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Asymptotic Distribution of the Discrete Transform of a Nonuniformly Sampled Multidimensional Process

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20. ABSTRACT (*Continue on reverse side if necessary and identify by block number*)

Multidimensional sampling of real data (for example, in space and time) often requires observations at nonuniformly spaced intervals. The discrete transform of a multidimensional stationary stochastic process transforms a multivariate problem into an asymptotically univariate one if the spacing is uniform in at least one dimension. For the two-dimensional case of N uniformly spaced observations obtained at each of M arbitrarily spaced points, $y(x_j, t)$, $j = 1, \ldots, M$, $t = 1, \ldots, N$, the correlation between the real and imaginary parts of

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20. Abstract (Continued)

the transformed variable $Y(m, n)$ is $O(N^{-1})$. The MN real variables $\{\mathrm{Re}[Y(m, n)], \mathrm{Im}[Y(m, n)]: m = 1,\ldots,M, n = 1,\ldots,N/2\}$ form a sufficient set. Their limiting joint distribution for $n \neq n'$ is that of mutually independent normally distributed random variables. Thus the limiting distribution of a collection of two-dimensional periodograms, defined as the squared modulus of the transformed variables, is that of mutually independent random variables, each proportional to a chi-square variate. If the spectral density is slowly varying, this proportionality varies similarly with n, $n = 1,\ldots,N$. In general however it may depend erratically on m, $m = 1,\ldots,M$. For a model of "signal" embedded in a "noise" process, the asymptotic independence justifies statistical testing, in each cell of the transofmred domain, of the hypothesis "noise alone" versus the alternative "signal plus noise."

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ASYMPTOTIC DISTRIBUTION OF THE DISCRETE TRANSFORM OF A NONUNIFORMLY SAMPLED MULTIDIMENSIONAL PROCESS

INTRODUCTION

Multidimensional discrete transforms that map arbitrarily spaced sampled data into arrays of coefficients of arbitrary basis functions were considered in Ref. 1. These studies are motivated by a model of observations uniformly spaced in time obtained simultaneously at a nonuniform set of spatial points. For the uniformly spaced samples the transformation becomes the familiar discrete finite Fourier transform (DFT), and fast-Fourier-transform processing is applicable. The nonuniformly spaced samples generally require a transformation matrix that is not as highly factorable. For a two-dimensional sample space consisting of M nonuniform spatial points and N uniform temporal points, an efficient transformation is possible if $M \ll N$. Under the same assumption this two-dimensional transformation will be shown to approximately diagonolize the covariance matrix. "Asymptotic" will refer here to the limit as $N \to \infty$, with M finite.

As with the conventional DFT the basis functions for the nonuniform transformation may also be imaginary exponentials. This provides an efficient representation of plane-wave signals, concentrating the information for discrimination between hypotheses in a particular region of the transformed space. The nonuniform transformation will not be restricted here to imaginary exponentials, since no simplification results from this specialization.

For the special case of uniform sampling in all dimensions (the multidimensional DFT) the asymptotic distribution of the transform of a zero-mean stationary stochastic process was considered in Ref. 2. The present report is intended to be an extention of the earlier work and to be a link between Refs. 1 and 2. It can however be read independently of the earlier reports. Minor changes in notation were required, and the relevant definitions and theorems will be restated as needed for clarification. Only one- and two-dimensional processes will be considered. The cumbersome notation required alone, not any conceptual problems, inhibits consideration of higher dimensionality.

THE TRANSFORMATION

Consider a collection $\{y_r(x_j, t): x_j \in X, t \in T, r = 1, \ldots, R\}$ of R observations of a two-dimensional wide-sense stationary stochastic process, where $X = \{x_1, \ldots, x_M\}$ and $T = \{1, \ldots, N\}$. Under the null hypothesis the process has zero mean and crosscorrelation functions represented as

$$R_{rr'}(x_j - x_{j'}, t - t') = E\{y_r(x_j, t) y_{r'}(x_{j'}, t')\}$$

$$= \delta_{rr'} \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{i[\kappa(x_j - x_{j'}) + \omega(t-t')]} f(\kappa, \omega)\, d\kappa\, d\omega, \qquad (1)$$

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where the two-dimensional spectral density $f(\kappa, \omega)$ is a bounded absolutely continuous function with continuous first partial derivatives for $\kappa, \omega \in [-\pi, \pi]$. If x_j and t are space and time variables, $k = \kappa/2\pi$ may represent wavenumber and $f = \omega/2\pi$ may represent frequency.

Under the alternative hypothesis,

$$y_r(x_j, t) = s_r(x_j, t) + n_r(x_j, t), \tag{2}$$

$j = 1, \ldots, M$, $t = 1, \ldots, N$, $r = 1, \ldots, R$, where $s_r(x_j, t)$ is either a deterministic signal function (regression function) with unknown parameters or a random signal conditioned on certain random variables (such as amplitude and phase) and where $n_r(x_j, t)$ is a zero-mean wide-sense stationary process.

For each realization $r = 1, \ldots, R$, let $\mathbf{y} = [y_r(x_j, t)]$ be the M-by-N matrix of observations. Let

$$\mathbf{y} = \mathbf{PYQ} + \epsilon, \tag{3}$$

where $\mathbf{P} = [p_{jk}]$ and $\mathbf{Q} = [q_{jk}]$ are respectively M-by-M and N-by-N transformation matrices chosen to minimize the error of representation of the class of signals $\{s_r(x_j, t)\}$ under an appropriate criterion and ϵ is an M-by-N error matrix. That is, $\mathbf{P}$ and $\mathbf{Q}$ are "matched" as closely as possible to the signal function. For example, if $s(x, t)$ is a superposition of two-dimensional plane-wave signals, each of the form $A \cos(\kappa x + \omega t + \phi)$, where $\kappa/2\pi$ and $\omega/2\pi$ are unknown wavenumber and frequency respectively and $\phi \in [0, 2\pi]$ is an unknown epoch, then we may let $p_{jk} = M^{-1/2}\exp(2\pi i x_j k/M)$, $j, k = 1, \ldots, M$, and $q_{jk} = N^{-1/2}\exp(2\pi ijk/N)$, $j, k = 1, \ldots, N$. In this case (which is fairly general in view of Fourier's theorem and its various extensions), $\mathbf{Q}$ is the matrix of a one-dimensional DFT, and the basis functions $\{q_{jk}\}$ form an orthonormal set. Because of the nonuniform spacing of the $\{x_j\}$, the subset of basis functions $\{p_{jk}\}$ are not orthogonal.

