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Normal Mode Theory for Combined Structures

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ABSTRACT

This report considers a linear elastic structure to consist of several parts: a primary structure which has a rigid base, and one or more substructures which are firmly attached to the primary structure. The normal modes and natural frequencies of the combined structure are determined from a knowledge of the primary structure's normal modes and frequencies and the mass and the elastic constants distribution of the secondary structure. The displacements of the total structure are expressed as a series expansion of the scaled normal modes of the primary structure alone, and a set of equations is used to find the normal coordinates of the secondary structures for the new natural frequencies. There is in general an increase in the number of degrees of freedom. The scaling factors involved are determined from the equilibrium equations of the primary structure, with the secondary structures removed and replaced by their internal forces of connection. Numerical examples are presented.

PROBLEM STATUS

This is an interim report on one phase of the problem; work is continuing on this and other phases.

AUTHORIZATION

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SYMBOLS

$C,\ D$	Arbitrary constants
$G_k(\lambda,\ k_{ij},\ m_i)$	Transfer function between internal connecting force and attachment displacement
k_{ij}	Stiffness matrix for secondary structure
m_i	Mass on primary structure
$\tilde{m}_i$	Mass on secondary structure
n	Number of degrees of freedom of primary structure
N	Number of degrees of freedom of total structure
P_k	Amplitude of connection force
q_a	Generalized coordinate as a function of time
$\tilde{q}_a$	Scaling factor in displacement expression
$\tilde{q}_{aa}$	Scaling factor associated with ath mode of vibration for total structure
S_o	Primary structure
$S_i,\ i = 1, 2, \text{etc.}$	Secondary structure
U_{ik}	Kronecker delta; 1 if $i = k$, 0 if $i \neq k$
x_i	Displacement of ith point of total structure
$\bar{x}_{ia}$	ath normal mode of primary structure
X_k	Displacement of attachment point
X_{jA}	Ath normal mode of total structure
ζ_{ij}	Influence coefficient
λ	Natural frequency of total structure
ω	Natural frequency of primary structure
$a,\ b$	(Subscripts) normal modes of primary structure
A	(Subscript) normal modes of total structure

NORMAL MODE THEORY FOR COMBINED STRUCTURES

INTRODUCTION

The response of linear elastic structures to dynamic loadings has been the subject of several earlier works including the NRL reports of Refs. 1 and 2. Recently normal mode theory has become widely used and accepted as a tool for structural design and analysis. This usage has brought to the fore the question of what relationships exist between the normal modes and natural frequencies of a structure and the normal modes and natural frequencies of its component parts.

This report is deliberately limited in its use of mathematical methods to those which are no more complex than necessary. No background knowledge of Fourier transforms, Laplace transforms, Hamilton's equations, or Lagrange's equations is assumed. These general methods are so powerful for this type of problem that solutions are produced with deceptive ease. The feeling of actually understanding the problem is lost while one is blindly following the operational rules.

It is assumed that the reader has some familiarity with elementary normal mode theory and with the concepts of characteristic value (eigenvalue/eigenvector) problems. With this assumption satisfied, this report is self-contained.

A literature search showed that the problem of predicting the normal mode shapes and natural frequencies of a freely vibrating complex structure from the normal modes and natural frequencies of its component parts has only partially been solved.

In Ref. 3, for example, the normal modes and natural frequencies are found for structures to which a mass or a spring has been added, but problems where additional generalized coordinates result from additions to the structure are not considered. More complicated structures than those presented in Ref. 3 are treated in Refs. 4 and 5. Once again, however, the total number of generalized coordinates is not changed by the additions to the structure, and even then only the natural frequencies of the combined structures were found.

In Refs. 6 and 7, changes in the natural frequencies are considered for the addition of mass-spring oscillators to beams in such a manner that the number of generalized coordinates is changed. No attempt was made, however, to find the normal modes of the combined structure. There seems to be a need for such a solution regardless of whether the number of generalized coordinates has been changed by the structural addition or not. The procedure outlined in this report was developed to predict the normal modes and natural frequencies of a freely vibrating complicated structure from the normal modes and natural frequencies of its primary part and a knowledge of the mass distribution and elastic constants of the secondary structure.

There are two basic approaches for the analytical representation of a linear elastic structure responding to dynamic forces. One method breaks the structure into a finite number of concentrated masses restrained by a weightless structure having the same strength properties as the real structure. Such systems are called lumped-parameter systems and have their governing equations of motion in the form of ordinary differential equations. The second method treats the structure as a continuous elastic body (an infinite number of masses) in which (at least segmentally) the material is assumed to be

homogeneous and isotropic, and to follow Hooke's law. These systems are called distributed-parameter systems and have partial differential equations for their equations of motion. Most engineering structures are too complex to be solved by this second method.

