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Ambiguity Resolution in the SPASUR Radio Interferometer Direction Finding System

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CONTENTS

Abstract	iv
Problem Status	iv
Authorization	iv
INTRODUCTION	1
FRAME-BY-FRAME TECHNIQUE OF AMBIGUITY RESOLUTION	2
Generalized Walkup Method	3
Generalized Pattern Method	6
Square Wave Combiner Method	7
Squared Wave Combiner	7
MULTIFRAME TECHNIQUE OF AMBIGUITY RESOLUTION	8
THE SPASUR SIMULATOR	11
COMPARISON OF THEORETICAL AND EXPERIMENTAL RESULTS OF AMBIGUITY RESOLUTION	13
CONCLUSIONS	15
ACKNOWLEDGMENTS	16
APPENDIX A — Use of Very Long Baselines at the Spasur Stations	18
APPENDIX B — Walkup Ambiguity Resolution	22
APPENDIX C — The Combiner and the Combiner Simulator	27
APPENDIX D — Proposal of a Joint Walkup Resolution and Rectification Procedure	30
APPENDIX E — Trajectory Generating Subroutine	32
APPENDIX F — Systematic Error Subroutine	34
APPENDIX G — Gaussian Correlated Noise Generator Subroutine	36
APPENDIX H — Subroutine for the Maximum Likelihood Method of Ambiguity Resolution	39
APPENDIX I — Subroutine for the Generalized Walkup Method of Ambiguity Resolution	42
APPENDIX J — Subroutine for the Generalized Pattern Method of Ambiguity Resolution	44
APPENDIX K — Evaluation of Point-by-Point Ambiguity Resolution Methods by NAREC Simulation	46

ABSTRACT

A number of methods for ambiguity resolution for use in the Spasur radio interferometer direction-finding system have been derived and have been studied using both Monte Carlo and operational data. Implementations of the methods in both hardware and software were analyzed. The conclusion was drawn that the optimum resolution results can be obtained only by the joint optimization of the antenna configuration and the resolution process to form an optimum processing system.

PROBLEM STATUS

This is an interim report on one phase of the problem; work on the problem is continuing.

AUTHORIZATION

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AMBIGUITY RESOLUTION IN THE SPASUR RADIO INTERFEROMETER DIRECTION FINDING SYSTEM

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INTRODUCTION

The Navy Space Surveillance (Spasur) System was primarily designed to detect the presence of nonradiating artificial earth satellites. The system consists of three transmitting and six receiving stations in a plane intersecting the southern part of the United States at about 33 degrees latitude. A continuous-wave signal is transmitted in a thin beam centered in the Spasur plane, and when a satellite enters the beam, the signal is reflected to one or more of the radio interferometer receiving sites. When the received signal-to-noise ratio exceeds a prescribed threshold, an alert is declared, and the phase angles from the various antenna pairs are sent every 1/20 second to the central computer at Dahlgren, Virginia.

For each receiving site the data are divided into one phase reading for each antenna pair, called a frame, and the phase ambiguity is resolved using a parallel ray assumption (see Appendix A) to yield a simple estimate of the zenith angle. These zenith angles are collected in data groups consisting of a block of contiguous alerted data frames, and an estimate of angle and angle rate at the time the satellite crossed the plane is determined. Such information from two or more stations is compared with predicted zenith angles for the known satellite population. If the detected signal correlates in both time and angle with a prediction for one of the known population, it is identified as an observation of that satellite and stored for future use in updating the orbit. Otherwise this observation is correlated with other unknown observations to determine if the orbit of a new satellite can be established.

The Spasur System has many baselines, some shorter than 1/2 wavelength but most substantially longer. When an interferometer direction-finding system uses baselines which are greater than 1/2 wavelength, the absolute phase difference between two antennas can only be measured modulo 2π, thereby creating a problem known as ambiguity resolution. This has led to studies of the ambiguity resolution problem, both theoretical and experimental using both simulated and operational data.

The problem is approached in two steps. First, a theoretical model is developed for a resolution process which would find the statistically most likely resolution. Then an algorithm is found which is within the computing capability of the processing system and which comes nearest to giving the most likely resolution. Using any algorithm, one can in principle derive what is called an optimum processing system by modifying the various parameters within the processing system to yield a "best" resolution based on any arbitrary definition of the word best. In all of our studies a single definition was used: the resolution is correct if the resolved phase on the longest baseline is within π radians of the true solution; otherwise it is incorrect. The problem then reduces to finding what processing technique yields the highest percent of correct resolutions for a specific type of environment.

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Theoretically, one would like to use all the phase readings from each station and each baseline in an algorithm which would find the most likely orbital parameters all in one step. Unfortunately, such a system is well beyond the capability of the Spasur computer system. Two techniques of ambiguity resolution were studied. The first technique was to derive the most likely zenith angle from a single frame (one phase reading per baseline) of data from a single station; we call this frame-by-frame resolution. The second technique was to use multiframe methods, using either phase angles or previously resolved zenith angles.

FRAME-BY-FRAME TECHNIQUE OF AMBIGUITY RESOLUTION

The theoretical model used for frame-by-frame ambiguity resolution was the maximum likelihood method, in which basically the phase noise was assumed to be Gaussian distributed, have a mean of zero, be independent, and have a predictable variance for each baseline. The likelihood function* takes the form

$$L = (2\pi)^{-N/2}\left(\prod_{i=1}^{N}\frac{1}{\sigma_{x_i}}\right)e^{-\left\{\sum_{i=1}^{N}\left[(x_i - x)/\sigma_{x_i}\right]^2\right\}/2}$$

where

x = sine of the zenith angle (the angle from vertical in the Spasur plane),

x_i = measured value of the sine of the zenith angle on the ith baseline,

σ_{x_i} = $(\lambda/d_i)\sigma_{\varphi_i}$ = standard deviation of x_i about the true zenith angle,

d_i = length of the ith baseline,

σ_{φ_i} = standard deviation of the phase angle of the ith baseline about its mean,

N = total number of baselines, and

λ = wavelength of the radio signal.

Since L is a monotonically increasing function of the exponent, remembering that σ_{x_i} is a (possibly unknown) constant, we may solve the problem by maximizing the exponent. Of course, the value of x is unknown, but for fixed values of x_i we can find its maximum likelihood estimate $\bar{x}$:

$$\bar{x} = \frac{\sum_{i=1}^{N} d_i^2 x_i/\sigma_{\varphi_i}^2}{\sum_{i=1}^{N} d_i^2/\sigma_{\varphi_i}^2} .$$

The exponent in the likelihood function becomes proportional to

*H. Cramer, "Elements of Probability Theory and Some of Its Applications," New York: Wiley, p. 194, 1955.

$$\bar{x}_{k-1} = \frac{\sum_{i=1}^{k-1} (d_i^2 x_i \sigma^2_{\dot{\varphi}_i})}{\sum_{i=1}^{k-1} (d_i^2 \sigma^2_{\dot{\varphi}_i})} .$$

In the unweighted method it is assumed that

$$\frac{d_1^2}{\sigma^2_{\dot{\varphi}_1}} << \frac{d_2^2}{\sigma^2_{\dot{\varphi}_2}} << \cdots << \frac{d_N^2}{\sigma^2_{\dot{\varphi}_N}}$$

in which case

$$\bar{x}_{k-1} \approx \frac{d_{k-1}^2 x_{k-1} \sigma^2_{\dot{\varphi}_{k-1}}}{d_{k-1}^2 \sigma^2_{\dot{\varphi}_{k-1}}} = x_{k-1} .$$

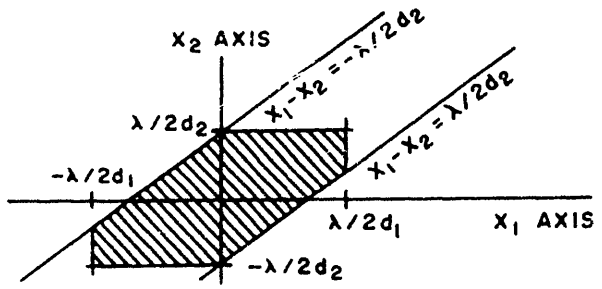
Referring to Eq. (2), when projecting to the next longer baseline we know λ, d_k, and φ_k, but do not know m_k and x_k. If it is assumed that $\bar{x}_{k-1} \approx x_k$, then

$$\hat{m}_k = \frac{d_k}{\lambda} x_{k-1} - \varphi_k + \frac{1}{2} \approx m_k .$$

The most likely value of m_k is the closest integer to $\hat{m}_k$. This integer is inserted for m_k in Eq. (2) to find the most likely value of x_k, and the process is repeated for all baselines.