Given the observations $\mathbf{y}$, we seek the least-squares estimator

$$\hat{\mathbf{Y}} = \mathbf{UyV}. \tag{4}$$

It is shown in Ref. 1 that $\mathbf{U} = \mathbf{P}^{-1}$ and $\mathbf{V} = \mathbf{Q}^{-1}$, provided the inverses exist. The matrix representation (3) provides a simple derivation of (4), which is given below, with the required lemmas on complex matrix differentiation given in Appendix A. If $\mathbf{P}$ or $\mathbf{Q}$ is complex, some additional restrictions are required, since $2MN$ real variables $\{Y_{\mathcal{R}}(m, n), Y_{\mathcal{J}}(m, n)\}$, where $\mathcal{R}$ and $\mathcal{J}$ refer to the real and imaginary parts respectively, cannot be linearly independent. In the case of the two-dimensional DFT, for example, with real $y(x, t)$,

$$Y(M\text{-}m, N\text{-}n) = \frac{1}{\sqrt{MN}} \sum_{x=1}^{M} \sum_{t=1}^{N} y(x, t) e^{-2\pi i[(M-m)x/M+(N-n)t/N]}$$

$$= \frac{1}{\sqrt{MN}} \sum_{x=1}^{M} \sum_{t=1}^{N} y(x, t) e^{2\pi i(mx/M+nt/N)}$$

$$= \overline{Y(m, n)},$$

where $\overline{Y(m, n)}$ is the complex conjugate of $Y(m, n)$. A set of sufficient statistics can easily be chosen and the redundant terms ignored. In the general case a linearly independent set of MN transformed variables cannot be as easily selected. It will be shown later (Eq. (12)) that if the basis functions are chosen as complex conjugate pairs, then with real observations the transformed variables will also occur as complex conjugate pairs. We deal only with the case where M and N are not both odd. We can then require that none of the transformed variables $Y(m, n)$ be purely real in general and hence avoid special consideration for particular values of m and n. Identification of redundant terms is particularly simple; if we take N even for convenience, then, for example, the MN real variables

$$\{Y_{\mathfrak{R}}(m, n) Y_{\mathfrak{I}}(m, n): \; m = 1, \ldots, M, \; n = 1, \ldots, N/2\} \tag{5}$$

form a sufficient set. These restrictions are easily imposed in practice and will be assumed in the following. (The case of odd M and odd N can be included, but the purely real $Y(m, n)$ must be treated separately both here and in the later statistical considerations.)

Let $\mathbf{A}^\dagger = \overline{\mathbf{A}}'$ denote the Hermitian conjugate of the matrix $\mathbf{A}$, where the overbar denotes the complex conjugate and the prime denotes the transpose. Let $tr(\mathbf{A})$ denote the trace of $\mathbf{A}$ when $\mathbf{A}$ is square. In (3) we wish to minimize

$$tr(\epsilon^\dagger \epsilon) = \sum_{j=1}^{M} \sum_{k=1}^{N} |\epsilon_{jk}|^2.$$

Now

$$tr(\epsilon^\dagger \epsilon) = tr[(\mathbf{y} - \mathbf{PYQ})^\dagger (\mathbf{y} - \mathbf{PYQ})]$$

$$= tr(\mathbf{y}^\dagger \mathbf{y}) - tr(\mathbf{Q y}^\dagger \mathbf{PY}) - tr(\mathbf{P}^\dagger \mathbf{y Q}^\dagger \mathbf{Y}^\dagger) + tr(\mathbf{PYQQ}^\dagger \mathbf{Y}^\dagger \mathbf{P}^\dagger)$$

by Theorems 9.1.1 and 9.1.4 of Graybill [3], which can easily be shown to be valid for complex matrices. Let $\mathbf{Y} = \mathbf{A} + i\mathbf{B}$, where the elements of $\mathbf{A}$ and $\mathbf{B}$ are real, and let $Y_{ij} = \overline{Y}_{i'j'}$ for some $i' \neq i$ and $j' \neq j$ but otherwise be independent of $Y_{i''j''}$ for $i \neq i'' \neq i'$ and $j \neq j'' \neq j'$. Assume that $p_{ki} = \overline{p}_{ki'}$ and $q_{jk} = \overline{q}_{j'k}$ for all k whenever $Y_{ij} = \overline{Y}_{i'j'}$. Then by Lemmas A1 through A3 (Appendix A)

$$\frac{\partial tr(\epsilon^\dagger \epsilon)}{\partial \mathbf{A}} + i\,\frac{\partial tr(\epsilon^\dagger \epsilon)}{\partial \mathbf{B}} = -2(\mathbf{Q}\mathbf{y}^\dagger \mathbf{P})^\dagger - 2\mathbf{P}^\dagger \mathbf{y}\mathbf{Q}^\dagger + 4\mathbf{P}^\dagger \mathbf{P}\mathbf{Y}\mathbf{Q}\mathbf{Q}^\dagger .$$

Setting this equal to zero yields

$$\mathbf{P}^\dagger \mathbf{P}\hat{\mathbf{Y}}\mathbf{Q}\mathbf{Q}^\dagger = \mathbf{P}^\dagger \mathbf{y}\mathbf{Q}^\dagger ,$$

which is Eq. (44) of Ref. 1. If $\mathbf{P}$ and $\mathbf{Q}$ are nonsingular, we have

$$\hat{\mathbf{Y}} = \mathbf{P}^{-1}\mathbf{y}\mathbf{Q}^{-1} = \mathbf{U}\mathbf{y}\mathbf{V}, \tag{6}$$

where $\mathbf{U} = \mathbf{P}^{-1}$ and $\mathbf{V} = \mathbf{Q}^{-1}$.

