NRL Report 6943

Scattering by Spherically Symmetric Inhomogeneities

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October 7, 1969

NAVAL RESEARCH LABORATORY
Washington, D.C.

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This report is a facsimile of a thesis submitted to Brown University in partial fulfillment for the degree of Master of Science, conferred in June 1969.

TABLE OF CONTENTS

Section		Page
1.	Introduction	1
2.	The Method of Partial Waves	8
3.	Scattering from a Constant Spherical Inhomogeneity: The Method of Partial Waves	12
4.	Scattering from Two Constant Spherical Inhomogeneous Layers: The Method of Partial Waves	15
5.	The Jost Function Formulation of Scattering Theory	16
6.	Scattering from a Constant Spherical Inhomogeneity: The Method of Jost Functions (Differential Equation Approach)	23
7.	Scattering from Two Constant Spherical Inhomogeneous Layers: The Method of Jost Functions (Differential Equation Approach)	25
8.	The Jost Integral Equation for $\lambda=1/2$ and Some Approximate Solutions	28
9.	General Jost Integral Equation Formulation for Arbitrary λ for Scattering from an Arbitrary Inhomogeneity	33
10.	Conclusion	35
	Acknowledgments	37
	Appendix A - Scattering from a Constant Spherical Inhomogeneity: The Jost Function for $\lambda=1/2$ (Differential Equation Approach)	38
	Appendix B - Scattering from Two Constant Spherical Inhomogeneous Layers: The Jost Function for $\lambda=1/2$ (Differential Equation Approach)	39
	Appendix C - Scattering from a Constant Spherical Inhomogeneity: A Laplace Transform Method for Solving the Jost Integral Equation for $\lambda=1/2$	40
	Figures	42
	Bibliography	50

ABSTRACT

The Jost function formulation of quantum scattering theory is applied to classical problems involving the scattering of a scalar plane wave by a medium in which the velocity is a function only of the spherical radial coordinate. This technique is used to solve the radial differential equation for scattering from a constant spherical inhomogeneity and from two constant concentric spherical inhomogeneous layers and is compared with the standard method of partial waves for these two cases. The radial equation can be converted into an integral equation incorporating the Jost boundary conditions. The $\ell=0$ partial wave integral equation for a constant inhomogeneity is solved using an iteration procedure (the first two iterations are considered). The Jost function and $\ell=0$ cross section σ_o are plotted as a function of kR_1, where k is the wave number in the surrounding medium and R_1 is the sphere radius. The iterative technique is good for long wavelengths ($kR_1 \ll 1$) and any ratio of wave numbers in the scattering and surrounding media. For shorter wavelengths and small ratio of wave numbers (e.g., $k_1/k = 1.1$, where k_1 is the wave number in the scattering medium), it gives a good approximation to σ_o for the entire range of kR_1 considered ($0 \leq kR_1 \leq 2\pi$). For shorter wavelengths and larger ratio of wave numbers (e.g., $k_1/k = 1.5, 2.0$), it gives a good approximation to σ_o out to approximately $kR_1 = 3\pi/4$. More general problems using this method are also discussed.

PROBLEM STATUS

This is an interim report on the problem; work is continuing.

AUTHORIZATION

NRL Problem S01-40
Project RF 05-121-401—4070

Manuscript submitted May 27, 1969.

SCATTERING BY SPHERICALLY SYMMETRIC INHOMOGENEITIES

1. Introduction

This paper is concerned with the scattering of a scalar plane wave by an inhomogeneous medium in which the velocity of propagation c is a function only of the spherical radial coordinate r, i.e., $c=c(r)$. This problem is of interest in various branches of physics: scattering by inhomogeneous spheres; scattering of acoustic waves in the ocean, electromagnetic waves in the atmosphere, and seismic waves in the earth. Although quantum-mechanical problems involving scattering by spherically symmetric potentials have been extensively investigated, the analogous classical problems involving scattering by spherically symmetric inhomogeneities have not been as thoroughly studied. It is the purpose of this thesis to show how a convenient quantum-mechanical method can be used to treat the classical problems and to apply this method to some simple solvable problems.

The theory of scattering of plane acoustic waves by spheres was first investigated by Rayleigh[1], who considered the limiting case of spheres which were small compared with the wavelength. Solutions for scattering by rigid, immovable spheres, not necessarily small compared with the wavelength, were obtained by Morse[2]. The scattering by elastic spheres immersed in a fluid has been considered by Faran[3], and the scattering by fluid spheres immersed in a fluid medium has been investigated by Anderson[4]. Frey and Goodman[5] have considered the scattering of acoustic waves and pulses from a sphere that has acoustic properties very similar to those of the surrounding medium. Hart and Montroll[6] have developed

an approximate theory for the scattering of plane waves by spherical obstacles with the property $\lambda_0/\lambda_1<1.5$, where λ_1 and λ_0 are, respectively, the wavelengths of a plane wave in the scattering medium and in the surrounding medium. Montroll and Greenberg[7] have devised a variational method for scattering by obstacles with spherical symmetry and have summarized their work and that of van de Hulst, Hart, and Glauber on scattering by spherical obstacles where the wavelength of the wave inside the scatterer does not differ much from that of the incident wave[8].

The extensive development in recent years of quantum scattering theory, which has included a thorough reexamination of classical methods such as the Watson-Sommerfeld transformation, has yielded techniques and results which can be of use in the solution of classical scattering problems. Thus, Überall[9] has applied the Saxon-Schiff[10,11] theory of potential scattering to the scattering of electromagnetic waves by weak scatterers with complex dielectric constant and permeability, using scattering by a homogeneous dielectric sphere as a particular example. Rockmore[12], who also treats the homogeneous dielectric sphere, has applied approximations of high-energy quantum scattering theory to a spinor formulation of electromagnetic scattering. Flammer[13] has used various approximation methods from potential scattering in his investigation of electromagnetic scattering by inhomogeneous dielectric media and has presented an exact formal solution for the problem of scattering by a sphere with dielectric constant $\varepsilon=1+(\kappa/r)$. He presents a

comprehensive review of applications of methods from quantum scattering theory to electromagnetic scattering problems.

In the case of scalar plane wave scattering by a spherically symmetric scatterer, the similarity in form between the inhomogeneous Helmholtz equation and the time-independent Schrödinger wave equation allows us to treat classical and quantum problems by analogous methods. The general expression for the classical scalar wave equation for a medium with spherically symmetric inhomogeneities is given by:

$$\nabla^2\Psi(\vec{r},t)-\frac{1}{c^2(r)}\frac{\partial^2\Psi(\vec{r},t)}{\partial t^2}=0 \tag{1.1}$$

Separating out the time dependence of the wave function and considering only one frequency component ω, we obtain:

$$\Psi(\vec{r},t)=\psi(\vec{r})e^{-i\omega t}, \tag{1.2}$$

where the spatial part $\psi(\vec{r})$ of $\Psi(\vec{r},t)$ satisfies the inhomogeneous Helmholtz equation:

$$\nabla^2\psi(\vec{r})+[k^2-V(r)]\psi(\vec{r})=0 \tag{1.3}$$

In Eq. (1.3), we have defined:

$$V(r)=k^2-k^2(r), \tag{1.4}$$

where $k(r)=\omega/c(r)$ is the wave number in the scattering medium, and $k=\omega/c$ is the wave number in the surrounding medium. Equation (1.3) corresponds to the Schrödinger equation where $k^2=2mE/\hbar^2$, $V(r)=2mU(r)/\hbar^2$, $\hbar=h/2\pi$, h is Planck's constant, and m is the mass of a particle of total energy E moving in a potential U(r). Thus, the wave function $\psi(\vec{r})$ can be interpreted either classically, as the spatial part of the acoustic velocity potential, for example, or quantum mechanically as the Schrödinger wave function.

Nussenzveig has treated the problem of high-frequency scalar plane wave scattering by an impenetrable sphere[14] (hard-core potential) and by a penetrable sphere[15] (square well potential) using a modified Watson-Sommerfeld transformation. Using the one-dimensional analogue of Eq. (1.3) and the method of partial waves, which is an exact classical technique for solving scattering problems that was originally used in quantum scattering theory by Faxén and Holtsmark[16], Harrison and Plutchok[17] have investigated one-dimensional scattering problems in inhomogeneous media, where the one-dimensional partial waves are "parity" waves. The integral equation thus obtained is solved using an iteration procedure.

The present work is concerned with the application of the Jost function formulation of quantum scattering theory as presented by DeAlfaro and Regge[18] to the scattering of a scalar plane wave by a medium with spherically symmetric inhomogeneities. Both the scattering and surrounding media are assumed to have the same constant density. If we separate Eq. (1.3) in spherical coordinates, we obtain:

$$\psi(\vec{r}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} A_{\ell m} R_\ell(r) Y_{\ell m}(\theta,\phi), \tag{1.5}$$

where $R_\ell(r)=u_\ell(r)/r$ and $Y_{\ell m}(\theta,\phi)$ is a spherical harmonic. Then $u_\ell(r)$ satisfies the partial wave radial equation:

$$\frac{d^2u_\ell(r)}{dr^2} + \left[k^2 - \frac{\ell(\ell+1)}{r^2} - V(r)\right] u_\ell(r) = 0 \tag{1.6}$$

The Jost function technique is essentially a sophisticated version of the standard partial wave analysis and involves a detailed study of the solutions of the radial equation. The Jost function is

defined as the Wronskian of two solutions of the radial equation, one satisfying boundary conditions at $r=0$ (regular solution) and the other satisfying boundary conditions at $r=\infty$ (Jost solution). The differential and total scattering cross sections are obtained from the phase shifts δ_ℓ, which can be calculated easily once the Jost function is known. The Jost function can be calculated either from the original radial differential equation with Jost boundary conditions or from a Volterra integral equation which incorporates these boundary conditions. Although the former approach is more straightforward in the case of a constant inhomogeneity for $r<R_1$ (three-dimensional square well potential), for example, we feel that the latter approach may be more useful in situations where the inhomogeneity (potential) assumes a more complicated functional form and the radial equation does not have an exact solution. The solution of the integral equation is then written as a perturbation expansion, and an iteration procedure yields the solutions to the desired accuracy. The perturbation expansion for a Volterra integral equation has the useful property that it converges everywhere. This procedure is valid for an arbitrary inhomogeneity (potential) satisfying relatively weak requirements.

In Sect. 2, we review the standard partial wave analysis. In Sect. 3, using the partial wave analysis, we calculate $S_\ell(k)$, the scattering matrix (S matrix), for the ℓth partial wave for scattering from a constant spherical inhomogeneity $[V(r)=-V_1,\ r<R_1;\ V(r)=0,\ r>R_1]$. We also calculate the $\ell=0$ scattering amplitude F_0, differential scattering cross section $d\sigma_0/d\Omega$, and

total cross section σ_0. In Sect. 4, we outline how the partial wave method can be used to solve the problem of scattering from two constant concentric spherical inhomogeneous layers $[V(r)=-V_1,\ r<R_1;\ V(r)=-V_2,\ R_1<r<R_2;\ V(r)=0,\ r>R_2]$. In Sect. 5, we present the Jost function formulation of scattering theory. In Sect. 6, we calculate the Jost function from the radial differential equation for scattering from a constant spherical inhomogeneity for arbitrary ℓ and evaluate it for $\ell=0$. In Sect. 7, we calculate the Jost function from the radial differential equation for scattering from two constant concentric spherical inhomogeneous layers for arbitrary ℓ and evaluate it for $\ell=0$. In Sect. 8, we derive the $\ell=0$ Jost integral equation for scattering from an arbitrary inhomogeneity with a cutoff. We calculate the Jost function for $\ell=0$ from this integral equation using an iteration procedure (the first two iterations are considered) for scattering from a constant spherical inhomogeneity and compare it and the cross sections graphically with the exact results. In Sect. 9, we present a general Jost integral equation formulation for arbitrary ℓ for scattering from an arbitrary inhomogeneity and apply it formally to scattering from a constant spherical inhomogeneity and from two constant spherical inhomogeneous layers. In Appendix A, we calculate the Jost function for $\ell=0$ directly from the $\ell=0$ radial differential equation for scattering from a constant spherical inhomogeneity. This method is basically the same as that employed for the case of arbitrary ℓ in Sect. 6. However, by considering only the $\ell=0$ case, we are able to see more clearly how the usual

classical wave solutions with Jost boundary conditions can be used to construct the regular and Jost solutions. In Appendix B, we calculate the Jost function for $\ell=0$ directly from the $\ell=0$ radial differential equation for scattering from two constant spherical inhomogeneous layers. In Appendix C, we present a Laplace transform method for solving exactly the $\ell=0$ Jost integral equation for scattering from a constant spherical inhomogeneity.