The mathematics for this resolution method are substantially simpler to work with than those of the maximum likelihood method. Again we go back to the concept of an optimum or matched processing system where the various elements of the system are matched to maximize the information throughout. The antenna system, with their parameters d_i or d_i/λ, are an integral part of this processing system and therefore can be modified to increase the probability of correct resolution. This is very similar to the concept of matched filters. The mathematics has been worked out in detail in Appendix B but will be briefly restated here. These equations yield not only the performance using a specific set of baselines but also predict the number and lengths of the baselines in an optimum set. The basic assumptions used here are the same as those used in the maximum likelihood derivation. It is important, however, that $\sigma_\varphi < 1/6$ in the operational system or the alert antennas will not detect the presence of the signal. Note also that once the partial solution exceeds the correct bounds it is considered incorrect forever. This means the percent of correct solutions reaches a maximum at a finite number of baselines.

Let us assume for the purposes of discussion that the true sine of the space angle $x_0 = \sin \theta = 0$ and that the baselines are written in monotonically increasing order of length $d_1 < d_2 < \cdots < d_n$. We make a basic assumption that we have an unambiguous estimate of x_1 from the shortest baseline, i.e., $|x_1| < \lambda/2d_1$. The fundamental question is now, "Given that $|x_1| < \lambda/2d_1$ and that x_2 is chosen such that $|x_2 - x_1| < \lambda/2d_2$, what is the probability that $|x_2| < \lambda/2d_2$?" This probability is given by the integral of the independent joint normal distribution over the area shown below:



As one can imagine, the detailed evaluation is complicated even using numerical methods on a digital computer. We must therefore make another assumption very similar to our previous assumptions. The assumption that $\sigma_{\tau_i} < 1/6$ is equivalent to $\sigma_{x_i} < \lambda/6d_i$; hence the error in neglecting the two constraints $|x_2| < \lambda/2d_2$ and $|x_1| < \lambda/2d_1$ but retaining the third constraint will be less than 0.003. The integral for the probability of "walking up" to the second baseline such that $|x_2| < \lambda/2d_2$ is then

$$P_{12} = \frac{1}{2\pi\sigma_1\sigma_2}\int_{-\infty}^{\infty} e^{-x_1^2/2\sigma_1^2}\int_{x_1-\frac{\lambda}{2d_2}}^{x_1+\frac{\lambda}{2d_2}} e^{-x_2^2/2\sigma_2^2}\,dx_2\,dx_1$$

which may be transformed into a single integration

$$P_{12} = \frac{1}{\sqrt{2\pi}\sqrt{\sigma_1^2+\sigma_2^2}}\int_{-\lambda/2d_2}^{+\lambda/2d_2} e^{-z^2/2(\sigma_1^2+\sigma_2^2)}\,dz$$

by the transformation $z = x_2 - x_1$.

This is equivalent to a well-known theorem which states that the difference between two independent Gaussian random variables, with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2, is a Gaussian random variable of mean $\mu_1-\mu_2$ and variance $\sigma_1^2+\sigma_2^2$. If the phase noise is independent on all baselines, the probability of "walking up" through N baselines is simply the product of the probabilities of "walking up" through each pair:

$$P_{1N} = \prod_{i=1}^{N-1} P_{i,i+1}\,.$$

In the unweighted walkup case where $\sigma_{x_i} = (\lambda/d_i)\sigma_{\tau_i}$ this function P_{1N} is a relatively simple function of the baseline lengths and is maximized by setting $\partial P_{1N}/\partial d_i = 0$, for $i = 2, 3, \ldots, N-1$, where d_1 and d_N are fixed. For the sake of a more compact solution, it is further assumed that $\sigma_{\tau_i} = \sigma$ = constant. This process yields the answer $d_i = cd_{i-1}$ (where $c^{N-1} = d_N/d_1$), which may be restated by saying that the probabilities $P_{i,i+1}$ must be all equal. The weighted walkup case is considerably more difficult, but it can be shown that the choice of baselines with

$$d_k^2 = c^2\sum_{j=1}^{k-1} d_j^2$$

will yield a higher probability of correct resolution than the corresponding probability in the unweighted case. It is this choice of baselines which is used to "pseudomaximize" the performance of the weighted resolution method.

The problem of maximizing the probability P_{1N} for fixed d_1 and d_N with respect to N is difficult in integer mathematics. The problem can quite easily be solved by iteration techniques. The results of this study will be presented in a later section.

Generalized Pattern Method

In the early days of Spasur the ambiguity resolution method used was called the pattern method. This method uses the possible values of the zenith angle as measured on the longest baseline as trial solutions and selects that trial solution which minimizes the sum of the absolute values of the phase errors, i.e.,

$$\sum_{i=1}^{N-1} |x_i - \bar{x}|\, d_i ,$$

where $\bar{x} = x_N$. It is constructive to consider this method in the light of the maximum likelihood theory both to present a unified approach and to show a simple method to improve the pattern method.

If one assumes that the variances of the phase errors on each baseline are equal and takes the limit of the maximum likelihood solution for which $d_N^2 >> d_i^2$ for $i \neq N$, then the mean approaches

$$\bar{x} = \frac{\sum_{i=1}^{N} (d_i^2 x_i / \sigma_{\varphi_i}^2)}{\sum_{i=1}^{N} (d_i^2 / \sigma_{\varphi_i}^2)} \to x_N$$

and the exponent to be minimized approaches

$$A = \sum_{i=1}^{N} \frac{(x_i - \bar{x})^2}{\sigma_{\varphi_i}} d_i^2 \to \sum_{i=1}^{N-1} (x_i - x_N)^2 d_i^2 .$$

If one further assumes that the value of x_N which minimizes the sum of the squares of the phase errors also minimizes the sum of the absolute values of the phase errors, then the simple pattern method can be justified. The validity of this last statement was studied (and found to be not true) with what is called the generalized pattern method. Here the exponent to be minimized was

$$A = \sum_{i=1}^{N-1} |x_i - x_N|^p d_1^q, \text{ for } p, q = 1, 2 .$$

The simulation runs with $p = q = 2$ had significantly better success than the runs with $p = q = 1$, which in turn had better success than runs with either combination of 1 and 2. The fact that squaring the phase errors, which requires very little additional running time, yields consistently better solutions has never been used in the operational programs. It is interesting to note that the running time for this program is approximately 2^{N-1} times smaller than the maximum likelihood technique due to the assumption $\bar{x} = x_N$.

A program was also developed called the modified pattern method which is one step nearer to the maximum likelihood method. In the modified pattern method, one takes a possible solution on the longest baseline X_N and finds the values X_i which are nearest to X_N. These values are combined to find the maximum likelihood estimate $\hat{\bar{r}}$ of the sine of the zenith angle. This estimate then replaces $\bar{X} = X_N$ in the regular pattern method with $p = q = 2$. As one might expect and as will be seen in a later section, the results of the modified pattern method were better than any form of the regular pattern method.

Square Wave Combiner Method

The implementation of the resolution techniques in hardware form can use either the exponent or the likelihood function itself. The function generations required are generally simpler in the exponent method to be described in the next subsection. The exception to this rule is the digital square wave combiner, where the (Gaussian) likelihood function on each baseline is approximated by a unit-height square wave of period proportional to d_i and phase shifted an amount proportional to the observed phase φ_i. Since the product of the individual likelihood functions is just the N input logical AND of the N waveforms, it can be handled very simply using standard digital techniques.

The major problem associated with the square wave combiner is the determination of the value of the duty cycle (gate width) of each square wave. While one would tend to think that the duty cycle should be proportional to the standard deviation of the phase errors on each baseline, this is generally not true. Two conflicting problems tend to make optimum duty cycles hard to evaluate. One would like to make the duty cycles as wide as possible to maximize the probability that the "correct" solution is an actual solution, and one would like to make the duty cycles as narrow as possible to minimize the probability of a false solution. In order to simplify the hardware the first solution found is the only solution reported, which makes the mathematics of optimum duty cycles difficult to solve. A square-wave-combiner simulator (see Appendix C) was developed where the duty cycles could be varied, and the "optimum" duty cycles were determined using Monte Carlo techniques.

