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The Static Equilibrium Configuration of Cable Arrays by Use of The Method of Imaginary Reactions

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ABSTRACT

A new technique, The Method of Imaginary Reactions, is introduced for determining the static equilibrium configuration of cable arrays. Throughout this technique, a discrete parameter representation of the system is employed. However, it should be noted that the analysis method used depends only on the number of redundant reactions encountered; thus, the system can be represented with arbitrary fineness without increasing the complexity of the problem. An iterative solution is generated for varying the unknown reactions, and convergence to the correct equilibrium configuration of the array is guaranteed. The rapidity of convergence is shown by several numerical examples. The solution for the case of position-dependent applied external forces is indicated.

PROBLEM STATUS

This is an interim report; work is continuing on other phases of the problem.

AUTHORIZATION

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THE STATIC EQUILIBRIUM CONFIGURATION OF CABLE ARRAYS BY USE OF THE METHOD OF IMAGINARY REACTIONS

SYMBOLS

The symbols used in this report are defined as they appear within the context. The most important ones are listed here for reference.

The Single Cable

$(a,\ b,\ c)$	the coordinates of the secondary anchor
$B(m)$	the extensional rigidity of the mth segment
$(F_x(m),\ F_y(m),\ F_z(m))$	the components of the external force acting at the mth station
$L(m)$	the stressed length of the mth segment
$L_0(m)$	the unstressed length of the mth segment
m	the index of a cable station or cable segment
M	the total number of stations or segments
Q_m	a functional expressing the stressed length of the mth segment in terms of the unstressed length of and tension in that segment
$(R_x(m),\ R_y(m),\ R_z(m))$	the components of the resultant force in the mth segment
$T(m)$	the tension in the mth segment
$(x(O),\ y(O),\ z(O))$	the coordinates of the primary anchor
$(x(m),\ y(m),\ z(m))$	the coordinates of the mth station

The Method of Imaginary Reactions for the Single Cable

E	a positive definite error function
$(F_x(M),\ F_y(M),\ F_z(M))$	the components of the imaginary reaction applied to the Mth station
$(\Delta F_x(M),\ \Delta F_y(M),\ \Delta F_z(M))$	the components of the additive force applied to the Mth station

δ	a positive convergence factor having the dimension of force
$(\ )'$	denotes trial iteration values

Simple Cable Arrays

$(a(n),\ b(n),\ c(n))$	the coordinates of the secondary anchor of the nth cable
$B(m;n)$	the extensional rigidity of the $(m;n)$th segment
$(F_x(m;n),\ F_y(m;n),\ F_z(m;n))$	the components of the external force acting at the $(m;n)$th station
$L(m;n)$	the stressed length of the $(m;n)$th segment
$L_0(m;n)$	the unstressed length of the $(m;n)$th segment
L-end	the end of the nth cable "closest" to the primary anchor
m	the index of a cable station or cable segment
$(m;n)$	the mth station or segment on the nth cable
$M(n)$	the total number of stations or segments on the nth cable
n	the index of a cable
$Q_{(m;n)}$	a functional expressing the stressed length of the $(m;n)$th segment in terms of the unstressed length of and tension in that segment
R-end	the end of the nth cable "furthest" from the primary anchor
$(R_x(m;n),\ R_y(m;n),\ R_z(m;n))$	the components of the resultant force in the $(m;n)$th segment
$T(m;n)$	the tension in the $(m;n)$th segment
$(x(0;1),\ y(0;1),\ z(0;1))$	the coordinates of the primary anchor
$(x(0;n),\ y(0;n),\ z(0;n))$	the coordinates of the L-end of the nth cable
$(x(m;n),\ y(m;n),\ z(m;n))$	the coordinates of the $(m;n)$th station
$\{n_L \leftrightarrow k_R\}$	the set of cables having an L-end which originates at the R-end of the kth cable
$\{n_R: a\}$	the set of cables which have an R-end attached to a secondary anchor
$\{n_R: f\}$	the set of cables which have a free R-end

The Method of Imaginary Reactions for Simple Cable Arrays

E	a positive definite error function
$(F_x(M(n);n),\ F_y(M(n);n),\ F_z(M(n);n))$	the components of the imaginary reaction applied to the $M(n)$th station of the nth cable
$(\Delta F_x(M(n);n),\ \Delta F_y(M(n);n),\ \Delta F_z(M(n);n))$	the components of the additive force applied to the $M(n)$th station of the nth cable
δ	a positive convergence factor having the dimension of force
$(\ )'$	denotes trial iteration values

Mathematical Symbols

ϵ	to be read as "is included in"
$\equiv$	to be read as "is identically"

INTRODUCTION

In this report, a new technique for determining the static equilibrium configuration and tensions in a cable array acted on by arbitrary forces is developed. This method of solution is called "The Method of Imaginary Reactions."

The need for such a solution has been generated by the greatly accelerated interest in ocean structural problems. For example, prior to this development no adequate analysis techniques were available to predict deflections and tensions in a buoy-cable system subject to drag forces dependent on both the position and the orientation of the system.

The primary reason for the lack of adequate analysis techniques has been that the differential equations of equilibrium for a cable are inherently nonlinear. Consequently, though solutions to particular problems are known, no general method of solution, such as the matrix force-displacement method for linear structures, has previously been developed.

The problem of a free-ended cable (for example, a towed body or kite) has been treated from a continuum point of view by Pode (1) and Alekseev (2). Alekseev obtained the general three-dimensional solution of the equilibrium equations under the combined action of gravity forces and arbitrary forces that were functions of the orientation of the cable in space. Pode, in treating a towed body, considered a special case of this general problem. In both of these papers, an inextensible cable was assumed to allow the exact integration of the equilibrium equations. Thus, these solutions predicted the position of a towed vehicle to be the same whether the towing cable was constructed of steel or nylon.

The free-ended cable has also been treated from a lumped parameter point of view by Walton and Polachek (3) and by Paquette and Henderson (4). Both of these papers allowed for current profiles but were restricted to two-dimensional problems. The paper by Walton and Polachek was restricted to inextensible cables, and the paper by Paquette and Henderson placed constraints on the equilibrium positions of the cable stations.

O'Brien (5) has considered fixed-fixed cables, such as a transmission line. The elasticity of the cable was included in the formulation, and an exact continuum solution was obtained. However, the forces applied to the cable were restricted to be piecewise constant, so that the shapes of the cable segments could be expressed as elastic catenaries.

Structural engineers have, for many years, used the classical method of consistent deformation to calculate redundant reactions in complex structures. This technique is particularly suited to the modern matrix methods of analyzing structural problems. However, the formulation of the method requires that the structure under consideration be linear and that the release of a redundant reaction does not cause gross structural distortions or instabilities. Neither requirement is satisfied by cable arrays.

The Method of Imaginary Reactions, as developed in this report for the analysis of cable systems, is a natural extension of consistent deformation theory to a highly nonlinear problem. By prescribing a simple method of varying the redundants, an iteration technique, guaranteed to converge to the correct reactions and consequently the correct static equilibrium configuration, is generated.

Throughout this report, a lumped parameter representation of the external forces acting on the cable array is assumed, and the bending stiffness of the cables is ignored. Consequently, each cable segment between stations becomes a straight line, and the equilibrium configuration of the system, including the effect of stretching, is uniquely determined from formulas which are functions of only the applied forces and the imaginary reactions. As a result, the number of equations to be solved does not depend on the number of stations used to describe the array but only on the number of external redundant reactions. Thus, extreme fineness in the representation of the system increases only the number of necessary arithmetic calculations but does not increase the complexity of the problem. It is worthwhile to note that these formulas also determine the static equilibrium configuration of a free-ended cable array, which is defined as a cable structure containing no unknown reactions.

The principal restriction on the use of The Method of Imaginary Reactions is that no internal loops of cable exist in the system.

It is also assumed that the cable array is statically stable; that is, under the action of the applied forces, no cable segment has zero tension. However, if this situation of zero tension occurs, the method will still converge to some obviously incorrect equilibrium configuration. Thus, preliminary stability investigations are unnecessary.

DESCRIPTION OF THE METHOD

Basic Concepts

To construct a foundation on which the precise mathematical formulation of The Method of Imaginary Reactions can be understood, the basic physical ideas involved in the method are presented in this section.

Consider the two-dimensional problem of a single cable acted on by an array of applied forces as shown in Fig. 1. If the component forces — F_{1x}, F_{1y}, F_{2x}, F_{2y}, R_x, and R_y — are known, then the reaction components R_{0x} and R_{0y}, at the anchor position O, can readily be determined by the summation of forces in the x and y directions, respectively. Knowing these reaction components, the direction of, tension in, and new length of the segment OA are easily found, and the position of point A (with coordinates A_x and A_y) located. Similarly, the positions of points B and C are found by the elementary rules of

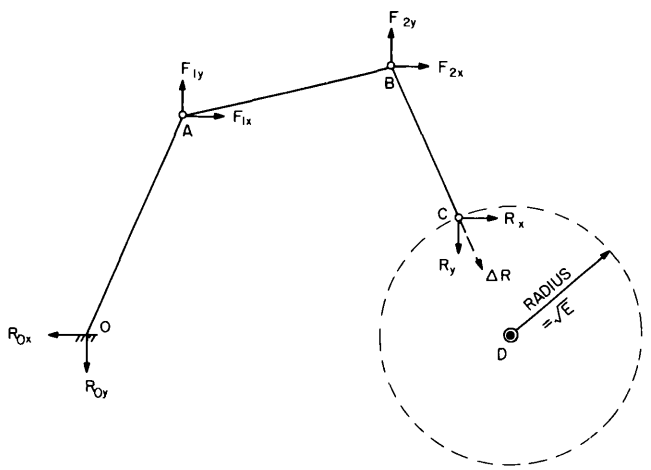

Fig. 1 - Basic concepts of the single cable

statics which determine the equivalent reactions at points A and B. As a result of these calculations, the equilibrium configuration of the free-ended cable is uniquely determined without solving the nonlinear force balance equations of the system.