2. The Method of Partial Waves

The presentation given here follows the treatments of Gottfried[19], Marion[20], Merzbacher[21], and Schiff[22]. $R_\ell(r)$ in Eq. (1.5) satisfies the following equation when $V(r)=0$:

$$\frac{d^2R_\ell(r)}{dr^2}+\frac{2}{r}\frac{dR_\ell(r)}{dr}+\left[k^2-\frac{\ell(\ell+1)}{r^2}\right]R_\ell(r)=0 \tag{2.1}$$

Making the substitution $R_\ell(r)=\chi_\ell(r)/\sqrt{r}$ in Eq. (2.1), we then have:

$$\frac{d^2\chi_\ell(r)}{dr^2}+\frac{1}{r}\frac{d\chi_\ell(r)}{dr}+\left[k^2-\frac{(\ell+1/2)^2}{r^2}\right]\chi_\ell(r)=0 \tag{2.2}$$

Changing variables, $z=kr$, we obtain:

$$z^2\frac{d^2\chi_\ell}{dz^2}+z\frac{d\chi_\ell}{dz}+[z^2-(\ell+1/2)^2]\chi_\ell=0 \tag{2.3}$$

The solutions of Eq. (2.3) are Bessel and Neumann functions of half-integral order, a particular solution being:

$$\chi_\ell(z)=A_\ell J_{\ell+1/2}(z)+B_\ell N_{\ell+1/2}(z), \tag{2.4}$$

where A_ℓ and B_ℓ are arbitrary constants. We define spherical Bessel, Neumann, and Hankel functions:

$$\begin{aligned} j_\ell(kr)&=\sqrt{\frac{\pi}{2kr}}J_{\ell+1/2}(kr)\\ n_\ell(kr)&=\sqrt{\frac{\pi}{2kr}}N_{\ell+1/2}(kr)\\ h_\ell^{(1)}(kr)&=j_\ell(kr)+in_\ell(kr)\\ h_\ell^{(2)}(kr)&=j_\ell(kr)-in_\ell(kr) \end{aligned} \tag{2.5}$$

The general expression for $\psi(\vec{r})$ can therefore be written as:

$$\psi(\vec{r})=\sum_{\ell,m}[A_{\ell m}j_\ell(kr)+B_{\ell m}n_\ell(kr)]Y_{\ell m}(\theta,\phi), \tag{2.6}$$

or as:

$$\psi(\vec{r})=\sum_{\ell,m}[C_{\ell m}h_\ell^{(1)}(kr)+D_{\ell m}h_\ell^{(2)}(kr)]Y_{\ell m}(\theta,\phi) \tag{2.7}$$

Using the following properties of the spherical Hankel functions:

$$h_\ell^{(1)}(kr) \sim (-1)^{\ell+1}\frac{e^{ikr}}{kr}, \quad kr \gg \ell+1, \tag{2.8a}$$

$$h_\ell^{(1)}(kr) = h_\ell^{(2)*}(kr), \tag{2.8b}$$

we find that the complete wave function has the asymptotic form:

$$\Psi(\vec{r},t) \sim \sum_{\ell,m} \frac{1}{kr}\left[E_{\ell m}e^{i(kr-\omega t)} + F_{\ell m}e^{i(-kr-\omega t)}\right] Y_{\ell m}(\theta,\phi), \quad r\to\infty \tag{2.9}$$

We identify the part of $\Psi(\vec{r},t)$ dependent on $h_\ell^{(1)}(kr)$ as an outgoing spherical wave $[e^{i(kr-\omega t)}/r]$ and the part dependent on $h_\ell^{(2)}(kr)$ as an incoming spherical wave $[e^{i(-kr-\omega t)}/r]$.

We now consider a plane wave of unit amplitude moving in the positive z-direction incident on a spherically symmetric scatterer. At large distances, the scattered waves will appear to be outgoing spherical waves. Since we are dealing with a spherically symmetric scatterer, we assume no ϕ-dependence in the scattered wave function. The asymptotic form of the total wave function is therefore:

$$\psi(\vec{r}) \sim e^{ikz} + F(\theta)\frac{e^{ikr}}{r}, \tag{2.10}$$

where $F(\theta)$ is the scattering amplitude. The differential scattering cross section is given by:

$$\frac{d\sigma(\theta)}{d\Omega} = |F(\theta)|^2 \tag{2.11}$$

The total scattering cross section is:

$$\sigma = \int_{4\pi} \frac{d\sigma(\theta)}{d\Omega} d\Omega = \int_{4\pi} |F(\theta)|^2 d\Omega \tag{2.12}$$

We can write $R_\ell(kr)$ in completely general form as:

$$R_\ell(kr) = C_\ell[A_\ell j_\ell(kr) - B_\ell n_\ell(kr)], \tag{2.13}$$

where we choose A_ℓ and B_ℓ to be real and C_ℓ to be complex. Defining $\cos\delta_\ell = A_\ell/\sqrt{A_\ell^2+B_\ell^2}$, $\sin\delta_\ell = B_\ell/\sqrt{A_\ell^2+B_\ell^2}$, and $D_\ell = C_\ell/\sqrt{A_\ell^2+B_\ell^2}$, we then

have:

$$R_\ell(kr)=D_\ell[\cos\delta_\ell j_\ell(kr)-\sin\delta_\ell n_\ell(kr)], \tag{2.14}$$

where the phase shift δ_ℓ is real and D_ℓ may be complex. Using the following properties of the spherical Bessel and Neumann functions:

$$\begin{aligned} j_\ell(kr) &\sim \frac{1}{kr}\sin(kr-\frac{\ell\pi}{2}), \quad kr>>\ell+1, \\ n_\ell(kr) &\sim -\frac{1}{kr}\cos(kr-\frac{\ell\pi}{2}), \quad kr>>\ell+1, \end{aligned} \tag{2.15}$$

we find that $R_\ell(kr)$ has the asymptotic form:

$$R_\ell(kr)\sim\frac{D_\ell}{kr}\sin(kr-\frac{\ell\pi}{2}+\delta_\ell) \tag{2.16}$$

The asymptotic form of $\psi(\vec{r})$ is therefore:

$$\psi(\vec{r})\sim\sum_{\ell=0}^{\infty}\frac{D_\ell}{kr}\sin(kr-\frac{\ell\pi}{2}+\delta_\ell)P_\ell(\cos\theta) \tag{2.17}$$

But we can expand the incoming plane wave in terms of spherical waves:

$$e^{ikz}=e^{ikr\cos\theta}=\sum_{\ell=0}^{\infty}(2\ell+1)i^\ell j_\ell(kr)P_\ell(\cos\theta), \tag{2.18}$$

and the asymptotic form of $\psi(\vec{r})$ can therefore also be written as:

$$\begin{aligned} \psi(\vec{r}) &\sim e^{ikz}+F(\theta)\frac{e^{ikr}}{r} \\ &\sim \sum_{\ell=0}^{\infty}\frac{(2\ell+1)i^\ell}{kr}\sin(kr-\frac{\ell\pi}{2})P_\ell(\cos\theta)+F(\theta)\frac{e^{ikr}}{r} \end{aligned} \tag{2.19}$$

Equating the two asymptotic forms [Eqs. (2.17) and (2.19)], we find that:

$$D_\ell=(2\ell+1)i^\ell e^{i\delta_\ell}, \tag{2.20}$$

$$F(\theta)=\frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)(e^{2i\delta_\ell}-1)P_\ell(\cos\theta) \tag{2.21}$$

$$=\frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)(e^{i\delta_\ell}\sin\delta_\ell)P_\ell(\cos\theta) \tag{2.22}$$

We therefore also have that:

$$\frac{d\sigma(\theta)}{d\Omega}=\frac{1}{k^2}\left|\sum_{\ell=0}^{\infty}(2\ell+1)(e^{i\delta_\ell}\sin\delta_\ell)P_\ell(\cos\theta)\right|^2 \tag{2.23}$$

$$\sigma=\frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell \tag{2.24}$$

and therefore:

$$\beta_\ell = k_1 \frac{j_\ell'(k_1R_1)}{j_\ell(k_1R_1)} \tag{3.6}$$

Solving Eq. (3.3) and using Eq. (3.6), we obtain:

$$S_\ell(k) = e^{2i\delta_\ell}$$

$$= -\left[\frac{j_\ell(kR_1) - in_\ell(kR_1)}{j_\ell(kR_1) + in_\ell(kR_1)}\right]\left\{1 + k\left[\frac{\dfrac{j_\ell'(kR_1) + in_\ell'(kR_1)}{j_\ell(kR_1) + in_\ell(kR_1)} - \dfrac{j_\ell'(kR_1) - in_\ell'(kR_1)}{j_\ell(kR_1) - in_\ell(kR_1)}}{\dfrac{k_1 j_\ell'(k_1R_1)}{j_\ell(k_1R_1)} - k\left[\dfrac{j_\ell'(kR_1) + in_\ell'(kR_1)}{j_\ell(kR_1) + in_\ell(kR_1)}\right]}\right]\right\} \tag{3.7}$$

$$= -\left[\frac{h_\ell^{(2)}(kR_1)}{h_\ell^{(1)}(kR_1)}\right]\left\{1 + k\left[\frac{\dfrac{h_\ell'^{(1)}(kR_1)}{h_\ell^{(1)}(kR_1)} - \dfrac{h_\ell'^{(2)}(kR_1)}{h_\ell^{(2)}(kR_1)}}{\dfrac{k_1 j_\ell'(k_1R_1)}{j_\ell(k_1R_1)} - \dfrac{k h_\ell'^{(1)}(kR_1)}{h_\ell^{(1)}(kR_1)}}\right]\right\}$$

$$= -\frac{k j_\ell(k_1R_1) h_\ell'^{(2)}(kR_1) - k_1 j_\ell'(k_1R_1) h_\ell^{(2)}(kR_1)}{k j_\ell(k_1R_1) h_\ell'^{(1)}(kR_1) - k_1 j_\ell'(k_1R_1) h_\ell^{(1)}(kR_1)} \tag{3.8}$$

In order to calculate F and $\frac{d\sigma}{d\Omega}$ for $\ell=o$, we use the following expressions for the spherical Bessel and Neumann functions and their derivatives[23]:

$$\begin{aligned} j_o(kr) &= \frac{\sin kr}{kr} \\ j_o'(kr) &= \frac{\cos kr}{kr} - \frac{\sin kr}{(kr)^2} \\ n_o(kr) &= -\frac{\cos kr}{kr} \\ n_o'(kr) &= \frac{\sin kr}{kr} + \frac{\cos kr}{(kr)^2} \end{aligned} \tag{3.9}$$

The various terms in Eq. (3.7) then become:

$$\frac{k_1 j_o'(k_1R_1)}{j_o(k_1R_1)} = \frac{k_1R_1 \cos k_1R_1 - \sin k_1R_1}{R_1 \sin k_1R_1} \tag{3.10a}$$

$$-\frac{j_o(kR_1) - in_o(kR_1)}{j_o(kR_1) + in_o(kR_1)} = e^{-2ikR_1} \tag{3.10b}$$

$$\frac{j_o'(kR_1)+in_o'(kR_1)}{j_o(kR_1)+in_o(kR_1)}=\frac{-1+ikR_1}{kR_1} \tag{3.10c}$$

$$\frac{j_o'(kR_1)-in_o'(kR_1)}{j_o(kR_1)-in_o(kR_1)}=\frac{-1-ikR_1}{kR_1} \tag{3.10d}$$

Noting that:

$$e^{i\delta_\ell}\sin\delta_\ell=\frac{e^{2i\delta_\ell}-1}{2i}, \tag{3.11}$$

we then have:

$$e^{i\delta_o}\sin\delta_o=e^{-2ikR_1}\left(\frac{k}{k_1\cot k_1R_1-ik}-e^{ikR_1}\sin kR_1\right) \tag{3.12}$$

But:

$$\begin{aligned} F_o&=\frac{1}{k}e^{i\delta_o}\sin\delta_o \\ &=\frac{e^{-2ikR_1}}{k}\left(\frac{k}{k_1\cot k_1R_1-ik}-e^{ikR_1}\sin kR_1\right), \end{aligned} \tag{3.13}$$

and:

$$\begin{aligned} \frac{d\sigma_o}{d\Omega}&=|F_o|^2 \\ &=\frac{1-\frac{k_1}{k}\sin 2kR_1\cot k_1R_1-2\sin^2 kR_1}{k_1^2\cot^2 k_1R_1+k^2}+\frac{\sin^2 kR_1}{k^2}, \end{aligned} \tag{3.14}$$

and:

$$\sigma_o=4\pi\frac{d\sigma_o}{d\Omega} \tag{3.15}$$

4. Scattering from Two Constant Spherical Inhomogeneous Layers: The Method of Partial Waves

We outline here how the method of partial waves can be used to solve the problem of scattering from two constant spherical inhomogeneous layers (see Fig. 2):

$$\begin{aligned} &\text{Region 1:} \quad V(r)=-V_1,\ k(r)=k_1,\ r<R_1 \\ &\text{Region 2:} \quad V(r)=-V_2,\ k(r)=k_2,\ R_1<r<R_2 \\ &\text{Region 3:} \quad V(r)=o,\ k(r)=k,\ r>R_2 \end{aligned} \tag{4.1}$$

The solutions in the three regions are:

$$\begin{aligned} &R_{\ell,1}(k_1r)=A_\ell j_\ell(k_1r) \\ &R_{\ell,2}(k_2r)=B_\ell h_\ell^{(1)}(k_2r)+C_\ell h_\ell^{(2)}(k_2r) \\ &R_{\ell,3}(kr)=D_\ell[\cos\delta_\ell j_\ell(kr)-\sin\delta_\ell n_\ell(kr)] \end{aligned} \tag{4.2}$$

We can eliminate two of the three constants A_ℓ, B_ℓ, and C_ℓ by using continuity boundary conditions at $r=R_1$:

$$\begin{aligned} &R_{\ell,1}(k_1r)=R_{\ell,2}(k_2r)\Big|_{r=R_1} \\ &\frac{dR_{\ell,1}(k_1r)}{dr}=\frac{dR_{\ell,2}(k_2r)}{dr}\Big|_{r=R_1} \end{aligned} \tag{4.3}$$

The remainder of the analysis is analogous to that of Sect. 3 with:

$$\beta_\ell=\frac{1}{R_{\ell,2}(k_2r)}\ \frac{dR_{\ell,2}(k_2r)}{dr}\Big|_{r=R_2}=\frac{1}{R_{\ell,3}(kr)}\ \frac{dR_{\ell,3}(kr)}{dr}\Big|_{r=R_2}$$

We solve this problem in detail using the method of Jost functions in Sect. 7.

