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Probability Density Functions of n Vectors Added With Random Directions

EDWARD L. WRIGHT

Propagation Branch
Acoustics Division

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NAVAL RESEARCH LABORATORY
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CONTENTS

Abstract	ii
Problem Status	ii
Authorization	ii
INTRODUCTION	1
HISTORY	1
METHOD	2
NUMERICAL CALCULATION USING A DIGITAL COMPUTER	4
FEATURES OF THE PROBABILITY DENSITY DISTRIBUTIONS	5
CONCLUSION	9
ACKNOWLEDGMENTS	9
REFERENCES	10

ABSTRACT

The probability density function of the resultant amplitude of n vectors in two dimensions, added with random directions, was calculated through an integral transformation which takes a probability density function for k components into a probability density function for $k + 1$ components. Examples were calculated to show the qualitative aspects of the density functions for 2, 3, and 4 components, with arbitrarily given magnitudes. For $n \geq 5$, the probability distribution rapidly approaches the Rayleigh distribution. The method also allows the use of a Rayleigh vector as one of the components.

PROBLEM STATUS

This is an interim report on the problem; work is continuing.

AUTHORIZATION

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PROBABILITY DENSITY FUNCTIONS OF n VECTORS ADDED WITH RANDOM DIRECTIONS

INTRODUCTION

When n vectors in two dimensions with amplitudes $A_1, A_2, \ldots A_n$ are added with random directions, the probability density of the resultant amplitude is the same as that for the resultant distance r_n when n lengths $A_1, A_2, \ldots A_n$ are joined with the angles $\theta_1, \theta_2, \ldots \theta_n$ taken from a uniform random distribution on $[0,2\pi]$ as illustrated in Fig. 1. This is a problem of the "random-walk" type and is the same as the resultant amplitude of the superposition of a number of sine waves with random phases. It is the concern of this report to determine the probability density for arbitrary sets $(A_1, A_2, A_3, \ldots A_n)$ and illustrate the form of the density function.

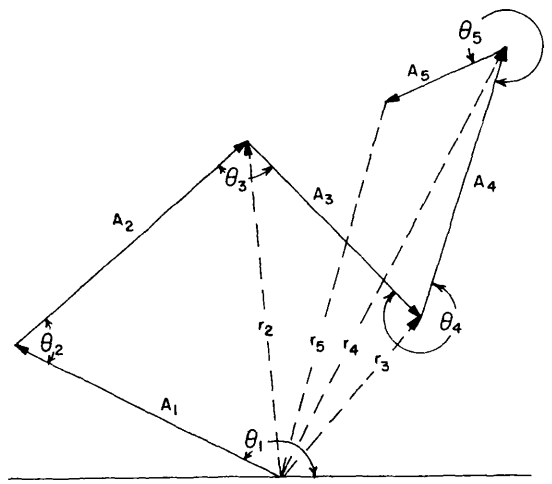

Fig. 1 - An example of the random phase addition of five vectors

HISTORY

The general solution of the probabilities associated with this problem was given by Kluyver (1) as

$$p_n(r_n) = r_n \int_0^\infty u J_0(ur_n) \prod_{i=1}^{n} J_0(uA_i)\, du \,, \tag{1}$$

where $p_n(r_n)$ is the probability density function that the resultant of n vectors or components $A_1, A_2, A_3, \ldots A_n$ has the length r_n and J_0 is a Bessel function of the first kind. Equation 1 is readily integrated when $n = 1$ or $n = 2$. When n approaches infinity, Eq. 1 is the Rayleigh (2) density function; i.e., when $n \to \infty$,

$$p_n(r_n) \to \frac{2r_n}{A^2} e^{-r_n^2/A^2} \tag{2}$$

where

$$A^2 = \sum_{i=1}^{n} A_i^2$$

Pearson (3) has provided a series expansion of Eq. 1 in terms of powers of $1/n$ that requires a large number of terms for small values of n. For $n > 6$ this expansion gives a good approximation with a reasonable number of terms. Beckmann (4) has provided a similar expansion. Bennett (5) solved Kluyver's equation in terms of an infinite series, which is most applicable in the region of intermediate and large values of n.