Imagine for a moment that the point C is actually the position of a second anchor. Then, under the action of the external forces F_1 and F_2, the fixed-fixed cable is in equilibrium in the configuration calculated above, and the reaction at C is given by the components R_x and R_y. In this case, the problem is solved.

Now, consider a similar problem where D is the position of the second anchor, and R_x and R_y are the components of a "guessed" (an imaginary) reaction. Since the point C does not coincide with the point D, the guessed reaction is not the actual reaction. The question then arises: Can the actual reaction, which places the point C at D, be found, so that the equilibrium configuration of the cable can be determined without recourse to solving the nonlinear force balance equations of the system? The Method of Imaginary Reactions answers this question affirmatively.

Let an additional force ΔR be applied at C *in the direction from* C *to* D. The ratio of the components of this additional force is given by

$$\frac{(\Delta R)_x}{(\Delta R)_y} = \frac{(D_x - C_x)}{(D_y - C_y)} , \tag{1a}$$

and the magnitudes of the components are defined by

$$(\Delta R)_x = \frac{\delta}{\sqrt{E}} (D_x - C_x) \tag{1b}$$

and

$$(\Delta R)_y = \frac{\delta}{\sqrt{E}} (D_y - C_y) \tag{1c}$$

In these equations,

$$E = (D_x - C_x)^2 + (D_y - C_y)^2 \tag{2}$$

Note that E is a measure of the error between the equilibrium position calculated using the guessed reaction and the correct equilibrium position. Since E is positive definite, it vanishes uniquely only when the correct equilibrium position has been obtained; that is, E is identically zero only when $D_x \equiv C_x$ and $D_y \equiv C_y$ simultaneously. The problem now has been reduced to finding a technique for making E become vanishingly small.

The symbol δ denotes a *positive* number, having the dimension of force, yet to be determined. In essence, it is a convergence parameter used to choose the magnitude of the additive force ΔR in a manner such that the cable array will approach the correct equilibrium position. It is important to note that since the ratios $(D_x - C_x)/\sqrt{E}$ and $(D_y - C_y)/\sqrt{E}$ are of bounded variation (between -1 and +1), the convergence parameter δ must approach zero as E becomes vanishingly small.

Iteration Procedure

The concept of the solution can now be laid out:

1. Make a reasonable engineering guess as to the components of the reaction at the redundant anchor.

2. Release the redundant anchor while maintaining the guessed component reaction forces. This creates a free-ended cable on which the imaginary reactions are the applied forces at its end.

3. The structure is now statically determinant. Calculate the equilibrium position and from this result the quantities $(D_x - C_x)$, $(D_y - C_y)$, and E.

4. Choose an initial value of δ to find candidates for $(\Delta R)_x$ and $(\Delta R)_y$. δ can be chosen to be large, since it will of necessity become smaller as the solution proceeds step by step. In fact, at first choose δ to be the order of magnitude of the guessed reaction.

5. Calculate the new equilibrium position of the cable when the force $R' = R + \Delta R$ is applied at the end assumed free.

6. If E', the new measure of error, is less than E, the old measure, then a successful step has been made. In this event, begin again from this *new position* by changing the retained values of R, $(D_x - C_x)$, $(D_y - C_y)$, and E to those values calculated from this *new* equilibrium position. Retain the same value of δ and proceed with another iteration by finding the new ΔR and the corresponding equilibrium position.

7. Possibly on the first step, but certainly on some subsequent step, the candidate E' will be found to be greater than or equal to E of the previous step. Then δ is too large. The candidate values should be rejected and the previous ones retained by returning to the former equilibrium position of $E < E'$.

8. Reduce δ. Halving is suggested. Proceed from the last acceptable position until again a candidate E' is found to succeed.

9. This iteration process continues until E becomes as small as desired. This is a function of the scaling used.

Since E vanishes only at the real rather than at an imaginary anchor position, a solution of the entire problem has been found by considering only statically determinant arrays. There is no need to be particular in the choice of δ once the process starts successfully. Let it remain constant until the candidate $E' \geq E$ and then halve it.

It is important to note that The Method of Imaginary Reactions is globally convergent. That is, the iteration process will converge to $E = 0$ from any set of initial guessed reactions consistent with the static stability of the system. Proofs of the existence of a positive δ at each iteration step, the convergence of E to zero, and the uniqueness of the reactions calculated are given in Appendix A.

Multicable Problems

The method is readily extended to arrays of more than one cable. Consider for example the two-dimensional problem of three cables which are acted on by an array of forces and which are to be anchored at three points as shown in Fig. 2.

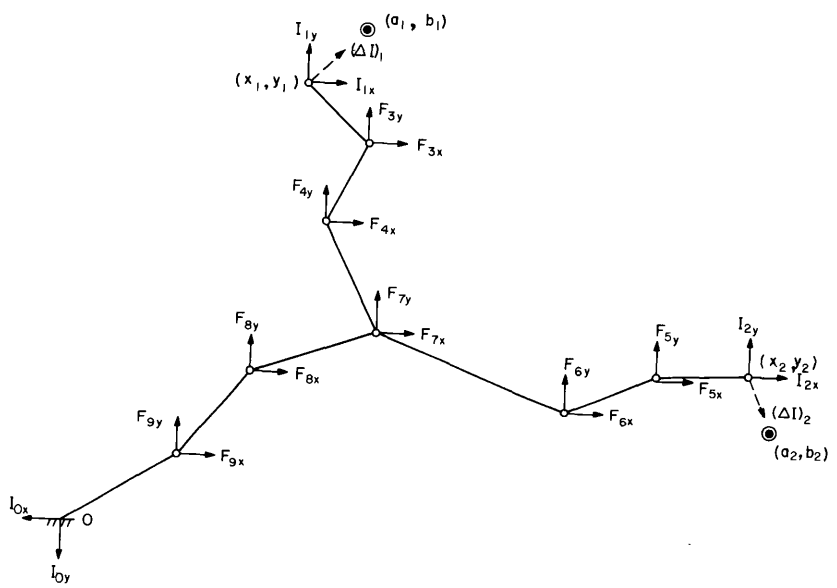

Fig. 2 - Basic concepts of multicable arrays

If, at the ends of two cables, imaginary reactions I_1 and I_2 are assumed, the reaction I_0 is easily calculated by summation of forces. By statics, and accounting for the changes of length in the cable segments, the coordinates of the free ends (x_1, y_1) and (x_2, y_2) are found.

Again, the basic problem is to find additive forces $(\Delta I)_1$ and $(\Delta I)_2$ to be applied at points 1 and 2, respectively, such that the cable array approaches its true equilibrium position.

The Method of Imaginary Reactions defines these forces as

$$(\Delta I)_{1x} = \frac{\delta}{\sqrt{E}}(a_1 - x_1) \ ,$$

$$(\Delta I)_{1y} = \frac{\delta}{\sqrt{E}}(b_1 - y_1) \ ,$$

$$(\Delta I)_{2x} = \frac{\delta}{\sqrt{E}}(a_2 - x_2) \ , \tag{3}$$

and

$$(\Delta I)_{2y} = \frac{\delta}{\sqrt{E}}(b_2 - y_2) \ ,$$

where the error function E is now defined by the sum of the squares of the coordinate errors as

$$E = (a_1 - x_1)^2 + (b_1 - y_1)^2 + (a_2 - x_2)^2 + (b_2 - y_2)^2 \ , \tag{4}$$

and δ is again a *positive* number, chosen to guarantee convergence to the correct reactions.

Note that, as in the previous problem, the incremental forces $(\Delta I)_1$ and $(\Delta I)_2$ are taken to act from point 1 to (a_1, b_1) and point 2 to (a_2, b_2), respectively; also, E vanishes uniquely only when the correct equilibrium configuration has been obtained. The relative magnitudes of these forces have been chosen so that the largest incremental force is applied at the end farthest away from the desired position at each step.

The iterative procedure to be followed is exactly the same as the procedure developed for the single-cable problem.

Lumped Parameter Representation

The reason for using a lumped parameter model to represent the cable array is now apparent. For, starting at the end of the array which is considered anchored, the paths to the free ends of the system are uniquely determined by elementary statics and formulas which express the elongation of the cables under tension. It is worthwhile to remark that the elongation formulas need not be linear; all that is required is that the cable lengths increase with increasing tension and be single-valued functions.

To obtain a successful lumped parameter representation, a few guidelines for modeling the array are presented. These are summarized as:

1. Each cable in the array is represented by at least two stations, one at each end.

2. Each point of discontinuity in a physical property of a cable (such as a change in cross-sectional area) is represented by a station. Consequently, each cable segment in the array has constant physical properties.

3. As many additional stations as are necessary to obtain a "satisfactory" approximation to the continuous equilibrium shape of the array are used. Thus, for example, a section of cable which is expected to have a small radius of curvature should be represented by more stations than a section which is expected to have a large radius of curvature. Note that the error function, the positive δ, and the various values of ΔR do not depend on the number of stations, and the structural analysis is always of a statically

determinant system. Thus, more stations do not grossly affect the amount of work necessary to find the unknown reactions and the resulting equilibrium position.

4. The method by which the external forces acting on the array are to be lumped at the cable stations is completely arbitrary. However, it is recommended that a half-segment technique, such as that used by Paquette and Henderson (4), be employed.

Position-Dependent External Forces

The ability to determine the equilibrium configuration of a cable array by simple statics depends totally on the lumped external forces being constant. Suppose, however, that a problem in which these forces are functions of position is encountered.