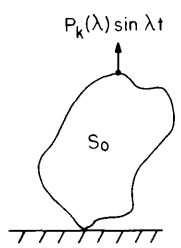
The primary concern of this report is to find the normal modes and natural frequencies of a structure which is formed by adding a linear elastic, undamped, lumped-parameter system to an existing linear elastic structure whose normal modes and natural frequencies are known.

The usual assumptions concerning linear elasticity are made in this report. In addition, unidirectional motion is assumed; e.g., all forces and deflections are parallel, and only primary structures which rest on infinitely massive and rigid bases are considered.

THEORY

Consider a primary structure S_0 (Fig. 1) which rests on an immovable base. Assume that this structure has been previously analyzed for its normal mode shapes, $\bar{X}_{ia}$, and for its fixed-base natural frequencies, ω_a, where a is the mode number. We assume that this set is complete. A new structure S_1 (Fig. 2) is to be added to S_0 at the mass point k. What then are the new normal mode shapes, X_{jA}, and the new natural frequencies, λ_A, of the combined structure $(S_0 + S_1)$?

Fig. 1 - Primary structure



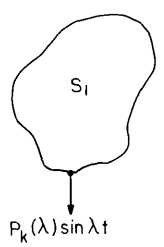


Fig. 2 - Secondary structure

Primary Structure

We assume that a force $[P_k(\lambda)\ \sin\lambda t]$ is applied to structure S_0 at the attachment point. The magnitude and sign of the force is unknown at the present but will later be chosen to correspond to the internal connecting force in the structure $(S_0 + S_1)$ during free vibrations.

Using D'Alembert's principle and influence coefficients, the distortion of the structure S_0 is described by the n equations

$$X_i = -\sum_{j=1}^{n} \zeta_{ij} m_j \ddot{X}_j + \zeta_{ik} P_k(\lambda) \sin \lambda t , \tag{1}$$

with n the number of degrees of freedom of the primary structure.

A solution of Eq. (1) is sought in the form of a series by

$$X_i = \sum_{a=1}^{n} \bar{X}_{ia} q_{aj} \qquad i = 1, \ldots, n , \tag{2}$$

where the $\bar{X}_{ia}$'s are the mode shapes for the structure S_0 and the q_a's (the generalized coordinates) are to be determined. Substitution of Eq. (2) into Eq. (1) yields

$$\sum_{a=1}^{n} (\ddot{q}_a \sum_{j=1}^{n} \zeta_{ij} m_j \bar{X}_{ja} + \bar{X}_{ia} q_a) = \zeta_{ik} P_k(\lambda) \sin \lambda t , \tag{3}$$

for all i.

From the definition of a normal mode,

$$\frac{\bar{X}_{ia}}{\omega_a^2} = \sum_{j=1}^{n} \zeta_{ij} m_j \bar{X}_{ja}; \quad i = 1, \ldots, n , \tag{4}$$

where ω_a is a fixed-base frequency of S_0 so that

$$\sum_{a=1}^{n} (\frac{\ddot{q}_a}{\omega_a^2} + q_a) \bar{X}_{ia} = \zeta_{ik} P_k (\lambda) \sin \lambda t, \quad i = 1, \ldots, n . \tag{5}$$

The influence coefficient times the force can be brought into the parentheses by expanding this expression into a series of mode shapes. It is well known (1) that

$$\zeta_{ik} P_k(\lambda) = P_k(\lambda) \sum_{a=1}^{n} \frac{\bar{X}_{ia} \bar{X}_{ka}}{\omega_a^2 \sum_{j=1}^{n} m_j \bar{X}_{ja}^2} , \tag{6}$$

and if this is substituted into Eq. (5) there results

$$\sum_{a=1}^{n} \left[\frac{\ddot{q}_a}{\omega_a^2} + q_a - \frac{\bar{X}_{ka} P_k(\lambda) \sin \lambda t}{\omega_a^2 \sum_{j=1}^{n} m_j \bar{X}_{ja}^2} \right] \bar{X}_{ia} = 0 . \tag{7}$$

The use of the orthogonality relationship reduces Eq. (7) to

$$\ddot{q}_a + \omega_a^2 q_a = \frac{\bar{X}_{ka} P_k(\lambda) \sin \lambda t}{\sum_{j=1}^{n} m_j \bar{X}_{ja}^2} \tag{8}$$

for each a, 1 to n.

No further progress can be made until the internal force $P_k(\lambda) \sin \lambda t$ is defined, and this requires analysis of the structure S_1.

Secondary Structure

Before we can write the equations of motion for the secondary structure, it is necessary to know the characteristics of the part of the structure being attached to S_0. If we assume that the connection point on structure S_1 is labeled point k, one of two possibilities can exist — either the point k on structure S_1 has mass or it does not. See Fig. 3.

