

Monte Carlo Simulation of Emission Frequencies from Partial Frequency Redistribution Functions

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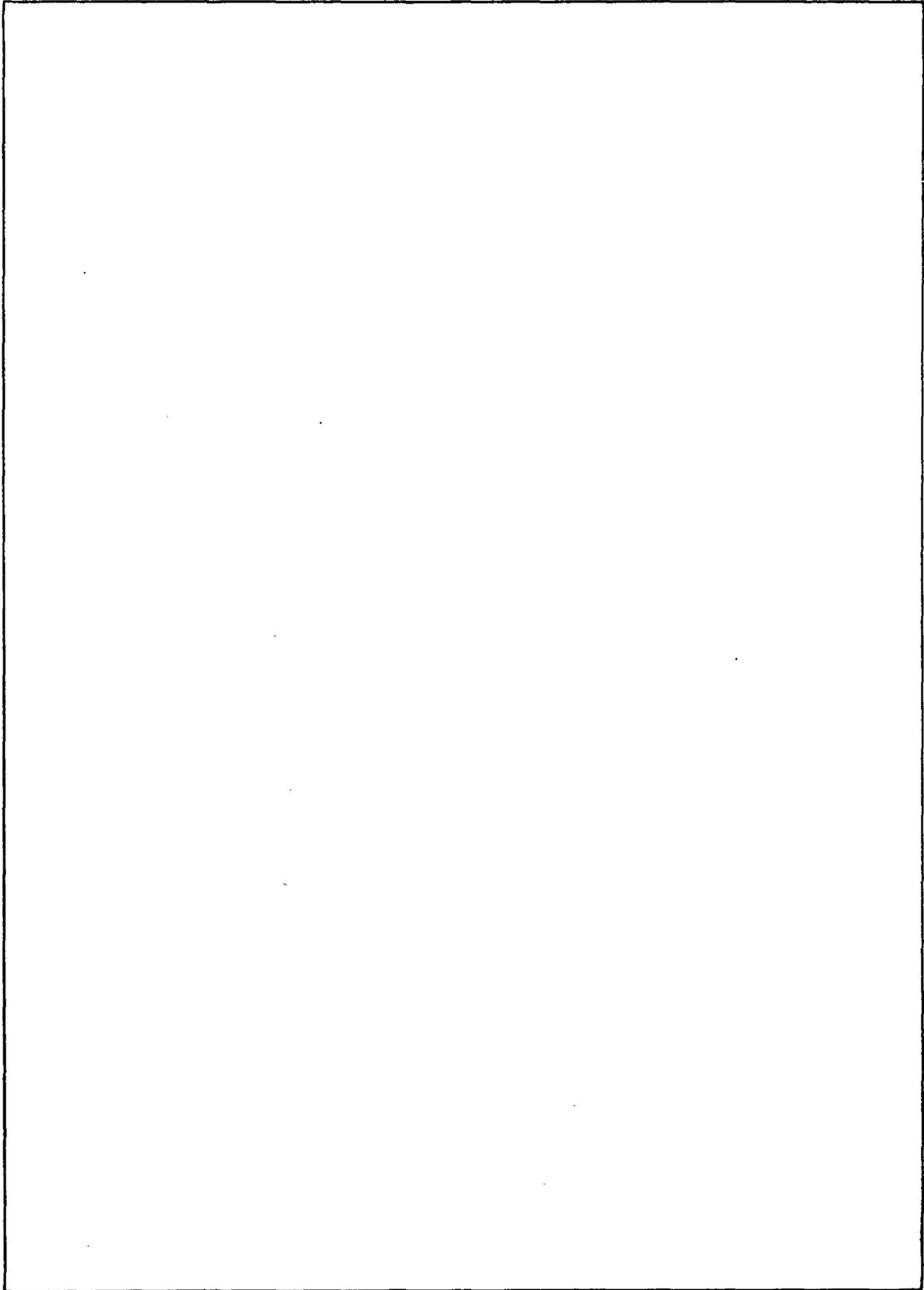
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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) Simple and efficient algorithms are presented for randomly selecting photon emission frequencies from a class of partial frequency redistribution functions. These formulas are developed by applying some basic rules of probability theory, in particular, Bayes' rule. Theoretical proofs of these formulas are given, and numerical values of the generated frequency distributions are presented to validate the algorithms. These algorithms are intended for Monte Carlo studies of photon diffusion problems.		