Under a large class of external forces, the equilibrium shape can still be determined by combining the static solution with the well-known mathematical technique of successive approximation (6).

This technique is briefly described. Suppose that in Fig. 1, the forces at A and B are functions of the locations of A and B. Let an estimate be made of the values of these forces. The problem is then solved with all forces being held constant. After the equilibrium position is determined, corrected values of the external forces are calculated, and the corrected equilibrium shape is found. This process is repeated until the equilibrium configuration has been obtained to within a specified accuracy and will converge if the external forces are continuous functions of position.

ELEMENTARY THEORY: THE SINGLE CABLE

Coordinate System and Notation

To describe the configuration of the cable in space, a right-handed Cartesian coordinate system is used, as shown in Fig. 3. The anchor position of the cable is denoted by

$$(x(0),\ y(0),\ z(0))$$

and the position of the mth station by

$$(x(m),\ y(m),\ z(m))\ ,$$

where $m = 1, 2, \ldots, M$, counting from the anchor position to the end of the cable. The external force which is acting at the mth station is defined by

$$(F_x(m),\ F_y(m),\ F_z(m))\ .$$

If the Mth station is also to be anchored, the desired point of anchorage is given by

$$(a,\ b,\ c)\ .$$

In this particular case, the force

$$(F_x(M),\ F_y(M),\ F_z(M))$$

represents the imaginary reaction which is applied to the Mth station.

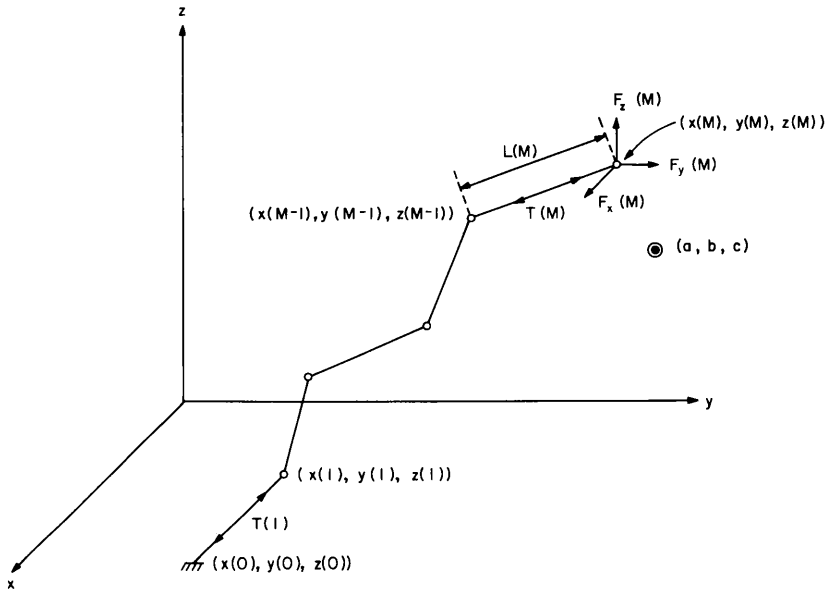

Fig. 3 - Single-cable nomenclature

The cable segment between the $(m-1)$ and mth stations is designated as the mth segment. The unstressed length of this segment is denoted by $L_0(m)$. In the stressed state, the tension in and length of this segment are denoted by $T(m)$ and $L(m)$, respectively.

Elongation Formulas

In the most general situation, the stressed length of a cable is an experimentally determined function of the unstressed length of the cable and the tension in the cable. For the mth segment, this relation is written as

$$L(m) = Q_m\{L_0(m),\ T(m)\} \ , \tag{5}$$

where the subscript m indicates that Q is the appropriate functional for the mth segment.

If all of the cable segments are operating in the elastic range of tension, the elongation formulas, Eq. 5, reduce to the engineering relation

$$L(m) = L_0(m)\left[1 + \frac{T(m)}{B(m)}\right] , \tag{6}$$

where $B(m)$ is the extensional rigidity of the mth segment.

Static Equilibrium Configuration

Since the external force acting at any cable station is known, the resultant force in any segment is readily found by the Method of Sections. Neglecting the bending stiffness of the cable, the tensions in and orientations of the cable segments are then uniquely determined. The segmental lengths are calculated from the elongation relations, Eq. (5).

Knowing the length and orientation of every segment and starting from the anchor position, the determination of the static equilibrium configuration of the cable becomes a simple trigonometric problem.

These results are expressed mathematically in the following formulas.

Let the resultant force in the mth segment be denoted as

$$(R_x(m),\ R_y(m),\ R_z(m)) \ .$$

This resultant is defined by the external forces acting on the cable through the equations

$$\begin{aligned} R_x(m) &= F_x(m) + R_x(m+1) \ , \\ R_y(m) &= F_y(m) + R_y(m+1) \ , \\ R_z(m) &= F_z(m) + R_z(m+1) \ , \qquad \text{for } m = 1, 2, \ldots, M-1 \ ; \end{aligned} \tag{7a}$$

and

$$\begin{aligned} R_x(M) &= F_x(M) \ , \\ R_y(M) &= F_y(M) \ , \\ R_z(M) &= F_z(M) \ , \qquad \text{for } m = M \ . \end{aligned} \tag{7b}$$

Note that starting from the free end of the cable the resultant forces are readily determined.

The tension in the mth segment is calculated as

$$T(m) = \sqrt{R_x^2(m) + R_y^2(m) + R_z^2(m)} \ , \tag{8}$$

where $T(m)$ is always positive, and the stretched length of this segment is found from Eq. (5).

The positions of the cable stations are then obtained by using

$$x(m) = \frac{L(m)}{T(m)} R_x(m) + x(m-1) \ , \tag{9a}$$

$$y(m) = \frac{L(m)}{T(m)} R_y(m) + y(m-1) \ , \tag{9b}$$

and

$$z(m) = \frac{L(m)}{T(m)} R_z(m) + z(m-1) \ . \tag{9c}$$

Method of Imaginary Reactions

If now the Mth station is also to be anchored, the applied force at this station becomes the guessed imaginary reaction. Using this guessed imaginary reaction, the static equilibrium configuration of the cable is found. As a measure of the error of the calculated end position $(x(M),\ y(M),\ z(M))$ from the desired end position (a, b, c) the positive definite error function is defined as

$$E = [a - x(M)]^2 + [b - y(M)]^2 + [c - z(M)]^2 \ . \tag{10}$$

The Method of Imaginary Reactions then states that if the iteration procedure determined by the substitutions

$$\begin{aligned} F'_x(M) &= F_x(M) + \Delta F_x(M) \ , \\ F'_y(M) &= F_y(M) + \Delta F_y(M) \ , \\ F'_z(M) &= F_z(M) + \Delta F_z(M) \end{aligned} \tag{11}$$

is followed, where

$$\begin{aligned} \Delta F_x(M) &= \frac{\delta}{\sqrt{E}} [a - x(M)] \ , \\ \Delta F_y(M) &= \frac{\delta}{\sqrt{E}} [b - y(M)] \ , \\ \Delta F_z(M) &= \frac{\delta}{\sqrt{E}} [c - z(M)] \ , \end{aligned} \tag{12}$$

and where δ is a *positive* number chosen at each iteration so that $E' < E$; then, after a large enough number of iterations, $E \to 0$ arbitrarily closely and the static equilibrium configuration of the cable is obtained.

It is instructive to note that The Method of Imaginary Reactions is readily generalized to a cable having an end which is not uniquely anchored. In this case, for the problem to be properly formulated, information must be given about the force at this end or the geometrical position constraint. Using the information thus supplied, the equations of the method are reformulated to conform to the constraints by eliminating the known forces and free coordinates from Eqs. 10, 11, and 12.

Numerical Example I

To illustrate the rapidity of the convergence of The Method of Imaginary Reactions and the application of the equilibrium formulas, Eq. 5, 7, 8, and 9, consider the single cable in two-dimensional space, acted on by the array of forces (in pounds) shown in Fig. 4.

The cable is taken to be uniform and elastic, with an extensional rigidity of 40 pounds. The initial length of each segment is 10 feet, and the cable is anchored at the coordinates (0, 0) feet and (24, 0) feet.

The anchor at the origin is taken as the "0" station. Consequently,

$$x(0) = 0, \ z(0) = 0 \ .$$

From Eq. 7, the resultant forces are calculated as

$$R_x(1) = 1.504461 + F_x(4) \ , \qquad R_z(1) = 2.673706 + F_z(4) \ ;$$

$$R_x(2) = 3.287020 + F_x(4) \ , \qquad R_z(2) = 5.202684 + F_z(4) \ ;$$

$$R_x(3) = 1.741578 + F_x(4) \ , \qquad R_z(3) = -0.883043 + F_z(4) \ ;$$

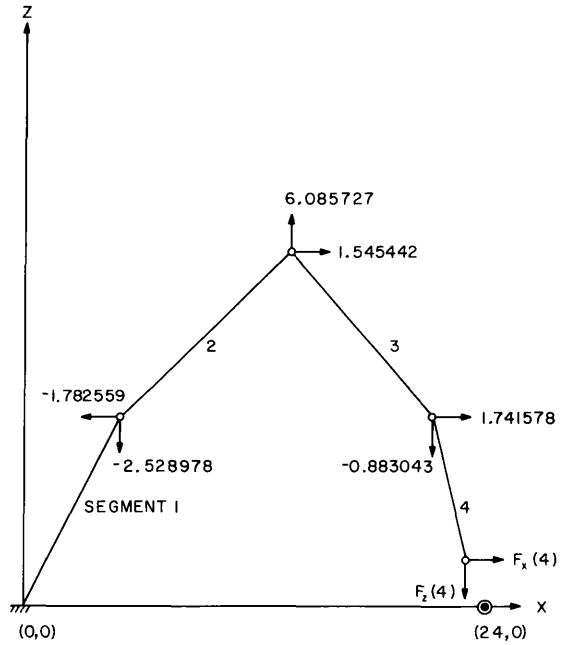

Fig. 4 - Array of forces

and

$$R_x(4) = F_x(4) \quad , \quad R_z(4) = F_z(4) \ .$$

The tensions are defined by Eq. 8, and the stressed lengths are calculated from Eq. 6 as

$$L(m) = 10 \left[1 + \frac{T(m)}{40}\right] \ .$$

For any imaginary reaction, $(F_x(4), F_z(4))$, the static equilibrium coordinates are then determined from Eq. 9.