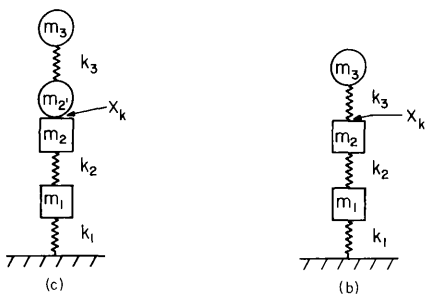


Fig. 3 - Possible characteristics of point of attachment of secondary structure: (a) Attachment point has mass and (b) attachment point has no mass

Case 1: Point k has Mass — For this case, point k has mass and the differential equations of motion of the secondary structure S_1 can be written as

$$k_{ik} X_k + \sum_{j=m+1}^{N} k_{ij} X_j = - \tilde{m}_i \ddot{X}_i + P_k(\lambda)(\sin \lambda t) \; U_{ik} \; , \tag{9}$$

where $i = k, \; n+1, n+2, \ldots,$ and $U_{ik} = 0$ unless $i = k$, at which time $U_{kk} = 1$; $(N-n)$ is the number of degrees of freedom of structure S_1; and $\tilde{m}_i$ are the masses making up the secondary structure (shown as circles in all figures).

Since we are only seeking sinusoidal solutions of the form

$$X_j = D_j \sin \lambda t$$

or

$$\ddot{X}_j = - \lambda^2 X_j \; ,$$

Eq. (9) can be written

$$\sum_{j=n+1}^{N} k_{ij} X_j - \lambda^2 \tilde{m}_i X_i - P_k(\lambda)(\sin \lambda t) \; U_{ik} = - k_{ik} X_k \; , \tag{10}$$

where $i = k, \; n+1, \; \ldots \; N.$

Case 2: Point k has no Mass — For the case in which point k has no mass the differential equations of motion of the structure S_1 can be written as

$$\sum_{j=n+1}^{N} k_{ij} X_j = - \tilde{m}_i \ddot{X}_i + P_k(\lambda)(\sin \lambda t) \; U_{i,n+1} \; , \tag{11}$$

where $i = n+1, \ldots, N$, which, along with the force-elongation equation for the spring attached to point k,

$$P_k(\lambda) \sin \lambda t = (X_{n+1} - X_k) k_{n+1,n+1} \; , \tag{12}$$

form the set for S_1

$$\sum_{j=n+1}^{N} k_{ij} X_j - \lambda^2 \tilde{m}_i X_i - P_k(\lambda)(\sin \lambda t) \; U_{i,n+1} = 0 \tag{13a}$$

and

$$P_k(\lambda) \sin \lambda t - k_{n+1,n+1} X_{n+1} = - k_{n+1,n+1} X_k, \quad i = n+1, \; \ldots, \; N. \tag{13b}$$

We note that $U_{i,n+1}$ has the same definition as in Eq. (9). Equations (10) or (13) — depending upon the case involved — are solved for $[P_k(\lambda) \sin \lambda t]$ as

$$P_k(\lambda) \sin \lambda t = G_k(\lambda, k_{ij}, \tilde{m}_i) X_k \; . \tag{14}$$

That is to say $G_k(\lambda, k_{ij}, \tilde{m}_i)$ is the transfer function between $[P_k(\lambda) \sin \lambda t]$ and X_i. In general this will have the form

$$P_k(\lambda) \sin \lambda t = \frac{|\text{Det B}|}{|\text{Det C}|} X_k \ . \tag{15}$$

Here Det C is the determinant of the left-hand side of Eq. (10) or (13) with $[P_k(\lambda) \sin \lambda t]$ being considered an unknown, and Det B is the determinant of the left-hand side of Eq. (10) or (13) with the $[P_k(\lambda) \sin \lambda t]$ coefficient column replaced by the column $-k_{ik}$.

More Than One Secondary Structure

All of the foregoing has been based on the assumption that only one structure S_1 was attached to one mass point on structure S_0, and therefore only one internal force was involved. If more than one structure is added to the structure S_0, then the right-hand side of Eq. (8) would become a summation of terms (similar to the term on the right-hand side of Eq. (8)) — one term for each structure attached. There will exist P sets of independent equations such as Eqs. (10) or (13) — where P is the number of structures added — which will enable the force in each term to be evaluated. If, on the other hand, the same structure is attached to more than one mass point of S_0, more than one internal force will again be present and again the right-hand side of Eq. (8) will become a summation of terms. In this case, however, only one set of equations such as Eqs. (10) or (13) will be available but will nonetheless yield expressions for all the internal forces.

