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Theory on Afterburning in a Low-Altitude Rocket Exhaust Plume

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ABSTRACT

An improved aerodynamic model is proposed for the exhaust plume issuing from a low-altitude rocket nozzle exit into quiescent ambient air in which chemical reactions and compressibility predominate. The model is based on the presently available knowledge and established aerodynamic principles for the various flow regimes involved. The major assumptions made are individually substantiated by related experimental evidence in existing literature.

A theoretical analysis is carried over the whole flow region based on the Kármán's integral approach to find solutions for the gross aerodynamic and thermodynamic behavior of the plume. Theory incorporating on empirically derived constants provides the required radial distribution function for the integration. Solutions are obtained for the velocity, density, temperature and chemical composition fields for extended regions of the plume. These results will provide the much needed framework for the understanding of the elements which contribute to the phenomenon of radar attenuation of rocket exhaust plumes.

PROBLEM STATUS

This is an interim report on a continuing problem.

AUTHORIZATION

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INTRODUCTION

In some of the aerodynamic models used in the study of rocket exhaust plumes in the existing literature, the flow field is first artificially broken down into several pieces which are analyzed independently, and the results from these uncoupled analyses are then artificially patched together. The validity of these models extends to a region downstream of the rocket nozzle only as far as a distance of the order of the diameter of the nozzle exit at most, which is orders of magnitude smaller than the total range of the rocket exhaust plume. In some other models, the concept of a fully developed turbulent jet is used. As a consequence, they can be expected to be valid very far downstream of the plume. The important processes of afterburning and air dilution usually become severe at axial stations throughout the whole downstream region of the rocket exit plane as was pointed out by Balwanz and Lee (1). For this reason, Lee and Balwanz (2) took the first step to establish a more realistic and justifiable aerodynamic model of rocket exhaust plume for all axial ranges, even if unavoidably at the sacrifice of detailed accuracy. Their theoretical analysis based on the Kármán's integral approach applied to their proposed simple aerodynamic model has been found to agree closely with related experimental data existing in the literature. However, their model serves adequately only for those cases where chemical reactions and compressibility do not exert a controlling influence and consequently their accompanying analysis does not yield adequate information on the temperature, density and composition fields. An improved model required for the modern high-energy systems in which chemical reactions and compressibility predominate is thus in order.

EXHAUST GAS: CO, H_2, CO_2, H_2O, N_2, HCl
AIR : O_2, N_2

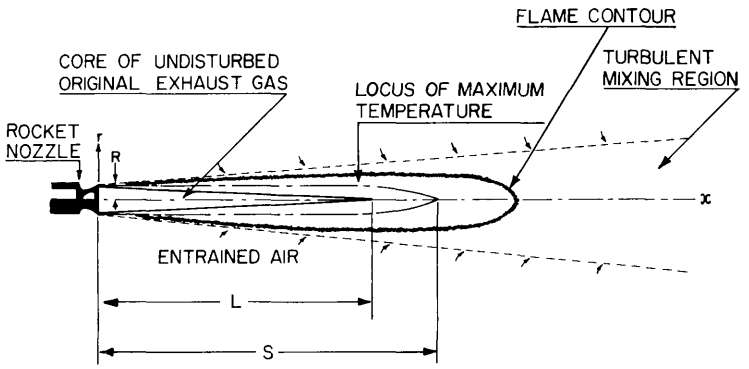

Fig. 1 - General structure of flow field

FORMULATION OF FUNDAMENTAL EQUATIONS

Let us assume the equalization of pressure between the exhaust and the surroundings and ignore the radiation loss. The supersonic gas stream leaving the nozzle is assumed to remain undisturbed within a cone of base diameter 2R and length L, which, from existing experimental results is a known function of R and the exit Mach number (2). Surrounding the cone is an increasing annular turbulent mixing region of exhaust gas and air, and the burning causes the temperature profile to deviate from the decaying velocity profile and to have its maxima somewhere therein. The locus of this temperature maxima is a circular ring which shrinks in size due to decreased supply of original gas beyond $x = L$ and finally to a point along the axis at $x = S$, beyond which the plume can be considered fully developed. The exhaust gas considered consists of CO, H_2, CO_2, H_2O, N_2, and $HC\ell$, and the ambient air O_2 and N_2. The predominant chemical reactions considered are:

$$2CO + O_2 \longrightarrow 2CO_2$$

$$2H_2 + O_2 \longrightarrow 2H_2O$$

Other components and reactions including ionizing reactions may be incorporated, but are ignored here for simplicity in presentation. A sketch of the general structure of flow field of a rocket exhaust plume is shown in Fig. 1.

If we let u and v be the components in the x - (axial) and r - (radial) directions respectively of the time-mean velocity of the gas mixture in the turbulent mixing region of the exhaust plume, the governing equations for such a region are then as follows:

Global Continuity Equation:

$$\frac{\partial}{\partial x}(\rho r u) + \frac{\partial}{\partial r}(\rho r v) = 0 \tag{1}$$

which is a statement of the conservation of mass of the reactive gas mixture as a whole.

Boundary-Layer Equation of Conservation of Species:

$$\rho r \left(u \frac{\partial y_i}{\partial x} + v \frac{\partial y_i}{\partial r}\right) = r w_i - \frac{\partial}{\partial r}\left(\rho r y_i J_{r_i}\right) \tag{2}$$

which states that the local convection transport of species i, as represented by the left-hand side of the equation, is balanced out by the local creation of that species and the transverse transport of that species due to turbulent mass diffusion, as shown by the first and the second terms on the right-hand side respectively.

Global Boundary-Layer Momentum Equation:

$$\rho r \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r}\right) = \frac{\partial}{\partial r}\left(r \tau\right) - \frac{\partial}{\partial r}\left[\rho r u \sum_i y_i J_i\right] \tag{3}$$

which states that the local convection transport of linear momentum for the mixture, as shown by the left-hand side of the equation, is balanced out by the action of the turbulent shear stress and the transverse transport of linear momentum due to turbulent mass diffusions of all the species in the mixture, as shown by the first and the second terms on the right-hand side respectively.

Global Boundary-Layer Energy Equation:

$$\rho r \left(u \frac{\partial h^o}{\partial x} + v \frac{\partial h^o}{\partial r} \right) = \frac{\partial}{\partial r} (rq) - \frac{\partial}{\partial r} \left[\rho r \sum_i y_i h_i^o \; J_{r_i} \right]$$

$$- r \left[\frac{\Delta H_{CO}}{M_{CO}} \; w_{CO} + \frac{\Delta H_{H_2}}{M_{H_2}} \; w_{H_2} \right] \qquad (4)$$

which states that the local convection transport of thermal and kinetic energies for the mixture, as shown by the left-hand side of the equation, is balanced out by the transverse turbulent eddy heat transfer, the transverse transport of thermal and kinetic energies due to turbulent mass diffusions of all the species in the mixture, and heat released from chemical reactions involving CO and H_2, as shown by the first, the second and the third terms on the right-hand side respectively.

Nomenclature:

ρ = local time-mean density

y_i = local time-mean mass fraction of species i

w_i = local time-mean mass creation rate of species i

J_{r_i} = local time-mean turbulent diffusion velocity of species i

τ = Reynolds turbulent stress

$h^o = h + \frac{u^2}{2}$, the local time-mean stagnation enthalpy

h = local time-mean enthalpy

q = eddy heat transfer

h_i^o = local time-mean stagnation enthalpy of species i

ΔH_{CO}, ΔH_{H_2} = molar heats of reaction of CO and H_2 respectively

M_i = molecular weight of species i

Eqs. (1) - (4) are also the governing equations for the inviscid core flow region of the original exhaust gas mixture, if the previously defined local time-mean quantities are identified just as the corresponding local quantities and all the terms on the right-hand side of these equations are put to zero.

Integrating Eq. (1) from 0 to ∞ with respect to r with the condition $v(x,0) = 0$ introduced, we have

$$\frac{d}{dx}\int_0^\infty (\rho r u)\,dr = -\lim_{r \to \infty} (\rho r v) \tag{5}$$

which states that the increase of the mass flux in the axial direction is supplied by the mass entrainment from the ambient air in the radial direction at the edge of the plume.

Integrating Eq. (2) from 0 to ∞ with respect to r with Eq. (1) and the conditions $v(x,0) = 0$ and $J_{r_i}(x,\infty) = 0$ introduced, we have

$$\frac{d}{dx}\int_0^\infty (\rho r u y_i)\,dr = \int_0^\infty (r w_i)\,dr - \lim_{r \to \infty} (\rho r v y_i) \tag{6}$$

which states that the increase of the mass flux of species i in the axial direction is supplied by the mass creation of species i due to chemical reactions in the plume and by the mass concentration of species i, if any, of the ambient air entrained in the radial direction at the edge of the plume.

Integrating Eq. (3) from 0 to ∞ with respect to r with Eq. (1) and the conditions $v(x,0) = 0$, $\tau(x,\infty) = 0$ and $J_{r_1}(x,\infty) = 0$ introduced, we have

$$\frac{d}{dx}\int_0^\infty (\rho r u^2)dr = 0 \tag{7}$$

which states that the linear momentum in the flow field of the plume is conserved.