The error function, Eq. 10, is defined by

$$E = [24 - x(4)]^2 + [-z(4)]^2 \ ,$$

and the formula for the change in the imaginary reaction is given by Eq. 12 as

$$\Delta F_x(4) = \frac{\delta}{\sqrt{E}} [24 - x(4)]$$

and

$$\Delta F_z(4) = \frac{\delta}{\sqrt{E}} [-z(4)] \ .$$

This problem was solved using a Wang Model 370 electronic desk calculator. A high-speed CDC digital computer was used for a three-dimensional, three-cable, 22-station problem illustrated later.

The convergence of The Method of Imaginary Reactions is illustrated in Table 1 for a "good" initial guess and in Table 2 for a "bad" initial guess of the actual reaction. In

Table 1
Convergence of the Method of Imaginary Reactions: "Good" Initial Guess

Iteration	$F_x(4)$	$F_z(4)$	$x(4)$	$z(4)$	E	$\frac{1}{2}\frac{n}{\infty}$ = n	$\Delta F_x(4)$	$\Delta F_z(4)$
0	0.5	-1.0	28.632348	-0.789850	2.208251×10^{1}	1	-0.492887	0.084041
1	0.007113	-0.915959	21.301520	-0.826674	7.965185×10^{0}	2	0.239035	0.073228
2	0.246148	-0.842731	25.440816	-0.252147	2.139528×10^{0}	3	-0.123129	0.021548
3	0.123019	-0.821183	23.361096	-0.304760	5.010772×10^{-1}	4	0.056412	0.026908
4	0.179431	-0.794275	24.403098	-0.068535	1.671849×10^{-1}	5	-0.030808	0.005238
5	0.148623	-0.789037	23.859544	-0.084635	2.689096×10^{-2}	6	0.013383	0.008064
6	0.162006	-0.780973	24.114002	-0.015213	1.322801×10^{-2}	7	-0.007744	0.001033
7	0.154262	-0.779940	23.975217	-0.021060	1.057742×10^{-3}	9	0.001488	0.001265
8	0.155750	-0.778675	24.004234	-0.011029	1.395627×10^{-4}	9	-0.000700	0.001824
9	0.155050	-0.776851	23.994332	-0.000953	3.30384×10^{-5}	11	0.000482	0.000081
10	0.155532	-0.776770	24.003230	0.000295	1.05206×10^{-5}	12	-0.000243	-0.000022
11	0.155289	-0.776792	23.998767	-0.000221	1.5704×10^{-6}	13	0.000120	0.000022
12	0.155409	-0.776770	24.000989	0.000099	9.879×10^{-7}	14	-0.000061	-0.000007
13	0.155348	-0.776777	23.999873	-0.000033	1.72×10^{-8}	17	0.000008	0.000002
14	0.155356	-0.776775	24.000010	-0.000010	2.00×10^{-10}			

Table 2
Convergence of the Method of Imaginary Reactions: "Bad" Initial Guess

Iteration	$F_x(4)$	$F_z(4)$	$x(4)$	$z(4)$	E	$\frac{1}{2}\sum_{n=0}^{\infty}$	$\Delta F_x(4)$	$\Delta F_z(4)$
						n		
0	-2.0	3.0	-6.445490	42.836365	2.761882×10^3	0	0.579323	-0.815098
1	-1.420677	2.184902	-0.223530	41.720098	2.327346×10^3	0	0.502119	-0.864798
2	-0.918558	1.320104	8.699543	35.260605	1.477414×10^3	0	0.398064	-0.917358
3	-0.520494	0.402746	9.982054	23.102359	7.302218×10^2	0	0.518749	-0.854927
4	-0.001745	-0.452181	20.816861	1.715791	1.307631×10^1	2	0.220066	-0.118621
5	0.218321	-0.570802	25.827369	1.589184	5.864781×10^0	2	-0.188643	-0.164055
6	0.029678	-0.734857	21.615613	0.148800	5.707252×10^0	2	0.249515	-0.015571
7	0.279193	-0.750428	26.227912	0.476242	5.190397×10^0	3	-0.122239	-0.026130
8	0.156954	-0.776558	24.029462	0.003840	8.827771×10^{-4}	9	-0.001937	-0.000252
9	0.155017	-0.776810	23.993797	-0.000753	3.90399×10^{-5}	*		

*Clearly, the method is now converging rapidly to the equilibrium solution. To check this, note Table 1, Step 9.

both schemes, the initial value of δ is one. In Table 3 a comparison is made between the exact equilibrium configuration, from which the problem was derived, and the equilibrium configuration calculated during the final iteration in Table 1.

GENERAL THEORY: SIMPLE CABLE ARRAYS

Definitions

A simple cable array is defined as a cable system which contains no closed loops. Physically, this means that a simple cable array possesses no internal redundants. Thus, for example, the cable-buoy structure depicted in Fig. 5a is a simple cable array, while the cable-buoy structure depicted in Fig. 5b is not.

Table 3
Comparison Between Exact and Calculated Equilibrium Configurations

m	Exact				Calculated			
	$x(m)$	$z(m)$	$L(m)$	$T(m)$	$x(m)$	$z(m)$	$L(m)$	$T(m)$
1	7	8	10.630146	2.520583	7.000005	7.999996	10.630146	2.520583
2	14	17	11.401754	5.607017	14.000007	16.999993	11.401754	5.607015
3	22	10	10.630146	2.520583	22.000004	9.999989	10.630147	2.520586
4	24	0	10.198039	0.792156	24.000010	-0.000010	10.198040	0.792159

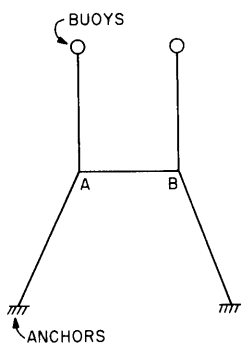

Fig. 5a - Simple cable array

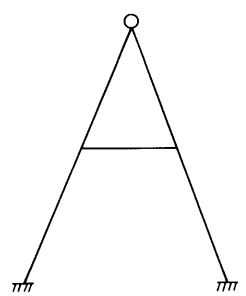

Fig. 5b - Nonsimple cable array

A branch point of a simple cable array is defined as a point at which a single cable "splits" into two or more cables. In Fig. 5a, the points A and B are the branch points of the particular system shown.

System Diagrams

To index the simple cable array, each cable in the system is denoted by a number n, where the convention is that no cable extends beyond a branch point. The index $n = 1$ is reserved for the cable which is attached to the anchor point of the system. (If more than one cable is anchored, the choice of cable 1 becomes arbitrary. In this case, the point to which cable 1 is attached is called the primary anchor; the other anchors are called secondary anchors.)

Any simple cable array is conveniently represented by a two-dimensional "system diagram." This diagram illustrates, for ready reference, the cable numbers, the primary and secondary anchor points, and the general branching network of the system under consideration. For example, if the cable-buoy structure depicted in Fig. 5a is numbered as shown in Fig. 6a, then the system diagram would be as illustrated in Fig. 6b. The system diagram for a particular ten-cable array is shown in Fig. 7.

In the expression of the general theory, it is important to distinguish between the two ends of a cable. The end of a cable "closest" to the primary anchor of the system

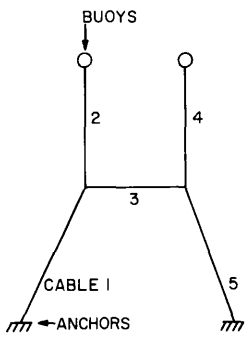

Fig. 6a - Indexing a simple cable array

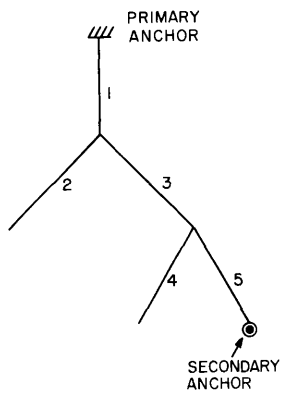

Fig. 6b - System diagram for the simple array

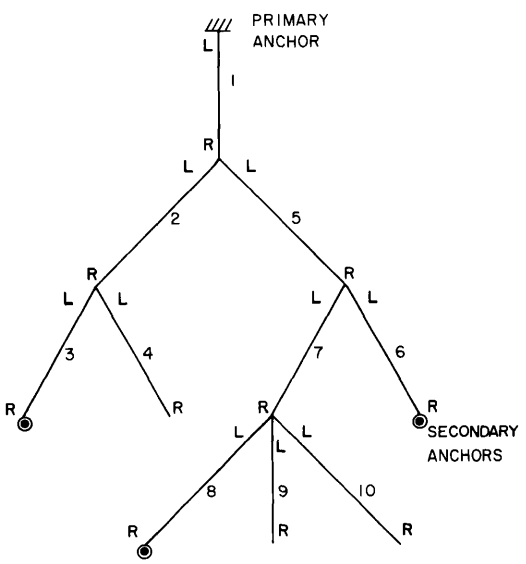

Fig. 7 - System diagram for a ten-cable array: the L-end and R-end nomenclature

diagram is denoted as the L-end; the other end is termed the R-end. Note that the R-end of a cable is either a free end, a secondary anchor, or a branch point of the simple cable array. The L-end of cable 1 is, by definition, the primary anchor point of the system.

