqslant V(x_0; p, e,^*\varphi) \quad \text{for all } p \in \mathcal{P}(e^*, x_0)$$
$$\text{for all } \varphi \in T(x_0; p, e^*) , \tag{4}$$

and

$$V(x_0; p^*, e, \varphi) \leqslant V(x_0; p^*, e^*, \varphi^*) \quad \text{for all } e \in \mathcal{E}(p^*, x_0)$$
$$\text{for all } \varphi \in T(x_0; p^*, e) \tag{5}$$

If and only if $(p^*, e^*) \in \mathcal{T}(X)$ and Eqs. (4) *and* (5) *hold for all $x_0 \in X$, the pair constitutes a* saddlepoint strategy pair over X.

Let the pair (p^*, e^*) be a saddlepoint strategy pair over X and let $x_0 \in X$. The value $V(x_0; p^*, e^*, \varphi^*)$ that satisfies Eqs. (4) and (5) is termed a saddlepoint game value at x_0. It follows immediately from Eqs. (4) and (5) that if φ^* and $\varphi^{**} \in T(x_0; p^*, e^*)$, then $V(x_0; p^*, e^*, \varphi^*) = V(x_0; p^*, e^*, \varphi^{**})$. Since the trajectory as an argument is redundant, we make the definition

$$V^*(x_0; p^*, e^*) = V(x_0; p^*, e^*, \varphi^*) \text{ for all } x_0 \in X \tag{6}$$

For $x_0 \in X$, the upper value $\overline{V}(x_0)$ of the game is defined by Eq. (7) and the lower value $\underline{V}(x_0)$ by Eq. (8):

$$\overline{V}(x_0) = \operatorname*{Inf}_{p \in \mathcal{P}(x_0)} \quad \operatorname*{Sup}_{\substack{e \in \mathcal{E}(p, x_0) \\ \varphi \in T(x_0; p, e)}} \quad V(x_0; p, e, \varphi) \tag{7}$$

$$\underline{V}(x_0) = \operatorname*{Sup}_{e \in \mathcal{E}(x_0)} \quad \operatorname*{Inf}_{\substack{p \in \mathcal{P}(e, x_0) \\ \varphi \in T(x_0; p, e)}} \quad V(x_0; p, e, \varphi)$$

We make the assumption that $\overline{V}(x_0)$ and $\underline{V}(x_0)$ are finite in value.

A CATEGORIZATION OF DIFFERENTIAL GAMES

The upper and lower values can be utilized to classify distinct types of differential games. In doing this we need some lemmas and a theorem.

For use in the following lemmas, theorem, and definition 1 let $x_0 \in X$ and let $\delta > 0$.

Lemma 1. If P, playing first plays $p \in \mathcal{P}(x_0)$ then E, going second, can play a strategy $e \in \mathcal{E}(p, x_0)$ so that

$$V(x_0; p, e, \varphi) \geqslant \overline{V}(x_0) - \delta \tag{9}$$

for some $\varphi \in T(x_0; p, e)$. If P plays first then P can play a strategy $p \in \mathcal{P}(x_0)$ assuring that

$$V(x_0; p, e, \varphi) \leqslant \overline{V}(x_0) + \delta \tag{10}$$

for all $e \in \mathcal{E}(p, x_0)$ and for all $\varphi \in T(x_0; p, e)$.

Proof. Define the function

$$\tilde{V} : \bigcup_{x_0 \in X} \left(x_0, \mathcal{P}(x_0)\right) \to R$$

so that

$$\tilde{V}(x_0, p) = \operatorname*{Sup}_{\substack{e \in \mathcal{E}(p, x_0) \\ \varphi \in T(x_0; p, e)}} \quad V(x_0; p, e, \varphi).$$

Note that

$$\overline{V}(x_0) = \operatorname*{Inf}_{p \in \mathcal{P}(x_0)} \quad \tilde{V}(x_0, p). \tag{11}$$

If P, playing first, plays $p \in \mathcal{P}(x_0)$ then

$$\overline{V}(x_0) \leqslant \tilde{V}(x_0, p) .$$

By definition of $\tilde{V}$ player E can play a strategy e such that

$$\tilde{V}(x_0, p) - \delta \leqslant V(x_0; p, e, \varphi)$$

for some $\varphi \in T(x_0; p, e)$. The two letter inequalities imply(9).