Integrating Eq. (4) from 0 to ∞ with respect to r with Eq. (1) and the conditions $v(x,0) = 0$, $q(x,\infty) = 0$ and $J_{r_1}(x,\infty) = 0$ introduced, we have

$$\frac{d}{dx}\int_0^\infty (\rho r u h^o)dr = -\int_0^\infty \left[\frac{\Delta H_{CO}}{M_{CO}} w_{CO} + \frac{\Delta H_{H_2}}{M_{H_2}} w_{H_2}\right] r dr - \lim_{r \to \infty} (\rho r v h) \tag{8}$$

which states that the increase of the stagnation enthalpy flux in the axial direction is supplied by the heat release due to chemical reactions in the plume in which CO and H_2 are consumed and by the enthalpy of the ambient air entrained in the radial direction at the edge of the plume.

Eqs. (5) - (8) form the fundamental set of integrated governing equations. The entrainment terms in Eqs. (5), (6) and (8) are not specified. However, they can be eliminated among those equations to give equations free from complications due to entrainment. For instance, by combining the equation formed by multiplying $(y_{O_2})_o$, the value of y_{O_2} in the ambient air, on to Eq. (5),

$$\frac{d}{dx}\int_0^\infty \left[\rho r u (y_{O_2})_o \right] dr = - \lim_{r \to \infty} \left[\rho r v y_{O_2} \right]$$

and Eq. (6) with the species i identified as O_2, we obtain

$$\frac{d}{dx}\int_0^\infty \left\{ \rho u \left[(y_{O_2})_o - y_{O_2} \right] \right\} r dr = - \int_0^\infty (w_{O_2}) r dr \tag{9}$$

Similarly, we can have

$$\frac{d}{dx}\int_0^\infty \left\{ \rho u \left[(y_{N_2})_o - y_{N_2} \right] \right\} r dr = 0 \tag{10}$$

where $(y_{N_2})_o$ = value of y_{N_2} in the ambient air. Since entrained air contains only O_2 and N_2, Eq. (6) gives

$$\frac{d}{dx}\int_0^\infty \left\{ \rho u y_{CO} \right\} r dr = \int_0^\infty (w_{CO}) r dr \tag{11}$$

$$\frac{d}{dx}\int_0^\infty \left\{ \rho u y_{H_2} \right\} r dr = \int_0^\infty (w_{H_2}) r dr \tag{12}$$

$$\frac{d}{dx}\int_0^\infty \left\{ \rho u y_{CO_2} \right\} r dr = \int_0^\infty (w_{CO_2}) r dr \tag{13}$$

$$\frac{d}{dx}\int_0^\infty \left\{ \rho u y_{H_2O} \right\} r dr = \int_0^\infty (w_{H_2O}) r dr \tag{14}$$

and

$$\frac{d}{dx}\int_0^\infty \left\{ \rho u y_{HC\ell} \right\} r dr = 0 \tag{15}$$

By combining the equation formed by multiplying T_o, the ambient temperature, on to Eq. (5)

$$\frac{d}{dx}\int_0^\infty (\rho r u T_o) dr = - \lim_{r \to \infty} (\rho r v T_o)$$

and Eq. (8) with the assumption of an ideal gas for the gas mixture, we obtain

$$\frac{d}{dx}\int_0^\infty (\rho u \Delta T^o) r dr = -\frac{1}{C_p}\int_0^\infty \left[\frac{\Delta H_{CO}}{M_{CO}} w_{CO} + \frac{\Delta H_{H_2}}{M_{H_2}} w_{H_2} \right] r dr \qquad (16)$$

where

$$\Delta T^o = T^o - T_o$$

$$T^o = T + \frac{u^2}{2C_p}$$, the time-mean stagnation temperature.

C_p = average specific heat at constant pressure for the mixture.

Furthermore, the mass creation rates for the various species, the w_i's, are related to one another through the chemical reactions and therefore Eqs. (9), (11) - (14) and (16) can be used to generate equations which govern the changes of the fluxes of the related species. For instance, by using the relation

$$\frac{w_{O_2}}{M_{O_2}} = \frac{1}{2}\frac{w_{CO}}{M_{CO}} + \frac{1}{2}\frac{w_{H_2}}{M_{H_2}}$$

we can combine Eqs. (9), (11) and (12) to give

$$\frac{d}{dx}\int_0^\infty \left\{ \frac{1}{M_{O_2}} \left(\rho u \left[\left(y_{O_2} \right)_o - y_{O_2} \right] \right) + \frac{1}{2M_{CO}} \left(\rho u y_{CO} \right) \right.$$

$$\left. + \frac{1}{2M_{H_2}} \left(\rho u y_{H_2} \right) \right\} r dr = 0 \tag{17}$$

In a similar fashion, we can also obtain the following equations:

$$\frac{d}{dx}\int_0^\infty \left\{ \frac{1}{M_{CO_2}} \left(\rho u y_{CO_2} \right) + \frac{1}{M_{CO}} \left(\rho u y_{CO} \right) \right\} r dr = 0 \tag{18}$$

$$\frac{d}{dx}\int_0^\infty \left\{ \frac{1}{M_{H_2O}} \left(\rho u y_{H_2O} \right) + \frac{1}{M_{H_2}} \left(\rho u y_{H_2} \right) \right\} r dr = 0 \tag{19}$$

Finally, eliminating $\left(\int_0^\infty w_{CO}\, r dr \right)$ and $\left(\int_0^\infty w_{H_2}\, r dr \right)$ among Eqs. (11), (12) and (16), we have

$$\frac{d}{dx}\int_0^\infty \rho u \left(\Delta T^o + \frac{\Delta H_{CO}}{C_p M_{CO}} y_{CO} + \frac{\Delta H_{H_2}}{C_p M_{H_2}} y_{H_2}\right) r dr = 0 \tag{20}$$

REGION EXTENDING TO THE DISAPPEARANCE OF UNDISTURBED CORE OF ORIGINAL EXHAUST GAS ($0 \leq x \leq L$)

At this point, the radial distributions of the mass fluxes of the various species across the turbulent mixing layer must be determined. For turbulent mixing without chemical reaction, the species profiles should follow the velocity profile closely. However, for the case with chemical reactions, the former can differ considerably from the latter. Baron (3) studied the flow field of a fully developed axisymmetrical turbulent diffusion flame of gas fuel and formulated the radial distributions of both the fuel gas and the axial velocity which, in terms of the present notation, read

$$\rho u y_f = \rho_a u_a (y_f)_a \exp\left(-\frac{r^2}{\lambda b^2}\right)$$

$$\rho^{\frac{1}{2}} u = \rho_a^{\frac{1}{2}} u_a \exp\left(-\frac{r^2}{b^2}\right)$$

where the subscript "a" refers to the condition along the flame axis, $b \equiv b(x)$ is a characteristic thickness of the turbulent mixing region, and the constant λ, which reflects the extent of the effect of chemical reaction

on the distribution of the fuel gas, would have to be supplied from experiment. It should be noted here that in a non-reactive mixture, the profile of the flux of fuel should coincide with that of the flux of momentum and λ should equal 0.5. In a reactive mixture, λ in general should be slightly below 0.5. The appearance of the density ρ brings in the effect due to compressibility. These distributions have been found to agree closely with experimental results by Hinze and Van Der Hegge Zignen (4) and the constant λ has been found to lie in the neighborhood of 0.4.

For the annular turbulent mixing region, $r_1 \leq r < \infty$, similar profiles for the fluxes of the fuel gases, CO and H_2, and the axial velocity will be made

$$\rho u y_{CO} = \rho_1 u_1 (y_{CO})_1 \exp\left[- \frac{(r - r_1)^2}{\lambda b^2} \right] \tag{21}$$

$$\rho u y_{H_2} = \rho_1 u_1 (y_{H_2})_1 \exp\left[- \frac{(r - r_1)^2}{\lambda b^2} \right] \tag{22}$$

$$\rho^{1/2} u = \rho_1^{1/2} u_1 \exp\left[- \frac{(r - r_1)^2}{b^2} \right] \tag{23}$$

where

$$r_1 \equiv r_1 (x) = R \left(1 - \frac{x}{L}\right) \tag{23}$$

is the radius of the undisturbed core and the length of the core L is supplied from comparisons of aerodynamic measurements on air jets and

rocket exhaust plumes and the corresponding theoretical results based on this type of flow model. For instance, as a first approximation (2) it has been established that, for the exit plane Mach number M_e in the range between 1.4 and 3.6,

$$\frac{L}{R} \doteq \frac{30}{8} (M_e - 1).$$

Substituting Eq. (23) together with the conditions of $\rho = \rho_1$ and $u = u_1$ in the undisturbed core, $0 \leq r \leq r_1$, in Eq. (7), we have

$$\frac{d}{dx} \left\{ r_1^2 + \left[\frac{b^2}{2} + \frac{\sqrt{\pi}}{\sqrt{2}} r_1 b \right] \right\} = 0 \tag{24}$$

which with the expression for $r_1(x)$, Eq. (24), and the following series solution for $b(x)$

$$\frac{b(x)}{R} = K_1 \left(\frac{x}{L} \right) + K_2 \left(\frac{x}{L} \right)^2 + K_3 \left(\frac{x}{L} \right)^3 + \ldots . \tag{25}$$

gives

$$K_1 = \frac{2\sqrt{2}}{\sqrt{\pi}}$$

$$K_2 = \frac{\sqrt{2}}{\sqrt{\pi}} \left(1 - \frac{4}{\pi} \right)$$

It should be noted that this series solution converges rather rapidly. Since $|K_2/K_1| = 0.137$, the first term in the series dominates.