Natural Frequencies and Normal Modes

The displacement X_k in Eq. (15) is expressed according to Eq. (2), and the equations that determine the generalized coordinates q_a are obtained by substituting Eq. (14) into Eq. (8):

$$\ddot{q}_a + \omega_a^2 q_a = \frac{\bar{X}_{ka} G_k(\lambda, k_{ij}, m_i)}{\sum_{j=1}^{n} m_j \bar{X}_{ja}^2} \sum_{b=1}^{n} \bar{X}_{kb} q_b, \quad a = 1, \ldots, n \ . \tag{16}$$

Since we are only interested in solutions at frequencies λ, we write

$$q_a = \tilde{q}_a \sin \lambda t \ , \tag{17}$$

and Eq. (16) becomes

$$\tilde{q}_a(\omega_a^2 - \lambda^2) - \frac{\bar{X}_{ka} G_k(\lambda, k_{ij}, \tilde{m}_i)}{\sum_{j=1}^{n} m_j \bar{X}_{ja}^2} \sum_{b=1}^{n} \bar{X}_{kb} \tilde{q}_b = 0, \quad a = 1, \ldots, n \ . \tag{18}$$

Equation (18) is a set of n homogeneous algebraic equations linear in the generalized coordinates $\tilde{q}_a$. If other than the trivial solution is to exist, the determinant of the coefficients must vanish. This fact yields an Nth-order polynomial equation in λ^2 from which the natural frequencies of the total $(S_0 + S_1)$ structure $(\lambda_A^2, A = 1, 2 \ldots, N)$ are determined.

These natural frequencies are then substituted back into Eq. (18) and for each of the N λ_A's, we obtain $(n-1)$ of the n q_a's in terms of the one remaining q_a. We arbitrarily

choose one of the $\tilde{q}_a$'s to be unity and substitute the resulting values of the $\tilde{q}_a$'s into Eq. (2) which now should be rewritten as

$$X_{iA} = \sum_{a=1}^{n} \bar{X}_{ia}\tilde{q}_{aA}, \quad i = 1, \; n \text{ and } A = 1, N \; . \tag{19}$$

Equation (19) gives N mode shapes for n of the N discrete mass points of the structure $(S_0 + S_1)$. The mode shapes for the remaining $(N-n)$ mass points are found by solving Eq. (10) or (13) for the X_{n+1} through X_N remaining unknowns in terms of the now known coordinate X_k:

$$X_{n+j,A} = \frac{|\text{Det } D_j|}{|\text{Det } C|} X_{kA} \; , \tag{20}$$

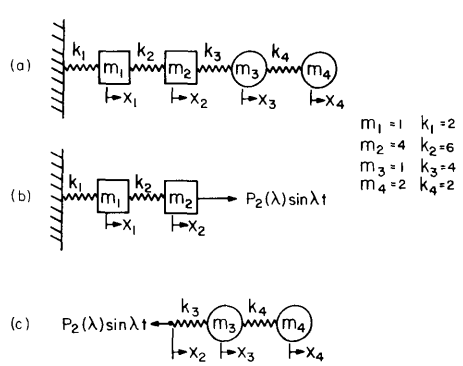
where Det C is the same as in Eq. (15) and Det D_j is the lefthand side of Eq. (10) or (13) with the coefficient column of X_{n+j} replaced with the $-k_{ik}$ column into which, of course, the particular values of λ_A^2 have been substituted.

We have thus determined the mode shapes and natural frequencies of the composite structure from a knowledge of the normal modes and natural frequencies of the primary structure and a knowledge of the parameters of the added structure.

SAMPLE PROBLEMS

General Case

We desire to know the normal modes and natural frequencies of the structure shown in Fig. 4a. The structure shown in Fig. 4b is arbitrarily chosen as the primary structure, leaving the structure 4c as the secondary structure. The solutions for the primary and total structure obtained by ordinary means are shown in Table 1.



$m_1 = 1$ $k_1 = 2$
$m_2 = 4$ $k_2 = 6$
$m_3 = 1$ $k_3 = 4$
$m_4 = 2$ $k_4 = 2$

Fig. 4 - Sample structure: (a) Complete structure, (b) primary structure, and (c) secondary structure

Since point X_2 on the secondary structure has no mass, this is an example of a Case 2 problem, and we write the equations of motion for S_1 as

$$\begin{aligned} k_3 X_3 \qquad\qquad\qquad\qquad &- P_2 \sin\lambda t = k_3 X_2 \\ (k_4 - m_3\lambda^2)X_3 - k_4 X_4 &+ P_2 \sin\lambda t = 0 \\ k_4 X_3 - (k_4 - m_4\lambda^2)X_4 &= 0 \; , \end{aligned} \tag{21}$$

which are a form of Eq. (13).