5. The Jost Function Formulation of Scattering Theory

This treatment follows that of DeAlfaro and Regge[24]. We choose only those portions of their discussion which are relevant to our investigation, at times perhaps at the expense of mathematical detail and rigor, but hopefully not at the expense of clarity. Since we are concerned primarily with the $\ell=0$ partial wave, we do not discuss analytic properties, which are important in a discussion of higher order partial waves. Interested readers are urged to pursue the reference.

We begin by writing down the radial equation again:

$$\frac{d^2u_\ell(r)}{dr^2}+\left[k^2-\frac{\ell(\ell+1)}{r^2}-V(r)\right]u_\ell(r)=0 \tag{1.6}$$

For the following analysis to hold, we must impose certain requirements on $V(r)$ in addition to its being a real function vanishing at $r=\infty$:

$V(r)$ must be almost everywhere continuous;

$$\begin{aligned} &\int_c^\infty |V(r)|dr=M(c)<\infty, \\ &\int_o^{c'} r|V(r)|dr=N(c')<\infty, \end{aligned} \tag{5.1}$$

where c and c' are arbitrary constants>o.

a. Jost Boundary Conditions at r=o

For small r, $\left|\frac{-\ell(\ell+1)}{r^2}\right| >> |k^2-V(r)|$, and we begin our discussion of Eq. (1.6) by neglecting the term $[k^2-V(r)]$. Eq. (1.6) then becomes:

$$\frac{d^2u_\ell(r)}{dr^2}-\frac{\ell(\ell+1)}{r^2}u_\ell(r)=o, \tag{5.2}$$

which has a regular point* at r=o (hence the term "regular solution") and the exact solution:

$$u_\ell(r)=\alpha r^{\ell+1}+\beta r^{-\ell} \tag{5.3}$$

Using Eq. (5.3) as a guide, we define two linearly independent solutions of Eq. (1.6) with the behavior:

$$\phi(r)=r^{\ell+1}[1+o(1)] \tag{5.4a}$$

$$\phi_1(r)=r^{-\ell}[1+o(1)] \tag{5.4b}$$

We also define $\lambda=\ell+1/2$, so that Eq. (1.6) becomes even in λ:

$$\frac{d^2u_{\lambda-1/2}(r)}{dr^2}+\left[k^2-\frac{\lambda^2-1/4}{r^2}-V(r)\right]u_{\lambda-1/2}(r)=o \tag{5.5}$$

We now, in a sense, take Eq. (5.5) out of its previous mathematical context and consider it and its solutions to be a function also of the parameters λ and k, which, in the general case, may be complex variables. Thus $\phi(r)\to\phi_1(r)$ as $\lambda\to-\lambda$, and we therefore replace $\phi(r)$ and $\phi_1(r)$ by $\phi(\lambda,k,r)$ and $\phi(-\lambda,k,r)$, respectively. We write $u_{\lambda-1/2}(r)$ instead of $u_\ell(r)$ in order that our notation be consistent. We solve Eq. (5.5) exactly by converting it into

---

*If we write the general linear second order homogeneous differential equation as $\frac{d^2f(r)}{dr^2}+p(r)\frac{df(r)}{dr}+q(r)f(r)=o$, then a regular point r=c is such that, although p(r) and q(r) may not be finite at c, both $\lim_{r\to c}(r-c)p(r)$ and $\lim_{r\to c}(r-c)^2q(r)$ exist.

an integral equation using the method of variation of parameters with the boundary conditions (5.4). We therefore let:

$$\phi(\lambda,k,r)=\alpha(r)r^{\lambda+1/2}+\beta(r)r^{-\lambda+1/2} \tag{5.6}$$

and impose the standard additional condition:

$$\alpha'(r)r^{\lambda+1/2}+\beta'(r)r^{-\lambda+1/2}=o \tag{5.7}$$

Computing $\phi'(\lambda,k,r)$ and $\phi''(\lambda,k,r)$ and substituting these values into Eq. (5.5), we obtain:

$$(\lambda+1/2)\alpha'(r)r^{\lambda-1/2}-(\lambda-1/2)\beta'(r)r^{-\lambda-1/2}=[V(r)-k^2]\phi(\lambda,k,r), \tag{5.8}$$

which, with Eq. (5.7), has the solution:

$$\begin{aligned}\alpha'(r)&=\frac{1}{2\lambda}[V(r)-k^2]\phi(\lambda,k,r)r^{-\lambda+1/2}\\ \beta'(r)&=\frac{1}{2\lambda}[k^2-V(r)]\phi(\lambda,k,r)r^{\lambda+1/2}\end{aligned} \tag{5.9}$$

Conditions (5.4) now become:

$$\begin{aligned}&\lim_{r\to o}\alpha(r)=1\\ &\lim_{r\to o}\beta(r)=o,\end{aligned} \tag{5.10}$$

since $\phi(\lambda,k,r)$ has boundary value (5.4a).

From Eqs. (5.9) and (5.10) we then have:

$$\begin{aligned}\alpha(r)&=1+\frac{1}{2\lambda}\int_o^r[V(\xi)-k^2]\phi(\lambda,k,\xi)\xi^{-\lambda+1/2}d\xi\\ \beta(r)&=-\frac{1}{2\lambda}\int_o^r[V(\xi)-k^2]\phi(\lambda,k,\xi)\xi^{\lambda+1/2}d\xi\end{aligned} \tag{5.11}$$

Substituting Eqs. (5.11) into Eq. (5.6), we obtain:

$$\phi(\lambda,k,r)=r^{\lambda+1/2}+\frac{1}{2\lambda}\int_o^r\left[\left(\frac{\xi}{r}\right)^{\lambda}-\left(\frac{r}{\xi}\right)^{\lambda}\right]\sqrt{r\xi}\,[k^2-V(\xi)]\phi(\lambda,k,\xi)d\xi \tag{5.12}$$

From Eq. (5.12) it is clear that we can write the Jost boundary conditions at r=o as:

$$\lim_{r\to o}\phi(\lambda,k,r)=o$$
$$\lim_{r\to o}\frac{d\phi(\lambda,k,r)}{dr}=\lim_{r\to o}(\lambda+1/2)\,r^{\lambda-1/2} \tag{5.13}$$

Eq. (5.12) is a Volterra integral equation, and we write its solution as a perturbation expansion:

$$\phi(\lambda,k,r)=\sum_{n=o}^{\infty}\phi_n(\lambda,k,r), \tag{5.14}$$

where:

$$\phi_o(\lambda,k,r)=r^{\lambda+1/2}, \tag{5.15}$$

and:

$$\phi_{n+1}(\lambda,k,r)=\frac{1}{2\lambda}\int_o^r\left[\left(\frac{\xi}{r}\right)^{\lambda}-\left(\frac{r}{\xi}\right)^{\lambda}\right]\sqrt{r\xi}\,[k^2-V(\xi)]\,\phi_n(\lambda,k,\xi)\,d\xi \tag{5.16}$$

The perturbation expansion for $\phi(\lambda,k,r)$ is bounded term by term and is therefore unrestrictedly convergent.

Since the Wronskian of any two linearly independent solutions of Eq. (5.5) is non-zero and constant, we can evaluate the Wronskian of $\phi(\lambda,k,r)$ and $\phi(-\lambda,k,r)$ by replacing $\phi(\lambda,k,r)$ with $r^{\lambda+1/2}$:

$$W[\phi(\lambda,k,r),\phi(-\lambda,k,r)]=\phi(\lambda,k,r)\phi'(-\lambda,k,r)-\phi'(\lambda,k,r)\phi(-\lambda,k,r)=-2\lambda \tag{5.17}$$

b. Jost Boundary Conditions at $r=\infty$

For large r, we can neglect the term $\left[-\frac{\lambda^2-1/4}{r^2}-V(r)\right]$ in Eq. (5.5), which therefore becomes:

$$\frac{d^2u_{\lambda-1/2}(r)}{dr^2}+k^2u_{\lambda-1/2}(r)=o \tag{5.18}$$

Eq. (5.18) has the exact solution:

$$u_{\lambda-1/2}(r)=\alpha e^{-ikr}+\beta e^{ikr}, \tag{5.19}$$

and we therefore construct a solution $f(\lambda,k,r)$ (Jost solution) of Eq. (5.5) with the asymptotic behavior:

$$\lim_{r\to\infty} e^{ikr}f(\lambda,k,r)=1^* \tag{5.20}$$

We use the method of variation of parameters in a manner analogous to that of Sect. 5a and let:

$$f(\lambda,k,r)=\alpha(r)e^{-ikr}+\beta(r)e^{ikr}, \tag{5.21}$$

so that Eq. (5.5) with the boundary conditions (5.20) becomes:

$$f(\lambda,k,r)=e^{-ikr}+\frac{1}{k}\int_r^\infty \sin\left[k(r'-r)\right]\left[V(r')+\frac{\lambda^2-1/4}{r'^2}\right]f(\lambda,k,r')dr' \tag{5.22}$$

We write the solution of Eq. (5.22) as a perturbation expansion:

$$f(\lambda,k,r)=\sum_{n=o}^{\infty} g_n(\lambda,k,r), \tag{5.23}$$

---

*We note that our choice of asymptotic form (5.20) for $f(\lambda,k,r)$ corresponds to incoming waves, which is contrary to our previous assumption of outgoing spherical waves. This choice is a matter of convention, and we could equally well have chosen $\lim_{r\to\infty} e^{-ikr}f(\lambda,k,r)=1$. In both cases, the S matrix, which is the quantity of interest in scattering problems, is the same.

where:

$$g_o(\lambda,k,r)=e^{-ikr}, \tag{5.24}$$

and:

$$g_{n+1}(\lambda,k,r)=\frac{1}{k_r}\int^{\infty}\sin\left[k(r'-r)\right]\left[V(r')+\frac{\lambda^2-1/4}{r'^2}\right]g_n(\lambda,k,r')\,dr' \tag{5.25}$$

The perturbation expansion for $f(\lambda,k,r)$ is bounded for any λ.

We state the following relation:

$$W[f(\lambda,k,r),f(\lambda,-k,r)]=2ik, \tag{5.26}$$

where we have evaluated the Wronskian by substituting for $f(\lambda,\pm k,r)$ its asymptotic behavior, $e^{\mp ikr}$.

c. The Jost Function and the S Matrix

We define the Jost function $f(\lambda,k)$ as the Wronskian (which, again, is non-zero and constant) of $f(\lambda,k,r)$ and $\phi(\lambda,k,r)$:

$$f(\lambda,k)=W[f(\lambda,k,r),\phi(\lambda,k,r)]=f(\lambda,k,r)\phi'(\lambda,k,r)-f'(\lambda,k,r)\phi(\lambda,k,r) \tag{5.27}$$

We can write the general solution of Eq. (5.5) as a linear combination of any two linearly independent solutions, and, in fact, we have that:

$$\phi(\lambda,k,r)=\frac{1}{2ik}[f(\lambda,k)f(\lambda,-k,r)-f(\lambda,-k)f(\lambda,k,r)], \tag{5.28}$$

where we have used Eq. (5.26).

The asymptotic form of $\phi(\lambda,k,r)$ is:

$$\phi(\lambda,k,r)\sim\frac{1}{2ik}[f(\lambda,k)e^{ikr}-f(\lambda,-k)e^{-ikr}], \tag{5.29}$$

where we have replaced $f(\lambda,\pm k,r)$ with $e^{\mp ikr}$, respectively.

If we parametrize the Jost functions in Eq. (5.29) as:

$$\begin{aligned} f(\lambda,k)&=\tau(\lambda,k)\exp[i\delta(\lambda,k)-(1/2)i\pi(\lambda-1/2)] \\ f(\lambda,-k)&=\tau(\lambda,k)\exp[-i\delta(\lambda,k)+(1/2)i\pi(\lambda-1/2)], \end{aligned} \tag{5.30}$$

where $\tau(\lambda,k)$ is a complex amplitude, then Eq. (5.29) becomes:

$$\phi(\lambda,k,r)\sim\frac{1}{k}\tau(\lambda,k)\sin[kr+\delta(\lambda,k)-(1/2)\pi(\lambda-1/2)], \tag{5.31}$$

which agrees with Eq. (2.16) [we recall that $R_\ell(r)=u_\ell(r)/r$].

The S matrix is therefore given by:

$$S(\lambda,k)=e^{2i\delta(\lambda,k)}=\frac{f(\lambda,k)}{f(\lambda,-k)}e^{i\pi(\lambda-1/2)} \tag{5.32}$$

We note that the S matrix is proportional to the ratio of the coefficients of the outgoing and incoming waves in Eq. (5.29).