For similar reasons, in the turbulent mixing layer with chemical reactions, the profile of the oxygen deficiency can be expected to differ from that of the axial velocity profile. Let us assume for $r_1 \leq r < \infty$

$$\rho u \left[\left(y_{O_2} \right)_o - y_{O_2} \right] = \rho_1 u_1 \left(y_{O_2} \right)_o \exp \left[- \frac{(r-r_1)^2}{\gamma b^2} \right] \tag{26}$$

where γ is the constant which characterizes this difference. With Eqs. (21), (22), (24) and (25) together with the conditions of $\rho = \rho_1$, $u = u_1$, $y_{CO} = \left(y_{CO}\right)_1$, $y_{H_2} = \left(y_{H_2}\right)_1$ and $y_{O_2} = 0$ for $0 \leq r \leq r_1$, we can obtain from Eq. (17) after some manipulation

$$\gamma \doteq \left[\frac{1}{\sqrt{2}} \left(k_o + 1 \right) - k_o \sqrt{\lambda} \right]^2 \tag{27}$$

where $$k_o = \left[\frac{\left(y_{CO}\right)_1}{2M_{CO}} + \frac{\left(y_{H_2}\right)_1}{2M_{H_2}} \right] \Big/ \left[\frac{\left(y_{O_2}\right)_o}{M_{O_2}} \right]$$

Since neither N_2 nor $HC\ell$ participates in the chemical reactions, the radial distribution of the N_2 deficiency and that of $HC\ell$ can be expected to follow closely the radial distribution of the axial velocity. Therefore, we will assume

$$\rho u \left[\left(y_{N_2} \right)_0 - y_{N_2} \right] = \rho_1 u_1 \left[\left(y_{N_2} \right)_0 - \left(y_{N_2} \right)_1 \right] \exp\left[- \frac{2(r-r_1)^2}{b^2} \right] \tag{28}$$

$$\rho u y_{HC\ell} = \rho_1 u_1 \left(y_{HC\ell} \right)_1 \exp \left[- \frac{2(r-r_1)^2}{b^2} \right]$$

The radial distribution of the combined mass flux of CO_2 and H_2O, $\rho u \left[y_{CO_2} + y_{H_2O} \right]$, can be calculated from those of the other species since by definition

$$\sum_i y_i = 1 \tag{29}$$

The ratio of the local mass flux of CO_2 to that of H_2O, $\left(\rho u y_{CO_2}\right) \Big/ \left(\rho u y_{H_2O}\right)$ $\equiv y_{CO_2} \Big/ y_{H_2O}$, can be estimated by the ratio of the corresponding integrated mass fluxes across the turbulent mixing layer, $\int_{r_1}^{\infty} \rho u y_{CO_2} r dr \Big/ \int_{r_1}^{\infty} \rho u y_{H_2O} r dr$. The integrated mass flux of CO_2 can be readily obtained by integrating Eq. (18) after the substitution of the expression from Eq. (21) together with the conditions of $\rho = \rho_1$, $u = u_1$, $y_{CO} = \left(y_{CO}\right)_1$ and $y_{CO_2} = \left(y_{CO_2}\right)_1$ for the undisturbed core, $0 \le r \le r_1$.

Similarly, the result for the integrated mass flux can be obtained from

Eq. (19). Using these results, we can have the estimated local mass flux ratio

$$\frac{\left(\rho u y_{CO_2}\right)}{\left(\rho u y_{H_2O}\right)} = \frac{y_{CO_2}}{y_{H_2O}} \simeq \frac{\int_{r_1}^{\infty} \rho u y_{CO_2} r dr}{\int_{r_1}^{\infty} \rho u y_{H_2O} r dr}$$

$$= \frac{\left(y_{CO_2}\right)_1 + \left(y_{CO}\right)_1 \cdot \frac{M_{CO_2}}{M_{CO}} f\left(\frac{x}{L}\right)}{\left(y_{H_2O}\right)_1 + \left(y_{H_2}\right)_1 \cdot \frac{M_{H_2O}}{M_{H_2}} f\left(\frac{x}{L}\right)} , \tag{30}$$

where

$$f\left(\frac{x}{L}\right) = 1 - \left\{ \frac{\left(\frac{b}{R}\right)\left[\lambda\left(\frac{b}{R}\right) + \sqrt{\lambda \pi}\left(\frac{r_1}{R}\right)\right]}{\left(1 - \frac{r_1^2}{R^2}\right)} \right\}$$

$$\doteq 1 - \left\{ \frac{K_1\left[\left(\lambda K_1 - \sqrt{\lambda \pi}\right)\left(\frac{x}{L}\right) + \sqrt{\lambda \pi}\right]}{\left(2 - \frac{x}{L}\right)} \right\} .$$

It should be noted that in the special case in which

$$\frac{\left(y_{CO_2}\right)_1}{M_{CO_2}} \doteq \frac{\left(y_{CO}\right)_1}{M_{CO}} \quad \text{and} \quad \frac{\left(y_{H_2O}\right)_1}{M_{H_2O}} \doteq \frac{\left(y_{H_2}\right)_1}{M_{H_2}} ,$$

the above expression, Eq. (30), reduces to

$$\frac{\left(\rho u y_{CO_2}\right)}{\left(\rho u y_{H_2O}\right)} = \frac{y_{CO_2}}{y_{H_2O}} \simeq \frac{\left(y_{CO_2}\right)_1}{\left(y_{H_2O}\right)_1} . \tag{30a}$$

Since similarity solutions are being sought for the various quantities, an observation of Eq. (20) with the expressions from Eqs. (21), (22) and (23) in mind will show that the temperature-dominated function $\left(\rho^{1/2}\Delta T^{o}\right)$ is obviously the quantity which characterizes the temperature field of the plume. At this point, a simple but reasonable radial similarity profile of this quantity has to be determined. In the study of ordinary turbulent diffusion flames, it is a generally accepted practice to let the maximum temperature in the turbulent mixing layer be located at the radial station where the concentrations of fuel and oxygen are of stoichiometric proportion. In our present problem, complications might arise due to the fact that there are two different fuel gases, CO and H_2 and therefore there might be two points of maxima in the corresponding profile of an appropriate quantity. Fortunately for a wide range of modern rockets, the molar concentrations of CO and H_2 at the exit plane of the rocket nozzle, $\left(y_{CO}\right)_1 / M_{CO}$ and $\left(y_{H_2}\right)_1 / M_{H_2}$ respectively, are found to be approximately equal to each other. In view of Eqs. (21) and (22), it is easy to see that the molar concentrations

of CO and H_2, y_{CO} / M_{CO} and y_{H_2} / M_{H_2} respectively, should also be approximately equal to each other throughout the turbulent mixing layer. Therefore, it is reasonable to let the radial profile of the quantity $\left(\rho^{1/2} \Delta T^{o}\right)$ have its maxima in the turbulent mixing layer at a radial station r_m where the molar concentrations of CO, H_2 and O_2 are of stoichiometric proportion, i.e.

$$\frac{1}{4}\left(\frac{y_{CO}}{M_{CO}} + \frac{y_{H_2}}{M_{H_2}}\right) = \frac{y_{O_2}}{M_{O_2}} \tag{31}$$

at $r = r_m$. Introducing Eqs. (21), (22) and (26) in Eq. (31), we have

$$E_m^{\sigma} - E_m + \frac{k_o}{2} \doteq 0, \tag{32}$$

where

$$E_m = E_m\left(B_m\right) = \exp\left[\left(\frac{1}{\lambda} - 1\right) B_m^2\right], \tag{33}$$

$$B_m = \frac{\left(r_m - r_1\right)}{b}, \tag{34}$$

$$\sigma = \left(\frac{1 - \frac{\lambda}{\gamma}}{1 - \lambda}\right),$$

from which r_m can be evaluated. For instance, for most practical rocket exhaust plumes, $\lambda \approx \gamma \approx 0.4$ and therefore Eq. (32) will yield

$$E_m(B_m) \doteq \frac{k_o}{2} + \frac{\left(1 + \frac{k_o}{2}\right) - \sigma}{\left(1 + \frac{k_o}{2}\right)^{1-\sigma} - \sigma} \quad . \tag{35}$$