The cables which have a free R-end are designated by the set notation

$$\{n_R : f\} \ ,$$

and the cables which are attached to a secondary anchor are denoted by the set

$$\{n_R : a\} \ .$$

Referring to Fig. 7,

$$\{n_R : f\} = \{4, 9, 10\}$$

and

$$\{n_R : a\} = \{3,6,8\} \ .$$

To designate the set of cables having an L-end which originates at the R-end of the kth cable, the notation

$$\{n_L \leftrightarrow k_R\}$$

is employed. Thus, in Fig. 7, the set

$$\{n_L \leftrightarrow 1_R\} = \{2,5\}$$

and the set

$$\{n_L \leftrightarrow 7_R\} = \{8,9,10\}$$

Coordinate System and Notation

To describe the configuration of the cable array in space, a right-handed Cartesian coordinate system is used. The position of the L-end of each component cable is denoted by

$$(x(0;n),\ y(0;n),\ z(0;n))$$

and the position of the mth station on the nth cable by

$$(x(m;n),\ y(m;n),\ z(m;n)) \ ,$$

where $m = 1,2,\ldots,M(n)$ counting from the L-end to the R-end of the cable. The notation $M(n)$ is used to indicate that each cable in the array can be represented by a different number of stations.

By definition, the location of the primary anchor point of the system is given by

$$(x(0;1),\ y(0;1),\ z(0;1)) \ .$$

Furthermore, the station indices

$$(M(k);k) \quad \text{and} \quad (0;n)$$

represent the same system branch point for

$$n \epsilon \ \{n_L \leftrightarrow k_R\} \ .$$

The external force which is acting at the $(m;n)$th station is defined by

$$(F_x(m;n),\ F_y(m;n),\ F_z(m;n)) \ ,$$

where the convention is that an external force acting at a branch point is indexed by $(M(k);k)$ rather than $(0;n)$.

If the nth cable in the array is to be attached to a secondary anchor, the desired point of anchorage is given by

$$(a(n),\ b(n),\ c(n))\ .$$

In this particular case, the force

$$(F_x(M(n);n),\ F_y(M(n);n),\ F_z(M(n);n))$$

represents the imaginary reaction which is applied to the Mth station of this cable.

The segment between the $(m-1)$ and mth stations on the nth cable is designated as the $(m;n)$th cable segment.

The further extension of the single-cable notation to a simple cable array is now obvious.

Elongation Formulas

The relation expressing the stressed length of a segment as a function of the unstressed length of the segment and tension in the segment is written as

$$L(m;n) = Q_{(m;n)}\{L_0(m;n),\ T(m;n)\}\ .$$

In the elastic range of tension, Eq. 13 reduces to

$$L(m;n) = L_0(m;n)\left[1 + \frac{T(m;n)}{B(m;n)}\right], \tag{14}$$

where $B(m;n)$ is the extensional rigidity of the $(m;n)$th segment.

Static Equilibrium Configuration

The resultant force in the $(m;n)$th segment is defined by the applied external forces through the equations

$$\begin{aligned} R_x(m;n) &= F_x(m;n) + R_x(m+1;n)\ , \\ R_y(m;n) &= F_y(m;n) + R_y(m+1;n)\ , \\ R_z(m;n) &= F_z(m;n) + R_z(m+1;n)\ , \quad \text{for } m = 1, 2, \ldots, M(n)-1\ , \quad \text{all } n\ ; \end{aligned} \tag{15a}$$

and

$$\begin{aligned} R_x(M(n);n) &= F_x(M(n);n)\ , \\ R_y(M(n);n) &= F_y(M(n);n)\ , \\ R_z(M(n);n) &= F_z(M(n);n)\ , \quad \text{for } m = M(n)\ , \quad n \in \{n_R{:}\,a\} \text{ or } \{n_R{:}\,f\}\ ; \end{aligned} \tag{15b}$$

and

$$R_x(M(n);n) = F_x(M(n);n) + \sum_{k}^{k \in \{k_L \leftrightarrow n_R\}} R_x(1;k)\ , \tag{15c (Cont.)}$$

$$R_y(M(n);n) = F_y(M(n);n) + \sum_{k}^{k\epsilon\{k_L \leftrightarrow n_R\}} R_y(1;k) \ ,$$

$$R_z(M(n);n) = F_z(M(n);n) + \sum_{k}^{k\epsilon\{k_L \leftrightarrow n_R\}} R_z(1;k) \ , \tag{15c}$$

for $m = M(n)$, $(M(n);n)$ a branch point.

Note, that starting from the R-end points of the entire cable array, the resultant forces are readily determined by summing the applied external forces "toward" the primary anchor. This "direction of summation" is shown in Fig. 8 for the cable array depicted in Fig. 7.

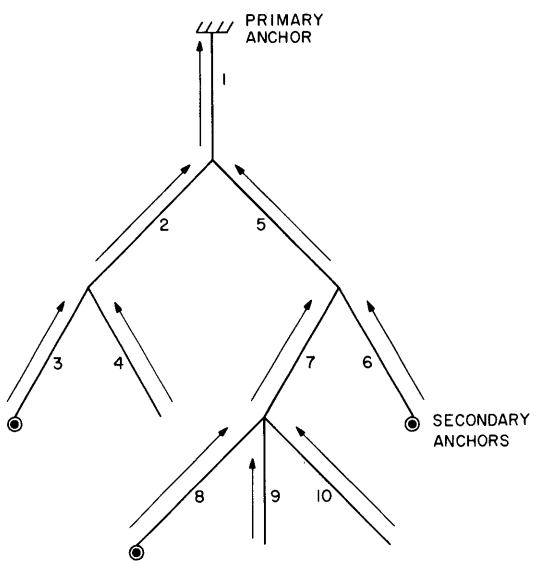

Fig. 8 - The "direction of summation"

In terms of the resultant force, the tension in the $(m;n)$th segment is given by

$$T(m;n) = \sqrt{R_x^2(m;n) + R_y^2(m;n) + R_z^2(m;n)} \ . \tag{16}$$

Starting from the primary anchor, the static equilibrium configuration is then calculated from the relations

$$x(m;n) = \frac{L(m;n)}{T(m;n)} R_x(m;n) + x(m-1;n) \ , \tag{17a}$$

$$y(m;n) = \frac{L(m;n)}{T(m;n)} R_y(m;n) + y(m-1;n) \ , \tag{17b}$$

$$z(m;n) = \frac{L(m;n)}{T(m;n)} R_z(m;n) + z(m-1;n) \ , \tag{17c}$$

where the identities

$$x(0;n) = x(M(k);k) \ , \tag{18a}$$

$$y(0;n) = y(M(k);k) \ , \tag{18b}$$

$$z(0;n) = z(M(k);k) \ , \tag{18c}$$

for $n \epsilon \{n_L \leftrightarrow k_R\}$, are used to extend the solution past a branch point.

Method of Imaginary Reactions

The Method of Imaginary Reactions is readily generalized to arbitrary simple cable arrays. The error function E is defined by

$$E = \sum_{n}^{n \epsilon \{n_R : a\}} \Big\{ [a(n) - x(M(n);n)]^2 + [b(n) - y(M(n);n)]^2 + [c(n) - z(M(n);n)]^2 \Big\} \ . \tag{19}$$

The theory then states that if the iteration procedure determined by the substitutions

$$F_x'(M(n);n) = F_x(M(n);n) + \Delta F_x(M(n);n) \ , \tag{20a}$$

$$F_y'(M(n);n) = F_y(M(n);n) + \Delta F_y(M(n);n) \ , \tag{20b}$$

$$F_z'(M(n);n) = F_z(M(n);n) + \Delta F_z(M(n);n) \ , \tag{20c}$$

for $n \epsilon \{n_R : a\}$, is followed, where

$$\Delta F_x(M(n);n) = \frac{\delta}{\sqrt{E}} [a(n) - x(M(n);n)] \ , \tag{21a}$$

$$\Delta F_y(M(n);n) = \frac{\delta}{\sqrt{E}} [b(n) - y(M(n);n)] \ , \tag{21b}$$

$$\Delta F_z(M(n);n) = \frac{\delta}{\sqrt{E}} [c(n) - z(M(n);n)] \tag{21c}$$

and where δ is a *positive* number chosen at each iteration so that $E' < E$; then, after a large enough number of iterations, $E \to 0$ arbitrarily closely, and the static equilibrium configuration of the cable array is obtained.

Numerical Example II

To demonstrate the ability of The Method of Imaginary Reactions to handle three-dimensional arrays, including cables which are skewed in the equilibrium configuration, two problems are considered for a three-cable arrangement. The system diagram for the array is illustrated in Fig. 9.

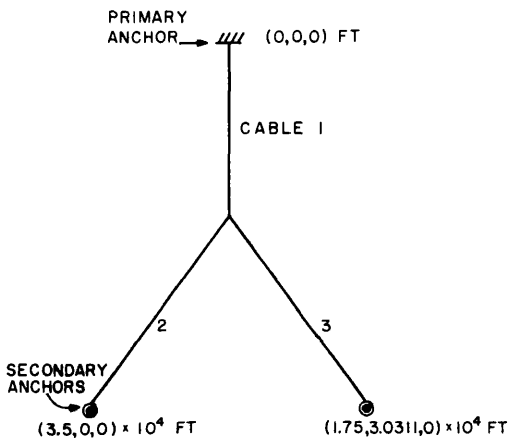

Fig. 9 - System diagram for example II

In the first problem, a net buoyancy force of 40,000 pounds (in the positive z direction) is applied to the branch point of the array, and all other external forces are taken as zero. In the second problem, a set of random drag forces are generated in addition to the acting buoyancy force. The cables comprising the system are taken to be elastic but with segmentally variable extensional rigidities.

