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A Probability Density Function for the Future Position of a Vessel at Sea

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ABSTRACT

A probability density function was derived which can be used to calculate the probability that the future position of a vessel at sea will be in a given region. The density function was derived for the following conditions: at the initial time the actual position of the vessel is a sample from a normal distribution with known mean and covariance matrix; the velocity of the vessel remains constant during the time period of interest; the speed of the vessel is a sample from a chi distribution; and the course of the vessel is a sample from an arbitrary distribution.

PROBLEM STATUS

This is an interim report; work on the problem is continuing.

AUTHORIZATION

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A PROBABILITY DENSITY FUNCTION FOR THE FUTURE POSITION OF A VESSEL AT SEA

INTRODUCTION

Assume that at an initial time t_0 the probability distribution of the position of a vessel on the surface of the ocean is known, as well as the distribution which governs the vessel's velocity. A natural question to ask is what can be said about the position of the vessel at a later time t_1, where $t_1 = t_0 + \tau$ (Fig. 1). Under the assumption that the random variables being considered are independent the problem of calculating the expected position of the vessel at time t_1 is not difficult. But the problem of deriving a probability density function which can be used to calculate the probability that the vessel will be in a given region at time t_1 is not trivial in all cases. This is the problem addressed in this report.

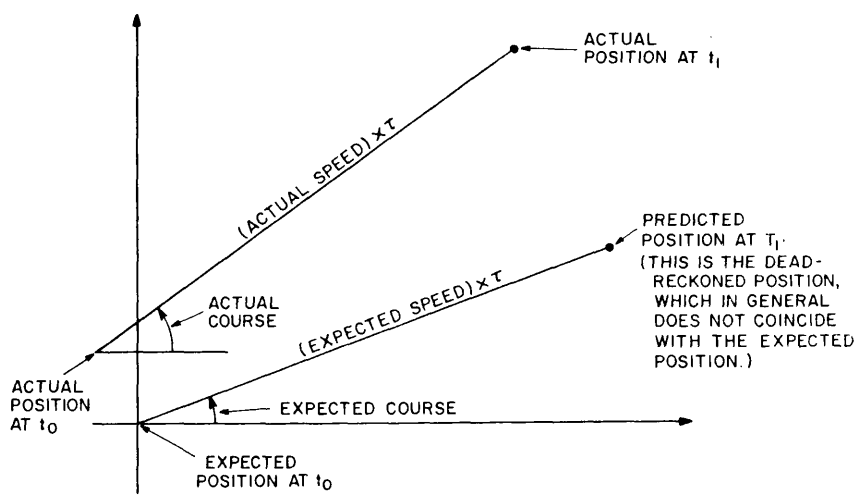

Fig. 1 - The geometry of the problem

The reader should note that there are a number of solutions to this problem; for example see Ref. 1. Since these available solutions use a normal distribution to approximate the distribution of the components of the velocity vector, it was determined that each of these solutions is inappropriate for the particular application for which this work was done.

ASSUMPTIONS

The probability density function which was discussed in the Introduction will be derived under the following assumptions:

1. The earth's surface is approximated by a plane.

2. The vessel maintains a constant velocity from time t_0 to time t_1.

3. The actual position of the vessel at time t_0 is a sample from a random vector $(\mathbf{X}_0, \mathbf{Y}_0)$ (random variables will be boldfaced and capitalized) where

$$(\mathbf{X}_0, \mathbf{Y}_0)^T \sim N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \Sigma\right],$$

and

$$\Sigma = \begin{pmatrix} \sigma_{\mathbf{X}}^2 & 0 \\ 0 & \sigma_{\mathbf{Y}}^2 \end{pmatrix}.$$

Note that any covariance matrix can be transformed into a diagonal matrix by a proper rotational transformation.

4. Let the actual course be a sample from the random variable $\mathbf{A}$, and let $f_{\mathbf{A}}$, the probability density function of $\mathbf{A}$, be such that

$$\int_{\pi - v}^{\pi + v} f_{\mathbf{A}}(a)\, da = 1$$

for some constant v.

5. Let the speed be a sample from the random variable $\mathbf{S}$ which is distributed according to the chi distribution* (see Ref. 2, page 181) which has the following density function:

$$f_{\mathbf{S}}(s) = \begin{cases} \dfrac{2(n/2)^{n/2}\, s^{n-1}}{\Gamma(n/2)\, \sigma^n} \exp\left(-ns^2/2\sigma^2\right), & s > 0, \\ 0, & \text{otherwise}, \end{cases}$$

where $n > 1$.