Table 1
Solutions for the Primary Structure and the Total Structure as Obtained by Ordinary Methods

Structure	Frequencies	Mode Shapes	
S_0	0.327049	$\bar{X}_{11} = 0.781968$	$\bar{X}_{21} = 1.0$
	9.172952	$\bar{X}_{12} = -5.115299$	$\bar{X}_{22} = 1.0$
$S_0 + S_1$	0.174454	$X_{11} = 1.0$	$X_{21} = 1.304258$
		$X_{31} = 1.533112$	$X_{41} = 1.857091$
	1.074521	$X_{12} = 1.0$	$X_{22} = 1.154247$
		$X_{32} = 0.145355$	$X_{42} = -1.950523$
	6.708285	$X_{13} = 1.0$	$X_{23} = 0.215286$
		$X_{33} = -2.405985$	$X_{43} = 0.421490$
	9.542740	$X_{14} = 1.0$	$X_{24} = -0.257123$
		$X_{34} = 0.310850$	$X_{44} = -0.036384$

In this case,

$$G_2(\lambda,\ k_{ij},\ \tilde{m}_i) = \frac{k_3 k_4 (m_3 + m_4)\lambda^2 - k_3 m_3 m_4 \lambda^4}{k_3 k_4 - \lambda^2 (k_4 m_3 + k_4 m_4 + k_3 m_4) + m_3 m_4 \lambda^4} . \tag{22}$$

From Eq. (21),

$$P_2 \sin \lambda t = \frac{\begin{vmatrix} k_3 & 0 & k_3 \\ (k_4 - m_3\lambda^2) & -k_4 & 0 \\ k_4 & -(k_4 - m_4\lambda^2) & 0 \end{vmatrix}}{\begin{vmatrix} k_3 & 0 & -1 \\ (k_4 - m_3\lambda^2) & -k_4 & +1 \\ k_4 & -(k_4 - m_4\lambda^2) & 0 \end{vmatrix}} . \tag{23}$$

Upon substituting values for the structure into Eqs. (22) and (23), we obtain

$$P_2 \sin \lambda t = G_2 X_2 , \tag{24}$$

where

$$G_2 = \frac{4\lambda^2 (3 - \lambda^2)}{\lambda^4 - 7\lambda^2 + 4} ,$$

which gives for Eq. (18)

$$\tilde{q}_a(\omega_a^2 - \lambda^2) - \frac{\bar{X}_{2a}\left[\dfrac{4\lambda^2(3-\lambda^2)}{\lambda^4 - 7\lambda^2 + 4}\right]\sum_{b=1}^{2}\bar{X}_{2b}\tilde{q}_b}{\sum_{j=1}^{2} m_j \bar{X}_{ja}^2} = 0$$

or, for the structure being studied,

$$\left[(0.327049 - \lambda^2) - \frac{0.867402\,(3-\lambda^2)}{\lambda^4 - 7\lambda^2 + 4}\right]\tilde{q}_1 - \frac{0.867403\,\lambda^2(3-\lambda^2)}{(\lambda^4 - 7\lambda^2 + 4)}\tilde{q}_2 = 0\,,$$

$$-\frac{0.132598\,\lambda^2(3-\lambda^2)}{\lambda^4 - 7\lambda^2 + 4}\tilde{q}_1 + \left[(9.172952 - \lambda^2) - \frac{0.132598(3-\lambda^2)}{\lambda^4 - 7\lambda^2 + 4}\right]\tilde{q}_2 = 0\,. \tag{25}$$

Expanding the determinant of the coefficients of Eq. (25), we arrive at the frequency equation

$$\lambda^8 - 17.500001\,\lambda^6 + 84.500014\,\lambda^4 - 83.000045\,\lambda^2 + 12.000020 = 0\,, \tag{26}$$

which gives λ^2 = 0.174455, 1.074521, 6.708286, and 9.542740. These agree with the natural frequencies as given in Table 1 for the total structure.

When these four values are substituted back into Eq. (25), taking $\tilde{q}_{1A} = 1$, we obtain

$$\begin{aligned}
\tilde{q}_{11} &= 1 & \tilde{q}_{21} &= 0.002602 & \text{for} \quad \lambda_1^2 &= 0.174455\\
\tilde{q}_{12} &= 1 & \tilde{q}_{22} &= -0.014109 & \text{for} \quad \lambda_2^2 &= 1.074521\\
\tilde{q}_{13} &= 1 & \tilde{q}_{23} &= -0.395790 & \text{for} \quad \lambda_3^2 &= 6.708296\\
\tilde{q}_{14} &= 1 & \tilde{q}_{24} &= 3.809717 & \text{for} \quad \lambda_4^2 &= 9.542740
\end{aligned} \tag{27}$$

which when substituted into Eq. (19) (taking $X_{1A} = 1.0$) yields

$$\begin{aligned}
X_{11} &= 1.0 & X_{21} &= 1.304354\\
X_{12} &= 1.0 & X_{22} &= 1.154250\\
X_{13} &= 1.0 & X_{23} &= 0.215286\\
X_{14} &= 1.0 & X_{24} &= -0.257123
\end{aligned} \tag{28}$$