The optimum duty cycle generally differs from baseline to baseline, and small changes made in duty cycles on one baseline necessitates changes on all the other baselines. The results obtained using three different sets of gates are compared with the unweighted walkup method of ambiguity resolution in Fig. 1. The duty cycles as indicated in Fig. 1 are in terms of degrees of rotation of the phase meter, 360 degrees being one full turn. For this set of baselines the best set uses wide gates on the shorter baselines and narrow gates on the longer baselines to filter out the multiple solutions.

Gate settings for the combiners installed in the system were selected using operational data. A series of experiments were run using these data in the combiner simulator. Different duty cycle settings were used in each experiment. The duty cycles that produced the most correct solution for the data sample were installed in the combiner. The results of the simulation runs were verified by comparing results obtained from the combiner and unweighted walkup methods of ambiguity resolution for several data samples. In general the combiner using optimum gate settings has a higher probability of obtaining the correct solution than the unweighted walkup method. In a typical 281-case sample of data the combiner got 263 correct resolutions as opposed to 239 correct solutions for the unweighted walkup method.

Squared Wave Combiner

The squared wave combiner uses the exponent of the likelihood function:

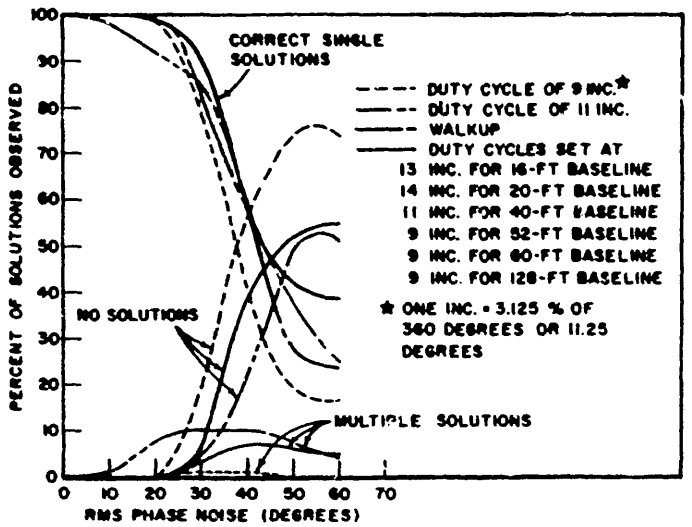


Fig. 1 - The effect of duty cycle on the resolution probability of the square wave combiner

$$A = \sum_{i=1}^{N} \frac{(x_i - \bar{x})^2}{\sigma_{x_i}^2} ,$$

where $\bar{x} = a + bt$. The squared function $(x_i - \bar{x})^2$ can be generated using a periodic squared wave of period inversely proportional to d_i and phase shifted proportional to the phase reading on the ith baseline. The separate waveforms are then combined using the weighting function $1/\sigma_{x_i}^2$ in a summing amplifier, and the absolute minimum of the sum is measured over one complete scan of $\bar{x}$ (from $\bar{x} = -1$ to $\bar{x} = +1$). While this system was proposed several years ago, it has never been built, even though it should exactly duplicate the maximum likelihood technique, which is the best known method of ambiguity resolution.

MULTIFRAME TECHNIQUE OF AMBIGUITY RESOLUTION

The frame-by-frame methods of ambiguity resolution lead to some nonanalytic problems when we consider the phase data as a time sequence. One method of handling these problems (to be described later in this section) is a process of combining the data by first resolving the angle for a fixed time and then combining the resolved angles as a function of time. Another method, which is much more likely to be analytic in the zenith angle and angular rate domain, would be to maximize the likelihood function of the time sequence, which is of the form

$$L = \prod_{i,j} \frac{1}{\sqrt{2\pi}\,\sigma_{i_j}} e^{-[x_{i_j} - x(t_j)]^2/2\sigma_{i_j}^2}$$

where $x(t_j)$, which is assumed to be of the form $x(t_j) = x(t_c) + x'j$, is the true sine of the space angle as a function of time, x_{i_j} is a possible estimate of the sine of the angle measured on baseline i at time t_j, and $\sigma_{i_j}^2$ is the variance of that estimate. One would like to vary both x and x' continuously over their ranges of conceivable values to find the

maximum of the likelihood function. This is entirely out of the question on a real-time digital computer, and some approximation methods are necessary.

Appendix D, "Proposal of a Joint Walkup Resolution and Rectification Procedure," is one example of an approximation method. This method maximizes the likelihood function if we consider one baseline at a time. Suppose there exists at least one baseline of length less than 1/2 wavelength; then we can calculate an unambiguous least mean square estimate of the mean angle and angle rate. With these estimates we may "rectify" the data on the second shortest baseline by choosing the ambiguous portion of the space angle on the second baseline to minimize the difference from the estimate on the first baseline. With the rectified data x_{2j} we calculate the estimate mean angle $x_2(t_0)$ and angle rate x_2' and the maximum likelihood estimate of $x(t_0)$ and x' using the information from both the first baseline and the second baseline. This procedure is continued for all baselines.

Suppose we have $2N+1$ equally spaced data points symmetric about the time t_0; then the equations for the least mean square fit of independent values x_i are

$$x = \frac{1}{2N+1} \sum_{i=-N}^{N} x_i$$

$$x' = \frac{3}{N(2N+1)(N+1)} \sum_{i=-N}^{N} i x_i \, .$$

These two coefficients can be combined to predict the value of x at $t = t_j$:

$$\bar{x}_j = x + j x' = \frac{1}{(2N+1)(N+1)(N)} \sum_{i=-N}^{N} [N(N+1) + 3ji] \, x_i \, .$$

The variance of the estimate $\bar{x}_j$ is

$$\sigma^2_{\bar{x}_j} = \frac{\sigma^2_{x_i} [N(N+1) + 3_j{}^2]}{(2N+1)(N+1)(N)} \text{ (for } \sigma_{x_i} = \text{constant)} .$$

If we consider only the worst case of $\sigma^2_{\bar{x}_j}$, namely, $j = N$,

$$\sigma^2_{\bar{x}_N} = \frac{\sigma^2_{x_i} (4N+1)}{(2N+1)(N+1)} \, .$$

When the curve $\bar{x}_j$ is projected on the data from the second baseline, it is of interest to find some statements about the probability that an ambiguity error will occur in the selection of ${}_2\hat{x}_N$, the observed value of the Nth point on the second baseline. Consider the variable $Z = {}_2\hat{x}_N - {}_1\bar{x}_N$. We follow the argument developed in the subsection "Generalized Walkup Method" by saying that $|Z| < \lambda/2d_2$, to define a correct rectification to baseline two of length d_2. If σ_φ is the phase noise of both baselines, then

$$\sigma_Z^2 = \sigma_2 \hat{x}_N^2 + \sigma_1 \bar{x}_N^2 = \frac{\lambda^2 \sigma_\varphi^2}{d_2^2} \left[1 + \frac{(4N+1)}{(2N+1)(N+1)} \frac{d_2^2}{d_1^2} \right] .$$

So rather than plot the solution in four dimensions corresponding to N, σ_φ, d_2/d_1, and probability of correct resolution P, let us choose $(d_2/d_1)^2 = 2$ and $P = 0.997$ and plot only the critical curve of σ_φ vs N. The condition $P = 0.997$ is equivalent to saying $\sigma_Z = \lambda/[3(2d_2)]$; so solving for σ_φ we get

$$\sigma_\varphi^2 = \frac{1}{36\left[1 + \frac{(4N+1)}{(2N+1)(N+1)}\frac{d_2^2}{d_1^2}\right]}.$$

A typical satellite pass might have 21 data points ($N = 10$), and one could resolve the ambiguity if $\sigma_\varphi \leq 51.54$ degrees with a probability of 0.997 for a single walkup step. For a long pass the maximum σ_φ approaches 60 degrees. One could further improve the probability of success by observing whether the phase residuals tend to increase and then decrease, indicating a 360-degree phase error on the extreme ends of the sample.

Once the ambiguity has been resolved on the frame-by-frame basis, some method must be devised for smoothing the data. Simple averaging schemes or least-squares fits will not suffice, since the resolved angles are nonanalytic functions of the phase angles in the region where resolution errors occur. The unweighted walkup method in particular will allow the "walking" off to a bad resolution as it chooses the reading on the $(i+1)$th baseline closest to the reading predicted from the ith baseline. A very noisy reading on one of the shorter baselines may cause a wrong reading on the longest baseline. It is also possible for more than one satellite to be observed in the same data group. The problem then becomes one of collecting the frames associated with a particular satellite into a group (cluster) and fitting in the group in order to obtain a smooth observation.