From (11) it follows that player P, playing first, can play a strategy p such that

$$\tilde{V}(x_0, p) \leqslant \overline{V}(x_0) + \delta .$$

By definition of $\tilde{V}$ we have

$$V(x_0; p, e, \varphi) \leqslant \tilde{V}(x_0, p)$$

for all $p \in \mathcal{P}(x_0)$, for all $e \in \mathcal{E}(p, x_0)$, and for all $\varphi \in T(x_0; p, e)$. Inequality (10) now follows.

The following lemma has a similar proof.

Lemma 2. If E, playing first, plays $e \in \mathcal{E}(x_0)$ then P, going second, can play a strategy $p \in \mathcal{P}(x_0, e)$ so that

$$V(x_0; p, e, \varphi) \leqslant \underline{V}(x_0) + \delta$$

for some $\varphi \in T(x_0; p, e)$. E, playing first, can play a strategy $e \in \mathcal{E}(x_0)$ so that

$$V(x_0; p, e, \varphi) \geqslant \underline{V}(x_0) - \delta$$

for all $p \in \mathcal{P}(e, x_0)$ and for all $\varphi \in T(x_0; p, e)$.

Definition 1. A player is said to play δ-*optimal* at x_0 if he plays a strategy resulting in a cost that is within δ of the most favorable cost possible to him at x_0, taking into account his playing order.

The above lemmas give rise to the following result.

Theorem 1. If P plays first and both players play δ-optimal then they play strategies p and e such that

$$|V(x_0; p, e, \varphi) - \overline{V}(x_0)| \leqslant \delta$$

for some $\varphi \in T(x_0; p, e)$. On the other hand, if E plays first and both players play δ-optimal then they play strategies p and e so that

$$|V(x_0; p, e, \varphi) - \underline{V}(x_0)| \leqslant \delta$$

for some $\varphi \in T(x_0; p, e)$.

We permit the player going second his choice of the trajectory $\varphi \in T(x_0; p, e)$. Thus, if P plays first the cost lies in a δ-neighborhood of $\overline{V}(x_0)$. If E plays first, however, then the cost lies in a δ-neighborhood of $\underline{V}(x_0)$.

Definition 2. Let g_1 and g_2 be real-valued functions defined on X. The function g_1 is said to be less than g_2, i.e., $g_1 < g_2$, if and only if $g_1(x) \leqslant g_2(x)$ for all $x \in X$ and there is at least one $x \in X$ such that $g_1(x) < g_2(x)$.

The inequality

$$\underline{V} < \overline{V} \tag{12}$$

characterizes those games in which each player desires to play last. Let $x_0 \in X$ be such that $\underline{V}(x_0) < \overline{V}(x_0)$. If P plays first then according to Theorem 1 he loses an amount arbitrarily close to $\overline{V}(x_0) - \underline{V}(x_0)$ over what he can profit by playing second. If E plays first he likewise loses the same amount. These games are termed *play-second.*

The inequality

$$\underline{V} > \overline{V} \tag{13}$$

represents those games in which each player desires to choose his strategy first. Let $x_0 \in X$ such that $\underline{V}(x_0) > \overline{V}(x_0)$. If player P plays first he gains an amount arbitrarily close to $\underline{V}(x_0) - \overline{V}(x_0)$ over what he can profit by playing second. The same gain hold for player E if he plays first. These games are termed *play-first.*

The equality

$$\underline{V} = \overline{V} \tag{14}$$

classifies those games in which the players are willing to play in any order. The player P loses an amount arbitrarily close to $\overline{V}(x_0) - \underline{V}(x_0)$, which is nothing in this case, if he plays first. He gains an amount arbitrarily close to $\underline{V}(x_0) - \overline{V}(x_0)$, which is again nothing, if he plays second. Also, there is no loss or gain for the player E. These games are termed *play-anytime.*

For a given game, let X_s denote the set of all states $x \in X$ such that $\underline{V}(x) < \overline{V}(x)$. Let X_f be the set of $x \in X$ such that $\underline{V}(x) > \overline{V}(x)$. And let X_a represent the set of $x \in X$ such that $\underline{V}(x) = \overline{V}(x)$. Note that

$$X = X_s \bigcup X_f \bigcup X_a \, .$$

The relations (12)–(14) refer only to cases in which either X_s is empty or X_f is empty. There are games in which both X_f and X_s are nonempty. Such games are not analyzed in this report.