For the special but interesting case of equally spaced samples in one dimension, let $q_{jk} = N^{-1/2} \exp[\pi ik(2j - N - 1)/N]$ and $v_{jk} = N^{-1/2} \exp[-\pi ij(2k - N - 1)/N]$, $j, k = 1,\ldots,N$. Then, for all j and all k,

$$q_{N+1-j,k} = N^{-1/2} e^{\pi ik(N+1-2j)/N} = \bar{q}_{jk}$$

and

$$v_{j,N+1-k} = N^{-1/2} e^{-\pi ij(N+1-2k)/N} = \bar{v}_{jk}. \tag{7}$$

Dropping the circumflex, we have the transformation

$$Y_r(m, n) = \frac{1}{\sqrt{N}} \sum_{j=1}^{M} \sum_{t=1}^{N} u_{mj} y_r(x_j, t) e^{-i\omega_n t}$$

$$= Y_{r\mathcal{R}}(m, n) - iY_{r\mathcal{I}}(m, n), \quad m = 1,\ldots,M,\ n = 1,\ldots,N, \tag{8}$$

where

$$\omega_n = \pi(2n_N - N - 1)/N \to \omega_{n_0} \quad \text{as} \quad N \to \infty \tag{9}$$

and where we have suppressed the dependence of n on N and where

$$Y_{r\mathcal{R}}(m, n) = \frac{1}{\sqrt{N}} \sum_{j=1}^{M} \sum_{t=1}^{N} y_r(x_j, t)[u_{mj\mathcal{R}} \cos \omega_n t - u_{mj\mathcal{I}} \sin \omega_n t],$$

$$Y_{r\mathcal{I}}(m, n) = \frac{1}{\sqrt{N}} \sum_{j=1}^{M} \sum_{t=1}^{N} y_r(x_j, t)[u_{mj\mathcal{I}} \cos \omega_n t + u_{mj\mathcal{R}} \sin \omega_n t]. \tag{10}$$

Provided, for all j and all k,

$$u_{M+1-m,j} = \overline{u}_{mj}, \tag{11}$$

we have for real $y_r(x_j, t)$, using (7),

$$Y_r(M+1-m, N+1-n) = \sum_{j=1}^{M} \sum_{t=1}^{N} u_{M+1-m,j}\, y_r(x_j, t) v_{t,N+1-n} = \overline{Y_r(m,n)}. \tag{12}$$

We can require that $p_{j,M+1-m} = \overline{p}_{jm}$ for all j and all m, but in general $\mathbf{U} = \mathbf{P}^{-1}$ cannot be written explicitly. To assure that the conditions (11) hold, we require the following

Definition: An M-by-M matrix $\mathbf{A} = [a_{ij}]$ is *row-Hermitian* if $a_{i,M+1-j} = \overline{a}_{ij}$ for all i and all j. A *column-Hermitian* matrix is similarly defined. (These definitions imply that a row- (column-) Hermitian matrix with more than one real column (row) is singular.)

Lemma 1. If A_{ij} *is the cofactor of the element* a_{ij} *of an* M-*by*-M *row-Hermitian matrix* $\mathbf{A}$, *then* $A_{i,M+1-j} = -\overline{A}_{ij}$.

Proof. Let m_{ij} be the minor of a_{ij} (the determinant of the submatrix obtained by deleting the ith row and jth column of $\mathbf{A}$). Then $m_{i,m+1-j} = \overline{m}_{ij}$ or $-\overline{m}_{ij}$ according as M is even or odd, since an even or odd number of column interchanges respectively make one the complex conjugate of the other. But

$$A_{i,M+1-j} = (-1)^{i+M+1-j} m_{i,M+1-j} = -(-1)^{i+j}\overline{m}_{ij} = -\overline{A}_{ij}. \quad \text{Q.E.D.}$$

Lemma 2. If det($\mathbf{A}$*) is the determinant of the* M-*by*-M *row-Hermitian matrix* $\mathbf{A}$, *then* $\overline{\det(\mathbf{A})} = -\det(\mathbf{A})$.

Proof.

Case 1, M *even.* For any i

$$\det(\mathbf{A}) = \sum_{j=1}^{M} a_{ij}A_{ij} = \left(\sum_{j=1}^{M/2} + \sum_{j=\frac{M}{2}+1}^{M} \right) a_{ij}A_{ij}$$

$$= \sum_{j=1}^{M/2} (a_{ij}A_{ij} + a_{i,M+1-j}A_{i,M+1-j}$$

$$= \sum_{j=1}^{M/2} (a_{ij}A_{ij} - \overline{a}_{ij}\overline{A}_{ij})$$

by the definition and Lemma 1. Therefore det(**A**) is purely imaginary.

Case 2, M *odd.* The proof is similar, with $a_{i,(M+1)/2}$ purely imaginary. Hence det(**A**) is always purely imaginary. Q.E.D.

Theorem 1. If a matrix is row- (column-) Hermitian, its inverse, if it exists, is column- (row-) Hermitian.

Proof. Let $\mathbf{A} = [a_{ij}]$ be an M-by-M row-Hermitian matrix and let $\mathbf{A}^{-1} = \mathbf{B} = [b_{ij}]$. Let A_{ij} be the cofactor of a_{ij}. Then $b_{ij} = A_{ji}/\det(\mathbf{A})$ and

$$b_{M+1-ij} = \frac{A_{j,M+1-i}}{\det(\mathbf{A})} = \frac{-\overline{A}_{ji}}{\det(\mathbf{A})} = \overline{b}_{ij}$$

using Lemmas 1 and 2. Q.E.D.