6. All of the random variables under consideration are independent.

Hereafter, the random variables $(\mathbf{X}_0, \mathbf{Y}_0)$, $\mathbf{A}$, and $\mathbf{S}$ will be referred to as the position, course, and speed respectively, but the reader should bear in mind that in reality the position, course, and speed are samples of these random variables.

DERIVATION

For mathematical convenience the transformation

*The chi distribution was chosen because it is intuitively appealing for approximating the distribution of a population of ship speeds and its functional form is convenient for this problem.

$$T = \begin{pmatrix} \sigma_X^{-1} & 0 \\ 0 & \sigma_Y^{-1} \end{pmatrix}$$

is applied to the reference coordinate system, and the random variables with respect to the transformed coordinate system are denoted by primes, as for example,

$$(\mathbf{X}_0', \mathbf{Y}_0')^T = T \cdot (\mathbf{X}_0, \mathbf{Y}_0)^T .$$

First observe that under this transformation the actual position of the vessel at time t_0 is such that

$$(\mathbf{X}_0', \mathbf{Y}_0')^T \sim N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad I\right] .$$

Although $\mathbf{A}$ and $\mathbf{S}$, the actual course and speed, are independent, it will now be shown that $\mathbf{A}$ and $\mathbf{S}'$ are dependent in the general case when $\sigma_X \neq \sigma_Y$. It is easily seen that for a given value of $\mathbf{A}$, say $\mathbf{A} = a$,

$$\begin{aligned} \mathbf{S}' &= \left| [T \cdot (\mathbf{S} \cos a, \ \mathbf{S} \sin a)^T]^T \right| \\ &= \mathbf{S} \cdot (\sigma_X^{-2} \cos^2 a + \sigma_Y^{-2} \sin^2 a)^{1/2} \\ &= \mathbf{S} q , \end{aligned}$$

where

$$q = q(a) = (\sigma_X^{-2} \cos^2 a + \sigma_Y^{-2} \sin^2 a)^{1/2} .$$

The conditional probability density function of $\mathbf{S}'$ given $\mathbf{A} = a$ is

$$f_{\mathbf{S}'|\mathbf{A}}(s|\mathbf{A} = a) = \frac{2(n/2)^{n/2} s^{n-1}}{\Gamma(n/2)\, \sigma^n q^n} \exp\left[-n s^2/(2\sigma^2 q^2)\right]$$

for s positive. In other words $\mathbf{S}'$ is distributed according to a chi distribution with parameters n and σq. Since q is a function of a, then $\mathbf{A}$ and $\mathbf{S}'$ are dependent as long as $\sigma_X \neq \sigma_Y$ and independent if $\sigma_X = \sigma_Y$.

Now define the random variable $\mathbf{W}$ to be the distance traveled by the vessel during the time interval (t_0, t_1). Thus,

$$\mathbf{W} = \tau \mathbf{S}$$

and

$$\mathbf{W}' = \tau \mathbf{S}' ,$$

$\mathbf{W}'$ being conditionally distributed given $\mathbf{A} = a$ according to the chi distribution with parameters n and $\sigma \tau q$.

If $(\mathbf{X}_1, \mathbf{Y}_1)$ is the random vector denoting the position of the vessel at time t_1, then

$$(\mathbf{X}_1, \mathbf{Y}_1) = (\mathbf{X}_0 + \mathbf{W} \cos \mathrm{A},\ \mathbf{Y}_0 + \mathbf{W} \sin \mathbf{A})$$

and

$$(\mathbf{X}_1', \mathbf{Y}_1') = (\mathbf{X}_0' + \mathbf{W}' \cos \mathbf{A}',\ \mathbf{Y}_0' + \mathbf{W}' \sin \mathbf{A}') \ .$$

Given $\mathbf{A}' = a$ and $\mathbf{W}' = w$ the transpose of $(\mathbf{X}_1', \mathbf{Y}_1')$ has the conditional distribution

$$N\left[\begin{pmatrix} w \cos a \\ w \sin a \end{pmatrix},\ I\right]$$

and

$$\left| E\left[(\mathbf{X}_1', \mathbf{Y}_1') \mid \mathbf{A}' = a \ \text{ and } \ \mathbf{W}' = w\right]\right| = w \ .$$

Next define the random variables

$$\mathbf{R} = \left|(\mathbf{X}_1, \mathbf{Y}_1)\right|$$

and

$$\mathbf{R}' = \left|(\mathbf{X}_1', \mathbf{Y}_1')\right| \ .$$

A special case of a well-known result (see Ref. 3, page 28) is that when $\mathbf{W}' = w$ is given, $\mathbf{R}'$ has the conditional probability density function

$$f_{\mathbf{R}'|\mathbf{W}'}(r|\mathbf{W}' = w) = r \exp\left[-(r^2 + w^2)/2\right] I_0(rw)$$

for positive r, where I_0 is the modified Bessel function of order zero. Observe that this density is independent of $\mathbf{A}'$.