Two methods for casting out bad observations from a cluster have been tried using operational data from Spasur. The first checked the difference between the predicted phase from the ith baseline and the recorded phase from the $(i+1)$th baseline. If the phase difference was greater than some specified tolerance, the resolution process was stopped. This method was found impractical for two reasons. It was found that the tolerance had to be quite large in order not to cast out all observations, and it still required some other method of collecting frames in a cluster.

The second method checked the difference between the resolved cosine on the longest baseline from one frame to the next in sequence. If the difference was less than some tolerance, the frame was collected in the cluster. If the difference was greater than the tolerance, the frame was cast out of the cluster. Since it is possible to have more than one satellite in the same data group, the cast-out frame was checked against the next frame to see if it belonged to a second cluster. An attempt can be made to connect like clusters together, thus obtaining a longer time sequence of data for rate information. The quality of the data obtained can be controlled by using a cluster tolerance of less than one revolution of the phase meter on the longest baseline, by setting the minimum number of frames that must exist in a cluster before it is called an observation at a reasonable number, or by a combination of both. If too few frames are used in a cluster and too large a difference is allowed between frames in a cluster, the preliminary data processing machinery will be saturated with nonsatellite observations that will be thrown out later on in the processing sequence. The processing of this nonsatellite data is time consuming. Still more important, allowing the data to have such a large variance can seriously affect the determination of rates. If the tolerance is too small or too many frames are required for a cluster, the loss of observations can be significant.

Some data containing 1109 satellite observations were processed using unweighted walkup. A minimum of two frames were used in a cluster, and no attempt was made to connect like clusters. If more than one cluster existed in a data group, the longest was used as the observation. If there were two or more clusters of equal length, the first was chosen. The results are shown below, where the tolerance is given in terms of the cosine:

Tolerance	Observations Missed	Percent Missed
0.01587	55	5.0
0.00795	63	5.7
0.0035	63	5.7

In the above experiment no provision was made to attempt to determine the presence of more than one satellite in a data group.

A larger sample of data containing 1268 observations was processed using the unweighted walkup method of ambiguity resolution. In this sample extrapolation between like clusters was used. Again no attempt was made to detect more than one satellite in a data group. The results obtained are shown below:

Tolerance	Observations Missed	Percent Missed
0.01587	11	0.9
0.0035	20	1.6

In both the above cases the number of actual satellite observations gained by using a large tolerance is less than 1%.

THE SPASUR SIMULATOR

Before incorporating any of the methods described in the preceding section, a means was needed to determine which of the methods was best for the system. The "best" in this sense, is the method that has a high probability of resolving the ambiguity correctly and can be employed in a real-time system with a large satellite population. The Spasur simulator provides a means for running numerous experiments of different sets of parameters to provide a basis for making this decision.

The Spasur simulator consists of a dozen or so subroutines, each of which is a model of some part of the Spasur system. These subroutines were written on the NAREC computer, which offers about a thousand-to-one time speedup in the generation of data. In addition to the time factor, the simulator allows for well-controlled reproducible experiments, in which almost every parameter is allowed to vary within what are considered to be reasonable limits. With such a large number of variables, only a small number of the possible combinations were actually tested.

Figure 2 is a block diagram of the final version of the Spasur simulator. The subroutines which have been programmed include the executive program, the trajectory generator, the systematic error generator, the Gaussian correlated random noise generator, frame-by-frame ambiguity resolution programs each with its many options, and the statistical analysis program. The details of these are in Appendixes E through J, but a brief description of some of them will follow.

The trajectory generator subroutine (Appendix E) assumes that the satellite is traveling along a path which intersects at right angles to the vertical from the receiver at an altitude h. The parameters are: v/h, the ratio of the satellite velocity to the aforementioned height; θ, the angle with respect to the vertical from the receiver at the time of the first data sample; n, the sample of data points desired; Δt, the time between observations; and a list of the baseline lengths measured in units of the transmitter wavelength. The output of the subroutine consists of the phase data along with the sine of the zenith angle in the middle of the interval and the phase rate as measured on a 1-wavelength baseline.

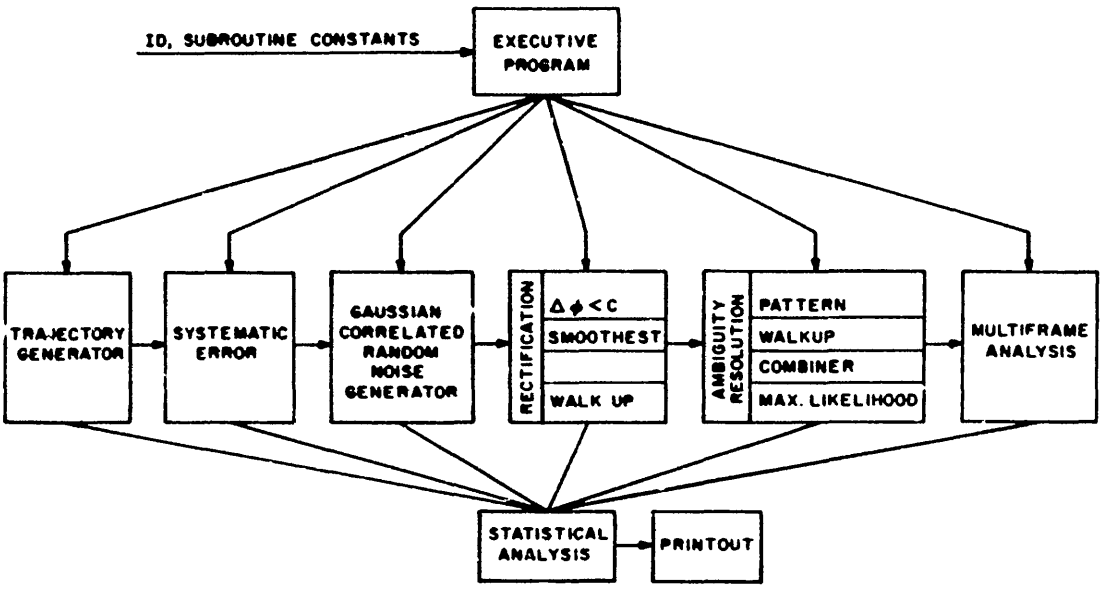


Fig. 2 - Final version of the Spasur simulator

Two subroutines add typical errors to the phase data. The systematic error subroutine (Appendix F) simply adds a systematic error to the phase data in the same manner as the errors due to miscalibration. The second error subroutine is the Gaussian correlated random noise generator (Appendix G). The white noise appears to have come through a low-pass filter with a corner frequency f_0, a sampling interval Δt, and a specified standard deviation. This routine allows us not only to run a controlled experiment because the data has a known standard deviation but also, since this is actually a pseudorandom generator, to use the identical generated data to test various methods of analysis.

The statistical analysis routine is used to control the number of times that a resolution method is tested. The criterion was chosen that a resolution would be called a correct one if the measured sine of the zenith angle was within 1/2 phase rotation on the longest baseline from the correct angle. A reasonable number of test resolutions are run, and the probability p of getting a correct resolution is estimated by dividing the number of correct resolutions in the trial run by the number of trials. The standard deviation of this estimate of the probability of getting a correct resolution is given by $\sqrt{p(1-p)/n}$, where n is the number of trials run and p is the probability of correct solution which has just been estimated. Prior to the start of the runs an upper limit to this standard deviation is set, and the runs will continue until this limit has been met.

The common interface for all the subroutines is a list of possible baseline lengths and a code word to tell which of them are to be used, a list of phase angles in order of increasing baseline length, and the output, which is a resolved space angle. The maximum likelihood subroutine (Appendix H) requires a list of the standard deviations of the phase data. The walkup ambiguity subroutine (Appendix I) requires in addition a table of the maximum allowable phase residuals both in order of increasing baseline length; if the phase residual is above the specified tolerance, an indicator is set and the resolution proceeds. The generalized pattern method subroutine (Appendix J) requires two weighted numbers p and q, which are the powers in the equation to be minimized, i.e.,

$$A = \sum_{i=1}^{N} |x_i - \bar{x}|^p d^q ,$$

and stores the maximum likelihood estimate of the sine of the zenith angle and the estimate using only the longest baseline. The maximum likelihood method requires the standard deviations of the phase angles and, because this method is so time consuming, the upper and lower limits of the sine of the space to be tested.