The solution of Eq. (21) for X_3 and X_4 gives

$$\begin{aligned}
X_{3A} &= \frac{k_3k_4 - k_3m_4\lambda_A^2}{2\lambda_A^4 - 14\lambda_A^2 + 8}X_{2A}\\
X_{4A} &= \frac{k_3k_4}{2\lambda_A^4 - 14\lambda_A^2 + 8}X_{2A}\,;
\end{aligned} \tag{29}$$

the mode shapes are found to be

$$
\begin{aligned}
X_{31} &= 1.533224 & X_{41} &= 1.857224 \\
X_{32} &= 0.145356 & X_{42} &= -1.950528 \\
X_{33} &= -2.405988 & X_{43} &= 0.421490 \\
X_{34} &= 0.310852 & X_{44} &= -0.036388 \ ,
\end{aligned}
\tag{30}
$$

and all of the mode shapes are in agreement with the values given in Table 1 as found by ordinary means.

Special Case — Node at Point of Attachment

If a fixed-base natural frequency of the secondary structure has the same value as one of the natural frequencies of the total structure, a node appears at the point of attachment. At this frequency, the secondary structure behaves as if attached to a fixed base. Equations (20) are now indeterminant and of little value in determining mode shapes for this mode of vibration. The ratios of displacements for points on the secondary structure (mode shapes) are seen to be the same as those obtained from the solution of the secondary structure as if supported on a fixed base. The values of the displacements must still be chosen, however, to be compatible with the displacements of the primary structure. For this reason it is felt that these displacements for the secondary structure are best found from the orthogonality conditions

$$
\sum_{i=1}^{N} m_i X_{iA} X_{iB} = 0, \quad A \neq B \ . \tag{31}
$$

If the normal modes for all other frequencies are first found according to the procedure outlined in the section on theory, Eq. (31) will always supply the number of independent equations required to find the mode shapes of the secondary structure for this nodal frequency.

The structure shown in Fig. 5a results in a situation wherein a point of attachment becomes a node.

The solutions for the primary structure and the total structure are given in Table 2.

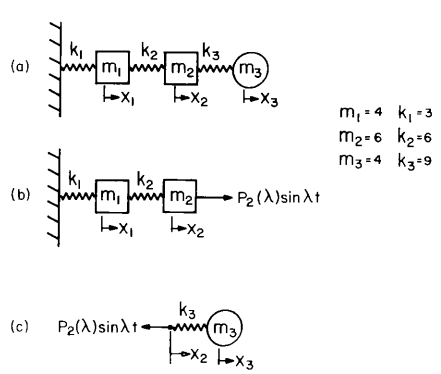


Fig. 5 - Structure with a new node located at attachment point: (a) Complete structure, (b) primary structure, and (c) secondary structure

The generalized coordinates are found from

Table 2
Solutions Obtained by Ordinary Methods for the
Nodal Attachment Point Problem

Structure	Frequencies	Mode Shapes		
S_0	0.250000	$\bar{X}_{11} = 1.0$	$\bar{X}_{21} = 1.333333$	
	3.000000	$\bar{X}_{12} = 1.0$	$\bar{X}_{22} = -0.500000$	
$S_0 + S_1$	0.163524	$X_{11} = 1.0$	$X_{21} = 1.390984$	$X_{31} = 1.500000$
	2.250000	$X_{12} = 1.0$	$X_{22} = 0$	$X_{32} = -0.666667$
	4.586476	$X_{13} = 1.0$	$X_{23} = -0.557651$	$X_{33} = 1.500000$

$$\left[(0.25 - \lambda^2) - \frac{4.363636\lambda^2}{9 - 4\lambda^2}\right] \tilde{q}_1 + \frac{1.636364}{9 - 4\lambda^2} \tilde{q}_2 = 0$$
$$\frac{4.363636\lambda^2}{9 - 4\lambda^2} \tilde{q}_1 + \left[(3 - \lambda^2) - \frac{1.636364\lambda^2}{9 - 4\lambda^2}\right] \tilde{q}_2 = 0 , \tag{32}$$

which yield the frequency equation

$$\lambda^6 - 7.000000\lambda^4 + 11.437500\lambda^2 - 1.687500 = 0 ,$$

whose roots $\lambda^2 = 0.163524$, 2.250000, and 4.586476 are the same as the values given in Table 2.

When these values are substituted back into Eqs. (32), taking $\tilde{q}_{1A} = 1$, we obtain

$$\begin{aligned} \tilde{q}_{11} &= 1 & \tilde{q}_{21} &= -0.030499 \\ \tilde{q}_{12} &= 1 & \tilde{q}_{22} &= 2.666667 \\ \tilde{q}_{13} &= 1 & \tilde{q}_{23} &= 2.733405 . \end{aligned} \tag{33}$$

These $\tilde{q}_{iA}$ values give in turn the mode shapes of the points on structure S_0 as, taking $X_{1A} = 1$,

$$\begin{aligned} \bar{X}_{11} &= 1 & \bar{X}_{21} &= 1.391008 \\ \bar{X}_{12} &= 1 & \bar{X}_{22} &= 0 \\ \bar{X}_{13} &= 1 & \bar{X}_{23} &= -0.557649 , \end{aligned} \tag{34}$$

and we find from Eq. (20) that

$$X_{31} = 1.500025,$$
$$X_{33} = 1.499998,$$

and that

$$X_{32} = \frac{k_3 X_{22}}{k_3 - m_3\lambda_2^2} = \frac{0}{0} ,$$

is indeterminant.