A standard integral formula from the theory of conditional probability yields

$$f_{\mathbf{R}'|\mathbf{A}}(r|\mathbf{A} = a) = \int_{-\infty}^{\infty} f_{\mathbf{R}'|\mathbf{W}'}(r|\mathbf{W}' = w)\ f_{\mathbf{W}'|\mathbf{A}}(w|\mathbf{A} = a)\ dw$$

$$= \frac{2(n/2)^{n/2}\, r \exp(-r^2/2)}{\Gamma(n/2)(\sigma\tau q)^n} \int_0^{\infty} w^{n-1} \exp\left\{-\frac{w^2}{2}\left[1 + \frac{n}{(\sigma\tau q)^2}\right]\right\} I_0(rw)\ dw \ . \tag{1}$$

From Ref. 4, page 198, it is known that

$$\int_0^{\infty} \exp(-c^2x^2)\ x^{\mu-1} J_\nu(bx)\ dx = \frac{b^\nu \Gamma\left(\frac{\mu+\nu}{2}\right)}{2^{\nu+1} c^{\mu+\nu} \Gamma(\nu+1)}\ F\left(\frac{\mu+\nu}{2}\ ;\ \nu+1\ ;-\frac{b^2}{4c^2}\right), \tag{2}$$

where

$$F(a_1;\ a_2;\ a_3) = \sum_{k=0}^{\infty} \frac{\Gamma(a_1+k)\,\Gamma(a_2)}{\Gamma(a_1)\,\Gamma(a_2+k)} \frac{a_3^{\,k}}{k!}$$

and the real parts of $\mu + \nu$ and c^2 are required to be positive. To integrate Eq. (1) recall that

$$I_0(x) = J_0(ix) \ ,$$

and in Eq. (2) set

$$c^2 = \left[1 + \frac{n}{(\sigma\tau q)^2}\right]\Big/2 \ ,$$

$$\mu = n \ ,$$

$$\nu = 0 \ ,$$

and

$$b = ir \ .$$

The substitutions yield

$$f_{\mathbf{R}'|\mathbf{A}}(r|\mathbf{A}=a) = \frac{2\,(n/2)^{n/2}\, r \exp(-r^2/2)}{[(\sigma\tau q)^2 + n]^n} \frac{\Gamma(n/2)\ 2^{n/2}}{2c^n} F(n/2;\ 1;\ r^2/4c^2)$$

$$= \frac{(n)^{n/2}\, r\, e\,(-r^2/2)}{[(\sigma\tau q)^2 + n]^{n/2}} F(n/2;\ 1;\ r^2/4c^2)$$

$$= \left[\frac{n}{(\sigma\tau q)^2 + n}\right]^{n/2} r \exp(-r^2/2) \left\{1 + \sum_{k=1}^{\infty} \frac{n(n+2)\ldots(n+2k-2)}{(2\cdot k!)^2} \left[\frac{(r\sigma\tau q)^2}{(\sigma\tau q)^2 + n}\right]^k\right\} \tag{3}$$

for r positive.

Since for a given value of $\mathbf{A}$, say $\mathbf{A} = a$,

$$\mathbf{R}' = q\mathbf{R} \ ,$$

a change of variables in Eq. (3) yields

$$f_{\mathbf{R}|\mathbf{A}}(r|\mathbf{A}=a) = f_{\mathbf{R}'|\mathbf{A}}(rq|\mathbf{A}=a)\ q \ . \tag{4}$$

It is interesting to note that if $n = 2$, then Eq. (4) reduces to a chi density function given by

$$f_{\mathbf{R}|\mathbf{A}}(r|\mathbf{A}=a) = \frac{2q^2 r}{(\sigma\tau q)^2 + 2} \exp\{-r^2 q^2/[(\sigma\tau q)^2 + 2]\}$$

for r positive.