6. Scattering from a Constant Spherical Inhomogeneity: The Method of Jost Functions (Differential Equation Approach)

We apply the method outlined in Sect. 5 to the problem of scattering from the V(r) specified in Eq. (3.1). The solutions in the two regions are:

$$\begin{aligned} &\text{Region 1:} \quad u_{\lambda-1/2,1}(k_1 r)=r\left[Aj_{\lambda-1/2}(k_1 r)+Bn_{\lambda-1/2}(k_1 r)\right] \\ &\text{Region 3:} \quad u_{\lambda-1/2,3}(kr)=r\left[Ch^{(1)}_{\lambda-1/2}(kr)+Dh^{(2)}_{\lambda-1/2}(kr)\right] \end{aligned} \tag{6.1}$$

Choosing $u_{\lambda-1/2,1}(k_1 r)$ to be $\phi(\lambda,k,r)$ and imposing the boundary conditions (5.13), we find that B=0 and:

$$\begin{aligned} \phi(\lambda,k,r)&=2^{\lambda+1/2}\pi^{-1/2}k_1^{-\lambda+1/2}r\Gamma(\lambda+1)j_{\lambda-1/2}(k_1 r) \\ \phi'(\lambda,k,r)&=2^{\lambda+1/2}\pi^{-1/2}k_1^{-\lambda+1/2}\Gamma(\lambda+1)\left[j_{\lambda-1/2}(k_1 r)+k_1 r j'_{\lambda-1/2}(k_1 r)\right], \end{aligned} \tag{6.2}$$

where we have used the following series representation[25] for $j_{\lambda-1/2}(k_1 r)$:

$$j_{\lambda-1/2}(k_1 r)=\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{1/2}(k_1 r)^{\lambda-1/2+2n}}{2^{\lambda+1/2+2n}n!\,\Gamma(\lambda+n+1)}, \quad \lambda-1/2\neq-1,-2,-3,\ .\ .\ . \tag{6.3}$$

Choosing $u_{\lambda-1/2,3}(kr)$ to be $f(\lambda,k,r)$ and imposing the boundary conditions (5.20), we find that C=0 and:

$$\begin{aligned} f(\lambda,k,r)&=ke^{-i\frac{\pi}{2}(\lambda+1/2)}rh^{(2)}_{\lambda-1/2}(kr) \\ f'(\lambda,k,r)&=ke^{-i\frac{\pi}{2}(\lambda+1/2)}\left[h^{(2)}_{\lambda-1/2}(kr)+krh'^{(2)}_{\lambda-1/2}(kr)\right], \end{aligned} \tag{6.4}$$

where we have used the following asymptotic form[26] for $h^{(2)}_{\lambda-1/2}(kr)$:

$$\lim_{kr\to\infty} h^{(2)}_{\lambda-1/2}(kr)=\frac{1}{kr}e^{-i\left[kr-(\frac{\pi}{2})(\lambda+1/2)\right]} \tag{6.5}$$

Since the point $r=R_1$ is the common domain of $\phi(\lambda,k,r)$ and $f(\lambda,k,r)$, we evaluate the Jost function at $r=R_1$ and thus obtain:

$$f(\lambda,k)=W[f(\lambda,k,r),\phi(\lambda,k,r)]_{r=R_1}$$
$$=2^{\lambda+1/2}\pi^{-1/2}kk_1^{-\lambda+1/2}\Gamma(\lambda+1)e^{-\frac{i\pi}{2}(\lambda+1/2)}r\circ \tag{6.6}$$
$$\bullet\left[k_1 j'_{\lambda-1/2}(k_1R_1)h^{(2)}_{\lambda-1/2}(kR_1)-kj_{\lambda-1/2}(k_1R_1)h'^{(2)}_{\lambda-1/2}(kR_1)\right]$$

We also have that:

$$f(\lambda,-k)=2^{\lambda+1/2}\pi^{-1/2}kk_1^{-\lambda+1/2}\Gamma(\lambda+1)e^{-\frac{i\pi}{2}(\lambda+1/2)}re^{i\pi(\lambda-1/2)}\circ \tag{6.7}$$
$$\bullet\left[-k_1 j'_{\lambda-1/2}(k_1R_1)h^{(1)}_{\lambda-1/2}(kR_1)+kj_{\lambda-1/2}(k_1R_1)h'^{(1)}_{\lambda-1/2}(kR_1)\right],$$

where we have used the following identities[27]:

$$h^{(2)}_{\lambda-1/2}(kre^{i\pi})=h^{(2)}_{\lambda-1/2}(-kr)=(-1)^{\lambda-1/2}h^{(1)}_{\lambda-1/2}(kr)=e^{i\pi(\lambda-1/2)}h^{(1)}_{\lambda-1/2}(kr)$$
$$h'^{(2)}_{\lambda-1/2}(-kr)=(-1)^{\lambda+1/2}h'^{(1)}_{\lambda-1/2}(kr)=e^{i\pi(\lambda+1/2)}h'^{(1)}_{\lambda-1/2}(kr) \tag{6.8}$$
$$=-e^{i\pi(\lambda-1/2)}h'^{(1)}_{\lambda-1/2}(kr),\quad \lambda-1/2=0,1,2,\ .\ .\ .$$

The S matrix is then given by:

$$S(\lambda,k)=-\frac{kj_{\lambda-1/2}(k_1R_1)h'^{(2)}_{\lambda-1/2}(kR_1)-k_1j'_{\lambda-1/2}(k_1R_1)h^{(2)}_{\lambda-1/2}(kR_1)}{kj_{\lambda-1/2}(k_1R_1)h'^{(1)}_{\lambda-1/2}(kR_1)-k_1j'_{\lambda-1/2}(k_1R_1)h^{(1)}_{\lambda-1/2}(kR_1)}, \tag{6.9}$$

which is in agreement with Eq. (3.8).

We can calculate the Jost function for $\lambda=1/2$ from Eq. (6.6):

$$f(1/2,k)=\frac{e^{-ikR_1}}{2}\left[\left(1-\frac{k}{k_1}\right)e^{-ik_1R_1}+\left(1+\frac{k}{k_1}\right)e^{ik_1R_1}\right], \tag{6.10}$$

where we have used Eq. (3.9) and the following relations[28]:

$$h_o^{(2)}(kR_1)=-\frac{e^{-ikR_1}}{ikR_1}$$
$$h_o'^{(2)}(kR_1)=\frac{e^{-ikR_1}}{kR_1}\left(1+\frac{1}{ikR_1}\right) \tag{6.11}$$

7. Scattering from Two Constant Spherical Inhomogeneous Layers: The Method of Jost Functions (Differential Equation Approach)

We apply the method of Jost functions to the problem of scattering from the V(r) specified in Eq. (4.1). We let Region 4=Region 2 + Region 3 and let $\phi(\lambda,k,r)$ be the solution in Region 1 and $f(\lambda,k,r)$ be the solution in Region 4. Using our results from Sect. 6, we then have that:

$$\phi(\lambda,k,r)=2^{\lambda+1/2}\pi^{-1/2}k_1^{-\lambda+1/2}\Gamma(\lambda+1)rj_{\lambda-1/2}(k_1r),\quad r<R_1$$
$$f(\lambda,k,r)=\begin{cases} f_2(\lambda,k,r)=r\left[Ch^{(1)}_{\lambda-1/2}(k_2r)+Dh^{(2)}_{\lambda-1/2}(k_2r)\right], & R_1<r<R_2 \\ f_3(\lambda,k,r)=ke^{-\frac{i\pi}{2}(\lambda+1/2)}rh^{(2)}_{\lambda-1/2}(kr), & r>R_2 \end{cases} \tag{7.1}$$

We calculate the constants C and D using continuity boundary conditions at $r=R_2$:

$$f_2(\lambda,k,r)=f_3(\lambda,k,r)\Big|_{r=R_2}$$
$$\frac{df_2(\lambda,k,r)}{dr}=\frac{df_3(\lambda,k,r)}{dr}\Bigg|_{r=R_2} \tag{7.2}$$

We find that:

$$C=\frac{ke^{-\frac{i\pi}{2}(\lambda+1/2)}}{h^{(1)}_{\lambda-1/2}(k_2R_2)}\Bigg\{h^{(2)}_{\lambda-1/2}(kR_2)+\frac{k_2R_2^2}{2i}h^{(2)}_{\lambda-1/2}(k_2R_2)\cdot \tag{7.3}$$
$$\cdot\left[kh^{(1)}_{\lambda-1/2}(k_2R_2)h'^{(2)}_{\lambda-1/2}(kR_2)-k_2h^{(2)}_{\lambda-1/2}(kR_2)h'^{(1)}_{\lambda-1/2}(k_2R_2)\right]\Bigg\}$$

$$D=-\frac{kk_2R_2^2}{2i}e^{-\frac{i\pi}{2}(\lambda+1/2)}\Big[kh^{(1)}_{\lambda-1/2}(k_2R_2)h'^{(2)}_{\lambda-1/2}(kR_2) \tag{7.4}$$
$$-k_2h^{(2)}_{\lambda-1/2}(kR_2)h'^{(1)}_{\lambda-1/2}(k_2R_2)\Big],$$

where we have used the following relation[29]:

$$W\left[h^{(1)}_{\lambda-1/2}(k_2R_2),h^{(2)}_{\lambda-1/2}(k_2R_2)\right]=\frac{-2i}{(k_2R_2)^2},\quad \lambda-1/2=0,1,2,\ .\ .\ . \tag{7.5}$$

Since the point $r=R_1$ is the common domain of $\phi(\lambda,k,r)$ and $f(\lambda,k,r)$, we evaluate the Jost function at $r=R_1$ and thus obtain:

$$
\begin{aligned}
f(\lambda,k) &= W[f(\lambda,k,r),\phi(\lambda,k,r)]_{r=R_1} \\
&= [f_2(\lambda,k,r)\phi'(\lambda,k,r)-f_2'(\lambda,k,r)\phi(\lambda,k,r)]_{r=R_1} \\
&= 2^{\lambda+1/2}\pi^{-1/2}k_1^{-\lambda+1/2}R_1^2\Gamma(\lambda+1)\cdot \\
&\quad\cdot\Big\{Ck_1\Big[h^{(1)}_{\lambda-1/2}(k_2R_1)j'_{\lambda-1/2}(k_1R_1)+h^{(2)}_{\lambda-1/2}(k_2R_1)j'_{\lambda-1/2}(k_1R_1)\Big] \\
&\quad -Dk_2\Big[h'^{(1)}_{\lambda-1/2}(k_2R_1)j_{\lambda-1/2}(k_1R_1)+h'^{(2)}_{\lambda-1/2}(k_2R_1)j_{\lambda-1/2}(k_1R_1)\Big]\Big\},
\end{aligned}
\tag{7.6}
$$

where C and D are given in Eqs. (7.3) and (7.4).

For $\lambda=1/2$, we have that:

$$
\begin{aligned}
C &= \frac{i}{2}(k_2-k)e^{-i(k_2+k)R_2} \\
D &= -\frac{i}{2}(k_2+k)e^{i(k_2-k)R_2},
\end{aligned}
\tag{7.7}
$$

and therefore:

$$
\begin{aligned}
f(1/2,k) &= \frac{1}{4}\left(1-\frac{k}{k_2}\right)e^{ik_2R_1}e^{-i(k_2+k)R_2}\left[\left(1+\frac{k_2}{k_1}\right)e^{-ik_1R_1}+\left(1-\frac{k_2}{k_1}\right)e^{ik_1R_1}\right] \\
&\quad +\frac{1}{4}\left(1+\frac{k}{k_2}\right)e^{-ik_2R_1}e^{i(k_2-k)R_2}\left[\left(1-\frac{k_2}{k_1}\right)e^{-ik_1R_1}+\left(1+\frac{k_2}{k_1}\right)e^{ik_1R_1}\right],
\end{aligned}
\tag{7.8}
$$

where we have used Eqs. (2.8b), (3.9) and (6.11). As expected, the two-layer solution (7.8) reduces to the one-layer solution (6.10) for $k_2\to k_1$ and $R_2\to R_1$.

We note that we could have also found $f(\lambda,k)$ by choosing Region 4=Region 1 + Region 2 and letting $\phi(\lambda,k,r)$ be the solution in Region 4 and $f(\lambda,k,r)$ be the solution in Region 3. We would then write the solutions as:

$$
\phi(\lambda,k,r)=\begin{cases}\phi_1(\lambda,k,r)=2^{\lambda+1/2}\pi^{-1/2}k_1^{-\lambda+1/2}\Gamma(\lambda+1)rj_{\lambda-1/2}(k_1r), & r<R_1 \\ \phi_2(\lambda,k,r)=r\Big[Ch^{(1)}_{\lambda-1/2}(k_2r)+Dh^{(2)}_{\lambda-1/2}(k_2r)\Big], & R_1<r<R_2\end{cases}
\tag{7.9a}
$$

$$f(\lambda,k,r)=ke^{-\frac{i\pi}{2}(\lambda+1/2)}\,rh^{(2)}_{\lambda-1/2}(kr),\quad r>R_2 \tag{7.9b}$$

We would calculate the constants C and D using continuity boundary conditions at $r=R_1$:

$$\begin{aligned}\phi_1(\lambda,k,r)&=\phi_2(\lambda,k,r)\Big|_{r=R_1}\\ \frac{d\phi_1(\lambda,k,r)}{dr}&=\frac{d\phi_2(\lambda,k,r)}{dr}\Big|_{r=R_1}\end{aligned} \tag{7.10}$$

Since the point $r=R_2$ is the common domain of $\phi(\lambda,k,r)$ and $f(\lambda,k,r)$, we would evaluate the Jost function at $r=R_2$:

$$\begin{aligned}f(\lambda,k)&=W[f(\lambda,k,r),\phi(\lambda,k,r)]_{r=R_2}\\ &=[f(\lambda,k,r)\phi_2'(\lambda,k,r)-f'(\lambda,k,r)\phi_2(\lambda,k,r)]_{r=R_2}\end{aligned} \tag{7.11}$$

We also remark that the Jost function method involves the use of continuity boundary conditions at either R_1 or R_2, while the standard partial wave method (see Sect. 4) involves the use of continuity boundary conditions at both R_1 and R_2.