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MONTE CARLO SIMULATION OF EMISSION FREQUENCIES FROM PARTIAL FREQUENCY REDISTRIBUTION FUNCTIONS

1. INTRODUCTION

In recent Monte Carlo studies of photon diffusion problems by Auer (1), Wallace (2), and Avery and House (3), principal attention has been paid to comparisons between the results for complete and partial frequency redistributions. One of the major difficulties encountered in these studies was the question of randomly selecting emission frequencies from a partial frequency redistribution function. Such a function, first derived by Unno (4), is so complicated that the conventional methods of generating it fail. Consequently numerical integration, approximations, and inverse interpolation have been used, requiring significant programming effort as well as unnecessarily long computational times. The present report is devoted to overcoming these difficulties.

In the course of this work, algorithms were derived not only for the aforementioned partial frequency redistribution but also, with one exception, for other partial frequency redistribution functions of astrophysical interest discussed by Hummer (5). Our attempt to handle the case of $i \rightarrow j \rightarrow i$ resonance was unsuccessful. Fortunately, it is also a case of little practical interest.

The problem was approached by first developing a general formula for generating a frequency from an arbitrary redistribution function and by subsequently obtaining explicit formulas for the following three partial frequency redistribution functions:

- (I) Zero natural line width with coherence in the atom's rest frame
- (II) Radiation damping with coherence in the atom's rest frame
- (III) Radiation and collision damping with complete redistribution in the atom's rest frame

For each case, two phase functions are considered. They are the isotropic and the dipole phase functions, designated by suffixes A and B respectively.

To make the derivation conceptually clear, probability-theory notation is used throughout this report, and the theory of conditional probability, in particular Bayes' rule, is applied to obtain various random-number generating schemes. Ultimately numerical-simulation results are presented in the form of histograms to confirm the theoretical proof of the suggested generating algorithms.

2. NOTATION AND BAYES' RULE

Let $p(x, x'|a, b)$ be the conditional joint probability density function (*p.d.f.*) of x and x' for the given a and b . Let the joint *p.d.f.* of x and x' be given by

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$$p(x, x') = \iint p(x, x'|a, b)p(a, b) dadb, \quad (1)$$

where $p(a, b)$ is the joint *p.d.f.* of a and b . If x and x' represent respectively the absorption and emission frequency of a photon, then the *p.d.f.* of x' , when a photon of frequency x is absorbed, is by Bayes' rule (e.g., Ref. 6, p. 114)

$$p(x'|x) = \frac{p(x, x')}{p(x)}. \quad (2)$$

Equation (2) is in effect the statement of Bayes' rule.

Substitution of Eq. (1) into Eq. (2) yields

$$p(x'|x) = \frac{\iint p(x, x'|a, b)p(a, b) dadb}{\iiint p(x, x'|a, b)p(a, b) dadbdx'}. \quad (3)$$

Equations (1) through (3) are the principal probability formulas used in the subsequent derivation of the random-number generating schemes of this report.

3. FORMULATION

The frequency redistribution functions used herein follow closely Hummer's definitions (5). However, probability notation, introduced in section 2, is used to distinguish the conditional *p.d.f.* from the marginal *p.d.f.*

Consider a photon of frequency ν which is absorbed by an atom moving in the observer's frame with velocity v in the direction n . The reemitted photon has frequency ν' and direction n' . By a suitable choice of coordinate system (e.g., Ref. 5), we obtain the Doppler relations

$$\nu = \xi + \Delta\mu u \quad (4)$$

and

$$\nu' = \xi' + \Delta\mu' u, \quad (5)$$

where the reduced velocity u is given by

$$u = \frac{v}{\alpha}, \quad \alpha = \sqrt{\frac{2kT}{M}}.$$

Furthermore, ξ is the absorption frequency in the atom's frame, and μ is the cosine of the angle between u and n . The quantities ξ' and μ' are defined similarly. The Doppler width is denoted by Δ .

Let ν_0 be the central frequency of the line, and let

$$x = \frac{\nu - \nu_0}{\Delta}$$

and

$$x' = \frac{\nu' - \nu_0}{\Delta}$$

Using these definitions, Eq. (4) and (5) can be written as

$$x = \frac{\xi - \nu_0}{\Delta} + \mu u \quad (6)$$

$$x' = \frac{\xi' - \nu_0}{\Delta} + \mu' u \quad (7)$$

Absorption Probability Density Function

The absorption *p.d.f.* in the atom's frame, $p(\xi)$, is taken to be independent of μ and u . Two different $p(\xi)$ will be considered. The first corresponds to zero natural line width, with

$$p(\xi)d\xi = \delta(\xi - \nu_0)d\xi, \quad (8)$$

where $\delta(\cdot)$ is the Dirac delta function; the second corresponds to radiation damping, with

$$p(\xi)d\xi = \frac{\delta'}{\pi} \frac{1}{(\xi' - \nu_0)^2 + (\delta')^2} d\xi. \quad (9)$$

In Eq. (9), δ' describes the damping properties of the atom. Using Eq. (6) with x considered as a random variable and μ and u as given, we can transform Eq. (8) into

$$p(x|u, \mu) = \delta(x - \mu u)dx \quad (10)$$

and Eq. (9) into

$$p(x|u, \mu)dx = \frac{a}{\pi} \frac{1}{a^2 + (x - \mu u)^2} dx, \quad -\infty \leq x \leq \infty, \quad (11)$$

where $a = \delta'/\pi$.