Equation 1 can also be expressed in terms of complete elliptical functions for the case $n = 3$, when the magnitudes of the vectors are equal. This was reported by Pearson (3), who obtained this solution from Geoffery T. Bennett by private communication. Bennett's equation for three equal components is

$$p_3(r_3) = \frac{\sqrt{r_3}}{\pi^2 A^{3/2}} \alpha K(\alpha) , \quad 0 < r_3 < A ,$$

$$p_3(r_3) = \frac{\sqrt{r_3}}{\pi^2 A^{3/2}} K\left(\frac{1}{\alpha}\right) , \quad A < r_3 < 3A ,$$

and

$$p_3(r_3) = 0 , \qquad r_3 > 3A ,$$

where A = magnitude of contributing vectors, $\alpha^2 = 16A^3 r_3/[(r_3 + A)^3(3A - r_3)]$, and $K(\alpha)$ denotes the complete elliptic integral of the first kind with modulus α.

Using a graphical method Pearson (3)* obtained the density functions for a small number of vectors, between $n = 2$ and $n = 7$, for the special case where the magnitudes of the vectors are equal.

This report takes an approach similar to Pearson and extends the treatment to include arbitrarily given magnitudes of the vectors. It also allows one of the components to be a Rayleigh vector, i.e., a noise component.

METHOD

Consider a vector in a plane whose probability density of having length r_k is $p_k(r_k)$ and whose direction Φ_k has a probability density uniform on $[0,2\pi]$, independent of r_k. Then the joint probability density $G(\mathbf{r}_k)$ that the vector will have a length r_k and a direction Φ_k is obtained from the joint probability function $p_k(r_k,\phi_k) = P_k(r_k)/2\pi$, as follows,

$$G(\mathbf{r}_k)\, da \equiv p_k(r_k,\phi_k)\, dr_k d\phi_k = \frac{1}{2\pi} p_k(r_k)\, dr_k d\phi_k = \frac{1}{2\pi} \frac{p_k(r_k)}{r_k}\, da . \tag{3}$$

*Pearson's results are also presented by Sleck in Ref. 6.

If one now adds a vector $\mathbf{A}_{k+1}$ to $\mathbf{r}_k$ (for convenience add them as shown in Fig. 2), the length of the resultant r_{k+1} has a probability density calculable from p_k and A_{k+1}.

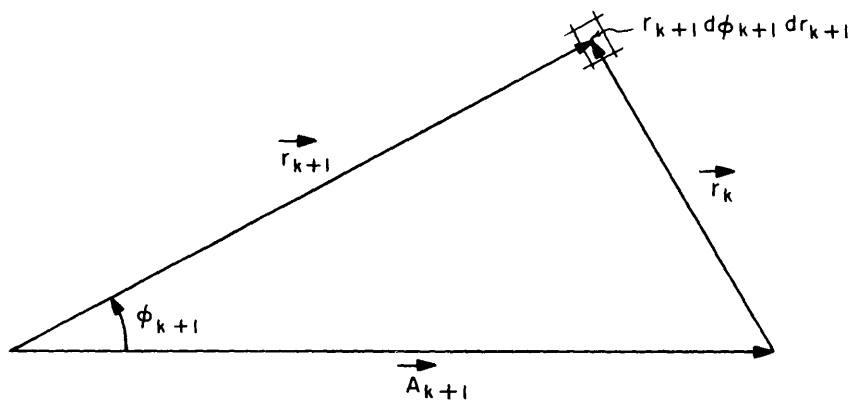

Fig. 2 - The geometry illustrating the method of vector addition

The probability that $r_{k+1} \leq c$, c being an arbitrary constant, is*

$$\int_0^c \int_0^{2\pi} G(\mathbf{r}_k)(r_{k+1}\, d\Phi_{k+1}\, dr_{k+1}) = \int_0^c p_{k+1}(r_{k+1})\, dr_{k+1} \tag{4}$$

This equation is an identity from which one obtains

$$p_{k+1}(r_{k+1}) = r_{k+1} \int_0^{2\pi} G(\mathbf{r}_k)\, d\Phi_{k+1} \ . \tag{5}$$