COMPARISON OF THEORETICAL AND EXPERIMENTAL RESULTS OF AMBIGUITY RESOLUTION

The results to be presented were accumulated over a number of years during which the antenna configuration and the operating frequency were changed. When this study was first started, the frequency was 108 Mc and the baseline lengths were 16, 20, 52, 60, and 128 feet, for which some preliminary results appear in Appendix K. In the walkup method, baselines of 4 and 8 feet were constructed by differencing the phase measurements for the baselines of 16 and 20 feet and of 52 and 60 feet. This differencing not only introduced nonindependent phase measurements but also increased the phase variance σ_φ^2 by a factor of 2 on the synthesized baselines (assuming equal variance on the physical baselines). This fact appears in Fig. 3, where both walkup techniques have generally a smaller probability of resolving the zenith angle than the other techniques. The actual data points used to plot these curves of probability of correct solution vs standard deviation of the phase noise were measured every 5 degrees from 20 to 60 degrees. In the maximum likelihood case, however, the long running time of the program forced us to use only

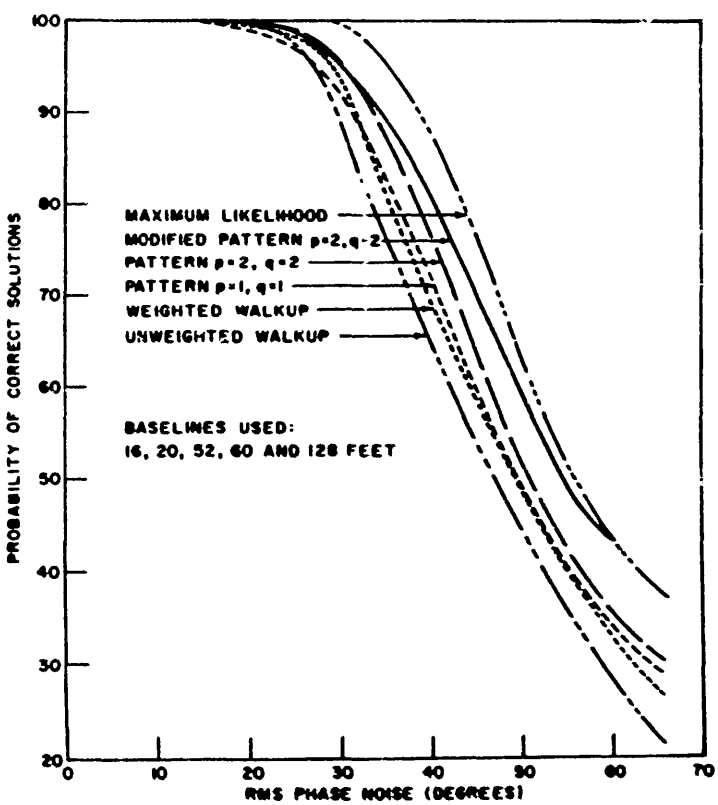


Fig. 3 - Comparison of the six ambiguity resolution methods for the original baseline set

three points, at 30, 45, and 60 degrees. The criteria for stopping the Monte Carlo program was $\sigma_p < 0.02$; i.e., each plotted point is within ±2% or better.

One could rate each of the methods in increasing order of accuracy (and incidentally running time). This order would be unweighted walkup method, weighted walkup method, pattern method with $p = 1$, $q = 1$, pattern method with $p = 2$, $q = 2$, modified pattern method, and maximum likelihood method. The general order of accuracy depends very strongly on the number of baselines and their length, so that the above order cannot be extrapolated to an arbitrary antenna configuration. One can only say that the maximum likelihood method will be most accurate, that the weighted walkup method will be better than the unweighted walkup method, and that the modified pattern method is better than the regular pattern method in the $p = 2$, $q = 2$ case, which in turn is better than the $p = 1$, $q = 1$ case.

Figure 4 demonstrates the value of physical 4 and 8-foot baselines in improving the probability of success for the walkup techniques. One would expect an improvement on the other resolution method but not to the degree of improvement gained by walkup methods. These antennas were constructed and used operationally prior to the time that the operating frequency was changed to 216 Mc. Figure 4 also demonstrates the closeness with which the theoretical calculations for the walkup methods outlined earlier agree with the Monte Carlo results; the answers are well within the experimental error inherent in the Monte Carlo calculations. The third item demonstrated in Fig. 4 is the value of maximizing the probability with respect to the number of baselines and in particular the relative value of the weighted vs the unweighted walkup. Of course it would be impractical

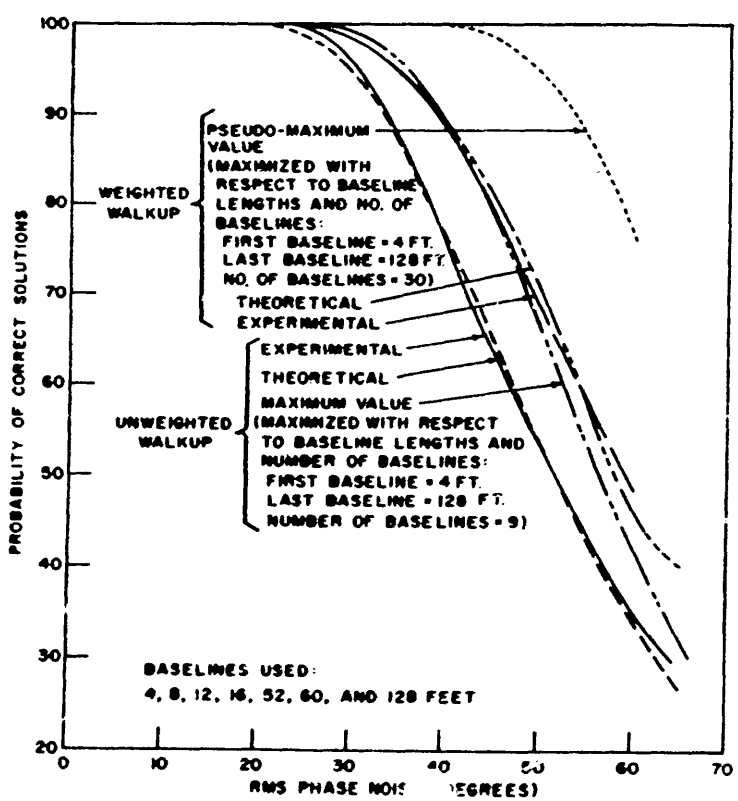


Fig. 4 - Comparison of the probability of correct resolution for the experimental and theoretical walkup methods and the theoretically optimum choice of baselines over the same range

to use 30 baselines within this range, but 15 or so would still yield a substantial improvement.

At the end of the section on the generalized walkup method it was stated that the optimum number of baselines could be found by calculating the probability for each successive value of N and choosing that value of N which maximizes the probability. These theoretical results are presented in Figs. 5 and 6 for the unweighted and weighted walkup respectively with an operating frequency of 216 Mc. It is interesting to note that the curves peak at a reasonably small value of N, particularly for the unweighted case, and that the optimum number of baselines depends on the noise level (or noise level distribution). The value of N at which the curves peak is small because the small increase in $P_{i,i+1}$ is more than overcome by the increase of the value of N; therefore the probability of success P_N must approach zero as N gets large. These graphs are in contrast to what one would expect for the other resolution techniques, where the probability of success should be a monotonically increasing function of the number of independent phase readings.

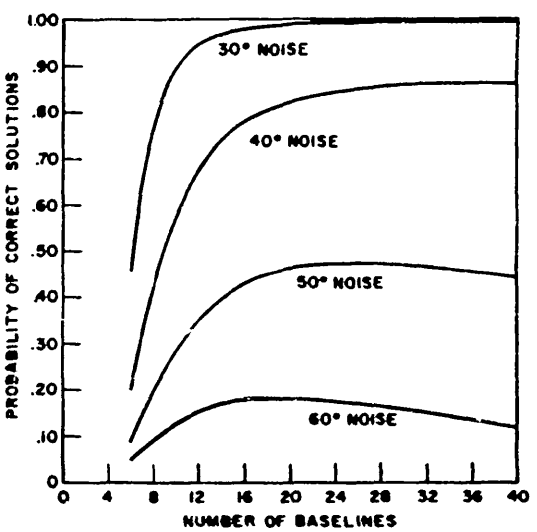


Fig. 5 - The effect of the number of baselines and rms phase noise on the probability of correct resolution in the theoretical unweighted walkup method

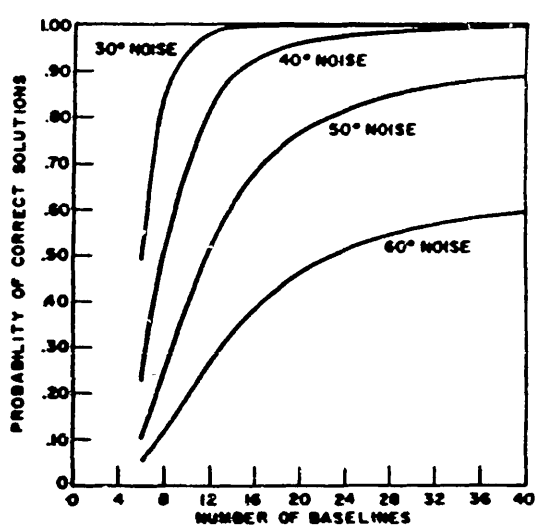


Fig. 6 - The effect of the number of baselines and rms phase noise on the probability of correct resolution in the theoretical weighted walkup method

Figures 7 and 8 are an analysis of two sets of 13 baselines with lengths between 2 and 2000 feet for an operating frequency of 216 Mc. The differences between the proposed set and the optimum sets are not great in terms of baseline lengths, but the optimum sets do yield significantly better results. It might be noted that the gain in probability by rearranging all the baselines in the unweighted case is essentially the same gain one would get by simply using the weighted method with the initial proposed set. Obviously the latter approach is more economically feasible.