8. The Jost Integral Equation for $\lambda=1/2$ and Some Approximate Solutions

If we assume there is an R such that $V(r)=0$ for $r>R$ and let $g(1/2,k,r)=e^{ikr}f(1/2,k,r)$, then Eq. (5.22) becomes the Jost integral equation for $\lambda=1/2$:

$$g(1/2,k,r)=1+\frac{1}{2ik}\int_r^R [1-e^{2ik(r-r')}]V(r')g(1/2,k,r')dr' \tag{8.1}$$

We write the solution of Eq. (8.1) as a perturbation expansion:

$$g(1/2,k,r)=\sum_{n=o}^{\infty} g_n(1/2,k,r), \tag{8.2}$$

where:

$$g_o(1/2,k,r)=1, \tag{8.3}$$

and:

$$g_n(1/2,k,r)=\frac{1}{2ik}\int_r^R [1-e^{2ik(r-r')}]V(r')g_{n-1}(1/2,k,r')dr' \tag{8.4}$$

If we assume that k is pure real (which is the case in the problems under consideration), we then have that:

$$|g_n(1/2,k,r)|\leq\frac{1}{|k|}\int_r^R |V(r')|\ |g_{n-1}(1/2,k,r')|dr', \tag{8.5}$$

and therefore:

$$|g_1(1/2,k,r)|\leq N_R(r)/|k|, \tag{8.6}$$

where:

$$N_R(r)=\theta(R-r)\int_r^R |V(r')|dr', \tag{8.7}$$

$$\theta(R-r)=\begin{cases}1, & r<R\\ 0, & r>R\end{cases}$$

By recursion, we obtain:

$$|g_n(1/2,k,r)|\leq N_R^n(r)/|k|^n n!, \tag{8.8}$$

and therefore:

$$|g(1/2,k,r)-1| \leq e^{N_R(r)/k}-1 \tag{8.9}$$

From Eq. (5.13), we have:

$$\begin{aligned} &\lim_{r\to o}\phi(1/2,k,r)=0 \\ &\lim_{r\to o}\phi'(1/2,k,r)=1 \end{aligned} \tag{8.10}$$

$f(1/2,k,0)$ and $f'(1/2,k,0)$ are finite (e.g., see Appendix A), and we can evaluate $f(1/2,k)$ at $r=0$ using Eq. (8.10), thus obtaining the useful relation:

$$f(1/2,k)=f(1/2,k,0)=g(1/2,k,0) \tag{8.11}$$

In order to illustrate the use and accuracy of this iteration procedure, we consider the case of scattering from a constant spherical inhomogeneity, for which we have already calculated $f(1/2,k)$ exactly in Sect. 7. We write down the solution again in the following form:

$$\begin{aligned} f(1/2,k)=&\cos[(k_1/k)kR_1]\cos kR_1+\frac{1}{(k_1/k)}\sin[(k_1/k)kR_1]\sin kR_1 \\ &+i\{-\cos[(k_1/k)kR_1]\sin kR_1+\frac{1}{(k_1/k)}\sin[(k_1/k)kR_1]\cos kR_1\} \end{aligned} \tag{8.12}$$

The $\lambda=1/2$ Jost integral Eq. (8.1) for this case is:

$$g(1/2,k,r)=1-\frac{V_1}{2ik}\int_r^{R_1}[1-e^{2ik(r-r')}]g(1/2,k,r')dr' \tag{8.13}$$

The first iteration $g_I(1/2,k,0)$ of Eq. (8.13) is:

$$\begin{aligned} g_I(1/2,k,0)&=g_o(1/2,k,0)+g_1(1/2,k,0) \\ &=1-\frac{1}{4}[(k_1/k)^2-1](1-\cos 2kR_1) \\ &\quad+\frac{i}{2}\{[(k_1/k)^2-1](kR_1-\frac{1}{2}\sin 2kR_1)\} \end{aligned} \tag{8.14}$$

The second iteration $g_{II}(1/2,k,0)$ is:

$$
\begin{aligned}
g_{II}(1/2,k,0) &= g_o(1/2,k,0)+g_1(1/2,k,0)+g_2(1/2,k,0) \\
&= g_I(1/2,k,0)+g_2(1/2,k,0) \\
&= 1-\frac{1}{4}[(k_1/k)^2-1](1-\cos 2kR_1) \\
&\quad -\frac{1}{8}[(k_1/k)^2-1]^2\{kR_1[kR_1+\sin 2kR_1]+\frac{3}{2}(\cos 2kR_1-1)\} \\
&\quad +\frac{i}{2}\{[(k_1/k)^2-1](kR_1-\frac{1}{2}\sin 2kR_1) \\
&\qquad -\frac{1}{2}[(k_1/k)^2-1]^2[kR_1(1+\frac{1}{2}\cos 2kR_1)-\frac{3}{4}\sin 2kR_1]\}
\end{aligned}
\tag{8.15}
$$

In Figs. 3-5, we have plotted $f(1/2,k)$, $g_I(1/2,k,0)$, and $g_{II}(1/2,k,0)$ as a function of kR_1 for $k_1/k=1.1$, 1.5, and 2.0, respectively. We see that the iteration procedure becomes less accurate with increasing k_1/k. The rapid convergence of the perturbation expansion for small k_1/k can be seen from the bound (8.8), since the n! term in the denominator begins to predominate over $N_R^n(r)/|k|^n$ for relatively small n. The slow convergence for large k_1/k is in agreement with the fact that the n! term does not begin to predominate until n is much larger.

For real λ and k, we have:

$$f(\lambda,-k)=f^*(\lambda,k), \tag{8.16}$$

and therefore:

$$
\begin{aligned}
\sigma_o/\pi R_1^2 &= |1-e^{2i\delta(1/2,k)}|^2/(kR_1)^2 \\
&= \left|1-\frac{f(1/2,k)}{f^*(1/2,k)}\right|^2/(kR_1)^2
\end{aligned}
\tag{8.17}
$$

Using Eqs. (8.12), (8.14), (8.15), and (8.17), in Figs. 6-8 we have plotted the exact result, first iteration, and second iteration for $\sigma_o/\pi R_1^2$ for $k_1/k=1.1$, 1.5, and 2.0, respectively. The iteration procedure is, in general, more accurate for the cross section than for the Jost function itself. For $k_1/k=1.1$, it is good for the

entire range of kR_1 considered. For $k_1/k=1.5$ and $k_1/k=2.0$, it is good out to approximately $kR_1=3\pi/4$.

From impact parameter arguments, the largest $\ell=\lambda-1/2$ which contributes significantly to the partial wave expansion (2.21) is on the order of kR_1. We can therefore conclude that, in terms of solving long-wavelength ($kR_1<<1$) scattering problems (with σ_o the only significant term) completely, this iteration procedure is good for any k_1/k. For shorter wavelengths and small k_1/k (e.g., $k_1/k=1.1$), it gives a good approximation to the $\ell=0$ contribution to the total cross section for the entire range of kR_1 considered. For shorter wavelengths and larger k_1/k (e.g., $k_1/k=1.5$, 2.0), it gives a good approximation to σ_o out to approximately $kR_1=3\pi/4$. We point out that, because Figs. 6-8 are an accurate measure of the total scattered field only for $kR_1<<1$, the behavior for larger kR_1 (including the seemingly strange behavior between $kR_1=\pi$ and 2π for $k_1/k=2.0$) will be overshadowed by the contributions of the higher order partial waves and should not be interpreted as an accurate description of the total scattered field.

We note that our plots of the $\ell=0$ cross section differ from those for the square well potential presented in standard quantum mechanics texts (e.g., Merzbacher[30]). The reason for this difference is the fact that in quantum scattering theory, we are concerned with the behavior of the cross section as a function of kR_1 for a fixed value of $V_1R_1^2$ (i.e., we have fixed the parameters of the potential), while in classical scattering theory, we are concerned with the behavior of the cross section as a function of

kR_1 for a fixed value of the relative index of refraction k_1/k. Also, in classical scattering theory, the phase shift and cross section are zero as $kR_1 \to 0$, while in quantum scattering theory, the phase shift and therefore the cross section for $kR_1 \to 0$ are dependent on the value of $V_1R_1^2$ and can, in fact, be non-zero, depending on the presence of bound states and the phase shift normalization at infinite energy.

9. General Jost Integral Equation Formulation for Arbitrary λ for Scattering from an Arbitrary Inhomogeneity

The general Jost integral equation formulation for arbitrary λ for scattering from an arbitrary inhomogeneity is based on the integral equations for $\phi(\lambda,k,r)$ and $f(\lambda,k,r)$ presented in Sect. 5:

$$\phi(\lambda,k,r)=r^{\lambda+1/2}+\frac{1}{2\lambda}\int_0^r\left[\left(\frac{\xi}{r}\right)^{\lambda}-\left(\frac{r}{\xi}\right)^{\lambda}\right]\sqrt{r\xi}\,[k^2-V(\xi)]\,\phi(\lambda,k,\xi)\,d\xi \tag{5.12}$$

$$f(\lambda,k,r)=e^{-ikr}+\frac{1}{k}\int_r^{\infty}\sin\left[k(r'-r)\right]\left[V(r')+\frac{\lambda^2-1/4}{r'^2}\right]f(\lambda,k,r')\,dr' \tag{5.22}$$

For any $V(r)$ satisfying requirements (5.1), we can solve Eqs. (5.12) and (5.22) using an iteration procedure (or other applicable method) analogous to that used in Sect. 8. Knowing $\phi(\lambda,k,r)$ and $f(\lambda,k,r)$, we can calculate $\phi'(\lambda,k,r)$ and $f'(\lambda,k,r)$ and therefore $f(\lambda,k)$, which can be evaluated at any point in the common domain of $\phi(\lambda,k,r)$ and $f(\lambda,k,r)$.

In the case of scattering from a constant spherical inhomogeneity, we have the following two integral equations:

$$\begin{aligned}\phi(\lambda,k,r)=r^{\lambda+1/2}&+\frac{1}{2\lambda}\int_0^{R_1}\left[\left(\frac{\xi}{r}\right)^{\lambda}-\left(\frac{r}{\xi}\right)^{\lambda}\right]\sqrt{r\xi}\,[k^2+V_1]\,\phi(\lambda,k,\xi)\,d\xi\\&+\frac{1}{2\lambda}\int_{R_1}^{r}\left[\left(\frac{\xi}{r}\right)^{\lambda}-\left(\frac{r}{\xi}\right)^{\lambda}\right]\sqrt{r\xi}\,k^2\,\phi(\lambda,k,\xi)\,d\xi\end{aligned} \tag{9.1}$$

$$\begin{aligned}f(\lambda,k,r)=e^{-ikr}&+\frac{1}{k}\int_r^{R_1}\sin\left[k(r'-r)\right]\left[-V_1+\frac{\lambda^2-1/4}{r'^2}\right]f(\lambda,k,r')\,dr'\\&+\frac{1}{k}\int_{R_1}^{\infty}\sin\left[k(r'-r)\right]\left[\frac{\lambda^2-1/4}{r'^2}\right]f(\lambda,k,r')\,dr'\end{aligned} \tag{9.2}$$

In the case of scattering from two constant spherical inhomogeneous layers, we have the following two integral equations:

$$\phi(\lambda,k,r)=r^{\lambda+1/2}+\frac{1}{2\lambda}\int_0^{R_1}\left[\left(\frac{\xi}{r}\right)^{\lambda}-\left(\frac{r}{\xi}\right)^{\lambda}\right]\sqrt{r\xi}\,[k^2+V_1]\,\phi(\lambda,k,\xi)\,d\xi$$

$$+\frac{1}{2\lambda}\int_{R_1}^{R_2}\left[\left(\frac{\xi}{r}\right)^{\lambda}-\left(\frac{r}{\xi}\right)^{\lambda}\right]\sqrt{r\xi}\,[k^2+V_2]\,\phi(\lambda,k,\xi)\,d\xi \tag{9.3}$$

$$+\frac{1}{2\lambda}\int_{R_2}^{r}\left[\left(\frac{\xi}{r}\right)^{\lambda}-\left(\frac{r}{\xi}\right)^{\lambda}\right]\sqrt{r\xi}\,k^2\,\phi(\lambda,k,\xi)\,d\xi$$

$$f(\lambda,k,r)=e^{-ikr}+\frac{1}{k}\int_r^{R_1}\sin\left[k(r'-r)\right]\left[-V_1+\frac{\lambda^2-1/4}{r'^2}\right]f(\lambda,k,r')\,dr'$$

$$+\frac{1}{k}\int_{R_1}^{R_2}\sin\left[k(r'-r)\right]\left[-V_2+\frac{\lambda^2-1/4}{r'^2}\right]f(\lambda,k,r')\,dr' \tag{9.4}$$

$$+\frac{1}{k}\int_{R_2}^{\infty}\sin\left[k(r'-r)\right]\left[\frac{\lambda^2-1/4}{r'^2}\right]f(\lambda,k,r')\,dr'$$

In the case of a cutoff inhomogeneity [$V(r)=0$ for $r>R$], e.g., the examples just considered, we can always write the solution in the exterior region ($r>R$) as [see Eqs. (6.4), (7.1), and (7.9)]:

$$f(\lambda,k,r)=ke^{\frac{-i\pi}{2}(\lambda+1/2)}\,rh^{(2)}_{\lambda-1/2}(kr) \tag{9.5}$$

We thus need only solve the integral equation for $\phi(\lambda,k,r)$. The Jost function can be evaluated at any point $r\geq R$, since we have restricted $f(\lambda,k,r)$ to the region $r>R$.

10. Conclusion

In this thesis we have shown how the Jost function formulation of quantum scattering theory can be applied to classical problems involving the scattering of a scalar plane wave by a medium with spherically symmetric inhomogeneities. We have seen how this technique can be used to solve the radial differential equation for scattering from a constant spherical inhomogeneity and from two constant spherical inhomogeneous layers and have compared it with the standard method of partial waves for these cases.

In situations where the radial equation with Jost boundary conditions is not exactly solvable, it may be useful to convert it into an integral equation incorporating these boundary conditions. We have shown that the $\ell=0$ Jost integral equation for a cutoff inhomogeneity can be solved using an iteration procedure and have used the exactly solvable constant spherical inhomogeneity as an example to check the usefulness of the iterative procedure. Although the iterative technique is not the most straightforward method for the case of a constant spherical inhomogeneity, we feel that it may be more useful in situations where the inhomogeneity assumes a more complicated functional form. We have found that the iteration procedure is good for long-wavelength ($kR_1 \ll 1$) problems for any k_1/k. For shorter wavelengths and small k_1/k (e.g., $k_1/k=1.1$), it gives a good approximation to σ_0 for the entire range of kR_1 considered ($0 \leq kR_1 \leq 2\pi$). For shorter wavelengths and larger k_1/k (e.g., $k_1/k=1.5$, 2.0), it gives a good approximation to σ_0 out to approximately $kR_1=3\pi/4$.