We can, for example, let

$$p_{jk} = M^{-1/2} e^{\pi i x_j (2k-M-1)/M}, \tag{13}$$

for arbitrary $\{x_j\}$. Then

$$p_{j,M+1-k} = M^{-1/2} e^{\pi i x_j (M+1-2k)/M} = \overline{p}_{jk},\ j,\ k = 1, \ldots, M.$$

We are assured by Theorem 1 that (11) (and hence (12)) is valid even though u_{ij} cannot be explicitly written in this case.* The theorem holds of course for the orthogonal case where the inverse is explicit, as exemplified by (7). Henceforth we assume that **P** is row-Hermitian and restrict attention to the set of transformed variables indicated in (5).

ASYMPTOTIC CORRELATION OF THE TRANSFORMED VARIABLES

We consider the transformation defined by (8) through (10), applied to the observations **y**. Under the null hypothesis it follows from (1) that $\text{cov}[Y_{r\mathfrak{R}}(m, n), Y_{r'\mathfrak{R}}(m', n')] = \text{cov}[Y_{r\mathfrak{R}}(m, n), Y_{r'\mathfrak{J}}(m', n')] = \text{cov}[Y_{r\mathfrak{J}}(m, n), Y_{r'\mathfrak{J}}(m', n')] = 0$ if $r \neq r'$. For $r = r'$ we drop the subscript. Using (10) and (1), we have

*I am grateful to F. M. Young for pointing out that the choice of (13) led to (11).

$$E\{Y_{\mathcal{R}}(m,n)Y_{\mathcal{I}}(m',n')\}$$

$$= \frac{1}{(2\pi)^2 N}\sum_{j=1}^{M}\sum_{t=1}^{N}\sum_{j'=1}^{M}\sum_{t'=1}^{N}\left\{\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(\kappa,\omega)e^{i[\kappa(x_j-x_{j'})+\omega(t-t')]}\,d\kappa d\omega\right.$$

$$\left.\times\,(u_{mj\mathcal{R}}\cos\omega_n t - u_{mj\mathcal{I}}\sin\omega_n t)(u_{m'j'\mathcal{I}}\cos\omega_{n'}t' + u_{m'j'\mathcal{R}}\sin\omega_{n'}t')\right\}$$

$$= \frac{1}{2\pi}\sum_{j=1}^{M}\sum_{j'=1}^{M}\int_{-\pi}^{\pi} e^{i\kappa(x_j-x_{j'})}\int_{-\pi}^{\pi} f(\kappa,\omega)$$

$$\times\sum_{k=1}^{4} G_k(m,m';j,j')H_k(\omega,\omega_n,\omega_{n'})\,d\omega d\kappa, \qquad (14)$$

where each of the four terms H_k in the inner integral can be expressed as products of the form $D_N(\omega-\lambda)D_N(\omega-\lambda')/2\pi N$, where $D_N(x) = \sin(Nx/2)/\sin(x/2)$ is the Dirichlet kernel and where $\lambda = \pm\omega_n$, $\lambda' = \pm\omega_{n'}$. Typically

$$\sum_{t=1}^{N} e^{i\omega t}\cos\omega_n t = \frac{1}{2}D_N(\omega+\omega_n)e^{i(\omega+\omega_n)(N+1)/2} + \frac{1}{2}D_N(\omega-\omega_n)e^{i(\omega-\omega_n)(N+1)/2}.$$

Using Lemma 8.3.4 of Anderson [4a], if $f(\kappa,\omega)$ is bounded in $(\lambda-\delta, \lambda+\delta)$ and $(\lambda'-\delta, \lambda'+\delta)$ for some $\delta > 0$ and $\lambda \neq \lambda'$, $|\lambda-\lambda'| \neq 2\pi$, then we have

$$\frac{1}{2\pi N}\int_{-\pi}^{\pi} f(\kappa,\omega)D_N(\omega-\lambda)D_N(\omega+\lambda')\,d\omega = O\left(\frac{1}{N}\right).$$

By virtue of (9), if $n \neq n'$, then $\omega_n \neq \pm\omega_{n'}$ and $|\omega_n - \omega_{n'}| \neq 2\pi$ for $1 \leqslant n, n' \leqslant N/2$. Since the existence of $\mathbf{U} = \mathbf{P}^{-1}$ implies that u_{mj} is finite for all j and all m, we have

$$E\{Y_{\mathcal{R}}(m,n)Y_{\mathcal{I}}(m',n')\} = O\left(\frac{1}{N}\right), \quad n \neq n'. \qquad (15)$$

Similarly,

$$E\{Y_{\mathcal{R}}(m,n)Y_{\mathcal{R}}(m',n')\} = O\left(\frac{1}{N}\right), \quad n \neq n', \qquad (16)$$

and

$$E\{Y_{\mathcal{I}}(m,n)Y_{\mathcal{I}}(m',n')\} = O\left(\frac{1}{N}\right), \quad n \neq n'. \qquad (17)$$

For a single (m, n) pair the correlation between the real and imaginary parts is given by (14) with $m = m'$ and $n = n'$. For N even and $1 \leqslant n \leqslant N/2$ we have $\omega_n \neq 0, \pm\pi$, so that all terms involving the product $D_N(\omega - \omega_n)D_N(\omega + \omega_n)$ will be $O(1/N)$, as before. Writing only the remaining terms explicitly, with $F_N(x) = D_N{}^2(x)$ denoting the Fejér kernel, we have

$$E\{Y_{\mathcal{R}}(m, n)Y_{\mathcal{J}}(m, n)\} = O\left(\frac{1}{N}\right) + \frac{1}{4(2\pi)^2 N}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(\kappa, \omega)\sum_{j=1}^{M}\sum_{j'=1}^{M} e^{i\kappa(x_j - x_{j'})}$$

$$\times\Big\{(u_{mj\mathcal{R}}u_{mj'\mathcal{J}} - u_{mj\mathcal{J}}u_{mj'\mathcal{R}})[F_N(\omega + \omega_n) + F_N(\omega - \omega_n)]$$

$$+ i(u_{mj\mathcal{R}}u_{mj'\mathcal{R}} + u_{mj\mathcal{J}}u_{mj'\mathcal{J}})[F_N(\omega + \omega_n) - F_N(\omega - \omega_n)]\Big\}\,d\kappa d\omega.$$