For simplicity, the assumed profile of $(\rho^{1/2}\Delta T^o)$ will be specified by a different simple function in each of the two radial regions separated by $r = r_m$ in the turbulent mixing layer, $r_1 \leq r \leq r_m$ and $r_m \leq r < \infty$ respectively. In the region of $r_1 \leq r \leq r_m$, the quantity $(\rho^{1/2}\Delta T^o)$ is assumed to increase from its value in the undisturbed core to its maximum value at $r = r_m$ in the following fashion:

$$\rho^{1/2}\Delta T^o = \rho_1{}^{1/2}\Delta T_1{}^o + \frac{(r - r_1)^2}{(r_m - r_1)^2} \left[\rho_m{}^{1/2}\Delta T_m{}^o - \rho_1{}^{1/2}\Delta T_1{}^o \right] \tag{36}$$

for $r_1 \leq r \leq r_m$, where the subscript "m" refers to condition at $r = r_m$. In the region of $r_m \leq r < \infty$, this quantity $(\rho^{1/2}\Delta T^o)$ is assumed to decrease from its maximum value at $r = r_m$ to zero in the ambient atmosphere following closely the decaying function $(\rho^{1/2}{}_u)$ in that same radial region, i.e.

$$\rho^{1/2}\Delta T^o = \rho_m{}^{1/2}\Delta T_m{}^o \exp\left[- \frac{(r - r_1)^2}{b^2} + B_m{}^2 \right] \tag{37}$$

for $r_m \leq r < \infty$. Plots of assumed radial profiles for this region are shown in Fig. 2.

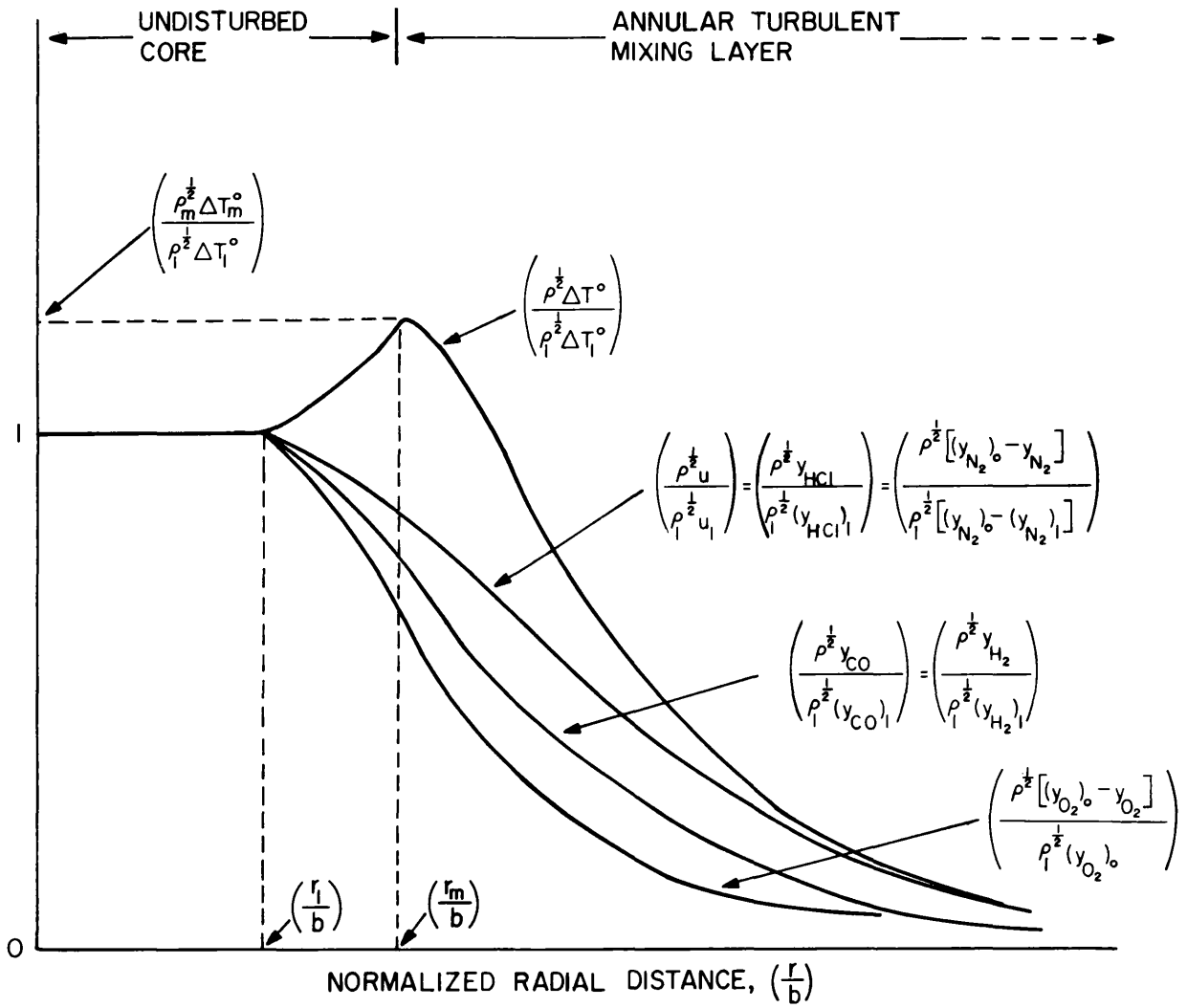

Fig. 2 - Assumed Radial Profiles for Region Extending to the Disappearance of Undisturbed Core of Original Exhaust Gas ($0 \leq x \leq L$)

Integrating Eq. (20) with the use of Eqs. (21), (22), (23), (36) and (37) together with the conditions that $\rho = \rho_1$, $u = u_1$, $\Delta T^o = \Delta T_1{}^o$, $y_{CO} = (y_{CO})_1$ and $y_{H_2} = (y_{H_2})_1$ for the undisturbed core region, $0 \le r \le r_1$, we have

$$Y_{T_m} = \frac{k_8\left(\frac{b}{R}\right)^2 + k_9\left(\frac{r_1}{R}\right)\left(\frac{b}{R}\right) - k_{10}\left(\frac{r_1{}^2}{R^2} - 1\right)}{k_6\left(\frac{b}{R}\right)^2 + k_7\left(\frac{r_1}{R}\right)\left(\frac{b}{R}\right)} , \tag{38}$$

where

$$Y_{T_m} = \left(\frac{\rho_m{}^{1/2}\Delta T_m{}^o}{\rho_1{}^{1/2}\Delta T_1{}^o}\right) ,$$

$$k_1 = \frac{(y_{CO})_1 \Delta H_{CO}}{C_p M_{CO} \Delta T_1{}^o} ,$$

$$k_2 = \frac{(y_{H_2})_1 \Delta H_{H_2}}{C_p M_{H_2} \Delta T_1{}^o} ,$$

$$k_3 = \exp\left(- B_m{}^2 \right) ,$$

$$k_4 = \int_{\sqrt{2}\,B_m}^{\infty} \exp\,(-z^2)\,dz\,,$$

$$k_5 = \int_{0}^{B_m} \exp\,(-z^2)\,dz\,,$$

$$k_6 = \left[\frac{(1 - k_3)}{2B_m^{\,2}} - \frac{k_3}{4}\right]\,,$$

$$k_7 = \left[\frac{k_4}{\sqrt{2}k_3} + \frac{(k_5 - k_3 B_m)}{2B_m^{\,2}}\right]\,,$$

$$k_8 = \frac{1}{2}\left[\frac{(1 - k_3)}{B_m^{\,2}} - \lambda(k_1 + k_2) - 1\right]\,,$$

$$k_9 = \left[\frac{(k_5 - k_3 B_m)}{2B_m^{\,2}} - k_5 - \frac{\sqrt{\pi\lambda}\,(k_1 + k_2)}{2}\right]\,,$$

$$k_{10} = \left[\frac{1 + k_1 + k_2}{2}\right]\,.$$

If we consider the gas mixture as an ideal gas, the equation of state of which under the constant pressure condition of the present problem is simply

$$\rho T = \text{constant} = \rho_1 T_1, \tag{39}$$

we can obtain the result for the maximum local temperature T_m, after some manipulation, from the definition of ΔT^o:

$$\frac{T_m}{T_1} = \frac{1}{4}\left\{ (k_{11} - k_{12}) Y_{T_m} + \left[(k_{11} - k_{12})^2 Y_{T_m}^2 + 4k_{12} \right]^{1/2} \right\}^2 \tag{40}$$

where

$$k_{11} = \left(\frac{1 + \dfrac{u_1^2}{2C_pT_1}}{1 + \dfrac{k_3^2 u_1^2}{2C_pT_1}} \right) ,$$

$$k_{12} = \left(\frac{\dfrac{T_o}{T_1}}{1 + \dfrac{k_3^2 u_1^2}{2C_pT_1}} \right) .$$

Similarly, the temperature distribution in the two radial regions $r_1 \leq r \leq r_m$ and $r_m \leq r < \infty$ can be obtained by the use of Eqs. (36) and (37) respectively.