To evaluate the modal component X_{32}, we write Eq. (31) for $A = 1$ and $B = 2$,

$$m_1 \bar{X}_{11} \bar{X}_{12} + m_2 \bar{X}_{21} \bar{X}_{22} + m_3 \bar{X}_{31} \bar{X}_{32} = 0$$

or (36)

$$\bar{X}_{32} = -\frac{m_1 \bar{X}_{11}}{m_3 \bar{X}_{31}} \bar{X}_{12} = -0.666656 ,$$

and all of the mode shapes are therefore determined and agree with the values presented in Table 2.

Another problem of special interest is what happens when the point of attachment is a node on the primary structure before the structure S_1 is added. This will be covered now, as part of the discussion that follows.

Continuous Primary Structure

As one further example of the procedure developed, we wish to examine a continuous-type structure as our primary structure. Consider the structure shown in Fig. 6a where, as our basic structure, we chose the simply supported beam of length ℓ. Our secondary structure consists of a mass-spring oscillator.

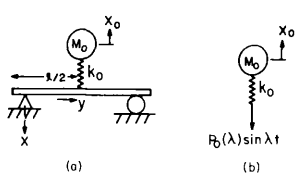


Fig. 6 - Continuous primary structure: (a) Complete structure and (b) secondary structure

The primary structure has as its natural frequencies

$$\omega_a^2 = \frac{a^2 \pi^4 EI}{M \ell^4}, \quad a = 1, 2, 3, \ldots,$$

where M is the mass of the beam per unit length. The normal modes of the primary structure are

$$\phi_a(X) = \sqrt{\frac{2}{M\ell}} \sin \frac{a\pi X}{\ell} .$$

We arbitrarily pick

$$\frac{k_0 \ell^3}{48EI} = 1.0, \; M\ell = 1.0, \text{ and } M_0 = 1.0 .$$

We rewrite Eq. (2) for a continuous system as

$$X_A = \sum_{a=1}^{\infty} q_{aA} \, \phi(y) . \tag{37}$$

Therefore, Eqs. (18) become

$$\tilde{q}_a(\omega_a^2 - \lambda^2) - \frac{\phi_a(\ell/2)\, G(\lambda, K_{ij}, \tilde{m}_i)}{\int_0^\ell \phi_a^2 \, dm} \sum_{b=1}^{\infty} \phi_b \tilde{q}_b = 0 \ . \tag{38}$$

The finite expressions in Eqs. (2) and (18) have been replaced with infinite summations in Eqs. (37) and (38). To make the solution tractable, we will terminate the infinite series with only four terms -- realizing of course that the answers will be only approximate and that a computer solution retaining more terms could very easily be accomplished not only to improve the answers but to check on rates of convergence and accuracy of answers as well.

Equations (38) are

$$\tilde{q}_1 \ (\omega_1^2 - \lambda^2) = \frac{48\lambda^2}{48 - \lambda^2} \sqrt{2} \sin \pi/2 \left(\tilde{q}_1 \sqrt{2} \sin \pi/2 + \tilde{q}_3 \sqrt{2} \sin 3\pi/2\right)$$

$$\tilde{q}_2 \ (\omega_2^2 - \lambda^2) = 0$$

$$\tilde{q}_3 \ (\omega_3^2 - \lambda^2) = \frac{48\lambda^2}{48 - \lambda^2} \sqrt{2} \sin 3\pi/2 \left(\tilde{q}_1 \sqrt{2} \sin \pi/2 + \tilde{q}_3 \sqrt{2} \sin 3\pi/2\right)$$

$$\tilde{q}_4 \ (\omega_4^2 - \lambda^2) = 0 \ .$$

The frequency equation is

$$(\omega_2^2 - \lambda^2)(\omega_4^2 - \lambda^2) \quad \left[(\omega_1^2 - \lambda^2)(48 - \lambda^2)(\omega_3^2 - \lambda^2) - 96\lambda^2(\omega_1^2 - \lambda^2) - 96\lambda^2(\omega_3^2 - \lambda^2)\right] = 0 \ , \tag{40}$$

giving

$$\lambda_1^2 = 21.1374$$

$$\lambda_2^2 = 218.4947$$

$$\lambda_3^2 = 1558.544$$

$$\lambda_4^2 = 7987.9079$$

$$\lambda_5^2 = 24936.7065.$$

These are in good agreement with the values for the natural frequencies obtained in Ref. 7, considering the numbers involved and the number of places retained past the decimal in the calculation performed.