As an indication of further work that can be done in this field, we have written down the Jost integral equations for arbitrary ℓ for scattering from an arbitrary inhomogeneity with and without a cutoff. A complete solution of problems for arbitrary wavelength would involve solving these integral equations and summing the entire partial wave series using methods such as the Watson-Sommerfeld transformation.

ACKNOWLEDGMENTS

The author wishes to thank Professor Arthur O. Williams, Jr., his adviser at Brown University, and Mr. Burton G. Hurdle, Acting Head of the Propagation Branch, Acoustics Division, at the Naval Research Laboratory, for their cooperation and helpful suggestions. In particular, thanks are due to Mr. David Chang for helpful discussions and assistance in computer programming and to Mr. Kenneth D. Flowers for useful comments. The author is indebted to Mrs. Frances A Vandivier for her patient and careful typing of the manuscript. Above all, he would like to express his sincere gratitude to Dr. John A. DeSanto, his adviser at the Naval Research Laboratory, for his patient guidance, constant encouragement, and constructive criticism in the preparation of this thesis.

Appendix A

Scattering from a Constant Spherical Inhomogeneity: The Jost Function for $\lambda=1/2$ (Differential Equation Approach)

We can find $f(1/2,k)$ for scattering from a constant spherical inhomogeneity by letting $\lambda=1/2$ in Eq. (5.5), which then becomes:

$$\frac{d^2u_o(r)}{dr^2}+[k^2-V(r)]u_o(r)=0 \tag{A.1}$$

The solutions in the two regions are:

$$\begin{aligned} &\text{Region 1:} \quad \phi(1/2,k,r)=Ae^{ik_1r}+Be^{-ik_1r} \\ &\text{Region 3:} \quad f(1/2,k,r)=Ce^{ikr}+De^{-ikr} \end{aligned} \tag{A.2}$$

Imposing the boundary conditions (5.13), we find that:

$$A=-B=1/2ik_1 \tag{A.3}$$

Imposing the boundary conditions (5.20), we find that $C=0$ and $D=1$. Since the point $r=R_1$ is the common domain of $\phi(1/2,k,r)$ and $f(1/2,k,r)$, we evaluate the Jost function at $r=R_1$ and thus obtain:

$$f(1/2,k)=\frac{1}{2}e^{-ikR_1}\left[\left(1-\frac{k}{k_1}\right)e^{-ik_1R_1}+\left(1+\frac{k}{k_1}\right)e^{ik_1R_1}\right], \tag{A.4}$$

which agrees with Eq. (6.10).

Appendix B

Scattering from Two Constant Spherical Inhomogeneous Layers: The Jost Function for $\lambda=1/2$ (Differential Equation Approach)

We can find $f(1/2,k)$ for scattering from two constant spherical inhomogeneous layers by using Eq. (A.1) and our results from Appendix A. We thus write the solutions in the three regions as:

$$\phi(1/2,k,r)=\frac{1}{2ik_1}(e^{ik_1r}-e^{-ik_1r}),\ r<R_1$$
$$f(1/2,k,r)=\begin{cases} f_2(1/2,k,r)=Ce^{ik_2r}+De^{-ik_2r}, & R_1<r<R_2 \\ f_3(1/2,k,r)=e^{-ikr}, & r>R_2 \end{cases} \tag{B.1}$$

We calculate the constants C and D using continuity boundary conditions at $r=R_2$:

$$f_2(1/2,k,r)=f_3(1/2,k,r)\big|_{r=R_2}$$
$$\frac{df_2(1/2,k,r)}{dr}=\frac{df_3(1/2,k,r)}{dr}\bigg|_{r=R_2} \tag{B.2}$$

We find that:

$$C=\frac{1}{2}\left(1-\frac{k}{k_2}\right)e^{-i(k_2+k)R_2} \tag{B.3}$$

$$D=\frac{1}{2}\left(1+\frac{k}{k_2}\right)e^{i(k_2-k)R_2} \tag{B.4}$$

Since the point $r=R_1$ is the common domain of $\phi(1/2,k,r)$ and $f(1/2,k,r)$, we evaluate the Jost function at $r=R_1$ and thus obtain:

$$f(1/2,k)=\frac{1}{4}\left(1-\frac{k}{k_2}\right)e^{ik_2R_1}e^{-i(k_2+k)R_2}\left[\left(1+\frac{k_2}{k_1}\right)e^{-ik_1R_1}+\left(1-\frac{k_2}{k_1}\right)e^{ik_1R_1}\right]$$
$$+\frac{1}{4}\left(1+\frac{k}{k_2}\right)e^{-ik_2R_1}e^{i(k_2-k)R_2}\left[\left(1-\frac{k_2}{k_1}\right)e^{-ik_1R_1}+\left(1+\frac{k_2}{k_1}\right)e^{ik_1R_1}\right], \tag{B.5}$$

which agrees with Eq. (7.8).

Appendix C

Scattering from a Constant Spherical Inhomogeneity: A Laplace Transform Method for Solving the Jost Integral Equation for $\lambda=1/2$

We can solve exactly the $\lambda=1/2$ Jost integral equation (8.13) for scattering from a constant spherical inhomogeneity using Laplace transforms[31]. We state the convolution theorem for Laplace transforms:

$$L\left[\int^{r} w(r-r')g(r')dr'\right]=W(p)G(p), \tag{C.1}$$

where $W(p)$ and $G(p)$ are the Laplace transforms of $w(r)$ and $g(r)$, respectively, and apply it to the following Volterra integral equation:

$$g(r)=h(r)+\int_{o}^{r} w(r-r')g(r')dr', \quad r>o \tag{C.2}$$

We take the Laplace transform of both sides of Eq. (C.2) and assume that $H(p)$ [the transform of $h(r)$] is analytic for Re $p>\tau_o$ and that the region of analyticity of $W(p)$ has at least a strip, parallel to the imaginary axis of p, in common with the band for $H(p)$. Then, in that strip, we have that:

$$G(p)=H(p)+W(p)G(p), \tag{C.3}$$

and therefore:

$$G(p)=\frac{H(p)}{1-W(p)} \tag{C.4}$$

Inverting the Laplace transform, we obtain a solution of Eq. (C.2):

$$g(r)=\frac{1}{2\pi i}\int_{-i\infty+\tau_o}^{i\infty+\tau_o}\left[\frac{H(p)}{1-W(p)}\right]e^{pr}dp, \quad \text{Re } r>o \tag{C.5}$$

We change variables, $r''=r'-R_1$, $dr''=dr'$, in Eq. (8.13), which then becomes:

$$g(1/2,k,r)=1+\frac{V_1}{2ik}\int_0^{r-R_1}[1-e^{2ik(r-r''-R_1)}]g(1/2,k,r''+R_1)dr'' \qquad (C.6)$$

If we let $r'''=r-R_1$ and $G(1/2,k,r''')=g(1/2,k,r'''+R_1)$, then Eq. (C.6) becomes:

$$G(1/2,k,r''')=1+\frac{V_1}{2ik}\int_0^{r'''}[1-e^{2ik(r'''-r'')}]G(1/2,k,r'')dr'', \qquad (C.7)$$

which is in the form of Eq. (C.2). We thus have that:

$$\begin{aligned} h(r''')&=1\\ H(p)&=1/p\\ w(r''')&=\frac{V_1}{2ik}(1-e^{2ikr'''}) \qquad (C.8)\\ W(p)&=\frac{-V_1}{p(p-2ik)} \end{aligned}$$

If we use Eqs. (C.8) and change variables back, Eq. (C.5) then becomes:

$$g(1/2,k,r)=\frac{1}{2\pi i}\int_{-i\infty+\tau_0}^{i\infty+\tau_0}\left[\frac{p-2ik}{p^2-2ikp+V_1}\right]e^{p(r-R_1)}dp, \text{ Re } r>o \qquad (C.9)$$

We can evaluate the latter integral by using the calculus of residues, where, adding on an infinite semicircle, we have the contour of integration shown in Fig. 9. The integrand I(p) has simple poles at $p_1=i(k+k_1)$ and $p_2=i(k-k_1)$, so that:

$$g(1/2,k,r)=\text{Res}[I(p),p_1]+\text{Res}[I(p),p_2] \qquad (C.10)$$

Evaluating the residues, we obtain:

$$\begin{aligned} \text{Res}[I(p),p_1]&=\frac{1}{2}\left(1-\frac{k}{k_1}\right)e^{i(k+k_1)(r-R_1)}\\ \text{Res}[I(p),p_2]&=\frac{1}{2}\left(1+\frac{k}{k_1}\right)e^{i(k-k_1)(r-R_1)}, \end{aligned} \qquad (C.11)$$

and we therefore have that:

$$f(1/2,k)=g(1/2,k,0)=\frac{1}{2}e^{-ikR_1}\left[\left(1-\frac{k}{k_1}\right)e^{-ik_1R_1}+\left(1+\frac{k}{k_1}\right)e^{ik_1R_1}\right], \qquad (C.12)$$

which agrees with Eqs. (6.10) and (A.4).