Recall that the density function of the actual course $\mathbf{A}$ is $f_{\mathbf{A}}$, and it follows that the joint density function of $\mathbf{R}$ and $\mathbf{A}$ is

$$f_{\mathbf{R},\mathbf{A}}(r,a) = f_{\mathbf{R}|\mathbf{A}}(r|\mathbf{A}=a)\ f_{\mathbf{A}}(a)\ , \tag{5}$$

where $f_{\mathbf{R}|\mathbf{A}}$ is given by Eq. (4).

The density function given by Eq. (5) is not the desired result, since $\mathbf{A}$ is the course as measured from the point $(\mathbf{X}_0, \mathbf{Y}_0)$, not (0,0). If Θ is the random variable denoting the angle corresponding to the point $(\mathbf{X}_1, \mathbf{Y}_1)$ as in Fig. 2, then a simple approximation can be made, namely,

$$f_{\mathbf{R},\Theta}(r,\theta) \approx f_{\mathbf{R},\mathbf{A}}(r,\theta)\ , \tag{6}$$

where $f_{\mathbf{R},\mathbf{A}}$ is given by Eq. (5).

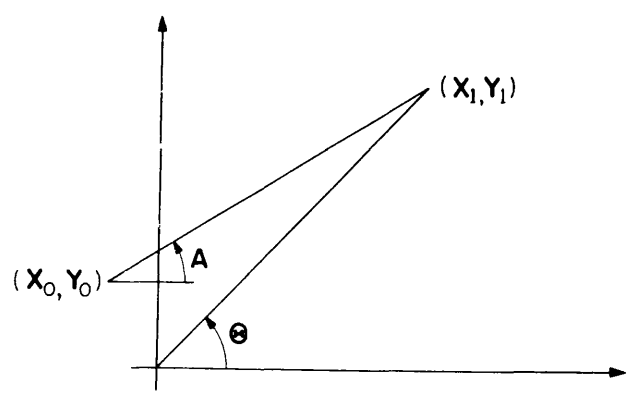

Fig. 2 - The relationship between A and Θ

Simulation has shown that if the expected distance traveled, $\tau \cdot E(\mathrm{S})$, is at least several times as large as the magnitude of the initial position error, then the probability calculations are relatively insensitive to the approximation made in Eq. (6).

In any case, consider a region Q in the plane. The probability that the vessel will be in Q at time t_1 is

$$\iint_Q f_{\mathbf{R},\Theta}(r,\theta)\ dr\ d\theta\ ,$$

and the integrand is given approximately by Eq. (6)

CONDITIONS FOR APPLICATION

The developed probability density function should be used when the variance of S, the speed, is not large (e.g., var (S) = 2 knots squared) and when the variance of $\mathbf{A}$, the course, is large (e.g., var ($\mathbf{A}$) = $\pi/4$ radians squared). Under these conditions a 90% probability region for the velocity vector (S, $\mathbf{A}$) might have the general appearance of Fig. 3. It is understood that the assumptions and conditions as stated in previous sections also apply.

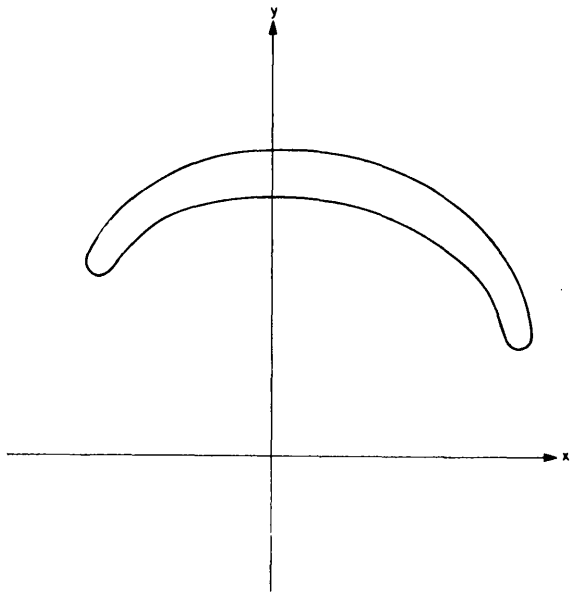

Fig. 3 - A possible 90% contour for the velocity vector (S, A) when var(S) is small and var(A) is large