For the radial region ($r_1 \leq r \leq r_m$):

$$\frac{T}{T_1} = \frac{1}{4}\left[A + (A^2 + 4B)^{1/2}\right]^2 , \tag{41}$$

where

$$A = A\,(x,r) = \frac{\left(\frac{\Delta T_1{}^{o}}{T_1}\right)\left\{1 + \left(\frac{1}{B_m{}^2}\right)\frac{(r - r_1)^2}{b^2}\left[Y_{T_m} - 1\right]\right\}}{\left\{1 + \frac{u_1{}^2}{2C_pT_1}\exp\left[-\frac{2(r - r_1)^2}{b^2}\right]\right\}} ,$$

$$B = B(x,r) = \frac{\left(\frac{T_o}{T_1}\right)}{\left\{1 + \frac{u_1{}^2}{2C_pT_1}\exp\left[-\frac{2(r - r_1)^2}{b^2}\right]\right\}} .$$

For the radial region ($r_m \leq r < \infty$):

$$\frac{T}{T_1} = \frac{1}{4}\left[C + (C^2 + 4B)^{1/2}\right]^2 , \tag{42}$$

where

$$C = C(x,r) = \frac{\left(\frac{\Delta T_1{}^{o}}{T_1}\right) \frac{1}{k_3} \cdot Y_{T_m} \cdot \exp\left[-\frac{(r - r_1)^2}{b^2}\right]}{\left\{1 + \frac{u_1{}^2}{2C_pT_1} \exp\left[-\frac{2(r - r_1)^2}{b^2}\right]\right\}} .$$

Using the solution for the temperature field from Eqs. (41) and (42), we can obtain the solution for the density field from Eq. (39). Using the solution for the density field, we can obtain the solution for the velocity field from Eq. (23). And, using the solutions of both the density and velocity fields, we can obtain the solutions for the concentration fields of the various species from Eqs. (21), (22), (26), (28) and (30).

TRANSITION REGION BETWEEN END OF UNDISTURBED CORE AND BEGINNING OF FULLY-DEVELOPED FLOW ($L \leq x \leq S$)

For similar reasons as those indicated in the case of the preceeding region, the radial profile of $(\rho^{1/2}u)$ in the axial region beyond the end of the undisturbed core will be assumed to be

$$\rho^{1/2}u = \rho_a^{1/2}u_a \exp\left(-\frac{r^2}{b^2}\right) \quad , \tag{43}$$

and since N_2 and $HC\ell$ do not participate in either of the two chemical reactions specified in the present problem and, consequently, the changes of their concentration are caused by the same turbulent diffusion mechanism that is responsible for the transfer of momentum, it is reasonable to assume that

$$\rho^{1/2}\left[\left(y_{N_2}\right)_0 - y_{N_2}\right] = \rho_a^{1/2}\left[\left(y_{N_2}\right)_0 - \left(y_{N_2}\right)_a\right] \exp\left(-\frac{r^2}{b^2}\right) \tag{44}$$

and

$$\rho^{1/2}y_{HC\ell} = \rho_a^{1/2}\left(y_{HC\ell}\right)_a \exp\left(-\frac{r^2}{b^2}\right) \quad . \tag{45}$$

Integrating Eq. (7) with respect to x, from $x = L$ to $x = x$, with the use of Eq. (43) and the conditions $\rho_a = \rho_1$, $u_a = u_1$ at $x = L$ and $b = b_L = b(L)$ from the solution for the precious region, we have

$$Y_u Y_b = 1 \quad , \tag{46}$$

where

$$Y_u = \frac{\rho_a^{1/2} u_a}{\rho_1^{1/2} u_1} \quad , \tag{46a}$$

$$Y_b = \frac{b}{b_L} \quad . \tag{46b}$$

Similarly, integrating Eqs. (10) and (15) with the use of Eqs. (44) and (45), respectively, together with Eq. (46), we have

$$Y_{N_2} = Y_u \; , \tag{47}$$

$$Y_{HC\ell} = Y_u \; , \tag{48}$$

where

$$Y_{N_2} = \frac{\rho_a^{1/2} \left[\left(y_{N_2} \right)_o - \left(y_{N_2} \right)_a \right]}{\rho_1^{1/2} \left[\left(y_{N_2} \right)_o - \left(y_{N_2} \right)_1 \right]} \quad , \tag{48a}$$

$$Y_{HC\ell} = \frac{\rho_a^{1/2}\left(y_{HC\ell}\right)_a}{\rho_1^{1/2}\left(y_{HC\ell}\right)_1} \quad . \tag{48b}$$

The equation of the conservation of mass flux, Eq. (5), can be approximated by

$$\frac{d}{dx}\int_0^\infty (\rho r u)dr = -\rho_0 b v_b \,, \tag{49}$$

where v_b is the entrainment velocity of ambient air at the characteristic radius $b(x)$. At this stage, further assumption would have to be made on the nature of the entrainment velocity v_b. Use will be made of an assumption first introduced by Taylor (5) on the lateral entrainment of ambient fluid to an axisymmetrical turbulent free-convection plume caused by the buoyancy force due to temperature difference between the plume fluid and the ambient fluid. His entrainment assumption deals strictly with the velocity field and, aside from some minor differences due to a different axial velocity profile that he assumed, it essentially takes the form

$$v_b \equiv v(x,b) = -\alpha u_a(x) \,,$$

where α is the entrainment coefficient, whose value depends solely on the geometry of the problem and the shape of the axial velocity profile assumed. With a Gaussian velocity profile assumed, a value $\alpha = 0.08$ for an axisymmetrical geometry and a value of $\alpha = 0.16$ for a two-dimensional geometry

have been firmly established by theoretical and experimental works by Morton, Taylor, and Turner (6), Morton (7,8), Rouse, Yih, and Humphreys (9), Lee and Emmons (10), and Lee (11 - 14). The value $\alpha = 0.08$ for an axisymmetrical geometry has been confirmed by a comparison of solutions based on the entrainment assumption and experimental results on axisymmetrical turbulent incompressible jets by Zimm (15), Ruden (16), Reichardt (17) and Wuest (18). Based on experimental results on jet entrainment by Ricon and Spalding (19), Morton (20) extended Taylor's lateral entrainment assumption to the case of flow with significant density variation. This extension, in terms of the present notation, states that

$$\rho_o^{1/2} v_b = -\alpha \rho_a^{1/2} u_a \quad , \tag{50}$$

where α is the same entrainment coefficient for both the compressible and incompressible cases.

Combining Eqs. (49) and (50), we have

$$\frac{d}{dx} \int_0^\infty (\rho r u) dr = \alpha \rho_o^{1/2} \rho_a^{1/2} u_a b ,$$

which, upon the use of Eqs. (43) and (46), becomes

$$\frac{d}{dx} \left\{ b \int_0^\infty \left(\frac{\rho}{\rho_o} \right)^{1/2} \exp\left(-\frac{r^2}{b^2} \right) \cdot \left(\frac{r}{b} \right) d\left(\frac{r}{b} \right) \right\} = \alpha \quad . \tag{51}$$

For the evaluation of the integral in Eq. (51), the following preliminary approximation for the density field is assumed

$$\left.\begin{aligned} \left(\frac{\rho}{\rho_o}\right)^{1/2} &\doteq 1 - \left[1 - \left(\frac{\rho_1}{\rho_o}\right)^{1/2}\right] \exp\left[\frac{r_m^2 - r^2}{b^2}\right] \quad \text{for } r_m \leq r < \infty \\ \left(\frac{\rho}{\rho_o}\right)^{1/2} &\doteq \left(\frac{\rho_1}{\rho_o}\right)^{1/2} \quad \text{for } 0 \leq r \leq r_m \,, \end{aligned}\right\} \tag{52}$$

where r_m, as in the previous axial region, is the radial station where the quantity $\rho^{1/2}\Delta T^o$ has its maxima. Substituting expressions from Eqs. (52) in Eq. (51), we have after some manipulation

$$Y_b \exp\left(-Y_{r_m}^2\right) \doteq f(X) \,, \tag{53}$$

where

$$Y_{r_m} = \frac{r_m}{b} \,,$$

$$f(X) = 4\alpha X + \exp\left(-B_m^2\right),$$

$$X = \frac{x - L}{b_L} \,.$$

Similar to the previous axial region, we will assume the following radial distributions:

$$\rho u y_{CO} = \rho_a u_a (y_{CO})_a \exp\left[-\frac{r^2}{\lambda b^2}\right] , \tag{56}$$

$$\rho u y_{H_2} = \rho_a u_a (y_{H_2})_a \exp\left[-\frac{r^2}{\lambda b^2}\right] , \tag{57}$$

$$\rho u \left[(y_{O_2})_o - y_{O_2}\right] = \rho_a u_a (y_{O_2})_o \exp\left[-\frac{r^2}{\gamma b^2}\right] , \tag{58}$$

$$\rho^{1/2} u = \rho_a^{1/2} u_a \exp\left[-\frac{r^2}{b^2}\right] , \tag{59}$$

where the constants λ and γ are the same as those for the corresponding expressions in the previous region.