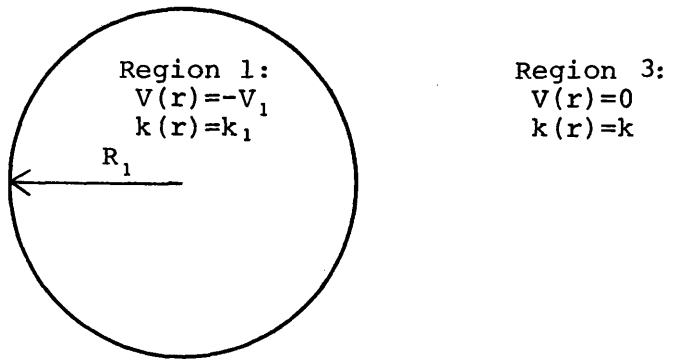

Fig. 1 - Constant spherical inhomogeneity

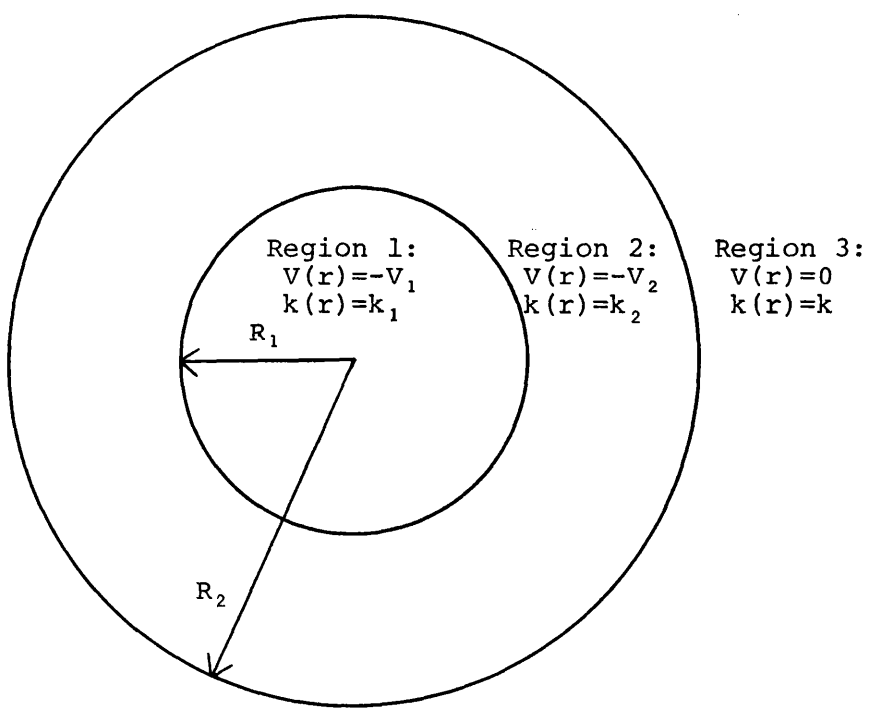

Fig. 2 - Two constant spherical inhomogeneous layers

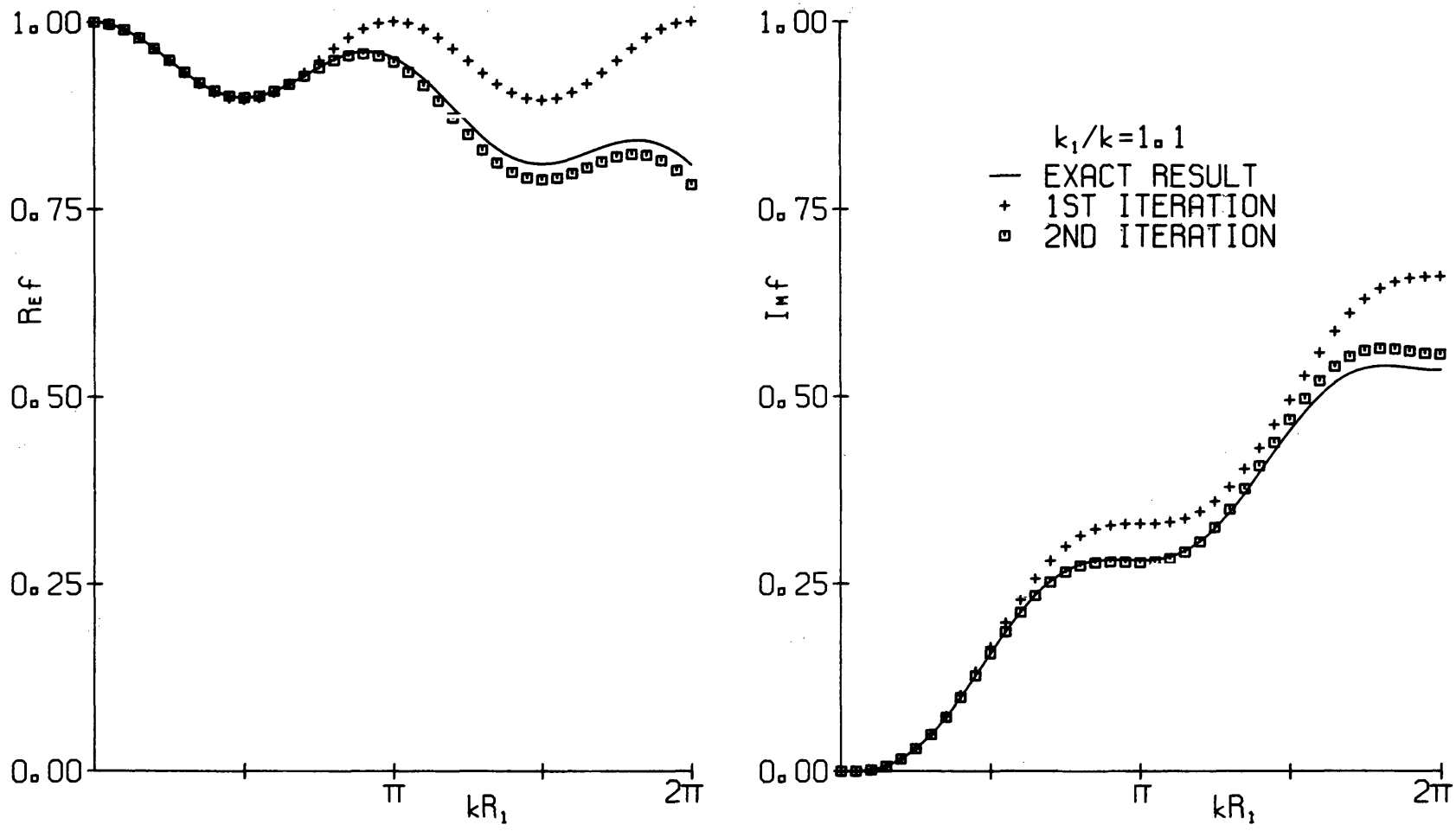

Fig. 3 - f(1/2,k) VS. kR_1 for scattering from a constant spherical inhomogeneity, $k_1/k=1.1$

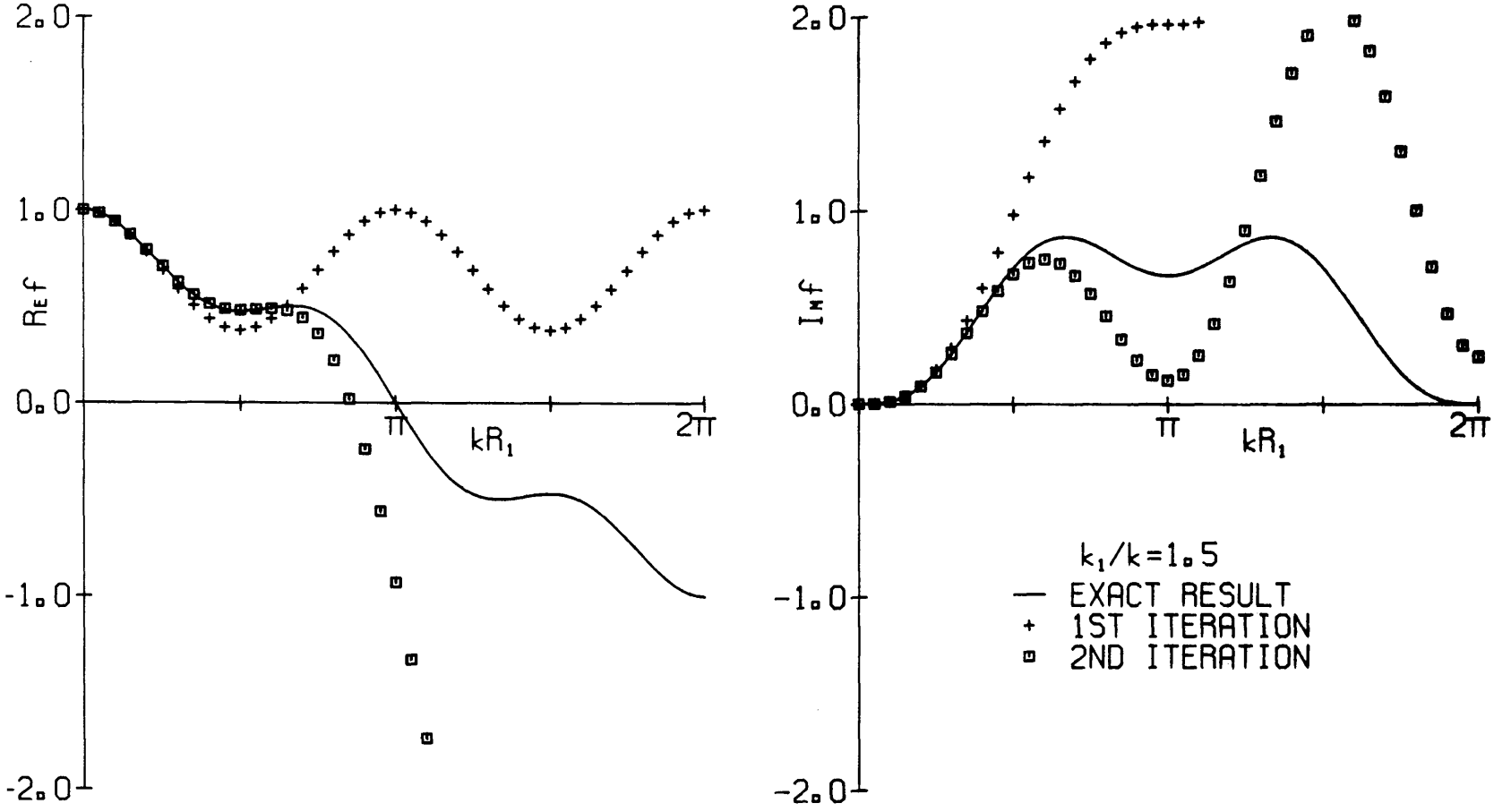

Fig. 4 - f(1/2,k) VS. kR_1 for scattering from a constant spherical inhomogeneity, $k_1/k=1.5$

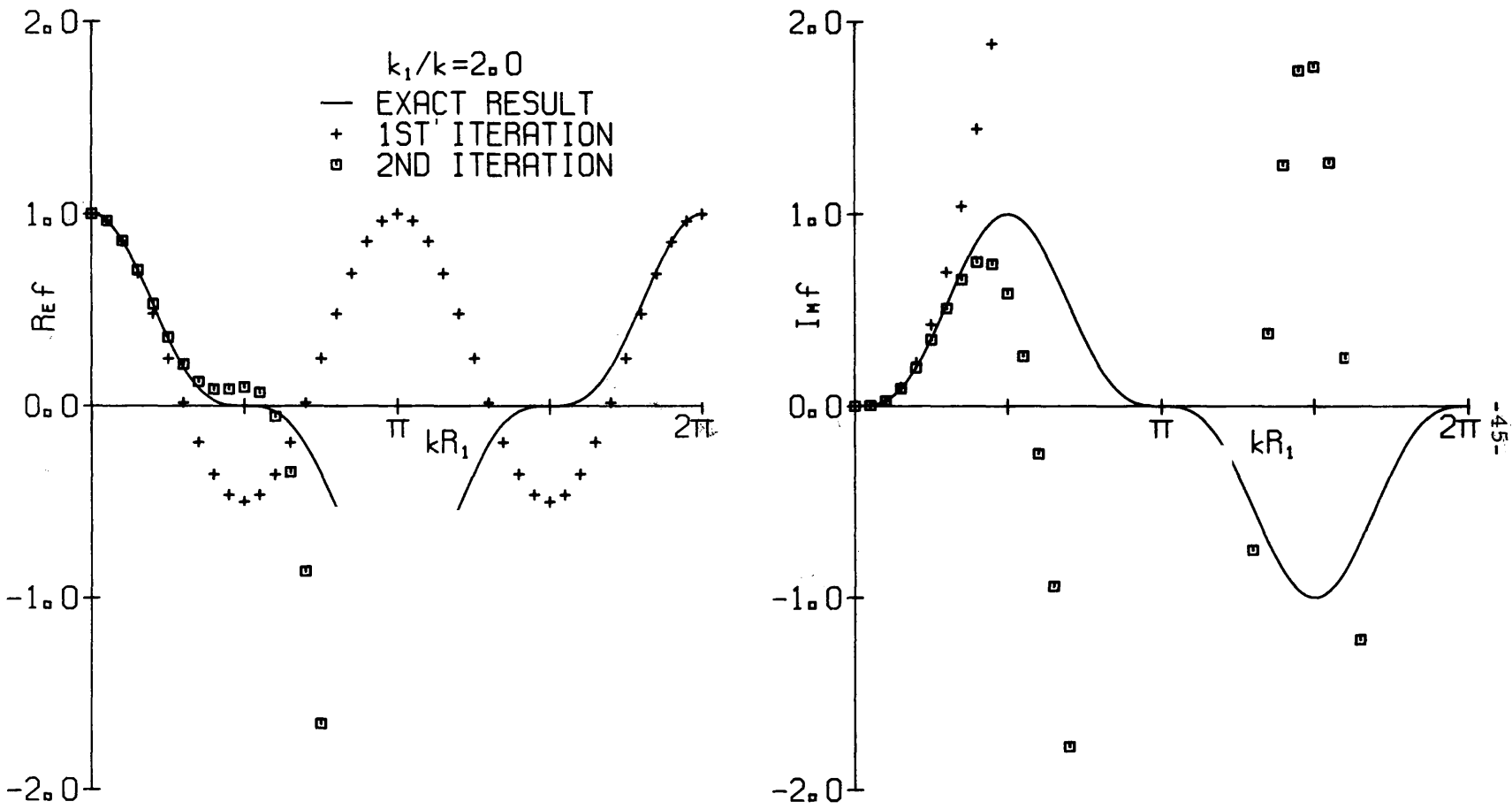

Fig. 5 - f(1/2,k) VS. kR_1 for scattering from a constant spherical inhomogeneity, $k_1/k=2.0$

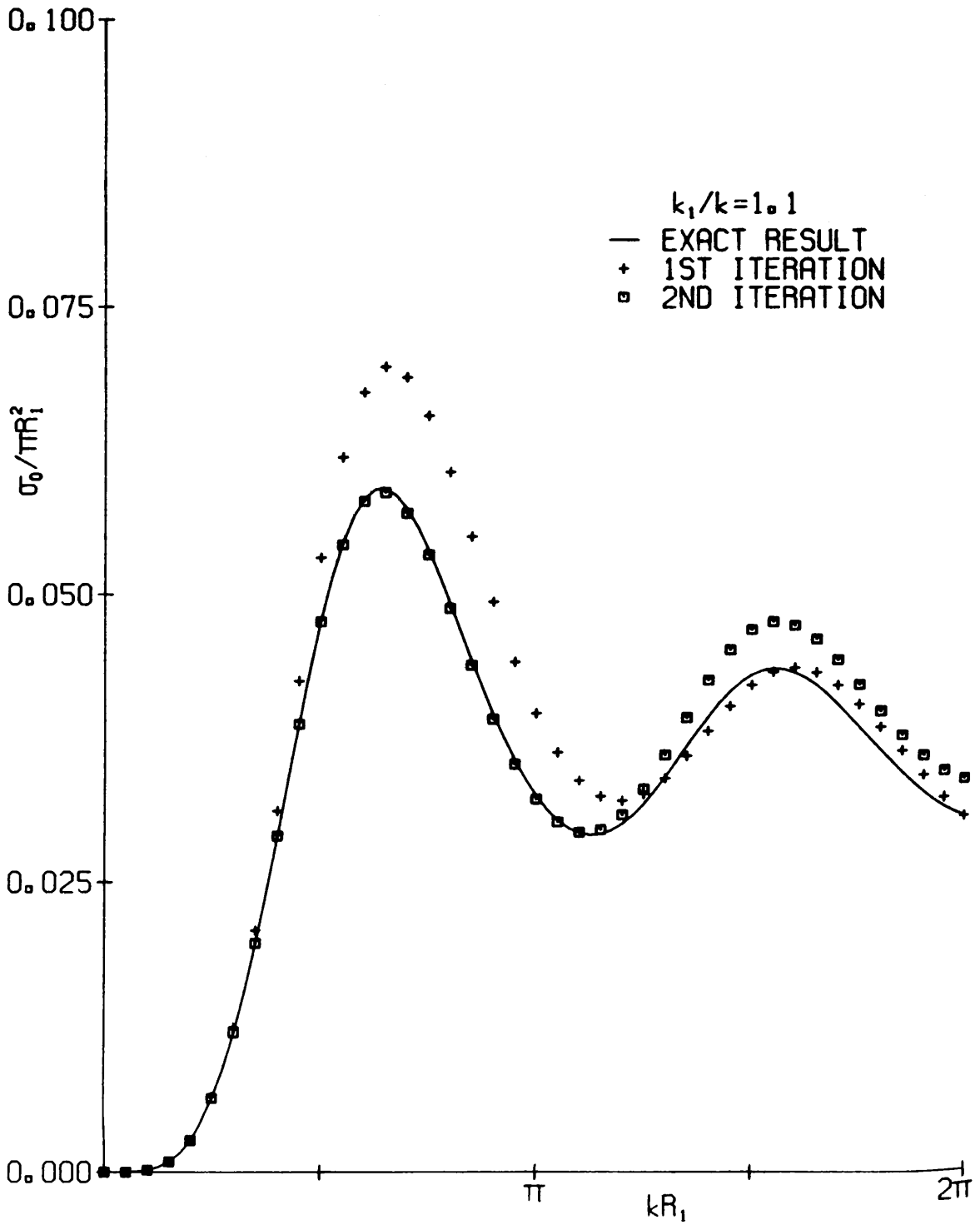

Fig. 6 - $\sigma_o/\pi R_1^2$ VS. kR_1 for scattering from a constant spherical inhomogeneity, $k_1/k=1.1$