CONCLUSIONS

The maximum likelihood method has been demonstrated to be the best resolution technique. Since the real time constraints within the digital computer preclude use of

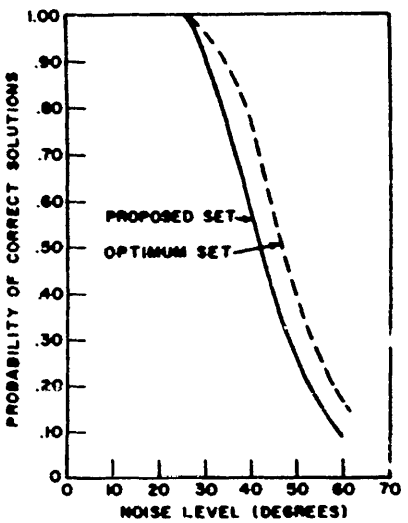


Fig. 7 - Comparison of the probability of correct resolution for the proposed (operational) set of 13 baselines and the theoretically optimum set using the unweighted walkup method

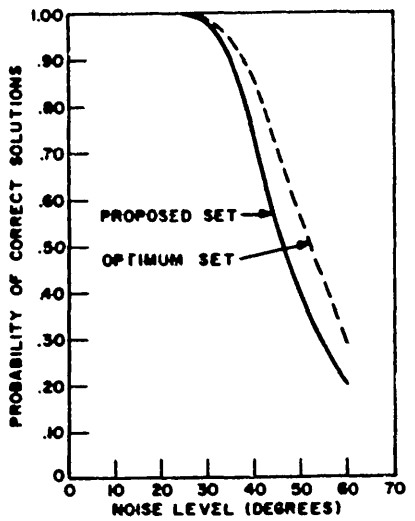


Fig. 8 - Comparison of the probability of correct resolution for the proposed (operational) set of 13 baselines and the theoretically optimum set using the weighted walkup method

the maximum likelihood method, the authors propose the construction of an analog combiner which is mathematically equivalent to this method. In addition to relieving the digital computer of the problem of ambiguity resolution, this device would probably yield significantly better answers than those now realized. Solutions to two additional problems would also be relatively easy to implement within this device, the problem of bearing error vs bearing (due to reradiation) and the problem of the clustering of data. In the latter case, only resolved angles after the second sample (or any other sample) in a cluster would be transmitted to Dahlgren. If this proposal were accepted, it would seem logical to spend the time that would be required to derive an optimum baseline set for the maximum likelihood method.

Prior to the time that such equipment is installed one of the other resolution methods would have to be used. The modified pattern method is then the best but again probably would cost too much computer time. Of the four remaining methods, the operational system has chosen to use the worst possible two methods. The authors recommend running another simulation run for the current baseline set (perhaps a one-man-week effort) to find whether the weighted walkup method or the pattern method with $p = 2$, $q = 2$ yields better results. The minor additional amount of computer time required is probably negligible but could certainly be obtained if the principles of cost/effective computer programming were practical (machine language programming of the resolution method).

ACKNOWLEDGMENTS

The authors thank Messrs. William L. Alford and William B. Poland, Jr., currently of NASA (Goddard Space Flight Center) for their help in defining the various simulation programs of actual Spasur equipment and Doris Ellis Gossett of the NRL Research Computation Center for help in programming these algorithms. The authors also acknowledge

the work of Robert R. Hohl, also of NASA, for his work in implementing some of these programs to use live data on the Dahlgren computer. We are particularly indebted to Dr. A. T. Hind of Clemson, S.C., for his efforts on the combiner simulator, to Dr. J. J. Freeman of the Electronics Division whose early work in maximum likelihood techniques set the direction of this project, and to both of these gentlemen for their help in reviewing this report.

APPENDIX A

USE OF VERY LONG BASELINES AT THE SPASUR STATIONS

In the processing of phase signals from the Spasur radio interferometer it has been assumed that the rays from the satellite to the interferometer are parallel. When the baseline length becomes appreciable with respect to the satellite height above the tangent plane, this assumption begins to lose its validity. Several questions about baseline length have been asked such as: "What is the longest baseline that should be used before the parallel ray assumption has to be discarded?" "What is the angular and height dependence of these errors?" And "How does this effect the measurement of phase and phase rate?"

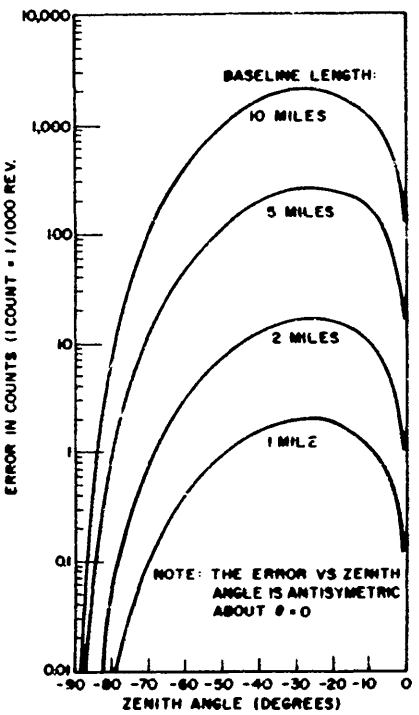


Fig. A1 - Angular dependence of the maximum error in difference in path length

From the following derivation we have concluded that the maximum error in difference in path length is equal to two counts (2/1000 revolution) on a 1-mile baseline, assuming a satellite at a height h of 100 miles (the minimum or worst height); the angular dependence is shown in Fig. A1. Figure A2 shows that the worst-worst case is h = 100 miles and θ = 26°34'; d (the baseline length) is the independent variable.

Of course, the maximum tolerable error one can use in a resolution system depends upon the data processing system used. However, in general, a dc error of 250 counts plus about 100 counts of random noise is a very rough upper limit. The figures indicate, therefore, that for baselines greater than 5 miles a new, nonparallel-ray-assumption technique of ambiguity resolution would have to be used, and even then it might be necessary to know something about the range of the satellite. Of heuristic interest is the fact that the inverse is also true; i.e., given an exact angle of the satellite (hence a noiseless continuous phase system) the range can be calculated exactly. The practicality of this idea is currently being studied.

The error in rate of change of x, the sine of the zenith angle, will be derived below, and its angular dependence is plotted in Fig. A3. In a worst-worst case (h = 100 miles, θ = 0) the phase rate error is only 1/8% of the true phase rate for a 10-mile baseline, which according to our present standards is negligible. It is important to note that the phase-rate-independent errors due to phase noise are inversely proportional to the baseline length; hence a long baseline would be very desirable as far as phase rate is concerned.

Note: This appendix was an internal memorandum (5130-34:FAP:yi) from NRL Code 5134 to NRL Code 5131 dated June 28, 1962.