For this region, it is similarly assumed that the radial profile of the temperature-dominated quantity $(\rho^{1/2}\Delta T^o)$ has its maxima at a radial

At this point, something must be said about the distribution of mass concentrations of the various species in the presence of chemical reactions. For those species which do not participate in the chemical reactions, the radial distribution of $\rho^{1/2} y_i$ will follow that of $\rho^{1/2} u$ closely since both are controlled by the same mechanism of turbulent diffusion. For those species which participate in the chemical reactions, the radial distribution of $\rho^{1/2} y_i$ will deviate from that of $\rho^{1/2} u$ since the mass concentration in this case is also influenced by the chemical reactions. If the axial decay of $\rho^{1/2} y_{CO}$ and $\rho^{1/2} y_{H_2}$ relative to $\rho^{1/2} u$ in this region is assumed to follow the pattern of the radial decay of the same quantities in the turbulent mixing annular layer in the previous region, we have

$$Y_{CO} = Y_u^{\left(\frac{1}{\lambda} - 1\right)} , \tag{54}$$

$$Y_{H_2} = Y_u^{\left(\frac{1}{\lambda} - 1\right)} , \tag{55}$$

where

$$Y_{CO} = \frac{\rho_a^{1/2} \left(y_{CO} \right)_a}{\rho_1^{1/2} \left(y_{CO} \right)_1} ,$$

$$Y_{H_2} = \frac{\rho_a^{1/2} \left(y_{H_2} \right)_a}{\rho_1^{1/2} \left(y_{H_2} \right)_1} .$$

Upon the use of Eqs. (54) and (55) and the approximation $\rho_m^{1/2} \approx \rho_1^{1/2}$, Eq. (60) becomes

$$\frac{k_o}{2} Y_u^{\left(\frac{1}{\lambda} - 1\right)} \exp\left[-\left(\frac{1}{\lambda} - 1\right) Y_{r_m}^2\right] = -Y_{O_2} \exp\left[-\left(\frac{1}{\gamma} - 1\right) Y_{r_m}^2\right] + 1 , \tag{61}$$

which can be further simplified, with Eq. (53) used, to give

$$Y_{O_2} = Y_b^{\left(\frac{1}{\gamma} - 1\right)} g(X) , \tag{62}$$

where

$$g(X) = \left[f(X)\right]^{\left(1 - \frac{1}{\gamma}\right)} \left\{1 - \frac{k_o}{2} \left[f(X)\right]^{\left(1 - \frac{1}{\lambda}\right)}\right\} .$$

Now, upon substituting expressions from Eqs. (56), (57) and (58) in Eq. (17), we have after integration with respect to x, from x = L to x = x,

$$\gamma \frac{(y_{O_2})_o}{M_{O_2}} \left[Y_{O_2} Y_u Y_b^2 - 1\right] + \lambda \left\{ \frac{(y_{CO})_1}{2M_{CO}} \left[Y_{CO} Y_u Y_b^2 - 1\right] + \frac{(y_{H_2})_1}{2M_{H_2}} \left[Y_{H_2} Y_u Y_b^2 - 1\right] \right\} = 0 , \tag{63}$$

station r_m where the molar concentrations of CO, H_2 and O_2 are of stoichiometric proportion, i.e.,

$$\frac{1}{4}\left(\frac{y_{CO}}{M_{CO}} + \frac{y_{H_2}}{M_{H_2}}\right) = \frac{y_{O_2}}{M_{O_2}} \tag{31}$$

at $r = r_m$. Substituting Eqs. (56), (57), (58) and (59) in Eq. (31), we have

$$\frac{1}{4}\frac{M_{O_2}}{\left(y_{O_2}\right)_o}\left[Y_{CO}\frac{\left(y_{CO}\right)_1}{M_{CO}} + Y_{H_2}\frac{\left(y_{H_2}\right)_1}{M_{H_2}}\right] \cdot \exp\left[-\left(\frac{1}{\lambda} - 1\right) Y_{r_m}{}^2\right]$$

$$= -Y_{O_2}\exp\left[-\left(\frac{1}{\gamma} - 1\right) Y_{r_m}{}^2\right] + \frac{\rho_m{}^{1/2}}{\rho_1{}^{1/2}}, \tag{60}$$

where

$$Y_{O_2} = \frac{\rho_a{}^{1/2}\left[\left(y_{O_2}\right)_o - \left(y_{O_2}\right)_a\right]}{\rho_1{}^{1/2}\left(y_{O_2}\right)_o} .$$

which, with the use of Eqs. (54) and (55), becomes

$$\gamma \left[Y_{O_2} Y_u Y_b^2 - 1 \right] + \lambda k_o \left[Y_u^{\frac{1}{\lambda}} Y_b^2 - 1 \right] = 0 \quad . \tag{64}$$

Eliminating Y_{O_2} and Y_u among Eqs. (46), (62) and (64), we have

$$\left[Y_b^{1/2} g(X) - 1 \right] + \frac{\lambda k_o}{\gamma} \left[Y_b^{2 - \frac{1}{\lambda}} - 1 \right] = 0 \quad , \tag{65}$$

which gives the solution for Y_b. For instance, for most practical rocket exhaust plumes $\lambda \approx \gamma \approx 0.4$ and therefore Eq. (65) will yield

$$Y_b \doteq \left\{ \frac{1}{g(X)} + \frac{\frac{\lambda k_o}{\gamma} \left\{ 1 - \left[g(X) \right]^{\gamma(\frac{1}{\lambda} - 2)} \right\}}{1 + k_o(2\lambda - 1) \left[g(X) \right]^{1 - \gamma\left(2 - \frac{1}{\lambda}\right)}} \right\}^{\gamma} \quad . \tag{66}$$

In order to calculate the temperature field of this region of the plume, some simple but reasonable radial distribution of the temperature-dominated

quantity $\rho^{1/2}\Delta T^{o}$ would have to be assumed. It can be readily reasoned that $\rho^{1/2}\Delta T^{o}$ will become $\rho_m^{1/2}\Delta T_m^{o}$ at $r = r_m$ and $\rho_a^{1/2}\Delta T_a^{o}$ at $r = 0$ and $\rho_m^{1/2}\Delta T_m^{o}$ will become identical with $\rho_a^{1/2}\Delta T_a^{o}$ when $r_m = 0$. Furthermore, it can also be reasoned as in the case of the previous region that the quantity $\rho^{1/2}\Delta T^{o}$ should decay from a value of $\rho_m^{1/2}\Delta T_m^{o}$ at $r = r_m$ to zero in the ambient air. Therefore, it will be assumed that <u>for the radial region $0 \le r \le r_m$:</u>

$$\rho^{1/2}\Delta T^{o} = \rho_a^{1/2}\Delta T_a^{o} + k_{13}\rho_1^{1/2}\Delta T_1^{o}\,\frac{r^2}{b^2}$$

$$= \rho_m^{1/2}\Delta T_m^{o} + k_{13}\rho_1^{1/2}\Delta T_1^{o}\left(\frac{r^2}{b^2} - \frac{r_m^2}{b^2}\right) \quad , \qquad (67)$$

where

$$k_{13} = \left(\frac{\rho_{m_L}^{1/2}\Delta T_{m_L}^{o}}{\rho_1^{1/2}\Delta T_1^{o}} - 1\right)\left(\frac{b_L^2}{r_{m_L}^2}\right)$$

and the subscript "L" refers to values evaluated from solutions of previous region at $x = L$, and <u>for the radial region $r_m \le r < \infty$:</u>

$$\rho^{1/2}\Delta T^{o} = \rho_m^{1/2}\Delta T_m^{o}\,\exp\left[-\frac{r^2}{b^2} + \frac{r_m^2}{b^2}\right] \quad . \qquad (68)$$

Plots of assumed radial profiles for this region are shown in Fig. 3.