The lumped parameter breakdown of the array and the random drag forces applied to the array stations are given in Table 4. All forces are nondimensionalized by dividing by 10^4 pounds and all lengths by dividing by 10^4 feet.

Both problems were done in one program on the NRL/CDC 3800 computer. The run time for the program, including compile, was 30 seconds. The nondimensional cutoff value of the error function was taken as $E \leq 2.5 \times 10^{-11}$; thus, all nondimensional secondary anchor positions are accurate to at least five decimal places.

The calculated equilibrium configurations are shown in Figs. 10 and 11.

STATICALLY UNSTABLE AND DIVISIBLE CABLE ARRAYS

Throughout this report, the assumption that the cable array is statically stable — that is, under the action of the applied external forces no cable segment has zero tension — has been tacitly made. It is of practical interest to note what happens to the technique when this assumption is not true for the system under investigation.

Consider a free-ended cable array in which the tension in a segment is zero. Mathematically, the equilibrium configuration of the array, given by Eq. 17, becomes indeterminate. Physically, what has occurred is that the external forces acting on the system are self-balanced; consequently, the position of the array is not unique. In this case, an examination of the computer printout would reveal which segment has gone into zero tension.

Suppose now that secondary anchors are present and that the tension in some segment approaches zero. Referring to the Appendix, the change dE in E then approaches negative infinity. As a result, it becomes impossible to find a δ such that $E' < E$, and

$$\lim_{\delta \to 0} E = K \neq 0 .$$

Table 4
Lumped Parameter Breakdown

Cable Properties			Drag Forces		
Segment or Station Number, m	$L_0(m;n)$ in 10^4 feet	$B(m;n)$ in 10^4 pounds	$F_x(m;n)$ in 10^4 pounds	$F_y(m;n)$ in 10^4 pounds	$F_z(m;n)$ in 10^4 pounds
Cable 1					
1	0.10	784	-0.005	0.005	-0.003
2	0.70	784	-0.031	0.029	-0.020
3	0.70	784	-0.032	0.028	-0.020
4	0.70	784	-0.033	0.027	-0.020
5	0.51	784	-0.050	0.040	-0.015
6	0.05	784	-0.005	0.003	-0.010
7	0.05	784	-0.030	0.070	-0.005
Cable 2					
1	0.05	784	-0.003	0.005	-0.010
2	0.05	784	-0.040	0.050	-0.015
3	0.51	784	-0.027	0.033	-0.020
4	0.70	784	-0.028	0.032	-0.020
5	0.70	784	-0.029	0.031	-0.020
6	0.70	784	-0.005	0.005	-0.003
7	0.10	784	Imaginary Reaction		
Cable 3					
1	0.05	280	-0.004	0.006	0.020
2	0.05	280	-0.041	0.061	0.015
3	0.32	280	-0.028	0.027	0.010
4	0.50	840	-0.034	0.032	0.010
5	0.60	840	-0.033	0.031	0.010
6	0.60	840	-0.032	0.030	0.010
7	0.60	840	-0.006	0.006	0.001
8	0.10	840	Imaginary Reaction		

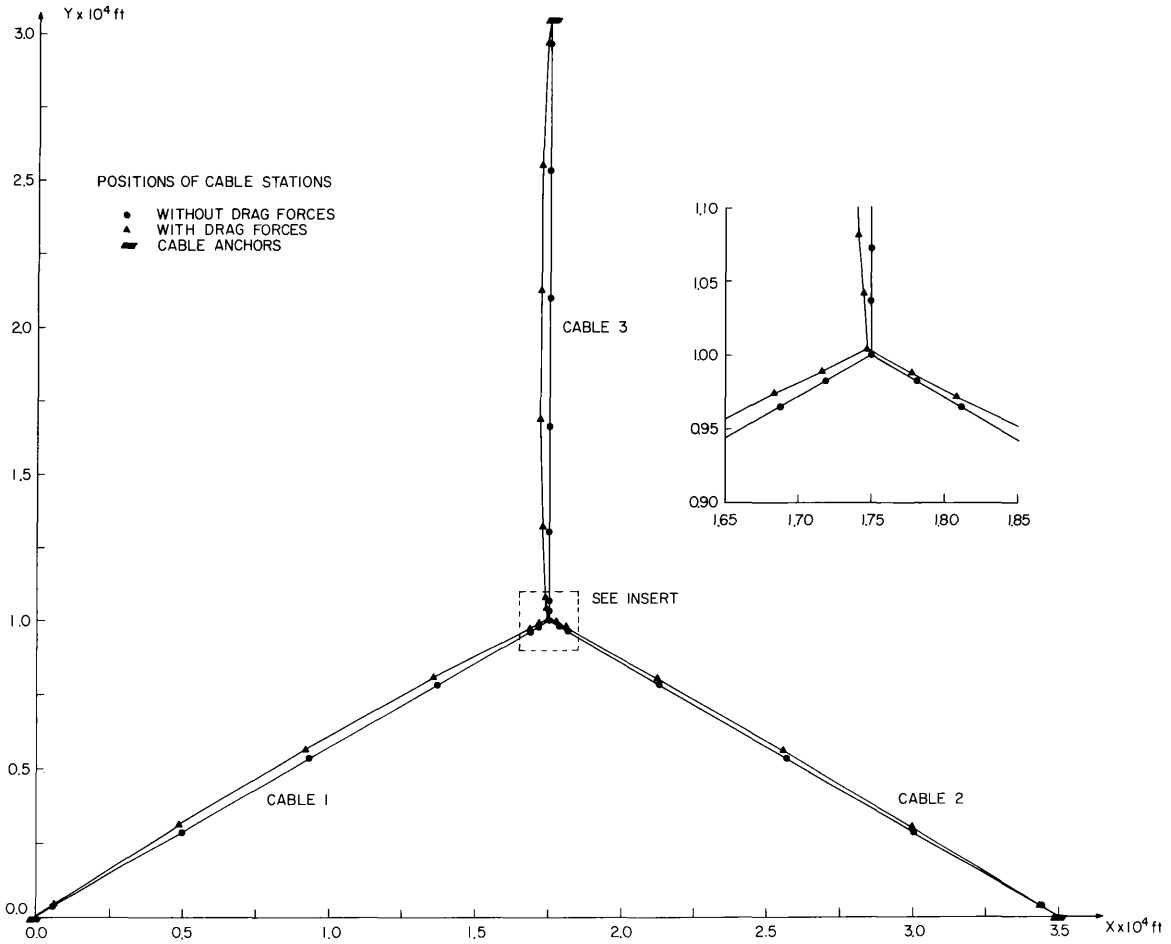

Fig. 10 - Equilibrium configurations of the three-cable array: top view

Two situations can occur. In the first case, an imaginary reaction approaches zero, so that the tension in an end segment becomes zero. The equilibrium position of the array would not be affected by this, and a solution would be found which stated that the particular secondary anchor is not necessary for the structural integrity of the system.

In the second case, when an internal segment approaches zero tension, an examination of the printout would show which segment is not acting. To proceed, it is possible now to divide up the array into two independent arrays and solve for the equilibrium configuration of each one.

SUMMARY AND CONCLUSIONS

The method presented in this report is applicable to a wide range of structural cabling problems. It will be particularly attractive to the practicing engineer who would otherwise be faced with the simultaneous solution of large interdependent sets of strength, geometrical, and force-balance equations, which are grossly nonlinear.

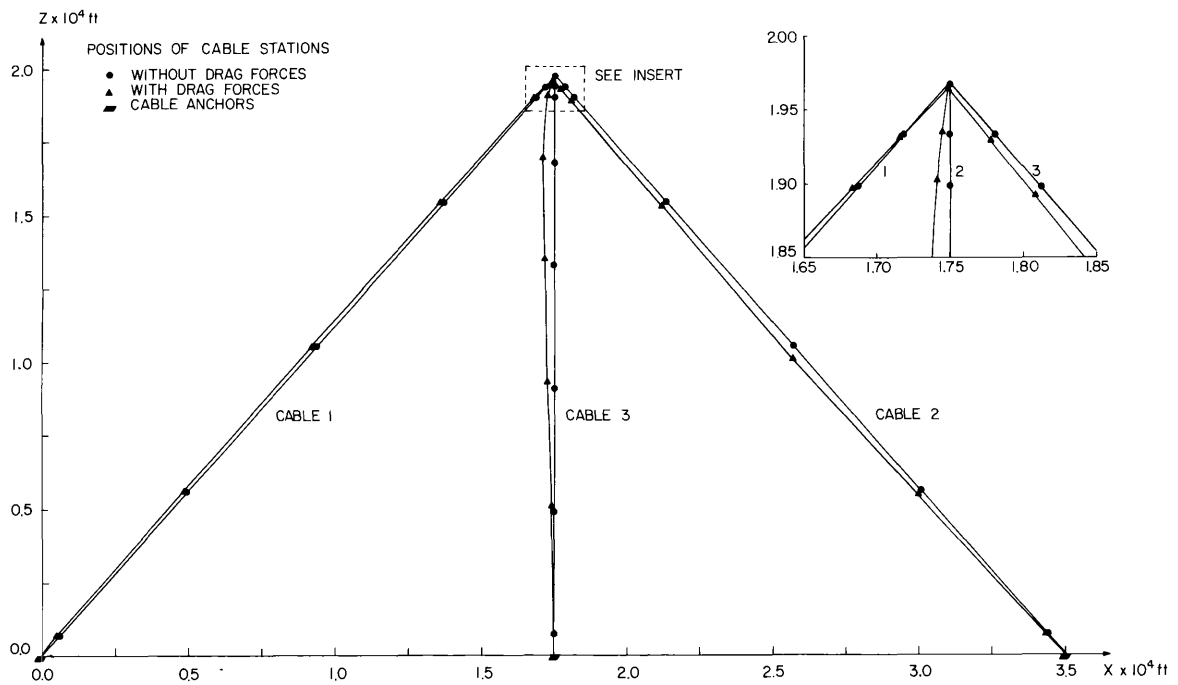

Fig. 11 - Equilibrium configurations of the three-cable array: side view

Although the formulation is of a discrete variable type, this should not be a serious drawback to its use. The fact that the technique is independent of the number of stations employed means that a fine distribution of stations can be made to approximate curvature and variations of external forces, without seriously affecting the rate of convergence.