Now

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(\kappa, \omega)\sum_{j=1}^{M}\sum_{j'=1}^{M} e^{i\kappa(x_j - x_{j'})}u_{mj\mathcal{J}}u_{mj'\mathcal{R}}[F_N[\omega + \omega_n) + F_N(\omega - \omega_n)]\ d\kappa d\omega$$

$$= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(-\kappa, -\omega)\sum_{j'=1}^{M}\sum_{j=1}^{M} e^{-i\kappa(x_{j'} - x_j)}u_{mj'\mathcal{J}}u_{mj\mathcal{R}}[F_N(-\omega + \omega_n) + F_N(-\omega - \omega_n)\ d\kappa d\omega$$

$$= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(\kappa, \omega)\sum_{j=1}^{M}\sum_{j'=1}^{M} e^{i\kappa(x_j - x_{j'})}u_{mj\mathcal{R}}u_{mj'\mathcal{J}}[F_N(\omega - \omega_n) + F_N(\omega + \omega_n)]\ d\kappa d\omega$$

and

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(\kappa, \omega)\left(\left|\sum_{j=1}^{M} u_{mj\mathcal{R}}e^{i\kappa x_j}\right|^2 + \left|\sum_{j=1}^{M} u_{mj\mathcal{J}}e^{i\kappa x_j}\right|^2\right)F_N(\omega + \omega_n)\ d\kappa d\omega$$

$$= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(\kappa, \omega)\left(\left|\sum_{j=1}^{M} u_{mj\mathcal{R}}e^{i\kappa x_j}\right|^2 + \left|\sum_{j=1}^{M} u_{mj\mathcal{J}}e^{i\kappa x_j}\right|^2\right)F_N(\omega - \omega_n)\ d\kappa d\omega,$$

so that

$$E\{Y_{\mathcal{R}}(m, n)Y_{\mathcal{J}}(m, n)\} = O\left(\frac{1}{N}\right),\quad \omega_n \neq 0, \pm\pi. \tag{18}$$

The variances also follow from (10) and (1):

$$\operatorname{var}[Y_{\mathcal{R}}(m,n)] = \frac{1}{(2\pi)^2 N} \sum_{j=1}^{M} \sum_{t=1}^{N} \sum_{j'=1}^{M} \sum_{t'=1}^{N} \left\{ \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(\kappa,\omega) e^{i[\kappa(x_j - x_{j'}) + \omega(t-t')]} \, d\kappa d\omega \right.$$

$$\left. \times \, (u_{mj\mathcal{R}} \cos \omega_n t - u_{mj\mathcal{I}} \sin \omega_n t)(u_{mj'\mathcal{R}} \cos \omega_n t - u_{mj'\mathcal{I}} \sin \omega_n t) \right\}$$

$$= O\left(\frac{1}{N}\right) + \frac{1}{4(2\pi)^2 N} \sum_{j=1}^{M} \sum_{j'=1}^{M} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(\kappa,\omega) e^{i\kappa(x_j - x_{j'})}$$

$$\times \left\{ (u_{mj\mathcal{R}} u_{mj'\mathcal{R}} + u_{mj\mathcal{I}} u_{mj'\mathcal{I}})[F_N(\omega + \omega_n) + F_N(\omega - \omega_n)] \right.$$

$$\left. + \, i(u_{mj\mathcal{I}} u_{mj'\mathcal{R}} - u_{mj\mathcal{R}} u_{mj'\mathcal{I}} [F_N(\omega + \omega_n) - F_N(\omega - \omega_n)] \right\} d\omega d\kappa$$

$$= \frac{1}{2\pi} \sum_{j=1}^{M} \sum_{j'=1}^{M} u_{mj} \bar{u}_{mj'} \int_{-\pi}^{\pi} e^{i\kappa(x_j - x_{j'})}$$

$$\times \left[\frac{1}{4\pi N} \int_{-\pi}^{\pi} F_N(\omega - \omega_n) f(\kappa,\omega) \, d\omega \right] d\kappa + O\left(\frac{1}{N}\right)$$

$$= \frac{1}{4\pi} \sum_{j=1}^{M} \sum_{j'=1}^{M} u_{mj} \bar{u}_{mj'} \int_{-\pi}^{\pi} e^{i\kappa(x_j - x_{j'})} f(\kappa, \omega_n) \, d\kappa + O\left(\frac{1}{N}\right) \tag{19}$$

by Theorem 2.1 of Ref. 2. Similarly

$$\operatorname{var}[Y_{\mathcal{I}}(m,n)] = \frac{1}{4\pi} \sum_{j=1}^{M} \sum_{j'=1}^{M} u_{mj} \bar{u}_{mj'} \int_{-\pi}^{\pi} e^{i\kappa(x_j - x_{j'})} f(\kappa, \omega_n) \, d\kappa + O\left(\frac{1}{N}\right). \tag{20}$$