Redistribution Function

The redistribution *p.d.f.* in the atom's frame, $p(\xi'|\xi)d\xi'$, is the probability that a photon absorbed at frequency ξ is reemitted at a frequency ξ' and $\xi' + d\xi'$. The two redistribution functions which will be considered correspond to coherence in the atom's frame, with

$$p(\xi'|\xi)d\xi' = \delta(\xi' - \xi)d\xi', \quad (12)$$

and to noncoherence in the rest frame, with

$$p(\xi'|\xi)d\xi' = \frac{\delta'}{\pi} \frac{1}{(\xi' - \nu_0)^2 + (\delta')^2} d\xi', \quad (13)$$

where δ' includes both the natural and collisional broadening. In Eq. (13), $p(\xi'|\xi)$ is statistically independent of ξ . Using Eqs. (6) and (7), we can transform Eq. (12) into

$$p(x'|x, \mu, \mu', u)dx' = \delta(x' - x + \mu u - \mu' u)dx' \quad (14)$$

and Eq. (13) into

$$p(x'|x, \mu, \mu', u)dx' = \frac{a}{\pi} \frac{1}{a^2 + (x' - \mu' u)^2} dx'. \quad (15)$$

Phase Function

For the isotropic phase function we have

$$p(\mu'|\mu)d\mu' = \frac{1}{2} d\mu', \quad -1 \leq \mu' \leq 1, \quad (16)$$

whereas for the dipole phase function the corresponding relation is

$$p(\mu'|\mu)d\mu' = \frac{3}{16} [3 - \mu^2 - (\mu')^2 + 3\mu^2(\mu')^2] d\mu', \quad -1 \leq \mu' \leq 1. \quad (17)$$

Velocity Distribution

The reduced velocity u is assumed to have the Maxwellian distribution

$$p(u)du = \frac{4}{\sqrt{\pi}} e^{-u^2} u^2 du, \quad 0 \leq u \leq \infty. \quad (18)$$

General Expressions for $p(x, x')$ and $p(x'|x)$

The derivation of $p(x, x')$ and $p(x'|x)$ can now be undertaken using the foregoing developments. The quantity $p(x'|x)$ is the partial frequency redistribution function from

which the random numbers will be generated. It should be noted that the special case of complete frequency redistribution corresponds to $p(x'|x) = p(x')$. A specific example of this situation has been considered by Lee (7) for the Voigt function. Recall now that by the general formula of section 2 we have

$$p(x, x') = \iiint p(x'|x, \mu, \mu', u)p(x, \mu, \mu', u)d\mu d\mu' du. \quad (19)$$

The second *p.d.f.* in the integral can be written as

$$p(x, \mu, \mu', u) = p(x|\mu, \mu', u)p(\mu, \mu', u). \quad (20)$$

Intuitively we can argue that the absorption probability will not be affected by the emission direction μ' . Consequently

$$p(x|\mu, \mu', u) = p(x|\mu, u). \quad (21)$$

Furthermore, the second *p.d.f.* on the right-hand side of Eq. (20) can be written as

$$p(\mu, \mu', u) = p(\mu, \mu'|u)p(u). \quad (22)$$

By an analogous argument, that the joint distribution of μ and μ' is independent of the velocity u , we have

$$p(\mu, \mu'|u) = p(\mu, \mu') = p(\mu'|\mu)p(\mu). \quad (23)$$

Substituting Eq. (20) through (23) into Eq. (19), we obtain

$$p(x, x') = \iiint p(x'|x, \mu, \mu', u)p(x|\mu, u)p(\mu'|\mu)p(\mu)p(u)d\mu d\mu' du. \quad (24)$$

All *p.d.f.*'s appearing in the integral of Eq. (24) are known and have been discussed in the earlier sections. Conceptually Eq. (24) is identical to the general formula derived by Hummer (5) using a different approach.