If var (A) is small, there is an attractive alternative to the density function derived in this report. In this case the distribution of (S, A) can be approximated reasonably well by a bivariate normal distribution in $(\dot{X}, \dot{Y})$ where

$$\dot{X} = \frac{dX}{dt} = S \cos A$$

and

$$\dot{Y} = \frac{dY}{dt} = S \sin A \ .$$

The approximate distribution of $(\dot{X}, \dot{Y})$ is

$$N\left[\begin{pmatrix} \mu_{\dot{X}} \\ \mu_{\dot{Y}} \end{pmatrix}, \quad \Sigma_0\right],$$

where

$$\mu_{\dot{X}} = E(S)\, E(\cos A) \ ,$$

$$\mu_{\dot{Y}} = E(S)\, E(\sin A) \ ,$$

and

$$\Sigma_0 = \begin{pmatrix} \sigma_{\dot{X}}^2 & \sigma_{\dot{X}\dot{Y}} \\ \sigma_{\dot{X}Y} & \sigma_{\dot{Y}}^2 \end{pmatrix},$$

with

$$\sigma_{\dot{X}}^2 = E(S^2)\, E(\cos^2 A) - \mu_{\dot{X}}^2 ,$$

$$\sigma_{\dot{X}\dot{Y}} = E(S^2)\, E(\sin A \cos A) - \mu_{\dot{X}}\mu_{\dot{Y}} ,$$

and

$$\sigma_{\dot{Y}}^2 = E(S^2)\, E(\sin^2 A) - \mu_{\dot{Y}}^2 .$$

The distribution of (X_1, Y_1) is approximately

$$N\left[\tau\begin{pmatrix}\mu_{\dot{X}} \\ \mu_{\dot{Y}}\end{pmatrix}, \quad \Sigma + \tau^2\, \Sigma_0\right] .$$

The reader is advised to exercise caution in applying the normal approximation, for if the appropriate conditions are not satisfied, this approximation can lead to erroneous results.

EXAMPLE

To illustrate the density function developed in this report the following values of the input parameters were used:

$$\tau = 4 \text{ hours} ,$$

$$\sigma_X^2 = 3 \text{ nautical miles squared} ,$$

$$\sigma_Y^2 = 5 \text{ nautical miles squared} ,$$

$$n = 24 ,$$

and

$$\sigma = 12 .$$

When the last two parameter values are substituted into the chi density function for the speed of the vessel, a simple calculation yields

$$E(S) \approx 11.88 \text{ knots}$$

and

$$var\,(S) \approx 2.97 \text{ knots squared} .$$

For the distribution of the course of the vessel a truncated normal distribution with mean $\pi/4$ and standard deviation $\pi/6$ was chosen.

Figure 4 represents the density function given by Eq. (5) when the approximation in Eq. (6) is used and the above substitutions have been made.

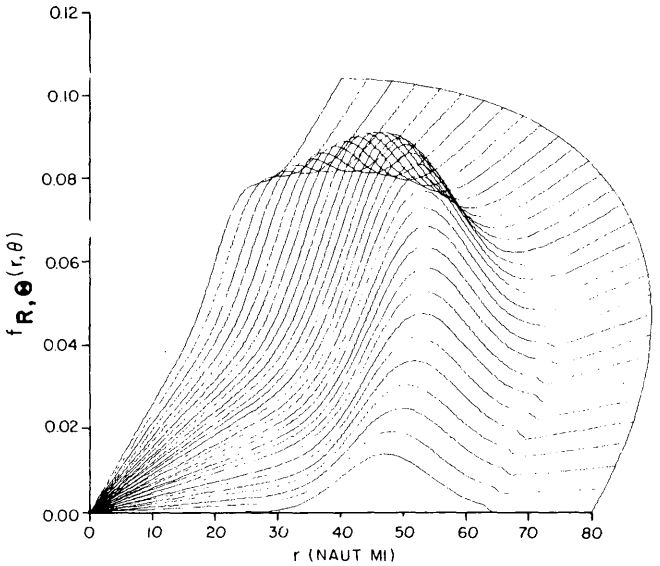

Fig. 4 - The graph of $f_{R,\Theta}$ when the approximation in Eq. (6) is used with the illustrative parameter values listed in the text

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13. ABSTRACT

A probability density function was derived which can be used to calculate the probability that the future position of a vessel at sea will be in a given region. The density function was derived for the following conditions: at the initial time the actual position of the vessel is a sample from a normal distribution with known mean and covariance matrix; the velocity of the vessel remains constant during the time period of interest; the speed of the vessel is a sample from a chi distribution; and the course of the vessel is a sample from an arbitrary distribution.

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