For the special case of the two-dimensional DFT (if $\{x_j\} = \{j\}$) we can let

$$\sum_{j=1}^{M} \sum_{j'=1}^{M} u_{mj} \bar{u}_{mj'} \, e^{i\kappa(j-j')} = \frac{1}{M} \sum_{j=1}^{M} \sum_{j'=1}^{M} e^{-\pi i(j-j')(2m-M-1)/M} e^{i\kappa(j-j')}$$

$$= \frac{1}{M} \left| \sum_{j=1}^{M} e^{i(\kappa - \kappa_m)j} \right|^2 = \frac{1}{M} F_M(\kappa - \kappa_m),$$

where $\kappa_m = \pi(2m_M - M - 1)/M \to \kappa_{m_0}$ as $M \to \infty$. In this case

$$\mathrm{var}[Y_{\mathcal{R}}(m,n)] \approx \mathrm{var}[Y_{\mathcal{I}}(m,n)] = \frac{1}{4\pi M}\int_{-\pi}^{\pi} F_M(\kappa, \kappa_m) f(\kappa, \omega_n)\, d\kappa + O\left(\frac{1}{N}\right)$$

$$= \frac{1}{2} f(\kappa_m, \omega_n) + O\left(\frac{1}{M}\right) + O\left(\frac{1}{N}\right),$$

as shown in Ref. 2. If $f(\kappa_m, \omega_n)$ varies slowly with m and n, then so too will $\mathrm{var}[Y(m,n)] = \mathrm{var}[Y_{\mathcal{R}}(m,n)] + \mathrm{var}[Y_{\mathcal{I}}(m,n)]$.

In the general case the m dependence of (19) and (20) will be erratic. If the spectral density is independent of wavenumber, that is, if $f(\kappa, \omega) = g(\omega)$, then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\kappa(x_j - x_{j'})} f(\kappa, \omega)\, d\kappa = g(\omega)\,\frac{\sin \pi(x_j - x_{j'})}{\pi(x_j - x_{j'})}\,. \tag{21}$$

If we now let

$$Z(m,n) = Y(m,n)\Bigg/\left[\sum_{j=1}^{M}\sum_{j'=1}^{M} u_{mj}\bar{u}_{mj'}\,\frac{\sin \pi(x_j - x_{j'})}{\pi(x_j - x_{j'})}\right]^{1/2}, \tag{22}$$

then from (19) through (21)

$$\mathrm{var}[Z_{\mathcal{R}}(m,n)] \approx \mathrm{var}[Z_{\mathcal{I}}(m,n)] = \frac{1}{2} g(\omega_n) + O\left(\frac{1}{N}\right). \tag{23}$$

One consequence of this "semiwhite noise" is that the correlation function given by (1) becomes

$$R(x_j - x_{j'}, t-t') = E\left\{y(x_j, t)y(x_{j'}, t')\right\} = \frac{\sin \pi(x_j - x_{j'})}{\pi(x_j - x_{j'})} R(0, t-t'),$$

where

$$R(0, t-t') = E\left\{y(x_j, t)y(x_j, t')\right\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega(t-t')} g(\omega)\, d\omega.$$

THE ASYMPTOTIC DISTRIBUTION

The transformation (8) may be written for each $r = 1, \ldots, R$ as

$$Y(m, n) = \sum_{j=1}^{M} u_{mj} \tilde{y}(x_j, n), \tag{24}$$

where

$$\tilde{y}(x_j, n) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} y(x_j, t) e^{-i\omega_n t}. \tag{25}$$

Equation (25) is the celebrated one-dimensional DFT. We define the one-dimensional *periodogram* at each sample point x_j as

$$I_{N,y}(x_j, \omega_n) = |\tilde{y}(x_j, n)|^2 \tag{26}$$

and let

$$g(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\kappa, \omega)\, d\kappa$$

denote the one-dimensional spectral density. The existence of the spectral density implies [4b,5] that the wide-sense stationary process $\{y(x_j, t)\}$ can be represented as a *moving average*

$$y(x_j, t) = \sum_{k=-\infty}^{\infty} \gamma(k) \lambda(x_j, t-k),$$

where $\sum_{k=-\infty}^{\infty} |\gamma(k)|^2 < \infty$. The residual process $\{\lambda(x_j, t)\}$ is orthonormal:

$$E\{\lambda(x_j, t)\} = 0 \quad \text{and} \quad E\{\lambda(x_j, t)\lambda(x_j, t')\} = \delta_{tt'}.$$

Theorem 2. Let $g[\omega_{n(k)}] \neq 0$, $\omega_{n(k)} \neq 0, \pi$, $k = 1, \ldots, K$. *Then as* $N \to \infty$, *under the null hypothesis the joint distribution of* $\tilde{y}_{\mathcal{R}}[x_j, n(k)]$, $\tilde{y}_{\mathcal{I}}[x_j, n(k)]$, $k = 1, \ldots, K$, *tends to that of* 2K *mutually independent zero-mean normally distributed random variables and the joint distribution of* $I_{N,y}[x_j, \omega_{n(k)}]$, $k = 1, \ldots, K$ *tends to that of* K *mutually independent random variables with*

$$I_{N,y}[x_j, \omega_{n(k)}] \xrightarrow{\mathcal{L}} \frac{1}{2} g[\omega_{n(k)}]\,\chi^2(2),$$

where $\chi^2(2)$ *denotes a random variable having the central chi-square distribution with two degrees of freedom.*

The proof is given by Walker [6] if $\Sigma|\gamma(k)| < \infty$. It is extended by Olshen [7], who requires only that $\Sigma|\gamma(k)|^2 < \infty$, and strengthens the needed convergence of $\tilde{y}(x_j, n)$ and $I_{N,y}(x_j, \omega_n)$ to variables proportional to the transform and periodogram respectively of the residual process $\{\lambda(x_j, t)\}$.