Since G is an even function of Φ, Eq. 5 becomes

$$p_{k+1}(r_{k+1}) = 2r_{k+1} \int_0^{\pi} G(\mathbf{r}_k)\, d\Phi_{k+1} \ . \tag{6}$$

Substituting the value of G from Eq. 3 into Eq. 6 one obtains the recursion formula

$$p_{k+1}(r_{k+1}) = \frac{r_{k+1}}{\pi} \int_0^{\pi} \frac{p_k(r_k)}{r_k}\, d\Phi_{k+1} \ . \tag{7}$$

*It can be shown that Eq. 4 is also equal to

$$\int_0^c \int_0^{2\pi} G(\mathbf{r}_{k+1})(r_{k+1}\, d\phi_{k+1}\, dr_{k+1}) \ ;$$

however, this does not imply equality of integrands.

The variable Φ in Eq. 7 may be expressed in terms of the variable r. By using the law of cosines, as illustrated in Fig. 2,

$$\cos \Phi_{k+1} = (A_{k+1}^2 + r_{k+1}^2 - r_k^2)/2A_{k+1} r_{k+1} \tag{8}$$

and the derivative

$$d\Phi_{k+1} = \frac{r_k \, dr_k}{A_{k+1} \, r_{k+1} \sin \Phi_{k+1}} , \tag{9}$$

the following equation is obtained,

$$p_{k+1}(r_{k+1}) = \frac{1}{\pi A_{k+1}} \int_{|A_{k+1} - r_{k+1}|}^{(A_{k+1} + r_{k+1})} \frac{p_k(r_k) \, dr_k}{\sin \Phi_{k+1}} , \tag{10}$$

where

$$\sin \Phi_{k+1} = \left[1 - (A_{k+1}^2 + r_{k+1}^2 - r_k^2)^2/4A_{k+1}^2 r_{k+1}^2\right]^{1/2} \tag{11}$$

by Eq. 8.

NUMERICAL CALCULATION USING A DIGITAL COMPUTER

If one lets $H_k(r_k) = p_k(r_k)/r_k$, Eq. 7 becomes

$$H_{k+1}(r_{k+1}) = \frac{1}{\pi} \int_0^{\pi} H_k(r_k) \, d\Phi_{k+1} \; ; \tag{12}$$

an evaluation is obtained by the finite sum

$$H_{k+1}(r_{k+1}) = \frac{1}{\pi} \sum_{r_k} \left[H_k(r_k) \, \Delta\Phi_{k+1}\right] .$$

For example, in this evaluation H_k is tabulated for 100 values and is assumed to be constant over intervals centered on the tabular values. The term $\Delta\Phi_{k+1}$ is found from Eq. 8 with an arc cosine subroutine.

The initial H is a rectangular approximation to a delta function, being zero for all values but one. Alternately a Gaussian noise component $H_0 = ce^{-ar^2}$ can be used as the starting distribution.

Using the incrementation of 100 values of H_k with the selected $\Delta\Phi$ indicated above, the transformation of a one-component distribution to a two-component distribution introduces an error of approximately 2% in the total probability. Further transformations to three or more component distributions do not add to this error. Furthermore, the distribution for two components plus noise has a very small error. This error is attributed to numerical difficulties in computing a function near a singularity. A comparison of the calculated density function with the known density function for two components

shows that points between these singularities (which are poles) have errors on the order of 0.1%. The above errors can be reduced by using finer incrementation. From the relation $p_n = r_n H_n$, one gets $p_n(0) = 0$, although $p_2(0) = 0$ is an error when $A_1 = A_2$ (Fig. 3a).*

FEATURES OF THE PROBABILITY DENSITY DISTRIBUTIONS

Equation 10 has been evaluated for distinct values of n. The relationships for $p_n(r_n)$ are listed below for these values of n and sketched in Figs. 3 through 6.