Back-substitution of these values into Eq. (39) results in the following values for the generalized coordinates:

$\tilde{q}_{11} = -103.1705$	$\tilde{q}_{21} = 0$	$\tilde{q}_{31} = 1.0$	$\tilde{q}_{41} = 0$
$\tilde{q}_{12} = 63.3638$	$\tilde{q}_{22} = 0$	$\tilde{q}_{32} = 1.0$	$\tilde{q}_{42} = 0$
$\tilde{q}_{13} = 0$	$\tilde{q}_{23} =$ arbitrary	$\tilde{q}_{33} = 0$	$\tilde{q}_{43} = 0$
$\tilde{q}_{14} = 0.01239$	$\tilde{q}_{24} = 0$	$\tilde{q}_{34} = 1.0$	$\tilde{q}_{44} = 0$
$\tilde{q}_{15} = 0$	$\tilde{q}_{25} = 0$	$\tilde{q}_{35} = 0$	$\tilde{q}_{45} =$ arbitrary.

These yield upon substitution into Eq. (37)

$$
\begin{aligned}
X_1 &= -103.1821\ \sqrt{2}\ \sin \pi X/\ell + \sqrt{2}\ \sin 3\pi X/\ell \\
X_2 &= \sqrt{2}\ \sin 3\pi X/\ell + 63.3630\ \sqrt{2}\ \sin \pi X/\ell \\
X_3 &= C \sin 2\pi X/\ell \\
X_4 &= -0.01239\ \sqrt{2}\ \sin \pi X/\ell + \sqrt{2}\ \sin 3\pi X/\ell \\
X_4 &= D \sin 4\pi X/\ell ,
\end{aligned}
\tag{41}
$$

with C and D arbitrary constants.

Now, from Eq. (20) written for the single oscillator, we find

$$
X_{oA} = \frac{k_0 X_A(\ell/2)}{k_0 - M_0 \lambda_A^2}
\tag{42}
$$

and

$$
\begin{aligned}
X_{01} &= -263.2670 \\
X_{02} &= -\ 24.8301 \\
X_{03} &= 0 \\
X_{04} &= +0.008655 \\
X_{05} &= 0\ .
\end{aligned}
\tag{43}
$$

The answers given in Eqs. (41) and (43) cannot be verified as being correct, but the orthogonality conditions for the total structure are for all practical purposes satisfied, and a modal effective mass calculation shows that all but 5 percent of the total mass is accounted for. This is good considering that the values obtained for the higher natural frequencies here and in Ref. 7 are known to be in slight error.

Note also that for two of the modes — three and five — the primary structure exhibited a node at the point of attachment. However, it was not necessary to use the orthogonality conditions, as the equations developed in the section on theory were adequate.

SUMMARY

A method has been demonstrated for obtaining the normal modes and natural frequencies of a structure from a knowledge of the uncombined modes and frequencies of a primary part and the mass distributions and elastic constants of the secondary part.

In passing we note that there are other, perhaps more efficient, ways of obtaining the natural frequencies of the total structure. We chose to use the determinant Eqs. (18) merely to show that all the needed information on the structure (i.e., normal modes and natural frequencies) can be obtained from the equations developed in the theory section.

This development is of great interest to an experimentalist. What we have shown is that a complex structure might possibly be tested by breaking it down into one primary structure and several secondary structures. To determine how the primary structure

will react to dynamic loading when it is a part of the total structure, one merely drives (at points of attachment) the primary structure with sinusoidal forces of arbitrary amplitude but with frequencies which correspond to the natural frequencies of the total structure. The displacements thus obtained (when normalized) are the normal modes of the primary structure when part of the total structure.

To determine how the secondary structures will react to dynamic forces when they are a part of the total structure, each secondary structure is individually driven at its attachment with sinusoidal forces. These forces are equal but opposite to those used to drive the primary structure. These displacements, when normalized, are the normal modes of the secondary structure when it is a part of the total structure.

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13. ABSTRACT

This report considers a linear elastic structure to consist of several parts; a primary structure which has a rigid base, and one or more substructures which are firmly attached to the primary structure. The normal modes and natural frequencies of the combined structure are determined from a knowledge of the primary structure's normal modes and frequencies and the mass and the elastic constants distribution of the secondary structure. The displacements of the total structure are expressed as a series expansion of the scaled normal modes of the primary structure alone, and a set of equations is used to find the normal coordinates of the secondary structures for the new natural frequencies. There is in general an increase in the number of degrees of freedom. The scaling factors involved are determined from the equilibrium equations of the primary structure, with the secondary structures removed and replaced by their internal forces of connection. Numerical examples are presented.

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