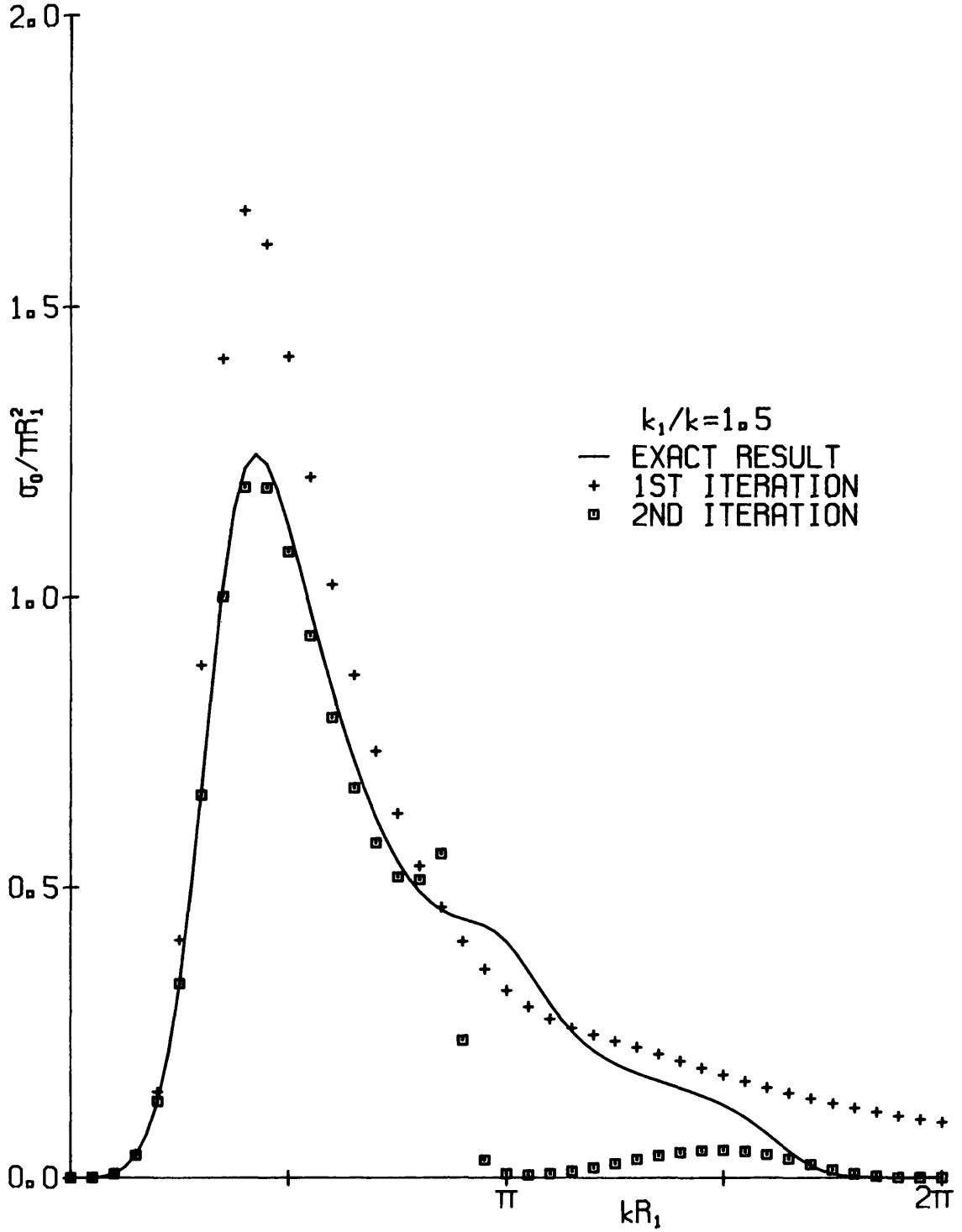

Fig. 7 - $\sigma_o/\pi R_1^2$ VS. kR_1 for scattering from a constant spherical inhomogeneity, $k_1/k=1.5$

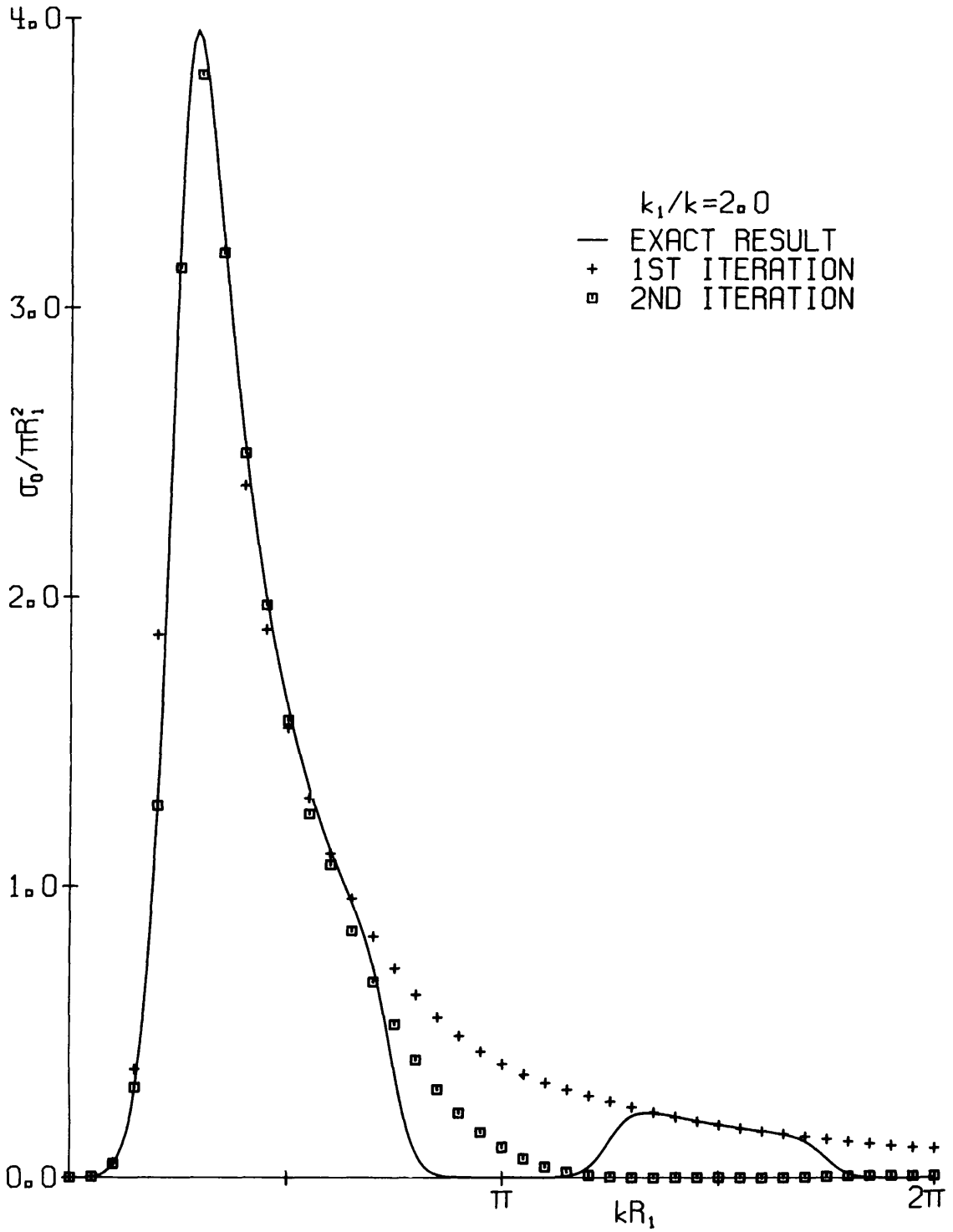

Fig. 8 - $\sigma_o / \pi R_1^2$ VS. kR_1 for scattering from a constant spherical inhomogeneity, $k_1/k=2.0$

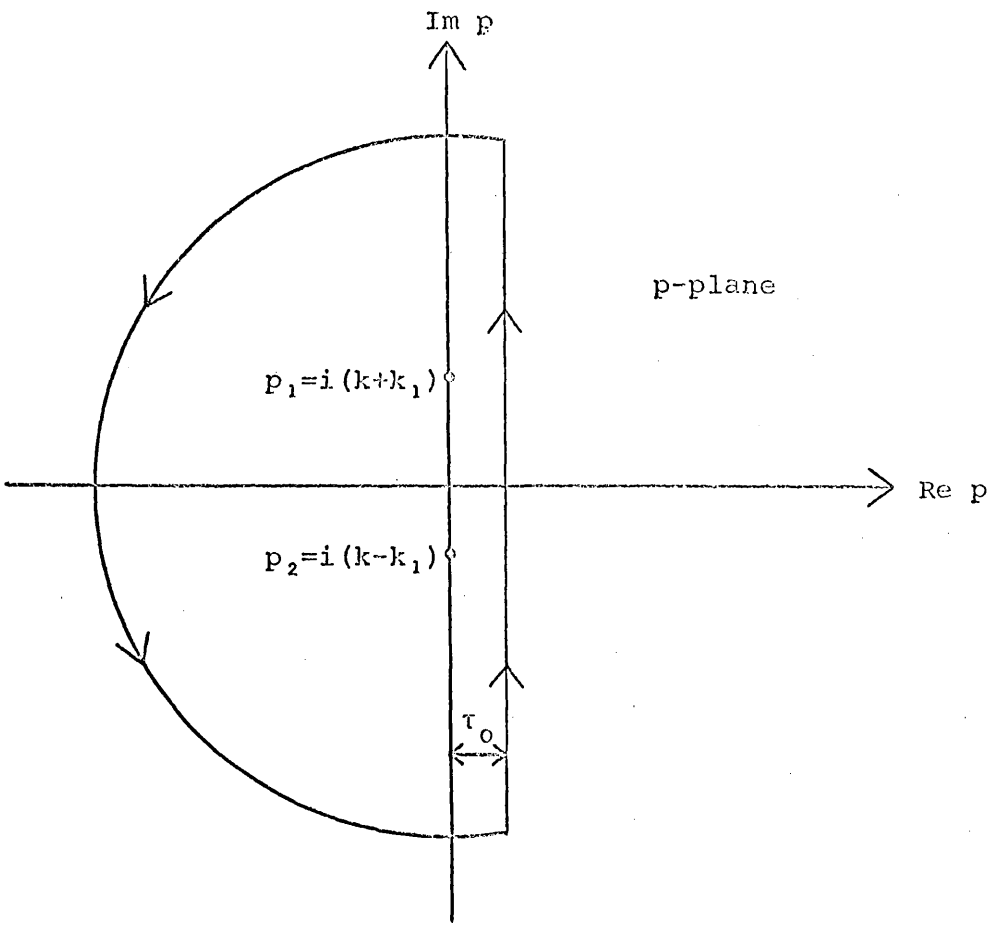

Fig. 9 - Contour of integration for Laplace transform method

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Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Naval Research Laboratory Washington, D.C. 20390	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

SCATTERING BY SPHERICALLY SYMMETRIC INHOMOGENEITIES

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

An interim report; work is continuing on the problem.

5. AUTHOR(S) *(First name, middle initial, last name)*

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6. REPORT DATE October 7, 1969	7a. TOTAL NO. OF PAGES 56	7b. NO. OF REFS 31
8a. CONTRACT OR GRANT NO. S01-40 b. PROJECT NO. RF 05-121-401-4070 c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) NRL Report 6943 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*	

10. DISTRIBUTION STATEMENT

This document has been approved for public release and sale; its distribution is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Department of the Navy (Office of Naval Research), Washington, D.C. 20360

13. ABSTRACT

The Jost function formulation of quantum scattering theory is applied to classical problems involving the scattering of a scalar plane wave by a medium in which the velocity is a function only of the spherical radial coordinate. This technique is used to solve the radial differential equation for scattering from a constant spherical inhomogeneity and from two constant concentric spherical inhomogeneous layers and is compared with the standard method of partial waves for these two cases. The radial equation can be converted into an integral equation incorporating the Jost boundary conditions. The $\ell=0$ partial wave integral equation for a constant inhomogeneity is solved using an iteration procedure (the first two iterations are considered). The Jost function and $\ell=0$ cross section σ_o are plotted as a function of kR_1, where k is the wave number in the surrounding medium and R_1 is the sphere radius. The iterative technique is good for long wavelengths ($kR_1 \ll 1$) and any ratio of wave numbers in the scattering and surrounding media. For shorter wavelengths and small ratio of wave numbers (e.g., $k_1/k = 1.1$, where k_1 is the wave number in the scattering medium), it gives a good approximation to σ_o for the entire range of kR_1 considered ($0 \leq kR_1 \leq 2\pi$). For shorter wavelengths and larger ratio of wave numbers (e.g., $k_1/k = 1.5, 2.0$), it gives a good approximation to σ_o out to approximately $kR_1 = 3\pi/4$. More general problems using this method are also discussed.

DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0101-807-6801

Security Classification

Security Classification

14. KEY WORDS	LINK A ROLE	LINK A WT	LINK B ROLE	LINK B WT	LINK C ROLE	LINK C WT
Quantum scattering theory						
Classical scattering theory						
Underwater sound propagation						
Acoustic scattering						
Potential scattering						
Wave propagation in inhomogeneous media						

DD FORM 1 NOV 65 1473 (BACK)
(PAGE 2)

Security Classification