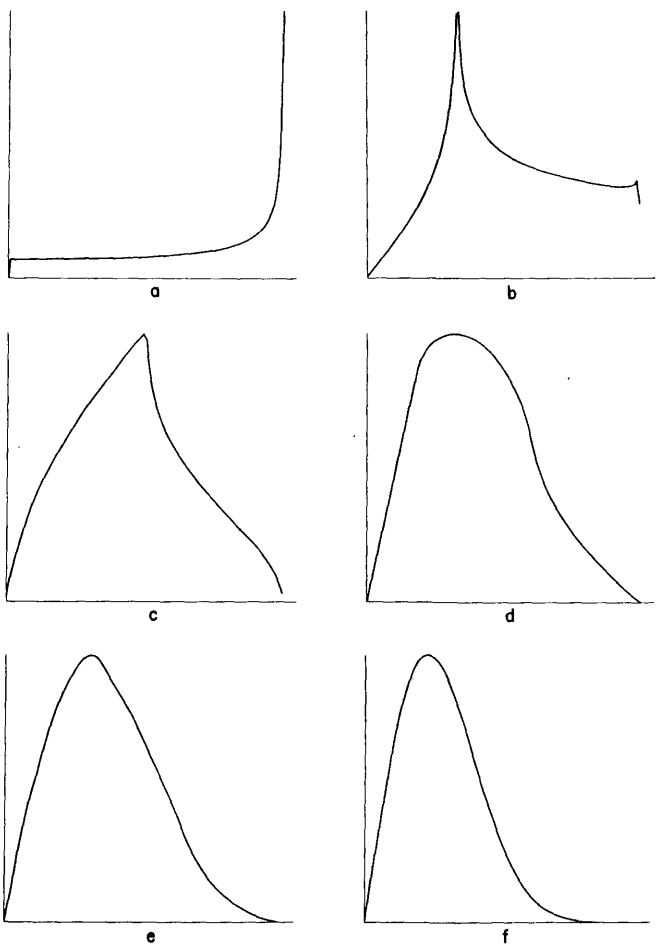

Fig. 3 - Probability density functions with 2, 3, 4, 5, 6, and 10 equal components, respectively

*The individual graphs of Fig. 3 through Fig. 6 are plotted with the probability density $p_n(r_n)$ as the ordinate. The abscissa is the amplitude with the maximum value equal to the sum of the amplitude of the components.