In Monte Carlo simulation of photon-diffusion problems, we need to select randomly an emission frequency x' from a redistribution function $p(x'|x)$ when a photon of frequency x is absorbed. Unfortunately we cannot use Eq. (2), with $p(x, x')$ given by Eq. (24), because of the difficulty in obtaining analytical formulas for random frequency generation. Hence we proceed to transform the basic expression for $p(x'|x)$,

$$p(x'|x) = \iiint p(x'|x, \mu, \mu', u)p(\mu, \mu', u|x)d\mu d\mu' du, \quad (25)$$

into a more convenient form. By virtue of the fundamental probability relations, the second *p.d.f.* in the integral is converted into

$$p(\mu, \mu', u|x) = p(\mu, \mu'|u, x)p(u|x), \quad (26)$$

and subsequently

$$p(\mu, \mu'|u, x) = p(\mu'|\mu, u, x)p(\mu|u, x). \quad (27)$$

Since the direction μ' of the reemitted photon, when μ is given, is independent of the velocity u of the atom and the absorption frequency x , we have

$$p(\mu'|\mu, u, x) = p(\mu'|\mu), \quad -1 \leq \mu \leq 1. \quad (28)$$

Substituting Eq. (26) through (28) into Eq. (25), we have the key equation of this report:

$$p(x'|x) = \iiint p(x'|x, \mu, \mu', u)p(\mu'|\mu)p(\mu|u, x)p(u|x)d\mu d\mu' du. \quad (29)$$

4. GENERAL ALGORITHM

A general random-frequency generating scheme based on Eq. (29) involves four steps:

1. Generate u when x is given from $p(u|x)$;
2. Generate μ from $p(\mu|u, x)$ using the u generated in step (1) and x ;
3. Generate μ' from $p(\mu'|\mu)$ using the μ generated in step (2);
4. Generate x' from $p(x'|x, \mu, \mu', u)$ using x and the μ' , μ , and u obtained in steps (1) through (3).

The generation of μ' and x' in steps 3 and 4 is simple because the phase function $p(\mu'|\mu)$ and the redistribution function $p(x'|x, \mu, \mu', u)$, discussed in section 2, can be easily integrated and inverted analytically when used in connection with the well-known random-number generating methods described, for instance, by Cashwell and Everett (8). The difficulties to be resolved reside in steps (1) and (2). We proceed to discuss these in detail. Note that $p(u|x)$ is not equal to $p(u)$. In other words the distribution of u does depend on the value of frequency x . Applying Bayes' rule, we have

$$p(u|x) = \frac{p(u, x)}{p(x)} = \frac{p(x|u)p(u)}{p(x)} = \frac{\int p(x|u, \mu)p(\mu)p(u) d\mu}{p(x)}, \quad (30)$$

where $p(x|u, \mu)$ is the absorption *p.d.f.* given by either Eq. (8) or (9), $p(u)$ is given by Eq. (18), and

$$p(\mu) = \frac{1}{2}, \quad -1 \leq \mu \leq 1.$$

The denominator $p(x)$ represents the absorption-line profile. If Eq. (8) is adopted, we have the Doppler absorption profile, and if Eq. (9) is used, we have the Voigt absorption profile. The proofs are straightforward, using the formula

$$p(x) = \phi(x) = \int_0^{\infty} \int_{-1}^1 p(x|\mu, u) p(\mu) p(u) d\mu du. \quad (31)$$

Similarly

$$p(\mu|u, x) = \frac{p(\mu, u, x)}{p(u, x)} = \frac{p(x|\mu, u)p(\mu)p(u)}{p(x|u)p(u)}.$$

Since $p(\mu|u) = p(\mu)$, we have

$$p(\mu|u, x) = \frac{p(x|\mu, u)p(\mu)}{p(x|u)} = \frac{p(x|\mu, u)p(\mu)}{\int p(x|\mu, u)p(\mu) d\mu}. \quad (32)$$

All *p.d.f.*'s are known and have been given in section 2. Hence μ can be simulated from Eq. (32).

In the next section we proceed to derive random-number generating algorithms for each case listed in the introductory section.

5. ALGORITHMS FOR GENERATING RANDOM NUMBERS; NUMERICAL RESULTS

Case I — Zero Natural Line Width with Coherent Redistribution in the Atom's Rest Frame

With $p(x|u, \mu)$ and $p(x'|x, \mu, \mu', u)$ given by Eqs. (10) and (14) respectively, we have, by applying Eqs. (30) and (31),

$$p(u|x) = 2ue^{-(u^2-x^2)}, \quad u > |x|, \quad (33)$$

and $p(x) = \exp(-x^2)/\sqrt{\pi}$. The latter relation describes the Doppler profile.