We shall now illustrate the three types of games by using simple examples.
Example 1: Consider the state equations

$$\dot{x}_1 = v_1 \qquad a \leqslant v_1 \leqslant b$$

$$\dot{x}_2 = u_1 \qquad a \leqslant u_1 \leqslant b ,$$

where a and b are real numbers with $a < b$. Let the target set be the straight line defined by

$$\Theta = \{(x_1, x_2) \in E^2 : x_1 + x_2 = 1\} .$$

The integrand of the cost, Eq. (3), is given by

$$f_0(x_1, x_2, u_1, v_1) = (u_1 - v_1)^2 \tag{15}$$

for all $(x_1, x_2) \in E^2$ and for all $u_1, v_1 \in [a, b]$. The state space X is taken to be the set $\{(x_1, x_2) \in E^2 : x_1 + x_2 \leqslant 1\}$.

If $a = 0$ and $b = 2$, one can verify that

$$\begin{aligned} \underline{V}(x_1^0, x_2^0) &= 0 \\ \overline{V}(x_1^0, x_2^0) &= \frac{2}{3}(1 - x_1^0 - x_2^0) \end{aligned} \tag{16}$$

for all $(x_1^0, x_2^0) \in X$; that is, the inequality (12) holds, and the game is of the *play-second* type.

Example 2: For $a = -1$ and $b = 2$ one has

$$\begin{aligned} \underline{V}(x_1^0, x_2^0) &= 9(1 - x_1^0 - x_2^0) \\ \overline{V}(x_1^0, x_2^0) &= 2(1 - x_1^0 - x_2^0) \end{aligned} \tag{17}$$

for all $(x_1^0, x_2^0) \in X$, so that inequality (13) holds. The game is of the *play-first* type.

Example 3: Suppose that, instead of f_0 given by Eq. (15),

$$f_0(x_1, x_2, u_1, v_1) = u_1^2 v_1 \tag{18}$$

for all $(x_1, x_2) \in E^2$ and for all $u_1, v_1 \in [a, b]$. For $a = -1$ and $b = 1$, it follows that

$$\underline{V}(x_0) = \overline{V}(x_0) = 0 \qquad \text{for all } x_0 \in X . \tag{19}$$

Thus, the equality (14) is met and the game is of *play-anytime* type.

A game of the play-first type is analyzed in greater detail in the fifth section.

GAME VALUE RELATIONS

As pointed out in the Introduction, using the notation of the previous sections, the four relations in question are given as I-IV below.

I. $\underline{V}(x_0) \leqslant \overline{V}(x_0)$ for all $x_0 \in X$.

II. $V^*(x_0; p_1^*, e_1^*) = V^*(x_0; p_2^*, e_2^*)$ for all $x_0 \in X$ and for all saddlepoint strategy pairs (p_1^*, e_1^*) and (p_2^*, e_2^*) over X.

If Condition II holds, then we make the definition

$V^*(x_0) = V^*(x_0; p_1^*, e_1^*)$ for all $x_0 \in X$.

III. $\underline{V}(x_0) = V^*(x_0) = \overline{V}(x_0)$ for all $x_0 \in X$.

IV. $V(x_0; p^*, e, \varphi) \leqslant \overline{V}(x_0)$ for all $x \in X$, for all $e \in \mathcal{E}(p^*, x_0)$, and for all $\varphi \in T(x_0; p^*, e)$.

$V(x_0; p, e^*, \varphi) \geqslant \underline{V}(x_0)$ for all $x_0 \in X$, for all $p \in \mathcal{P}(p^*, x_0)$ and for all $\varphi \in T(x_0; p, e^*)$,

where (p^*, e^*) is a saddlepoint strategy pair over X.

Condition I implies that each player prefers that his opponent be first to choose (and disclose) a strategy. For, if Condition I holds and player P chooses any $p \in \mathcal{P}(x_0)$, with player E optimizing his choice in $\mathcal{E}(p, x_0)$, then the resulting cost is no better than the upper value of the game. On the other hand, if player E chooses first e from $\mathcal{E}(x_0)$, followed by player P optimizing his choice in $\mathcal{P}(e, x_0)$, then the resulting cost is not above the lower value of the game. Consequently, if I holds, the player choosing first sustains a loss, the absolute value of which is $\overline{V}(x_0) - \underline{V}(x_0)$.

Condition II states that the saddlepoint value of the game is unique. Condition III is a statement about equality among the lower, saddlepoint, and upper values of the game. Condition IV states that if a player uses his strategy of a saddlepoint strategy pair, then he is guaranteed the upper value as player P and the lower value as player E.