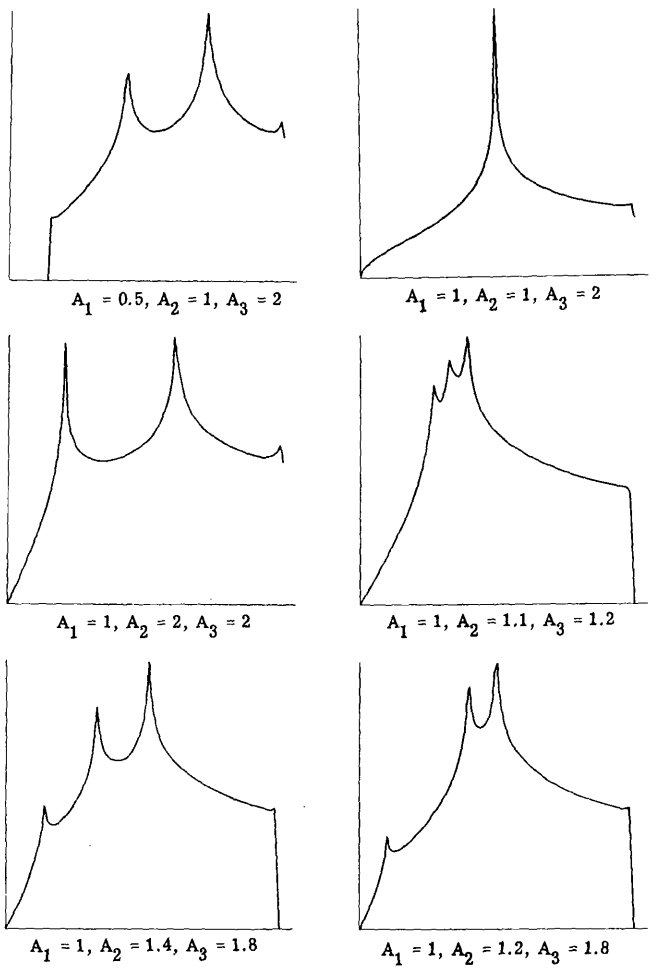

Fig. 4 - Probability density functions with three components

For $n = 1$,

$$p_1(r_1) = \delta(r_1 - A_1) \ . \tag{13}$$

For $n = 2$ (Fig. 3a),

$$p_2(r_2) = \frac{1}{\pi A_2} \int_{|A_2 - r_2|}^{A_2 + r_2} \frac{p_1(r_1)\, dr_1}{\sin \Phi_2} = \frac{1}{\pi A_2 \sin \Phi_2} = \frac{2r_2}{\pi} \left[\frac{1}{\sqrt{(A_1 + A_2)^2 - r_2^2} \cdot \sqrt{r_2^2 - (A_1 - A_2)^2}} \right], \tag{14}$$

when $|A_1 - A_2| < r_2 < (A_1 + A_2)$. Otherwise, $p_2(r_2) = 0$.

If $A_1 \neq A_2$, there are two poles of p_2, each of order $1/2$. If $A_1 = A_2$, there is one pole and a nonzero probability density at zero amplitude.

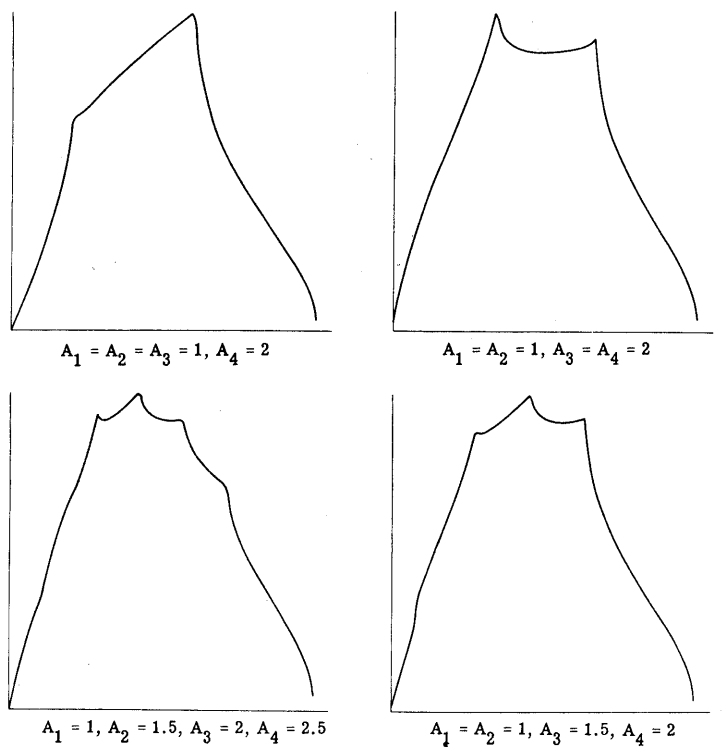

Fig. 5 - Probability density functions with four components

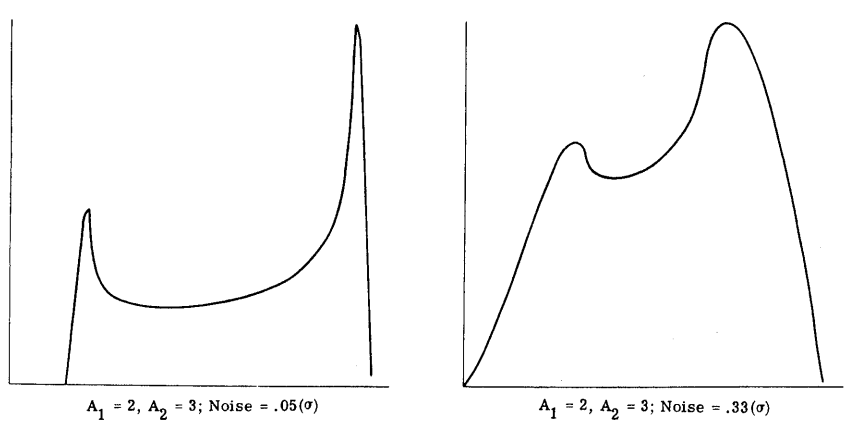

Fig. 6 - Probability density functions with two components plus noise

For $n = 3$ (Fig. 3b and Figs. 4a-f),

$$p_3(r_3) = \frac{1}{\pi A_3} \int_{|A_3 - r_3|}^{A_3 + r_3} \frac{p_2(r_2)}{\sin \Phi_3} dr_2 \ . \tag{15}$$

One can substitute p_2 and $\sin \Phi_3$ in terms of A_1, A_2, A_3, r_2, and r_3 into Eq. 15, but in general the resulting integral is intractable in closed form, except for the special case $A_1 = A_2 = A_3$ (see page 2).