The method aside from the mathematical proofs of the Appendix involves no more than elementary statics and strength of materials, once the external force distribution has been defined. Reduced to a simplistic view the method merely states that if you have a free-ended cable in a static equilibrium position with the end at the point C in Fig. 1 and wish to have this end to be located at point D, always pull toward point D, regardless of the path that C follows as it approaches D.

It is suggested to those who may have reservations on the efficiency of this method that they write the complete set of force-balance, elastic, and geometrical constraint equations for the simple numerical example I and attempt a simultaneous solution.

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Appendix A

PROOF OF THE CONVERGENCE OF THE METHOD OF IMAGINARY REACTIONS

In this appendix the proof that The Method of Imaginary Reactions insures convergence to the correct redundant reactions (and, consequently, to the equilibrium configuration of an arbitrary simple cable array) is given.

To facilitate this proof, a matrix notation is introduced. If the set of cables attached to secondary anchor positions is given by

$$\{n_R : a\} = \{n_1, n_2, \ldots, n_K\} ,$$

the error function, defined by Eq. 19, can be written in matrix form as

$$E = \{D\}^T\{D\} , \tag{A1}$$

where the column matrix $\{D\}$ is defined by

$$\{D\} = \begin{bmatrix} a(n_1) - x(M(n_1);n_1) \\ \vdots \\ a(n_K) - x(M(n_K);n_K) \\ b(n_1) - y(M(n_1);n_1) \\ \vdots \\ b(n_K) - y(M(n_K);n_K) \\ c(n_1) - z(M(n_1);n_1) \\ \vdots \\ c(n_K) - z(M(n_K);n_K) \end{bmatrix}$$

Consider now the total derivative of E. Since, through the equilibrium equations, the calculated end positions are functions of *only* the imaginary reactions, this derivative is

$$dE = -2\{D\}^T[J]\{dF\} , \tag{A2}$$

where

$$\{dF\} = \begin{bmatrix} dF_x(M(n_1);n_1) \\ \vdots \\ dF_x(M(n_K);n_K) \\ dF_y(M(n_1);n_1) \\ \vdots \\ dF_y(M(n_K);n_K) \\ dF_z(M(n_1);n_1) \\ \vdots \\ dF_z(M(n_K);n_K) \end{bmatrix}$$

and where

$$[J] = \begin{bmatrix} \dfrac{\partial x(M(n_1);n_1)}{\partial F_x(M(n_1);n_1)}, & \ldots, & \dfrac{\partial x(M(n_1);n_1)}{\partial F_z(M(n_K);n_K)} \\ \vdots & & \vdots \\ \dfrac{\partial z(M(n_K);n_K)}{\partial F_x(M(n_1);n_1)}, & \ldots, & \dfrac{\partial z(M(n_K);n_K)}{\partial F_z(M(n_K);n_K)} \end{bmatrix} . \tag{A3}$$

The determinant of $[J]$ is, of course, the Jacobian of the transformation from "end position" space to "imaginary reaction" space.

Substituting in Eq. A2 the expressions for the changes in the imaginary reactions, Eqs. 21, the total derivative of E becomes

$$dE = \frac{-2\delta}{\sqrt{E}} P , \tag{A4}$$

where P is the quadratic form

$$P = \{D\}^T [J] \{D\} . \tag{A5}$$

Note that since P is on the order of E, dE vanishes identically at $E = 0$; consequently, those sets of reactions for which $E = 0$ are points of absolute minimum of the error function in reaction space.

If it can now be shown that P is positive definite (vanishing only at $E = 0$), the convergence of The Method of Imaginary Reactions to an equilibrium set of reactions, and the uniqueness of this set of reactions, follows from the argument given below.

Suppose P is positive definite. Then dE, defined by Eq. A4, is negative (positive) definite for δ greater than (less than) zero. Since the error function is at least positive semidefinite in reaction space, the existence of a *positive* δ which makes $E' < E$ follows from the Mean Value Theorem. Since dE is then negative definite, the limit of the iteration process is necessarily $E = 0$; consequently, convergence to an equilibrium set of reactions is guaranteed. Further, this set of reactions is unique; for, if P is positive

definite, the Jacobian $|J|$ of the transformation from end space to reaction space is *positive*.* Thus, the transform is one-to-one;† that is, a unique set of anchor points determines a unique set of reactions.

Therefore, it remains only to show that P is indeed positive definite.

To this end, consider the resultant forces. Using the summation rule, Eq. 7, these forces can be expressed in terms of the imaginary reactions applied to the system as

$$R_x(m;n) = C_x(m;n) + \sum_{n_\ell}^{n_\ell \epsilon \{\{n_R:a\} \to n\}} F_x(M(n_\ell);n_\ell) \ , \tag{A6}$$

where the C_x's are constants, independent of the applied imaginary reactions and where the set $\{\{n_R:a\} \to n\}$ represents the subset of cables attached to secondary anchors and simultaneously directly "descended" from the nth cable on the system diagram.

For example, referring to Fig. 8,

$$\{\{n_R:a\} \to 5\} = \{6,8\}$$

and

$$\{\{n_R:a\} \to 6\} = \{6\} \ .$$

By definition of cable 1,

$$\{\{n_R:a\} \to 1\} = \{n_R:a\} \ .$$

Similar results are obtained when expressing the y and z components of the resultant forces in terms of a constant plus the y and z components of the imaginary reactions, respectively.

Using the above expressions for the resultant forces and Eq. 16 for the tensions and noting that $L(m;n)$ is a function of only $T(m;n)$ and $L_0(m;n)$, the following partial derivatives with respect to the imaginary reactions are obtained:

$$\frac{\partial\left[\frac{L(m;n)}{T(m;n)} R_x(m;n)\right]}{\partial F_x(M(n_\ell);n_\ell)} = \begin{cases} \frac{L(m;n)}{T(m;n)} + \frac{R_x^2(m;n)}{T(m;n)} A(m;n) \\ 0 \end{cases}$$

$$\text{if} \begin{cases} n_\ell \epsilon \{\{n_R:a\} \to n\} \\ n_\ell \not\epsilon \{\{n_R:a\} \to n\} \end{cases}$$

$$\frac{\partial\left[\frac{L(m;n)}{T(m;n)} R_x(m;n)\right]}{\partial F_y(M(n_\ell);n_\ell)} = \begin{cases} \frac{R_x(m;n)\ R_y(m;n)}{T(m;n)} A(m;n) \\ 0 \end{cases} \tag{A7 (Cont.)}$$

$$\text{if} \begin{cases} n_\ell \epsilon \{\{n_R:a\} \to n\} \\ n_\ell \not\epsilon \{\{n_R:a\} \to n\} \end{cases}$$

*R.A. Frazer, W.J. Duncan, and A.R. Collar, "Elementary Matrices," New York:MacMillan, 1947.
†A. E. Taylor, "Advanced Calculus," Boston:Ginn, 1955.

$$\frac{\partial\left[\frac{L(m;n)}{T(m;n)}R_x(m;n)\right]}{\partial F_z(M(n_\ell);n_\ell)} = \begin{cases} \dfrac{R_x(m;n)\,.\,R_z(m;n)}{T(m;n)}\,A(m;n) \\ 0 \end{cases} \tag{A7}$$

$$\text{if} \begin{cases} n_\ell \in \{\{n_R:a\} \to n\} \\ n_\ell \notin \{\{n_R:a\} \to n\} \end{cases}$$

with similar results for the permutations of x, y, and z. In these expressions

$$A(m;n) = \frac{d\left[\frac{L(m;n)}{T(m;n)}\right]}{dT(m;n)} \,. \tag{A8}$$

Consider now the three-cable array, for which the system diagram is illustrated in Fig. 9. Using the equilibrium equations 17 and 18, the positions of the end points, to which imaginary reactions are applied, can be written as

$$x(M(2);2) = x(0;1) + \sum_{m=1}^{M(1)} \frac{L(m;1)}{T(m;1)} R_x(m;1) + \sum_{m=1}^{M(2)} \frac{L(m;2)}{T(m;2)} R_x(m;2)$$

$$x(M(3);3) = x(0;1) + \sum_{m=1}^{M(1)} \frac{L(m;1)}{T(m;1)} R_x(m;1) + \sum_{m=1}^{M(3)} \frac{L(m;3)}{T(m;3)} R_x(m;3)$$

with similar expressions for the y and z coordinates. For this particular system

$$\{\{n_R:a\} \to 1\} = \{2,3\} \,,$$

$$\{\{n_R:a\} \to 2\} = \{2\} \,,$$

and

$$\{\{n_R:a\} \to 3\} = \{3\} \,.$$

Using the above expressions for the end coordinates and the sets $\{\{n_R:a\} \to n\}$, the expression in Eq. A6 for the resultant forces, and the partial derivatives defined in Eq. A7, the function P defined by Eq. A5 can be written as the sum