In two-person zero-sum differential games, the relations I-IV are usually presupposed because their analogs hold for classical games (e.g. Ref. 2), and hold also for differential games in which every strategy of one player is playable with every strategy of the other player. But, as we shall demonstrate by the example in the next section, conditions I-IV need not hold.

SADDLEPOINT STRATEGIES OF A PLAY-FIRST GAME

We borrow the following game from Ref. 14 and show that it is of the type described by inequality (13). In Ref. 14, open-loop strategies are presented. Closed-loop strategies are provided below for this game.

Consider the differential game with state equations

$$(\dot{x}_1, \dot{x}_2, \dot{x}_3) = (v_1, u_1, v_2 + u_2), \tag{20}$$

where $u_1, v_1, \in [0, 2]$ and $u_2, v_2 \in [-2, 2]$. The state space is defined by

$$X = \{(x_1, x_2, x_3) \in E^3 : x_1, x_2 \geqslant 0\}.$$

The target set is

$$\Theta = \{(x_1, x_2, x_3) \in X : x_1 + x_2 - x_3^2 - 1 = 0\},$$

and the integrand of the cost, Eq. (3), is

$$f_0(x_1, x_2, x_3, u_1, v_1, u_2, v_2) = 2(x_2 u_1 - x_1 v_1) \tag{21}$$

for all $(x_1, x_2, x_3) \in E^3$ and for all $u_1, v_1, \in [0, 2]$ and $u_2, v_2 \in [-2, 2]$.

In view of Eqs. (20) and (21), Eq. (3) becomes

$$V(x_0; p, e, \varphi) = x_2^2(t_f) - x_1^2(t_f) - x_2^2(t_0) + x_1^2(t_0). \tag{22}$$

In Eq. (22), $x_0 = (x_1(t_0), x_2(t_0), x_3(t_0))$, a playable strategy (p, e) transfers the state from x_0 to the terminal point $x_f = (x_1(t_f), x_2(t_f), x_3(t_f)) \in \Theta$, and φ is a path associated with (p, e). Note that Eq. (22) is path independent.

Let $c \in [0, 1]$. For each such c we define the strategy pair (p_c^*, e_c^*) as follows, where $p_c^* = (p_{1c}^*, p_{2c}^*)$ and $e_c^* = (e_{1c}^*, e_{2c}^*)$:

$$p_{1c}^*(x_1, x_2, x_3) = \begin{cases} 0 & \text{if } x_1 + x_2 \geqslant 1 \text{ or } x_2 \geqslant c \\ 2 & \text{if } x_1 + x_2 \leqslant 1 \text{ and } x_2 < c \end{cases} \tag{23}$$

$$p_{2c}^*(x_1, x_2, x_3) = \begin{cases} -2 & \text{if } x_1 + x_2 - x_3^2 < 1 \text{ and } x_3 > 0 \\ 0 & \text{if } x_1 + x_2 - x_3^2 < 1 \text{ and } x_3 = 0, \text{ or} \\ & \text{if } x_1 + x_2 - x_3^2 = 1 \text{ and all } x_3 \\ 2 & \text{if } x_1 + x_2 - x_3^2 < 1 \text{ and } x_3 < 0, \text{ or} \\ & \text{if } x_1 + x_2 - x_3^2 > 1 \text{ and all } x_3. \end{cases} \tag{24}$$

$$e_{1c}^*(x_1, x_2, x_3) = \begin{cases} 0 & \text{if } x_1 + x_2 \geqslant 1 \text{ or } x_1 \geqslant 1 - c \\ 2 & \text{if } x_1 + x_2 \leqslant 1 \text{ and } x_1 < 1 - c \end{cases} \tag{25}$$

$$e^*_{2c}(x_1, x_2, x_3) = p^*_{2c}(x_1, x_2, x_3) \quad \text{for all } (x_1, x_2, x_3) \in X\,. \tag{26}$$

We claim that the pair (p^*_c, e^*_c) is a saddlepoint strategy over X. This claim is established by introducing the auxiliary function V^*_c below and then applying the sufficiency theorem of Ref. 15 to show that V^*_c is, indeed, the function $V^*(\cdot\,; p^*_c, e^*_c)$ defined in Eq. (6). Thus, we make the definition, for

$$x = (x_1, x_2, x_3) \in X\,,$$

$$V^*_c(x) = \begin{cases} x_1^2 - (1 - x_2)^2 & \text{if } x_1 + x_2 < 1 \text{ and } x_2 \geq c \\ (1 - x_1)^2 - x_2^2 & \text{if } x_1 + x_2 < 1 \text{ and } x_1 \geq 1 - c \\ c^2 - (1 - c)^2 + x_1^2 - x_2^2 & \text{if } x_1 < 1 - c \text{ and } x_2 < c \\ 0 & \text{if } x_1 + x_2 \geq 1\,. \end{cases} \tag{27}$$

Note that V^*_c is a continuous function on X.