The generation of u when x is given follows the conventional inverse method based on the expressions

$$r_1 = \int_{|x|}^u p(u|x) dx = 1 - e^{-u^2+x^2}$$

and

$$u = \sqrt{-\ln(r_1 e^{-x^2})}, \quad (34)$$

where r_1 is a uniformly distributed random number in the interval $[0, 1]$. Ways of obtaining r_1 are readily available (for example, Ref. 9). For convenience, throughout this paper $\{r_i\}$ will denote a sequence of independent uniformly distributed random numbers in the interval $[0, 1]$. Similarly, using Eq. (32), we have

$$p(\mu|x, u) = \begin{cases} u\delta(x - u\mu) & \text{if } u > |x|; \\ 0 & \text{otherwise.} \end{cases} \quad (35)$$

The generating scheme is simply

$$\mu = \frac{x}{u}. \quad (36)$$

So far we have obtained generating schemes for steps (1) and (2) of the general algorithm. For step (3) we consider the isotropic and the dipole phase functions separately. For the isotropic phase function, $p(\mu'|\mu)$ is uniformly distributed in the interval $[-1, 1]$, as given by Eq. (16). The generating scheme is

$$\mu' = 2\left(r_2 - \frac{1}{2}\right), \quad (37)$$

where r_2 is a random number selected from a uniform distribution in the interval $[0, 1]$. For the dipole phase function the generating procedure is more complicated. The algorithm given for it is based on the well-known rejection method, which, for $p(\mu'|\mu)$ given by Eq. (17), proved to be very efficient. The procedure for selecting μ' for the dipole phase function involves the following steps:

- (a) Simulate μ' from $[-1, 1]$ by using Eq. (37).
- (b) Select a uniformly distributed random number r_3 from $[0, 1]$, and accept the value of μ' if

$$\frac{p(\mu'|\mu)}{M} > r_3,$$

where

$$M = \max \{p(\mu'|\mu)\}, \quad -1 \leq \mu' \leq 1.$$

If not, return to step a.

The value of M depends on μ and is given by

$$M = \begin{cases} 3(\mu^2 + 1)/8 & \text{if } |\mu| > 1/\sqrt{3}; \\ 3(3 - \mu^2)/16 & \text{if } |\mu| \leq 1/\sqrt{3}. \end{cases}$$

This previous algorithm is incorporated in all cases involving the dipole phase function. In summary we have the following simulation algorithms:

Simulation Algorithms I-A and I-B

For a given x the emitted frequency is generated by the following steps:

- (1) $u = \sqrt{-\ln(r_1 e^{-x^2})}$;
- (2) $\mu = x|u$;
- (3) For the isotropic phase function use Eq. (37); for the dipole phase function use the simulation algorithm for the dipole phase function;
- (4) $x' = x - \mu u + \mu' u$.

It should be noted that the equation in step (4) can be simplified into $x' = \mu' u$ by using the equation of step (2).

It is of some interest to compare the corresponding redistribution functions first obtained by Unno (4) and Field (10). For isotropic scattering their result is

$$p_{I-A}(x', x) = \frac{1}{2} \operatorname{erfc}(|\bar{x}|),$$

and for dipole scattering it is

$$p_{I-B}(x', x) = \frac{3}{8} \left\{ \frac{1}{2} \operatorname{erfc}(|\bar{x}|) [3 + 2(x^2 + (x')^2) + 4x^2(x')^2] - \frac{e^{-|\bar{x}|^2}}{\sqrt{\pi}} |\bar{x}| (2|\bar{x}|^2 + 1) \right\},$$

where

$$|\bar{x}| = \max \{ |x'|, |x| \}$$

and

$$|\underline{x}| = \min \{ |x'|, |x| \}.$$

Numerical-simulation results produced by algorithms I-A and I-B are presented in Fig. 1 to confirm the theoretical development. The points of each rectangular element in the histograms are connected by smooth curves, and only the positive half of x' is simulated, because $p(x'|x)$ is symmetrical in x' . Comparison of Fig. 1 with the curves plotted by Hummer (5) and Jefferies (11) further indicates that algorithms I-A and I-B are the correct mechanisms for selecting random frequency x' for these cases.

Case II — Radiation Damping with Coherence in the Atom's Rest Frame

With $p(x|u, \mu)$ and $p(x'|x, \mu, \mu', u)$ given by Eqs. (11) and (14) respectively, we compute $p(u|x)$ using Eqs. (30) and (31). Thus