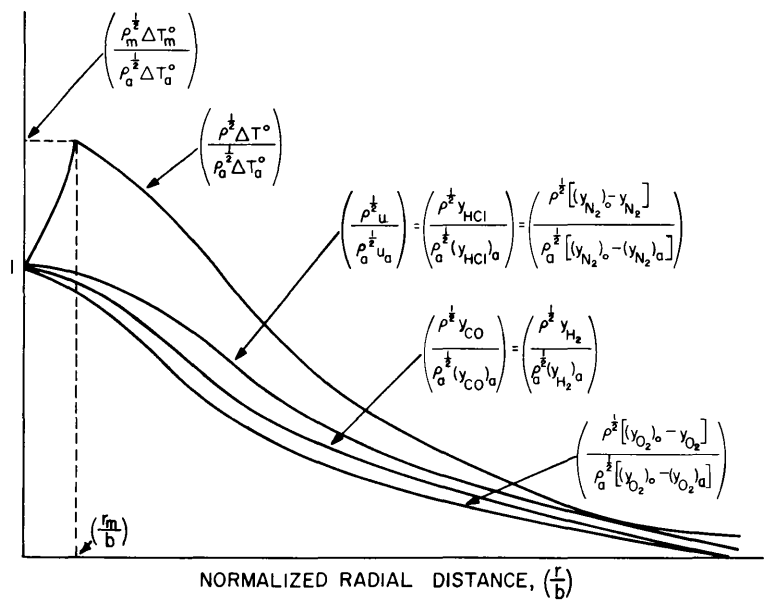

Fig. 3 - Assumed Radial Profiles for the Transition Region Between End of Undisturbed Core and Beginning of Fully-Developed Flow ($L \leq x \leq S$)

Integrating Eq. (20) with the use of Eqs. (56), (57) and (59), and the expressions from Eqs. (67) and (68), we have

$$\frac{1}{2} Y_b^2 \exp\left(- Y_{r_m}^2 \right) \cdot \left\{ Y_u Y_T + k_{13} Y_{r_m}^2 Y_u \right\}$$

$$+ \left\{ Y_u Y_T Y_b^2 \cdot \left[1 - \exp\left(- Y_{r_m}^2 \right) \right] + k_{13} Y_u Y_b^2 \left[1 - \left(1 + Y_{r_m}^2 \right) \exp\left(- Y_{r_m}^2 \right) \right] \right\}$$

$$+ \lambda Y_u Y_b^2 \left\{ k_1 Y_{CO} + k_2 Y_{H_2} \right\} = C_1 \quad , \tag{69}$$

where

$$Y_T = \frac{\rho_a^{1/2} \Delta T_a^{\circ}}{\rho_1^{1/2} \Delta T_1^{\circ}} \quad ,$$

$$k_1 = \frac{(y_{CO})_1 \, \Delta H_{CO}}{\Delta T_1^{\circ} M_{CO} C_p} \quad \text{as defined previously,}$$

$$k_2 = \frac{(y_{H_2})_1 \, \Delta H_{H_2}}{\Delta T_1^{\circ} M_{H_2} C_p} \quad \text{as defined previously,}$$

and C_1 is the integration constant

Eq. (69) can be simplified by the use of Eqs. (46), (53), (54) and (55), and the integration constant C_1 can be evaluated by the conditions at $x = L$. The resulting equation gives

$$Y_T = \left\{ \lambda k_{14} (Y_b^{2-\frac{1}{\lambda}} - 1) - \left[\left(\frac{1}{2} Y_{r_m}^2 + 1 \right) f(X) - \left(\frac{1}{2} B_m^2 + 1 \right) \exp\left(- B_m^2 \right) \right] k_{13} \right.$$

$$\left. + k_{13}(Y_b - 1) + \left[\frac{1}{2} \exp\left(- B_m^2 \right) - 1 \right] \right\} \Big/ \left\{ Y_b - \frac{1}{2} f(X) \right\} , \qquad (70)$$

where

$$k_{14} = k_1 + k_2 = \frac{1}{C_p \Delta T_1^{\circ}} \left[\frac{(y_{CO})_1 \, \Delta H_{CO}}{M_{CO}} + \frac{(y_{H_2})_1 \, \Delta H_{H_2}}{M_{H_2}} \right] .$$

With the solution for Y_b obtained in Eq. (66), the solution for Y_{r_m} can be obtained from Eq. (53). With these solutions established, Eq. (70) then gives the solution for the axial distribution of the temperature-dominated function Y_T. The assumed radial profile for the temperature-dominated quantity $\rho^{1/2}\Delta T^{\circ}$, Eq. (67), gives the solution for the change of the quantity Y_{T_m} which essentially represents the cross-sectional maximum temperature in the plume:

$$Y_{T_m} = k_{13} Y_{r_m}^2 + Y_T \quad . \qquad (71)$$

The cross-sectional maximum temperature can then be computed from Eqs. (70) and (71) and the definitions Y_T and ΔT^o. The resulting solution reads:

$$\frac{T_m}{T_1} = \frac{1}{4} \left[D + (D^2 + 4E)^{1/2} \right]^2 , \tag{72}$$

where

$$D = D(X) = \left[\frac{Y_{T_m}}{\frac{T_1}{\Delta T_1{}^o} + \frac{u_1{}^2}{2C_p \Delta T_1{}^o} \frac{f(X)}{Y_b{}^4}} \right] ,$$

and

$$E = E(X) = \left[\frac{1}{\frac{T_1}{T_o} + \frac{u_1{}^2}{2C_p T_o} \frac{f(X)}{Y_b{}^4}} \right] .$$

With these solutions obtained, the solutions for the temperature, density, velocity and concentration fields for this region of the plume can be obtained in exactly the same way they have been obtained for the preceding region of the plume.

FULLY-DEVELOPED FLOW REGION ($x \geq S$)

For this region, the radial profile of $(\rho^{1/2}u)$ will be similarly assumed to be

$$\rho^{1/2}u = \rho_a^{1/2}u_a \exp\left(-\frac{r^2}{b^2}\right) , \tag{73}$$

and those of $(\rho^{1/2}y_{N_2})$ and $(\rho^{1/2}y_{HC\ell})$,

$$\rho^{1/2}\left[(y_{N_2})_o - y_{N_2}\right] = \rho_a^{1/2}\left[(y_{N_2})_o - (y_{N_2})_a\right] \exp\left(-\frac{r^2}{b^2}\right), \tag{74}$$

$$\rho^{1/2}y_{HC\ell} = \rho_a^{1/2}(y_{HC\ell})_a \exp\left(-\frac{r^2}{b^2}\right) \tag{75}$$

respectively.

Integrating Eq. (7) with respect to x, from $x = S$ to $x = x$, with the use of Eq. (73) and the condition that $Y_uY_b = 1$ at $x = S$, we have again

$$Y_uY_b = 1 , \tag{76}$$

where

$$Y_u = \frac{\rho_a^{1/2}u_a}{\rho_1^{1/2}u_1} , \tag{76a}$$

$$Y_b = \frac{b}{b_L} \tag{76b}$$

as before. Similarly, integrating Eqs. (10) and (15) with the use of Eqs. (74) and (75) respectively together with Eq. (76), we have again

$$Y_{N_2} = Y_u \quad , \tag{77}$$

$$Y_{HC\ell} = Y_u \, , \tag{78}$$

where Y_{N_2} and $Y_{HC\ell}$ are defined as in Eqs. (48a) and (48b).

The equation of the conservation of mass flux, Eq. (5), for this region can be similarly integrated to give

$$\frac{d}{dx}\left\{ b\int_0^\infty \left(\frac{\rho}{\rho_o}\right)^{1/2} \exp\left(-\frac{r^2}{b^2}\right)\cdot\left(\frac{r}{b}\right) d\left(\frac{r}{b}\right)\right\} = \alpha \, . \tag{79}$$

In order to carry out the integral in Eq. (79), some preliminary approximation for the density distribution would have to be assumed. In consistance with the radial distribution of $(\rho/\rho_o)^{1/2}$ assumed in the first equation of Eqs.(52) for the previous region, we will assume the following radial distribution

$$\left(\frac{\rho}{\rho_o}\right)^{1/2} \doteq 1 - \left[1 - \left(\frac{\rho_a}{\rho_o}\right)^{1/2}\right] \cdot \exp\left(-\frac{r^2}{b^2}\right) \quad . \tag{80}$$

An observation of Eqs. (73) and (80) reveals that both the quantity $(\rho^{1/2}u)$ and the quantity $\left[1 - \left(\frac{\rho}{\rho_o}\right)^{1/2}\right]$ will decay in the same fashion in the radial direction from their maximum values at the axis. It will therefore be assumed that these quantities along the axis will also decay in a similar fashion in the axial direction from their respective maximum values at the axial station of x = S:

$$\left[1 - \left(\frac{\rho_a}{\rho_o}\right)^{1/2}\right] \Big/ \left[1 - \left(\frac{\rho_s}{\rho_o}\right)^{1/2}\right] \doteq \left(\rho_a{}^{1/2}u_a\right) \Big/ \left(\rho_s{}^{1/2}u_s\right), \tag{81}$$

where the subscript "s" refers to quantities evaluated at x = S from the solution of the previous region and $\rho_s \approx \rho_1 \ll \rho_o$ as previously assumed. Combining Eqs. (80) and (81), we have

$$\left(\frac{\rho}{\rho_o}\right)^{1/2} \doteq \left\{1 - \left[1 - \left(\frac{\rho_s}{\rho_o}\right)^{1/2}\right] \cdot \left(\frac{\rho_a{}^{1/2}u_a}{\rho_s{}^{1/2}u_s}\right) \exp\left(-\frac{r^2}{b^2}\right)\right\} . \tag{82}$$

Substituting Eq. (82) in Eq. (80) with Eq. (76) used, we have after some manipulation

$$Y_b = 2\alpha \mathbf{X} + (Y_b)_s \quad , \tag{83}$$

where

$$Y_b = \frac{b}{b_L} \quad ,$$

$$(Y_b)_s = \frac{b_s}{b_L} \, ,$$

$$\mathbf{X} = \frac{x - S}{b_L} \quad .$$

Using Eq. (83), we have from Eq. (76)

$$Y_u = \frac{1}{2\alpha \mathbf{X} + (Y_b)_s} \quad . \tag{84}$$

Again at this point, something must be said about the distribution of mass concentrations of the various species in the presence of chemical reactions. For those species which do not participate in the chemical reactions, the quantity $(\rho^{1/2} y_i)$ will follow $(\rho^{1/2} u)$ closely since both are controlled by the same mechanism of turbulent diffusion. The axial decay of $(\rho^{1/2} y_{CO})$ and $(\rho^{1/2} y_{H_2})$ relative to $(\rho^{1/2} u)$ in this region will be similarly assumed, as is in the previous region, to follow the pattern of the radial decay of the same quantities. Therefore, we have, after using Eq. (84)

$$Y_{CO} = Y_u^{\left(\frac{1}{\lambda} - 1\right)} = \left[2\alpha X + (Y_b)_s \right]^{\left(1 - \frac{1}{\lambda}\right)} , \tag{85}$$

$$Y_{H_2} = Y_u^{\left(\frac{1}{\lambda} - 1\right)} = \left[2\alpha X + (Y_b)_s \right]^{\left(1 - \frac{1}{\lambda}\right)} , \tag{86}$$

where Y_{CO} and Y_{H_2} are defined as in the previous region.