$$P = \sum_{m=1}^{M(1)} \{D\}^T \begin{bmatrix} B_{xx}(m;1) & B_{xx}(m;1) & B_{xy}(m;1) & B_{xy}(m;1) & B_{xz}(m;1) & B_{xz}(m;1) \\ B_{xx}(m;1) & B_{xx}(m;1) & B_{xy}(m;1) & B_{xy}(m;1) & B_{xz}(m;1) & B_{xz}(m;1) \\ B_{xy}(m;1) & B_{xy}(m;1) & B_{yy}(m;1) & B_{yy}(m;1) & B_{yz}(m;1) & B_{yz}(m;1) \\ B_{xy}(m;1) & B_{xy}(m;1) & B_{yy}(m;1) & B_{yy}(m;1) & B_{yz}(m;1) & B_{yz}(m;1) \\ B_{xz}(m;1) & B_{xz}(m;1) & B_{yz}(m;1) & B_{yz}(m;1) & B_{zz}(m;1) & B_{zz}(m;1) \\ B_{xz}(m;1) & B_{xz}(m;1) & B_{yz}(m;1) & B_{yz}(m;1) & B_{zz}(m;1) & B_{zz}(m;1) \end{bmatrix} \{D\} \tag{Cont.}$$

$$+ \sum_{m=2}^{M(2)} \{D\}^T \begin{bmatrix} B_{xx}(m;2) & 0 & B_{xy}(m;2) & 0 & B_{xz}(m;2) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ B_{xy}(m;2) & 0 & B_{yy}(m;2) & 0 & B_{yz}(m;2) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ B_{xz}(m;2) & 0 & B_{yz}(m;2) & 0 & B_{zz}(m;2) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \{D\}$$

$$+ \sum_{m=3}^{M(3)} \{D\}^T \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_{xx}(m;3) & 0 & B_{xy}(m;3) & 0 & B_{xz}(m;3) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_{xy}(m;3) & 0 & B_{yy}(m;3) & 0 & B_{yz}(m;3) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_{xz}(m;3) & 0 & B_{yz}(m;3) & 0 & B_{zz}(m;3) \end{bmatrix} \{D\}$$

where

$$B_{ab}(m;n) = \frac{R_a(m;n)R_b(m;n)}{T(m;n)} A(m;n) + \frac{L(m;n)}{T(m;n)} \delta_{ab}$$

and where the Kronecker delta

$$\delta_{ab} = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases} .$$

From this expression for P, the following theorem can be proven:

Theorem: If P^*, defined by

$$P^* = \{D\}^T \begin{bmatrix} B_{xx}(1;1) & B_{xx}(1;1) & B_{xy}(1;1) & B_{xy}(1;1) & B_{xz}(1;1) & B_{xz}(1;1) \\ B_{xx}(1;1) & B_{xx}(1;1) & B_{xy}(1;1) & B_{xy}(1;1) & B_{xz}(1;1) & B_{xz}(1;1) \\ B_{xy}(1;1) & B_{xy}(1;1) & B_{yy}(1;1) & B_{yy}(1;1) & B_{yz}(1;1) & B_{yz}(1;1) \\ B_{xy}(1;1) & B_{xy}(1;1) & B_{yy}(1;1) & B_{yy}(1;1) & B_{yz}(1;1) & B_{yz}(1;1) \\ B_{xz}(1;1) & B_{xz}(1;1) & B_{yz}(1;1) & B_{yz}(1;1) & B_{zz}(1;1) & B_{zz}(1;1) \\ B_{xz}(1;1) & B_{xz}(1;1) & B_{yz}(1;1) & B_{yz}(1;1) & B_{zz}(1;1) & B_{zz}(1;1) \end{bmatrix} \{D\}$$

is positive definite, vanishing only at $\{D\} = 0$, then P itself is positive definite, vanishing only at $\{D\} = 0$.

Proof: If P^* is positive definite, then each term in the summation over cable 1 is also positive definite by similarity. Thus, the summation over cable 1 is positive definite, vanishing only at $\{D\} = 0$. Furthermore, the *reduced* forms of P^* appearing in the summations over cables 2 and 3 are positive definite, vanishing only at $\{D\} = 0$*. Thus, the summations over cables 2 and 3 are also positive definite. Consequently, P is positive definite, vanishing only at $\{D\} = 0$, and the theorem is proved.

Obviously now, any simple cable array can be dissected in the same manner as the three-cable array and the function P reduced to a series of summations over the component cables. The most general summation, that is the summation in which no rows and columns of $[J]$ are zero, is that over cable 1, since only for this cable does $\{\{n_R : a\} \to n\} = \{n_R : a\}$.

The extension of the theorem proven above is then straightforward, the result being that P is positive definite for an arbitrary simple cable array if P^* is positive definite, where

$$P^* = \{D\}^T [J^*] \{D\} \tag{A9}$$

and where $[J^*]$ is defined by

$$[J^*] = \left[\begin{array}{c|c|c} \frac{L(1;1)}{T(1;1)} + \frac{R_x^2(1;1)}{T(1;1)} A(1;1) & \frac{R_x(1;1)R_y(1;1)}{T(1;1)} A(1;1) & \frac{R_x(1;1)R_z(1;1)}{T(1;1)} A(1;1) \\ \hline \frac{R_x(1;1)R_y(1;1)}{T(1;1)} A(1;1) & \frac{L(1;1)}{T(1;1)} + \frac{R_y^2(1;1)}{T(1;1)} A(1;1) & \frac{R_y(1;1)R_z(1;1)}{T(1;1)} A(1;1) \\ \hline \frac{R_x(1;1)R_z(1;1)}{T(1;1)} A(1;1) & \frac{R_y(1;1)R_z(1;1)}{T(1;1)} A(1;1) & \frac{L(1;1)}{T(1;1)} + \frac{R_z^2(1;1)}{T(1;1)} A(1;1) \end{array}\right] \tag{A10}$$

Each partitioned element of $[J^*]$ represents a $K \times K$ matrix of identical terms, K being the total number of secondary anchors.

Consider now $[J^*]$. Multiplying the terms in this matrix containing $L(1;1)$ by unity, given in the form

$$1 = \frac{R_x^2(1;1) + R_y^2(1;1) + R_z^2(1;1)}{T^2(1;1)} ,$$

and expanding $A(1;1)$, defined by Eq. A8, as

$$A(1;1) = \frac{1}{T(1;1)} \frac{d[L(1;1)]}{d[T(1;1)]} - \frac{L(1;1)}{T^2(1;1)} ,$$

$[J^*]$ can be written as

$$[J^*] = \frac{L(1;1)}{T^3(1;1)} \left[\begin{array}{c|c|c} R_y^2(1;1) + R_z^2(1;1) & -R_x(1;1)R_y(1;1) & -R_x(1;1)R_z(1;1) \\ \hline -R_x(1;1)R_y(1;1) & R_x^2(1;1) + R_z^2(1;1) & -R_y(1;1)R_z(1;1) \\ \hline -R_x(1;1)R_z(1;1) & -R_y(1;1)R_z(1;1) & R_x^2(1;1) + R_y^2(1;1) \end{array}\right] \tag{A11 (Cont.)}$$

*R.A. Frazer, W.J. Duncan, and A.R. Collar, "Elementary Matrices," New York:MacMillan, 1947.

$$+\frac{1}{T^2(1;1)}\frac{d[L(1;1)]}{d[T(1;1)]}\left[\begin{array}{c|c|c} R_x^2(1;1) & R_x(1;1)R_y(1;1) & R_x(1;1)R_z(1;1) \\ \hline R_x(1;1)R_y(1;1) & R_y^2(1;1) & R_y(1;1)R_z(1;1) \\ \hline R_x(1;1)R_z(1;1) & R_y(1;1)R_z(1;1) & R_z^2(1;1) \end{array}\right] \tag{A11}$$

Defining the square matrix

$$[S] = \left[\begin{array}{c|c|c} 0 & -\dfrac{R_z(1;1)}{K} & \dfrac{R_y(1;1)}{K} \\ \hline \dfrac{R_z(1;1)}{K} & 0 & -\dfrac{R_x(1;1)}{K} \\ \hline -\dfrac{R_y(1;1)}{K} & \dfrac{R_x(1;1)}{K} & 0 \end{array}\right]$$

and the row matrix

$$[R] = \left[\begin{array}{c|c|c} R_x(1;1) & R_y(1;1) & R_z(1;1) \end{array}\right],$$

the expression in Eq. A11 for $[J^*]$ can then be written as

$$[J^*] = \frac{L(1;1)}{T^3(1;1)}\,[S]^T[S] + \frac{1}{T^2(1;1)}\,\frac{d[L(1;1)]}{d[T(1;1)]}\,[R]^T[R]\,. \tag{A12}$$

But the forms $\{D\}^T[S]^T[S]\{D\}$ and $\{D\}^T[R]^T[R]\{D\}$ are necessarily positive definite, and since the coefficients $L(1;1)/T^3(1;1)$ and $[1/T^2(1;1)]\ \{d[L(1;1)]/d[T(1;1)]\}$ are both positive, P^* becomes the sum of two positive definite forms and thus is itself positive definite. Therefore, P is positive definite for an arbitrary simple cable array; consequently, The Method of Imaginary Reactions guarantees convergence to a unique set of equilibrium reactions.

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13. ABSTRACT

A new technique, The Method of Imaginary Reactions, is introduced for determining the static equilibrium configuration of cable arrays. Throughout this technique, a discrete parameter representation of the system is employed. However, it should be noted that the analysis method used depends only on the number of redundant reactions encountered; thus, the system can be represented with arbitrary fineness without increasing the complexity of the problem. An iterative solution is generated for varying the unknown reactions, and convergence to the correct equilibrium configuration of the array is guaranteed. The rapidity of convergence is shown by several numerical examples. The solution for the case of position-dependent applied external forces is indicated.

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