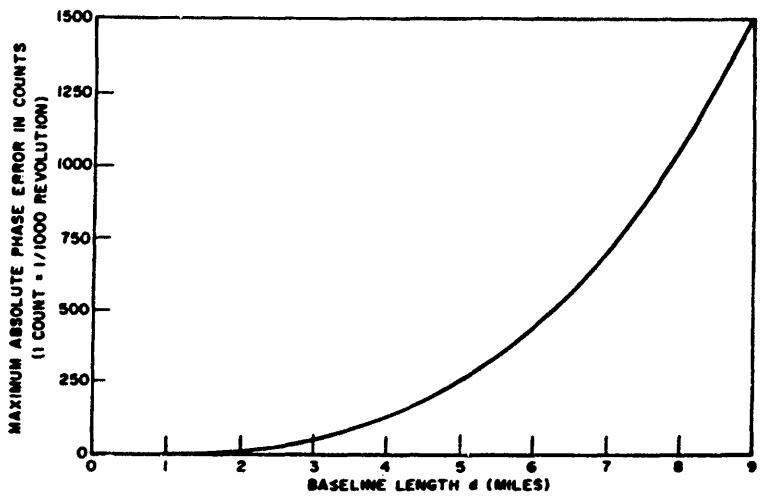


Fig. A2 - Maximum absolute phase error as a function of baseline length assuming h = 100 miles

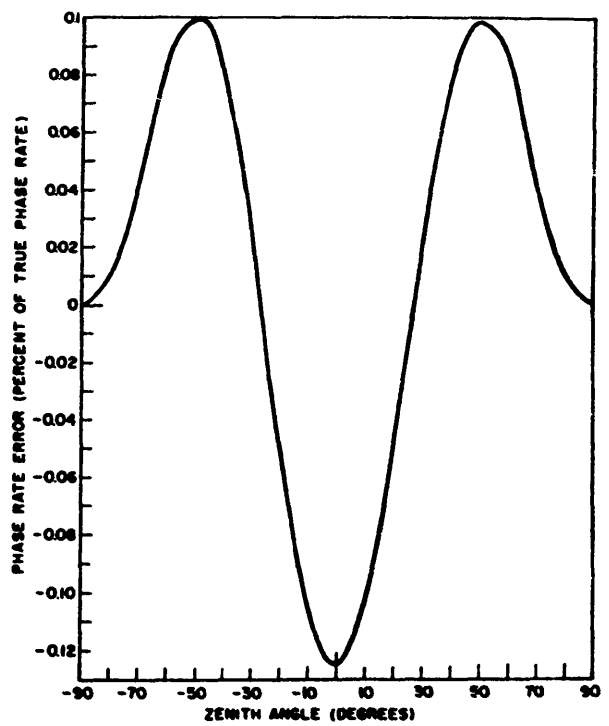


Fig. A3 - Angular dependence of the error in the rate of change of x, the sine of the zenith angle

The calculation of approximate phase error due to use of a very long baseline is as follows. As shown in Fig. A4,

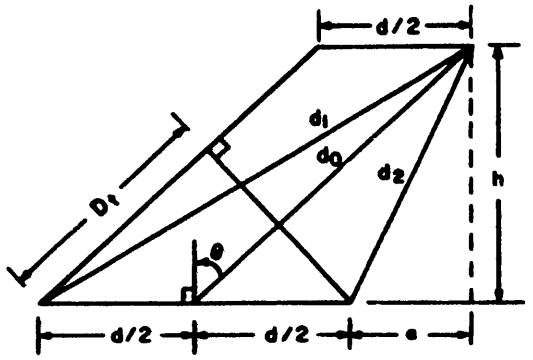


Fig. A4 - Geometry used in deriving the results shown in Fig. A3

$$d_1 = \frac{\sqrt{h^2 + hd \sin\theta \cos\theta + \frac{d^2}{4}\cos^2\theta}}{\cos\theta}$$

$$d_2 = \frac{\sqrt{h^2 - hd \sin\theta \cos\theta + \frac{d^2}{4}\cos^2\theta}}{\cos\theta}$$

$$D_t = d \sin\theta .$$

The error in difference of path length is

$$D = d_1 - d_2 - D_t .$$

Doing a power expansion in terms of d/h one gets

$$D = -\frac{d^3}{8h^2}\sin\theta\cos^4\theta + \frac{d^5}{h^4}\frac{\sin\theta\cos^4\theta}{128}\left[3 - 10\sin^2\theta + 7\sin^4\theta\right] + O\left(\frac{d^7}{h^6}\right) .$$

If d/h is small ($d/h \leq 1/10$), one is justified in keeping only the first-order term. This yields a maximum of $\sin\theta \pm (1/\sqrt{5})$. The results are plotted in Figs. A1 and A2.

The calculation of the error in the phase rate due to use of a very long baseline is as follows. Let $x = \sin\theta$ and

$$x_1 = \frac{1}{d}\left(d \sin\theta - \frac{d^3}{8h^2}\sin\theta\cos^4\theta\right)$$

$$= x - \frac{d^2}{8h^2}x(1 - x^2)^2 .$$

The true rate of change of the sine of the zenith angle is dx/dt and the measured rate is dx_1/dt:

$$\frac{dx_1}{dt} = \left[1 - \frac{d}{8h^2}(1 - x^2)(1 - 5x^2)\right]\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{dx_1/dt}{1 - \frac{d^2}{8h^2}(1 - x^2)(1 - 5x^2)} .$$

The percentage phase rate error is

$$\frac{\frac{dx_1}{dt} - \frac{dx}{dt}}{dx/dt}(100) = -\frac{d^2}{8h^2}(1 - x^2)(1 - 5x^2)100 .$$

The results are plotted in Fig. A3.

APPENDIX B

WALKUP AMBIGUITY RESOLUTION

The problem of ambiguity resolution has been treated experimentally (5130-1:FAP: yi). Now a much more satisfactory solution has arisen which not only tells how well this method will perform using a particular set of baselines but also yields under certain conditions an optimum choice of baseline lengths (say given the first, last, and total number of baselines) and also the optimum total number of baselines, assuming that the walkup ambiguity resolution scheme is used. This solution does have limitations, which will be evident in the derivation.

First a short review will be given of what walkup ambiguity resolution attempts to do. Suppose there exists at least one baseline which is less than 1/2 wavelength. On this baseline there exists a one-to-one mapping of electrical phase to sine of the space angle, i.e.,

$$\sin\theta = x = \frac{\lambda}{d}\left(\phi - \frac{1}{2}\right)$$

where

λ = wavelength = 9.105 feet,

d = baseline length (in feet), and

ϕ = electrical phase ($0 \leq \phi < 1$) = true phase plus a Gaussian random variable.

Reading only the electrical phase on the shortest baseline gives a rough estimate of the true $\sin\theta_0 = x_0$. The rms error in x is given by $\lambda\sigma_\varphi/d$ where σ_φ is the rms error in phase. This has a very definite limitation when $x_0 \approx \pm 1$ and $d \approx \lambda/2$. That is to say, assuming that the satellite is very near one horizon, a small phase error can make the satellite appear to be on the other horizon. With this rough estimate of x we can find the possible value of x derived from the next longest baseline which is nearest to our previous estimate. By walking up through all possible baselines, we get an answer which becomes more and more precise, with a decreasing probability of accuracy.

Let us assume we have walked up to the ith baseline correctly, i.e., $|x_i - x_0| < \lambda/2d_i$. Consider the random variable $x_{i+1} - x_i$; since x_{i+1} and x_i are both Gaussian random variables of mean x_0, $x_{i+1} - x_i$ is a Gaussian random variable of mean zero on the standard deviation

$$\sigma_{i+1,i} = \sqrt{\sigma^2_{x_{i+1}} + \sigma^2_{x_i} + 2\,\mathrm{cov}(x_1, x_2)} \tag{B1}$$

where

$$\sigma^2_{x_{i+1}} = \frac{\lambda^2}{d^2_{i+1}}\,\sigma^2_{\varphi_{i+1}}$$

Note: This appendix was an internal NRL Code 5130 memorandum (5130-62:FAP:yi) dated October 29, 1962.

and where $\sigma_{x_i}^2$ depends upon the particular method of walkup used, i.e., weighted or unweighted. The probability of walking up to the correct lobe on the $(i+1)$th baseline assuming that the value of x_i was in the correct lobe is given by the conditional probability

$$P\left[\left(|x_{i+1}-x_0| < \frac{\lambda}{2d_{i+1}}\right)\middle|\left(|x_i - x_0| < \frac{\lambda}{2d_i}\right)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\lambda/2d_{i+1}\sigma_{i+1,i}}^{\lambda/2d_{i+1}\sigma_{i+1,i}} e^{-x^2/2}\,dx\,. \tag{B2}$$

These values for fixed values of the parameters can be looked up in tables. The probability of walking up through all baselines is then

$$P_N = \prod_{i=1}^{N-1}\left(\sqrt{\frac{2}{\pi}}\int_0^{\lambda/2d_{i+1}\sigma_{i+1,i}} e^{-x/2}\,dx\right). \tag{B3}$$