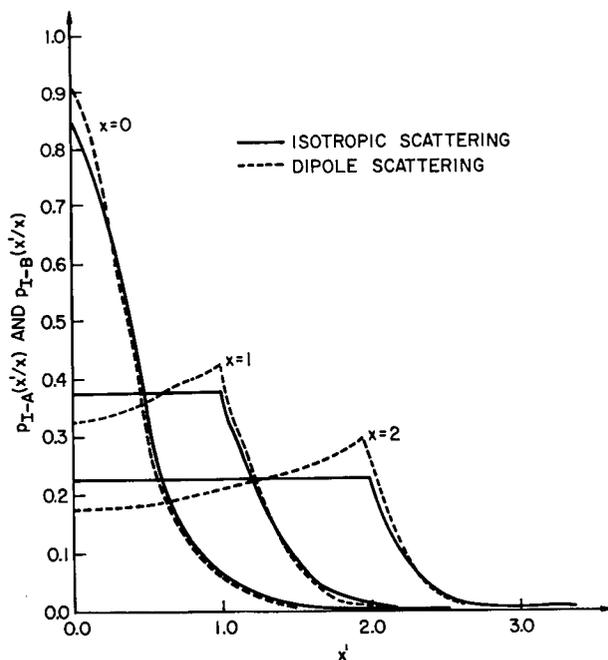


Fig. 1 — Histogram of simulated frequency distribution for Case I-A and Case I-B

$$p(u|x) = \frac{\pi^{-3/2}}{H(a, x)} \left[\tan^{-1} \left(\frac{u-x}{a} \right) + \tan^{-1} \left(\frac{u+x}{a} \right) \right] e^{-u^2} 2u, \quad 0 \leq u \leq \infty. \quad (38)$$

Here we have introduced the well-known Voigt distribution function

$$p(x) = H(a, x) = \frac{a}{\pi^{3/2}} \int_{-\infty}^{\infty} \frac{e^{-u^2}}{a^2 + (x-u)^2} du.$$

It is very difficult, and perhaps impossible, to obtain an efficient analytical formula to simulate u from this $p(u|x)$. Consequently numerical methods are used in its random-number generation. This involves computing a cumulative density function, solving for its inverse, tabulating the inverse, and then generating the random number from the tabulated data. For a given absorption frequency x between sampling points, interpolation must be used. Generally the large amounts of data would have to be precomputed and stored. However, for large x ($x > 6$ for $a = 10^{-2}$ to 10^{-4}) we can approximate Eq. (38) by the Maxwellian distribution

$$p(u|x) \approx \frac{4u^2 e^{-u^2}}{\sqrt{\pi}}, \quad 0 \leq u \leq \infty.$$

This approximation greatly simplifies the computation needed outside the main computer program and reduces its storage requirements. Furthermore, once the necessary information has been precomputed and stored, it offers a very efficient means of simulating random numbers.

Plots of $p(u|x)$ are shown in Fig. 2. It can be seen that when $x = 1$ or 2 , $p(u|x)$ behaves like a Dirac delta function centered at $u = 1$ or $u = 2$ respectively. Two peaks are observed for the $x = 3$ curve, which provides an explanation for the unusual behavior of $p_{\Pi-A}(x'|x)$ and $p_{\Pi-B}(x'|x)$ at $x = 3$ shown in Figs. 3 and 4. For $x > 4$, $p(u|x)$ is very close to the Maxwellian distribution, which is independent of x .