This theorem, together with (24) and the earlier results on asymptotic correlation, shows that if $n(k) \neq n(k')$ whenever $k \neq k'$, then under the null hypothesis the limiting joint distribution of $Y_{\mathcal{R}}[m(k), n(k)]$, $Y_{\mathcal{I}}[m(k), n(k)]$, $k = 1, \ldots, K$ is that of $2K$ mutually independent zero-mean normally distributed random variables, with variances given by (19) and (20). It follows that the two-dimensional periodogram, defined as

$$I_{M,N,y}(\kappa_m, \omega_n) = |Y(m, n)|^2, \tag{27}$$

is asymptotically distributed as $h(m, n)$ times a central chi-square variable with two degrees of freedom and is asymptotically independent of $I_{M,N,y}(\kappa_{m'}, \omega_{n'})$ for $n \neq n'$. In the case of uniformly spaced sampling in two dimensions, $h(m, n) = f(\kappa_m, \omega_n)/2$. In the general case however the dependence of $h(m, n)$ on m will be more erratic than that of $f(\kappa_m, \omega_n)$. If $f(\kappa, \omega)$ is independent of κ, the erratic behavior can be eliminated by "normalization" as in (22). Under the alternative hypothesis, $Y_{\mathcal{R}}(m, n)$ and $Y_{\mathcal{I}}(m, n)$ will be normally distributed with nonzero mean, so that $I_{M,N,y}(\kappa_m, \omega_n)$ will be proportional to a noncentral chi-square variable.

CONCLUSIONS

We have considered some properties of a discrete transform of two-dimensional sampled data. In a case of considerable practical interest, sampling in one dimension (time) is at uniformly spaced intervals, and that in the other dimension (space) is at non-uniformly spaced intervals. The two-dimensional discrete Fourier transform is applicable only to data uniformly sampled in both dimensions. In that case it is well known that nearly half of the transformed variables are redundant, since they occur as complex-conjugate pairs. In the general case the complex transformed variables are also interdependent. They can be separated into complex-conjugate pairs if the basis functions used for the transformation also have this relationship. The spatial transformation matrix considered here is the inverse of a matrix chosen to provide an efficient representation of plane-wave signals and in general cannot be written explicitly. It has been shown that if the elements of the representation matrix are chosen as complex-conjugate pairs, its inverse, the transformation matrix, will also have this desired property. These have been called here "row-Hermitian" and "column-Hermitian" matrices respectively.

The correlation between the real and imaginary parts of the transformed variables, both within the same cell and from different cells has been shown to be $O(1/N)$, where N is the number of time samples. The convergence is thus faster than the $O(\ln N/N)$ stated in the literature for the one-dimensional discrete Fourier transform. The price of this more rapid convergence is the requirement that the spectral density be continuously differentiable. This does not appear to be too restrictive, since a finite sampling interval implies that any spectral lines are broadened and are not delta functions.

In the special case of uniform sampling in space as well as in time, the correlation is $O(1/M) \times O(1/N)$, where M is the number of space samples. In this case the variances of the real and imaginary parts are $(1/2)f(\kappa, \omega) + O(1/M) + O(1/N)$, where $f(\kappa, \omega)$ is the spectral density. In the general case of nonuniform spatial sampling the variances depend on the transformation matrix and the sample spacing as well as the unknown spectral density.

By a central-limit type argument the limiting joint distribution of a collection of transformed variables was shown to be that of mutually independent normally distributed random variables. This limiting distribution does not depend on the selection of spatial sample points. The asymptotic normality and independence of the transofrmed variables $Y_{\Re}(m, n)$ and $Y_{\Im}(m, n)$ can be used to justify statistical tests of the hypothesis "noise alone" against the alternative hypothesis "signal plus noise." This can be done on a cell-by-cell basis, for each (m, n) pair, thus testing each cell for the presence of a signal component. For example, if the unknown spectral density $f(\kappa, \omega)$ varies slowly, data from nearby cells may be used to approximately "Studentize": The ratio of the mean of R observations of the periodogram (27) to the sum of K similar means of periodograms of nearby cells has asymptotically an approximate F distribution with $2R$ and $2KR$ degrees of freedom. In the general case however this ratio will not be approximately independent of the unknown variances (19) and (20) unless all K+1 periodograms have the same value of m. For the special case of uniformly spaced sampling in two dimensions, $I_{M,N,y}(\kappa_{m'}, \omega_{n'})$ with $m' \neq m$ may also be used, as shown in Ref. 2. In this case if $f(\kappa, \omega)$ is slowly varying, or in the general case if $f(\kappa, \omega)$ is also independent of κ, analysis of variance and methods of multiple comparison may be applied in the transformed domain.

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Appendix A
DERIVATIVE OF THE TRACE OF SPECIAL COMPLEX MATRICES WITH RESPECT TO A REAL MATRIX

In the following let $\mathbf{Z} = \mathbf{X} + i\mathbf{Y}$ denote an M-by-N matrix whose elements occur in complex-conjugate pairs but are otherwise independent. That is, for all i and j, there exists $i' \neq i$ and $j' \neq j$ such that $z_{ij} = \overline{z}_{i'j'}$, but z_{ij} is independent of $z_{i''j''}$ for $i \neq i'' \neq i'$ and $j \neq j'' \neq j'$. Here the overbar denotes the complex conjugate, and the elements of $\mathbf{X}$ and $\mathbf{Y}$ are real. Let $\mathbf{A}$ and $\mathbf{B}$ be matrices of complex constants such that $\mathbf{AZB}$ is defined. The derivative of a function f of the real matrix $\mathbf{X}$ is defined* as

$$\frac{\partial f}{\partial \mathbf{X}} = \left[\frac{\partial f}{\partial x_{ij}}\right],$$

where each partial derivative $\partial f/\partial x_{ij}$ is assumed to exist. Let $\mathbf{A}^\dagger = \overline{\mathbf{A}}'$ denote the Hermitian conjugate of the matrix $\mathbf{A}$, where the prime denotes the transpose, and let $\mathrm{tr}(\mathbf{C})$ denote the trace of the square matrix $\mathbf{C}$.