A decomposition D of the state space X is given by

$$D = \left\{ X_1, X_2, X_3, X_4 \right\},$$

where

$$\begin{aligned} X_1 &= \left\{x \in X : x_1 + x_2 < 1 \text{ and } x_2 \geq c \right\} \\ X_2 &= \left\{x \in X : x_1 + x_2 < 1 \text{ and } x_1 \geq 1 - c \right\} \\ X_3 &= \left\{x \in X : x_1 < 1 - c \text{ and } x_2 < c \right\} \\ X_4 &= \left\{x \in X : x_1 + x_2 \geq 1\right\}. \end{aligned}$$

Observe that

$$X = \bigcup_{i=1}^{4} X_i\,.$$

The function V^*_c is of class C^1 with respect to the decomposition D, Ref. 15, since V^*_c is continuously differentiable on X_i, $i = 1, 2, 3, 4$.

The pair (p^*_c, e^*_c) is playable for all initial states of X and V^*_c equals zero on Θ. One can verify that the following two conditions are met:

(i) $f_0(x, p^*_c(x), v) + \operatorname{grad} V^*_c(x) \cdot f(x, p^*(x), v) \leq 0$
for all $x \in X$ and all $v = (v_1, v_2) \in [0, 2] \times [-2, 2]$

(ii) $f_0(x, u, e_c^*(x)) + \text{grad } V_c^*(x) \cdot f(x, u, e_c^*(x)) \geqslant 0$
for all $x \in X$ and all $u = (u_1, u_2) \in [0, 2] \times [-2, 2]$,

where the function f is defined by Eqs. (1) and (20).

It follows, then, from Theorem 3.1 in Ref. 15 that the strategy pair (p_c^*, e_c^*) is a saddlepoint strategy over X.

Using Eqs. (7) and (8), we note that

$$\overline{V}(x_1, x_2, x_3) = x_1^2 - (1 - x_2)^2 \tag{28}$$

$$\underline{V}(x_1, x_2, x_3) = (1 - x_1)^2 - x_2^2 \tag{29}$$

for $x_1 + x_2 < 1$, and that

$$\overline{V}(x_1, x_2, x_3) = \underline{V}(x_1, x_2, x_3) = 0 \tag{30}$$

for $x_1 + x_2 \geqslant 1$. From Eqs. (28) and (29) we see that

$$\overline{V}(x_1, x_2, x_3) < \underline{V}(x_1, x_2, x_3) \tag{31}$$

whenever $x_1 + x_2 < 1$. Thus, the game under consideration is of the *play-first* type.

Finally, we note that the Condition I-IV do not hold for this game. For, from Eq. (27), it follows that relation II fails to hold for the saddlepoint strategy pairs (p_c^*, e_c^*), $c \in [0, 1]$. Relations II and III fail to hold because of Eq. (31). From Eqs. (27)–(29), it follows that relation IV fails to hold for all (p_c^*, e_c^*), $c \in [0, 1]$. Furthermore, if $c = 0$, then $V_c^* = \overline{V}$; and if $c = 1$, then $V_c^* = \underline{V}$.

CONCLUSION

Two-person zero-sum differential games can be classified according to whether it is best for a player to play first, second, or in any order. If the minimizing player P plays first then the cost of play lies in a neighborhood of the upper value of the game. On the other hand, if the maximizing player E plays first then the cost of play lies in a neighborhood of the lower value of the game. Consequently, if the lower value of the game is less than the upper value then both players desire to play last. If, however, the lower value is greater than the upper value then each player wants to be first. Thus, the difference between the upper and lower values of the game provides a criterion for classifying game types. If this difference is positive, the game is of the play-second type. If it is negative the game is of the play-first type. Games in which the difference is zero are termed play-anytime.

Each type of game is illustrated with a simple example. In addition a game of the play-first type is analyzed in some detail. Closed-loop strategies are presented for this game and a sufficiency theorem is applied to show that they are saddlepoint strategies. Furthermore, four classical relations are shown to be invalid for this game.

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