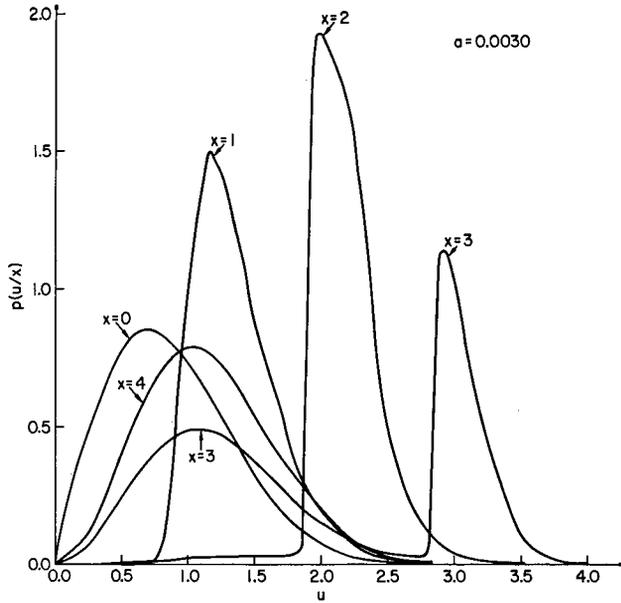


Fig. 2 — Probability density function of u given x

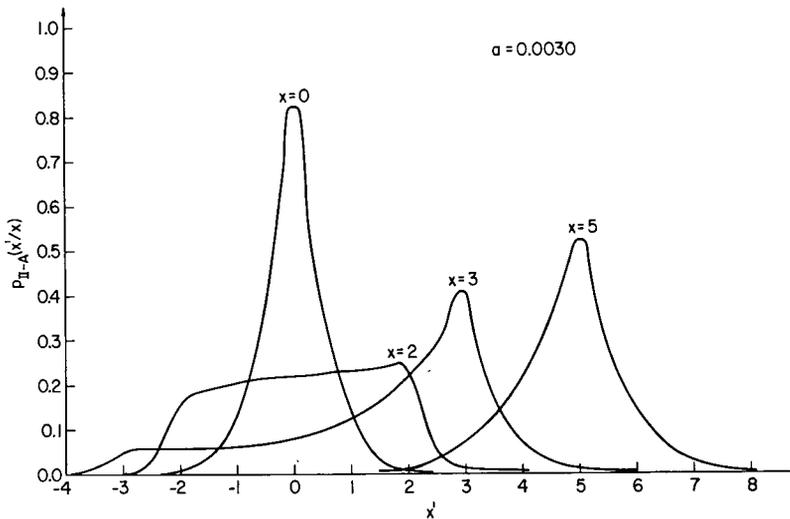


Fig. 3 — Histogram of simulated frequency distribution for Case II-A

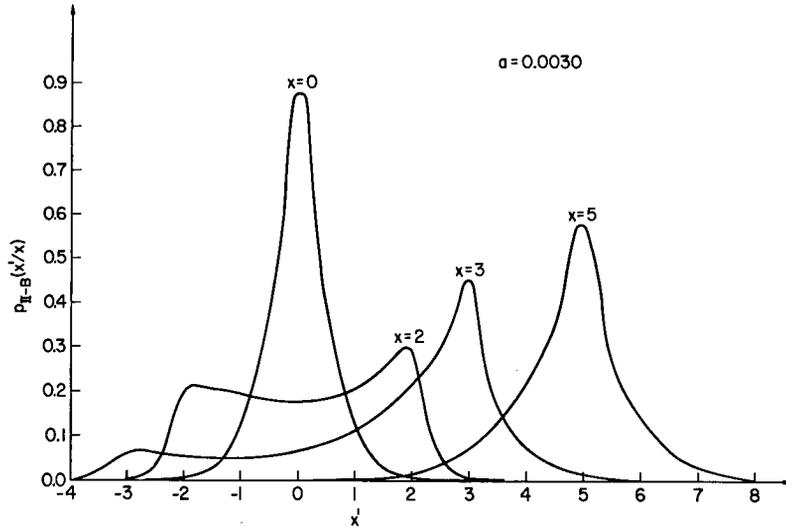


Fig. 4 — Histogram of simulated frequency distribution for Case II-B

We now proceed to derive simulation algorithms for $p(\mu|u, x)$. Substituting Eq. (11) into Eq. (32), we have

$$p(\mu|u, x) = \frac{\frac{a}{a^2 + (x - \mu u)^2}}{\frac{1}{u} \left[\tan^{-1} \left(\frac{u - x}{a} \right) + \tan^{-1} \left(\frac{u + x}{a} \right) \right]}, \quad -1 \leq \mu \leq 1. \quad (39)$$

Applying the conventional inverse method for generating μ , we have

$$r_2 = \int_{-1}^{\mu} p(\mu|u, x) d\mu$$

and

$$\mu = \frac{a}{u} \tan \left(\frac{r_2}{B} - A \right) + x, \quad (40)$$

where

$$A = \tan^{-1} \left(\frac{u + x}{a} \right)$$

and

$$B = \frac{u}{\left[\tan^{-1} \left(\frac{u - x}{a} \right) + A \right]}.$$

In summary, for case II we have the following simulation algorithms:

Simulation Algorithms II-A and II-B

For a given x , x' is selected by performing the following steps:

- (1) Simulate u from Eq. (38) by a numerical method.
- (2) Simulate μ using Eq. (40).
- (3) Simulate μ' for the isotropic phase function using Eq. (37) and for the dipole phase function using the simulation algorithm for the dipole phase function.
- (4) $x' = x - u(\mu - \mu')$.

For comparison, the corresponding $p(x', x)$ developed by Unno (4) and Sobolev (12) are:

$$p_{\text{II-A}}(x', x) = \pi^{-\frac{3}{2}} \int_{|\bar{x}-x|/2}^{\infty} e^{-u^2} \left[\tan^{-1} \left(\frac{x+u}{a} \right) - \tan^{-1} \left(\frac{\bar{x}-u}{a} \right) \right] du;$$

$$p_{\text{II-B}}(x', x) = \frac{3\pi^{-3/2}}{8} a \int_{|\bar{x}-x|/2}^{\infty} e^{-u^2} \int_{\bar{x}-u}^{x+u} \times \left[3 - \left(\frac{x-t}{u} \right)^2 - \left(\frac{x'-t}{u} \right)^2 + 3 \left(\frac{x-t}{u} \right)^2 \left(\frac{x'-t}{u} \right)^2 \right] \frac{dt du}{t^2 + a^2}.$$

Numerical results provided by algorithms II-A and II-B are plotted in Fig. 3 and 4 respectively. For each curve 40,000 runs have been simulated; and midpoints of each rectangular element in the histogram are fitted by a smooth curve. For $x = 2$ and 3, $p_{\text{II-B}}(x'|x)$ shows sagging in the center part of the curves compared with $p_{\text{II-A}}(x'|x)$.