Similar to the previous region, we will assume for this region the following radial distributions:

$$\rho u y_{CO} = \rho_a u_a (y_{CO})_a \exp\left[- \frac{r^2}{\lambda b^2} \right] , \tag{87}$$

$$\rho u y_{H_2} = \rho_a u_a (y_{H_2})_a \exp\left[-\frac{r^2}{\lambda b^2}\right] , \tag{88}$$

$$\rho u \left[(y_{O_2})_o - y_{O_2}\right] = \rho_a u_a (y_{O_2})_o \exp\left[-\frac{r^2}{\gamma b^2}\right] , \tag{89}$$

where the constants λ and γ are the same as those for the corresponding expressions in the previous regions.

Upon substituting expressions from Eqs. (87), (88) and (89) in Eq. (17), we have after integration with respect to x, from x = S to x = x, with Eqs. (85) and (86) used

$$\left[Y_{O_2} Y_b - (Y_{O_2})_s (Y_b)_s\right] + \frac{\lambda k_o}{\gamma}\left[Y_b^{\left(-\frac{1}{\lambda}+2\right)} - (Y_b)_s^{\left(-\frac{1}{\lambda}+2\right)}\right] = 0, \tag{90}$$

which yields the solution for Y_{O_2}:

$$Y_{O_2} = \frac{1}{Y_b}\left\{(Y_{O_2})_s (Y_b)_s + \frac{\lambda k_o}{\gamma}\left[(Y_{b_s})^{\left(-\frac{1}{\lambda}+2\right)} - Y_b^{\left(-\frac{1}{\lambda}+2\right)}\right]\right\} , \tag{91}$$

where the expression for Y_b is supplied from Eq. (83) and the subscript "s" as before refers to quantities evaluated at x = S from the solutions for the previous region.

For most practical rocket exhaust plumes $\lambda \approx 0.4$ and therefore Eq. (91) will give

$$Y_{O_2} = \frac{1}{Y_b} \left\{ (Y_{O_2})_s (Y_b)_s + \frac{k_o}{\gamma}(1 - 2\lambda)\ (Y_b)_s^{\left(-\frac{1}{\lambda} + 2\right)} \left[1 - \frac{(Y_b)_s}{Y_b} \right] \right\} . \tag{92}$$

Since in this region, the cross-sectional maximum value of the temperature-dominated quantity $\rho^{1/2}\Delta T^o$ will be located on the axis and the same turbulent mechanism controls both the momentum and energy transfers in the radial direction, it is reasonable to assume the following distribution

$$\rho^{1/2}\Delta T^o = \rho_a^{1/2}\Delta T_a^o \exp\left(- \frac{r^2}{b^2} \right), \tag{93}$$

which is consistant with the distribution assumed for the previous region as shown in Eq. (68) in which $r_m = 0$ and $(\rho_m^{1/2}\Delta T_m^o)$ becomes $(\rho_a^{1/2}\Delta T_a^o)$ at x = s. Plots of assumed radial profiles for this region are shown in Fig. 4.

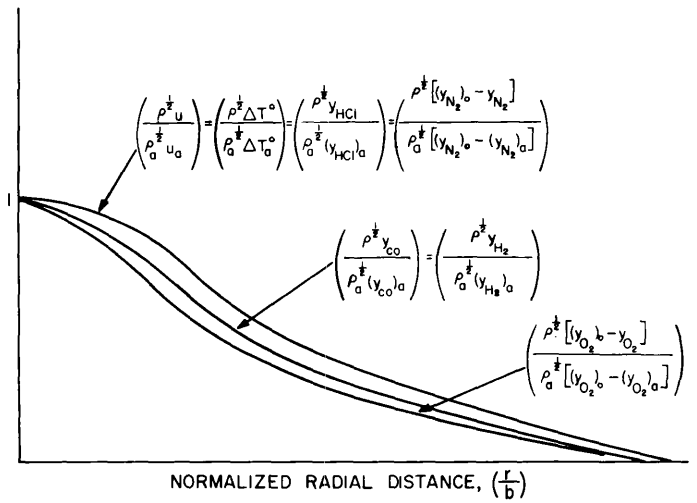

Fig. 4 - Assumbed Radial Profiles for the Fully-Developed Flow Region ($x \leq S$)

Integrating Eq. (20) with the use of Eqs. (73), (87), (88) and (93) together with Eqs. (76), (85) and (86) and the conditions of $Y_b = (Y_b)_s$ and $Y_T = (Y_T)_s$ at $x = S$, we have

$$\left[Y_T Y_b - (Y_T)_s (Y_b)_s\right] + 2\lambda k_{14}\left[Y_b^{\left(-\frac{1}{\lambda}+2\right)} - (Y_b)_s^{\left(-\frac{1}{\lambda}+2\right)}\right] = 0, \quad (94)$$

which yields the solution for Y_T:

$$Y_T = \frac{1}{Y_b}\left\{(Y_T)_s (Y_b)_s + 2\lambda k_{14}\left[(Y_b)_s^{\left(-\frac{1}{\lambda}+2\right)} - Y_b^{\left(-\frac{1}{\lambda}+2\right)}\right]\right\}, \quad (95)$$

where Y_b is given by Eq. (83).

For most practical rocket exhaust plumes $\lambda \approx 0.4$ and therefore Eq. (95) will give

$$Y_T = \frac{1}{Y_b}\left\{(Y_T)_s (Y_b)_s + 2k_{14}(1 - 2\lambda)(Y_b)_s^{\left(-\frac{1}{\lambda}+2\right)}\left[1 - \frac{(Y_b)_s}{Y_b}\right]\right\}. \quad (96)$$

With these solutions obtained, the solutions for the temperature, density, velocity and concentration fields for this region of the plume can be obtained from the preceding regions of the plume.

CONCLUSIONS:

The conclusions of this study can be summarized as follows:

1. The theoretical analysis based on the proposed aerodynamic model is developed from a rigorous set of governing aerodynamic equations and appropriate boundary conditions and therefore is valid throughout the total flow field of the plume. Emphasis of the adopted integral approach is on the gross aerodynamic behavior rather than the minute details of the flow. Solutions have been obtained for such important aerodynamic quantities as velocity, density, temperature and chemical composition fields for the whole plume. For visualization, it is possible to construct a reference flame contour, such as is shown in Fig. 1, based on the solution of the composition field.

2. The formulation of the present analysis depends on a few coefficients to be supplied from the related preliminary experiments. It would be highly desirable to have these experiments conducted for the various typical rocket exhaust gas systems. Furthermore, the present analysis has produced far more information about the flow field than has been brought to light from experimental investigations in the existing literature. The analysis should provide a guide to future experimental investigations and a tool for correlation of results of such investigations.

3. The model for the present analysis should serve adequately for those relatively large number of cases where the pressure variation does not exert a controlling influence. Improved models for the study of afterburning in a high-altitude rocket exhaust plume would still have to be developed. Furthermore, the more realistic models would also have to include any possible interaction between the plume flow field and the ambient flow field around the rocket and a region of ignition delay extending down-stream from the rocket exit plane in which the detailed chemical kinetics will be expected to play a significant role.

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13. ABSTRACT

An improved aerodynamic model is proposed for the exhaust plume issuing from allow-altitude rocket nozzle exit into quiescent ambient air in which chemical reactions and compressibility predominate. The model is based on the presently available knowledge and established aerodynamic principles for the various flow regimes involved. The major assumptions made are individually substantiated by related experimental evidence in existing literature.

A theoretical analysis is carried over the whole flow region based on the Karman's integral approach to find solutions for the gross aerodynamic and thermodynamic behavior of the plume. Theorgyincorporating on empirically derived constants provides the required radial distribution function for the integration. Solutions are obtained for the velocity, density, temperature and chemical composition fields for extended regions of the plume. These results will provide the much needed framework for the understanding of the elements which contribute to the phenomenon of radar attenuation of rocket exhaust plumes.

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