Inspection of the numerical results shows that $p_3(r_3)$ has peaks at

$$R_1 = A_1 + A_2 - A_3 \ ,$$

$$R_2 = A_1 + A_3 - A_2 \ ,$$

and

$$R_3 = A_2 + A_3 - A_1 \ .$$

Each of these peaks is a singularity.

If one of the values R_i is zero,

$$p_3(r_3) \approx \sqrt{r_3} \quad \text{for} \quad r_3 \approx 0 \ .$$

If $R_i < 0$,

$$p_3(r_3) = 0 \quad \text{for} \quad 0 \le r_3 \le |R_i| \ .$$

If all $R_i > 0$,

$$p_3(r_3) \approx r_3 \quad \text{for} \quad r_3 \approx 0 \ .$$

In general the magnitudes of the peaks are in the order of increasing amplitudes. Thus, $p_3(R_1) < p_3(R_2)$, if $R_1 < R_2$. However, if $R_1 = R_2 < R_3$, the coincident peaks at $R_1 = R_2$ may be larger than the single peak at R_3.

For $n = 4$ (Fig. 3c and Figs. 5a-d), if A_1, A_2, A_3, and A_4 are all distinct, one obtains discontinuities in the first derivative of $p_4(r_4)$ at

$$r_4 = A_1 + A_2 + A_3 - A_4 \ ,$$

$$r_4 = A_1 + A_2 + A_4 - A_3 \ ,$$

$$r_4 = A_1 + A_3 + A_4 - A_2 \ ,$$

and

$$r_4 = A_2 + A_3 = A_4 - A_1 \ ,$$

provided these four values are positive. Of these points, one, or occasionally two, is a local maxima of p_4.

When $A_1 = A_2 = A_3 = A$, the behavior of p_4 is dominated by the pole in p_3 at A. Between $|A_4 - A|$ and $A_4 + A$, p_4 is almost linear. If $A_4 = A$ also (Fig. 3c), p_4 resembles a wave about to break, with its peak at $2A$.

For $n = 5$ and 6 (Figs. 3d and 3e), the amount of structure on the curves for p_5 and p_6 is very small.

For large values of n

$$\lim_{n\to\infty} \quad p_n(r_n) = \frac{2r}{n} e^{-r^2/n} , \tag{16}$$

provided

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} A_i^2 = 1 .$$

For intermediate values of n, Beckmann (4) obtained

$$p_n(r_n) \approx \frac{2r}{n} e^{-r^2/n} \left[1 - \frac{3}{8n}\left(\frac{r^4}{2n^2} - \frac{2r^2}{n} + 1\right)\right] ,$$

where $A_1 = A_2 = \cdots = A_n = 1$.

The maximum difference between p_n obtained by Beckmann and a Rayleigh distribution is then

$$|\Delta p|_{max} \approx 0.14n^{-3/2} .$$

To distinguish a one-standard deviation difference in the above distribution from a Rayleigh distribution in a counting experiment with n equal contributing components, one would need $N > 2000n^3$ counts. Thus, for $n \geq 6$, p_n is for practical purposes a Rayleigh distribution.

For $n = 2$ + noise (Figs. 6a and 6b), it is noted that the poles of Eq. 14 are eliminated and that the peaks are broadened with increasing noise.

CONCLUSION

A numerical method is given for obtaining the probability density function of the resultant of a number of vectors of arbitrary magnitude added with random directions. The method has potential use in many fields, particularly in a number of acoustic and electromagnetic problems where the field statistics of a small number of sources is desired.

ACKNOWLEDGMENTS

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13. ABSTRACT

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DD FORM 1 NOV 65 1473 (BACK)
(PAGE 2)

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