Lemma A1. If $\mathbf{A} = [a_{ij}]$ *is* N *by* M *and* $a_{ij} = \overline{a}_{i'j'}$ *whenever* $z_{ij} = \overline{z}_{i'j'}$, *then*

$$\frac{\partial \mathrm{tr}(\mathbf{AZ})}{\partial \mathbf{X}} + i\frac{\partial \mathrm{tr}(\mathbf{AZ})}{\partial \mathbf{Y}} = 2\mathbf{A}^\dagger. \tag{A1}$$

Proof.

$$\frac{\partial \mathrm{tr}(\mathbf{AZ})}{\partial \mathbf{X}} + i\frac{\partial \mathrm{tr}(\mathbf{AZ})}{\partial \mathbf{Y}} = \left[\frac{\partial}{\partial x_{ij}}\sum_{n=1}^{N}\sum_{m=1}^{M} a_{nm}x_{mn}\right] + i\left[i\frac{\partial}{\partial y_{ij}}\sum_{n=1}^{N}\sum_{m=1}^{M} a_{nm}y_{mn}\right]$$

$$= [a_{ji} + \overline{a}_{ji} - a_{ji} + \overline{a}_{ji}] = 2[\overline{a}_{ji}] = 2\mathbf{A}^\dagger. \quad \text{Q.E.D.}$$

Lemma A2. If $\mathbf{B} = [b_{ij}]$ *is* N *by* M *and* $b_{ij} = \overline{b}_{i'j'}$ *whenever* $z_{ij} = \overline{z}_{i'j'}$, *then*

$$\frac{\partial \mathrm{tr}(\mathbf{B}^\dagger\mathbf{Z}^\dagger)}{\partial \mathbf{X}} + i\frac{\partial \mathrm{tr}(\mathbf{B}^\dagger\mathbf{Z}^\dagger)}{\partial \mathbf{Y}} = 2\mathbf{B}^\dagger. \tag{A2}$$

The proof is similar to that of Lemma A1.

*F. A. Graybill, *Introduction to Matrices with Applications in Statistics*, Wadsworth, Belmont, Calif., 1969, p. 263.

Lemma A3. If **A** *is* K *by* M *and* **B** *is* N *by* L, $a_{ki} = \overline{a}_{ki'}$ *and* $b_{jk} = \overline{b}_{j'k}$ *for all* k *whenever* $z_{ij} = \overline{z}_{i'j'}$ *and* **AZB** *is real, then*

$$\frac{\partial}{\partial \mathbf{X}} \operatorname{tr}(\mathbf{AZBB}^\dagger \mathbf{Z}^\dagger \mathbf{A}^\dagger) + i \frac{\partial}{\partial \mathbf{Y}} \operatorname{tr}(\mathbf{AZBB}^\dagger \mathbf{Z}^\dagger \mathbf{A}^\dagger) = 4\mathbf{A}^\dagger \mathbf{AZBB}^\dagger. \tag{A3}$$

Proof.

$$\frac{\partial}{\partial \mathbf{X}} \operatorname{tr}(\mathbf{AZBB}^\dagger \mathbf{Z}^\dagger \mathbf{A}^\dagger) = \left[\frac{\partial}{\partial x_{ij}} \sum_{k=1}^{K} \sum_{m=1}^{M} \sum_{n=1}^{N} \sum_{p=1}^{L} \sum_{q=1}^{N} \sum_{r=1}^{M} a_{km} z_{mn} b_{np} \overline{b}_{qp} \overline{z}_{rq} \overline{a}_{kr}\right]$$

$$= \left[\sum_{k} \sum_{p} \sum_{q} \sum_{r} (a_{ki} b_{jp} \overline{b}_{qp} \overline{z}_{rq} \overline{a}_{kr} + \overline{a}_{ki} \overline{b}_{jp} \overline{b}_{qp} \overline{z}_{rq} \overline{a}_{kr})\right.$$

$$\left. + \sum_{k} \sum_{m} \sum_{n} \sum_{p} (a_{km} z_{mn} b_{np} \overline{b}_{jp} \overline{a}_{ki} + a_{km} z_{mn} b_{np} b_{jp} a_{ki})\right]$$

$$= [(\mathbf{A}'\overline{\mathbf{A}}\,\overline{\mathbf{Z}}\,\overline{\mathbf{B}}\mathbf{B}')_{ij} + (\mathbf{A}^\dagger\overline{\mathbf{A}}\,\overline{\mathbf{Z}}\,\overline{\mathbf{B}}\mathbf{B}^\dagger)_{ij} + (\mathbf{A}^\dagger \mathbf{AZBB}^\dagger)_{ij}$$

$$+ (\mathbf{A}'\mathbf{AZBB}')_{ij}]$$

$$= 2\mathbf{A}'\mathbf{AZBB}' + 2\,\mathbf{A}^\dagger\mathbf{AZBB}^\dagger.$$

Also

$$i\frac{\partial}{\partial \mathbf{Y}} \operatorname{tr}(\mathbf{AZBB}^\dagger \mathbf{Z}^\dagger \mathbf{A}^\dagger) = i\left[\frac{\partial}{\partial y_{ij}} \sum_{k=1}^{K} \sum_{m=1}^{M} \sum_{n=1}^{N} \sum_{p=1}^{L} \sum_{q=1}^{N} \sum_{r=1}^{M} a_{km} z_{mn} b_{np} \overline{b}_{qp} \overline{z}_{rq} \overline{a}_{kr}\right]$$

$$= [-(\mathbf{A}'\overline{\mathbf{A}}\,\overline{\mathbf{Z}}\,\overline{\mathbf{B}}\mathbf{B}')_{ij} + (\mathbf{A}^\dagger\overline{\mathbf{A}}\,\overline{\mathbf{Z}}\,\overline{\mathbf{B}}\mathbf{B}^\dagger)_{ij} + (\mathbf{A}^\dagger \mathbf{AZBB}^\dagger)_{ij}$$

$$- (\mathbf{A}'\mathbf{AZBB}')_{ij}]$$

$$= -2\mathbf{A}'\mathbf{AZBB}' + 2\mathbf{A}^\dagger\mathbf{AZBB}^\dagger. \quad \text{Q.E.D.}$$