Case III — Radiation and Collision Damping with Complete Redistribution in the Atom's Rest Frame

In this case Eqs. (11) and (15) are adopted for $p(x|u, \mu)$ and $p(x'|x, \mu, \mu', u)$ respectively. The quantities $p(u|x)$ and $p(\mu'|u, x)$ are identical to those used in Case II. Hence the simulation algorithms for III-A and III-B are identical to algorithms II-A and II-B, except that step (4) is replaced by

$$x' = a \tan \left[\left(r_4 - \frac{1}{2} \right) \pi \right] + u\mu'. \quad (41)$$

This change is introduced because Eq. (15) is used rather than Eq. (14). Equation (41) constitutes the simulation algorithm for x' from $p(x'|x, \mu, \mu', u)$, given by Eq. (15).

Hummer (5) showed that for isotropic scattering,

$$p_{IV}(x', x) = \pi^{-5/2} \int_0^\infty e^{-u^2} \left[\tan^{-1} \left(\frac{u+x}{a} \right) + \tan^{-1} \left(\frac{u-x}{a} \right) \right] \\ \times \left[\tan^{-1} \left(\frac{u+x'}{a} \right) + \tan^{-1} \left(\frac{u-x'}{a} \right) \right] du.$$

However, its counterpart for dipole scattering did not appear in his paper.

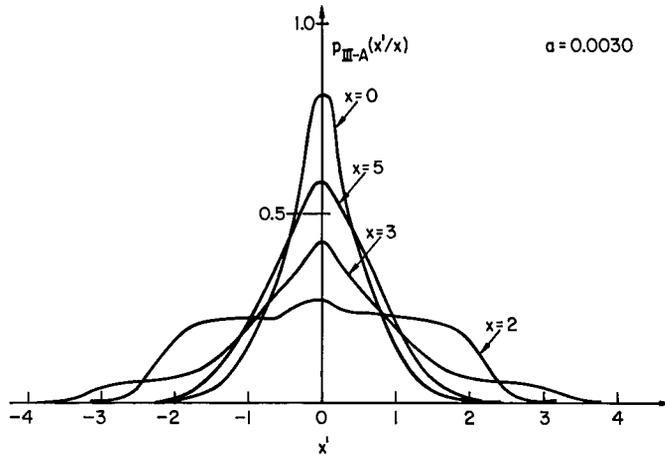


Fig. 5 — Histogram of simulated frequency distribution for Case III-A

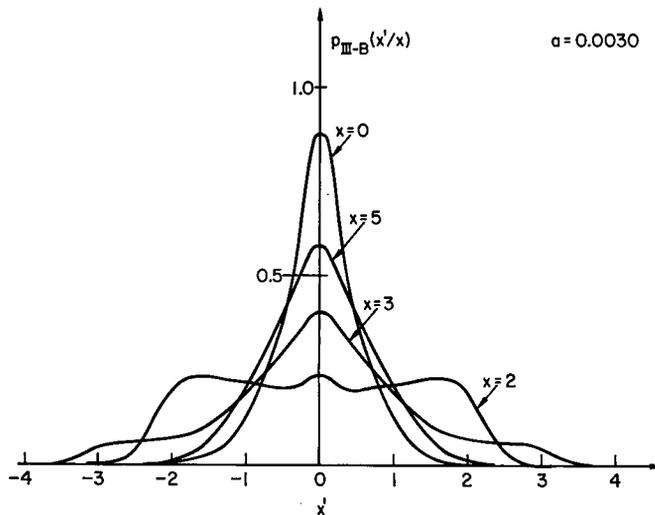


Fig. 6 — Histogram of simulated frequency distribution for Case III-B

Numerical-simulation results for both isotropic and dipole scattering are shown in Fig. 5 and 6. Unlike the results of case II, curves of $p_{\text{III-A}}(x'|x)$ and $p_{\text{III-B}}(x'|x)$ are centered at $x' = 0$. No significant difference is observed between curves of III-A and III-B except at $x = 2$, where sagging in the center part of the curve is observed for dipole scattering.

6. SUMMARY

We have presented simple and computationally efficient algorithms for randomly selecting emission frequencies based on a given absorption frequency for a class of partial frequency redistribution functions. Numerical-simulation results presented in the form of histograms substantiate the theoretical derivation of the algorithms. The author believes that these algorithms should be of great value in making Monte Carlo studies of photon-diffusion problems simpler and more efficient.

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