SPECIAL CASE I

Consider the case of N baselines (independent noise) with an unweighted walkup. Then $\sigma_{x_i} = (\lambda/d_i)\sigma_{\varphi_i}$ and

$$\sigma_{i+1,i} = \sqrt{\frac{\lambda^2}{d_{i+1}^2}\sigma_{\varphi_{i+1}}^2 + \frac{\lambda^2}{d_i^2}\sigma_{\varphi_i}^2}\,.$$

Furthermore, suppose $\sigma_{\varphi_{i+1}} = \sigma_{\varphi_i} = \sigma_\varphi$; then

$$\sigma_{i+1,i} = \frac{\lambda}{d_{i+1}}\sigma_\varphi\sqrt{1+\frac{d_{i+1}^2}{d_i^2}}$$

$$P\left[\left(|x_{i+1}-x_i| < \frac{\lambda}{2d_{i+1}}\right)\middle|\left(|x_i - x_0| < \frac{\lambda}{2d_i}\right)\right] = \frac{2}{\sqrt{2\pi}}\int_0^{1/2\sigma_\varphi\sqrt{1+(d_{i+1}^2/d_i^2)}} e^{-x^2/2}\,dx\,.$$

and the probability of walking from the first base through the last (nth) baseline all correctly is given by

$$P_N = \left(\sqrt{\frac{2}{\pi}}\right)^{N-1}\prod_{i=1}^{N-1}\int_0^{1/2\sigma_\varphi\sqrt{1+(d_{i+1}^2/d_i^2)}} e^{-x^2/2}\,dx\,.$$

Let us further suppose that d_1, d_N, and N are fixed constants and $d_2, d_3, \ldots, d_{N-1}$ are to be chosen to maximize P_N. For $j \neq 1$ or N,

$$\frac{\partial P_N}{\partial d_j} = 0$$

or

$$\int_0^{1/2\sigma_\varphi\sqrt{1+(d_j^2/d_{j-1}^2)}} e^{-x^2/2}\,dx \left[e^{\dfrac{1}{-8\sigma_\varphi^2\left(1+\dfrac{d_{j+1}^2}{d_j^2}\right)}}\right] \frac{d_{j+1}^2}{d_j^3\left(1+\dfrac{d_{j+1}^2}{d_j^2}\right)^{3/2}}$$

$$= \int_0^{1/2\sigma_\varphi\sqrt{1+(d_{j+1}^2/d_j^2)}} e^{-x^2/2}\,dx \left[e^{\dfrac{1}{-8\sigma_\varphi^2\left(1+\dfrac{d_j^2}{d_{j-1}^2}\right)^2}}\right] \frac{d_j}{d_{j-1}^2\left(1+\dfrac{d_j^2}{d_{j-1}^2}\right)^{3/2}}$$

which has the solution

$$d_j = cd_{j-1} = \frac{1}{c}\,d_{j+1}\,.$$

Then

$$d_N = c^{N-1}d_1 \quad \text{or} \quad c = \left(\frac{d_N}{d_1}\right)^{1/(N-1)}$$

and

$$P_{N\,\max} = \left(\sqrt{\frac{2}{\pi}}\int_0^{1/2\sigma_\varphi\sqrt{1+(d_N/d_1)^{2/(N-1)}}} e^{-x^2/2}\,dx\right)^{N-1}. \tag{B4}$$

Again fixing d_1 and d_N it is possible to formally differentiate P_N with respect to N to find the best value of N to maximize P_N. However, from a practical standpoint it is probably faster to try various values of N in Eq. (B4).

SPECIAL CASE II

Consider the case of N independent baselines with a weighted walkup, again assuming the noise is constant and independent in all baselines. Suppose we have walked up correctly to the jth baseline; then we can get an estimate of the mean value of x_j, say $\bar{x}_j$, which has a smaller variance than x_j by

$$\bar{x}_j = \sum_{k=1}^{j} d_k^2 x_k \Big/ \sum_{k=1}^{j} d_k^2\,.$$

We may now use this to walk up to the $(j+1)$th baseline, for which Eq. (B1) becomes

$$\sigma_{j+1,j} = \frac{\lambda\sigma_\varphi}{d_{j+1}}\sqrt{1+\frac{d_{j+1}^2}{\sum_{k=1}^{j} d_k^2}}\,.$$

And the probability of walking from the first to the nth baseline with no error is

$$P_N = \left(\sqrt{\frac{2}{\pi}}\right)^{N-1} \prod_{j=1}^{N-1} \int_0^{1/2\sigma_\varphi \sqrt{1+\left(d_{j+1}^2 \Big/ \sum_{k=1}^{j} d_k^2\right)}} e^{-x^2/2}\, dx\,.$$

Again this can be maximized with respect to the baselines, assuming d_1 and d_N are fixed.

If $k \neq 1$ or N, then $\partial P_N/\partial d_k = 0$. This yields the equation

$$\frac{e^{-1/8\sigma_\varphi^2\left[1+\left(d_k^2 \Big/ \sum_{j=1}^{k-1} d_j^2\right)\right]}}{\int_0^{1/2\sigma_\varphi \sqrt{1+\left(d_k^2 \Big/ \sum_{j=1}^{k-1} d_j^2\right)}} e^{-x^2/2}\, dx \left(\sum_{j=1}^{k-1} d_j^2\right)\left(1+\frac{d_k^2}{\sum_{j=1}^{k-1} d_j^2}\right)^{3/2}}$$

$$= \sum_{\ell=k+1}^{N} \frac{e^{-1/8\sigma_\varphi^2\left[1+\left(d_\ell^2 \Big/ \sum_{j=1}^{\ell-1} d_j^2\right)\right]}}{\int_0^{1/2\sigma_\varphi \sqrt{1+\left(d_\ell^2 \Big/ \sum_{j=1}^{\ell-1} d_j^2\right)}} e^{-x^2/2}\, dx \left(\sum_{j=1}^{\ell-1} d_j^2\right)^2 \left(1+\frac{d_\ell^2}{\sum_{j=1}^{\ell-1} d_j^2}\right)^{3/2}}\,. \tag{B5}$$

The solution to this equation is difficult. However, it was suggested that in the unweighted walkup P_N could be maximized by taking all the integrals in the product to be equal. In this case

$$d_k^2 = c^2 \sum_{j=1}^{k-1} d_j^2\,.$$

It can be shown that this is <u>not</u> a solution to Eq. (B5), but it is guaranteed to be better than the best choice in the unweighted walkup for $N > 2$.

The integral may be written in the form

$$\int_0^{1/2\sigma_\varphi \sqrt{1+c^2}} e^{x^2/2}\, dx\,.$$

Obviously, the N method with the smallest value of c yields the maximum value of the integral. From unweighted walkup

$$(c_0^2)^{N-1} = d_N^2/d_1^2$$

and from weighted walkup

$$c_1^2(1+c_1^2)^{N-2} = d_N^2/d_1^2\,.$$

Therefore

$$1 = \frac{c_1^2}{c_0^2}\left(\frac{1 + c_1^2}{c_0^2}\right)^{N-2} .$$

So $c_0 > c_1$ and the resulting weighted walkup will be better than the unweighted walkup.

Under these conditions,

$$P_N = \left(\sqrt{\frac{2}{\pi}}\int_0^{1/2\sigma_\varphi\sqrt{1+c^2}} e^{-x^2/2}\, dx\right)^{N-1}$$

where

$$c^2(1 + c^2)^{N-2} = d_N^2/d_1^2 .$$

This too can be maximized with respect to N, and, as before, the quickest solutions can be found by experimentation.

RESULTS

For the two special cases, the theory agrees very well with experimentation on the Spasur simulator. There remains much work to be done on the cases of nonindependent phase reading. The probability of a correct resolution when the phases are all of mean zero but different standard deviations can be calculated from Eq. (B3); however, any maximization with respect to baseline lengths or number of baselines requires additional information (i.e., a function dependence of phase noise to baseline length).

APPENDIX C

THE COMBINER AND THE COMBINER SIMULATOR

The principal problem in determining the zenith angle of a satellite passing through the Spasur fence is that of determining the integer n (as shown in Fig. C1) to add to the fractional value ϕ, called a phase reading, which is obtained from the radio interferometer system. The determination of n is made by obtaining phase readings from baselines of different lengths and using them in various mathematical procedures. One such procedure is that of the combiner.

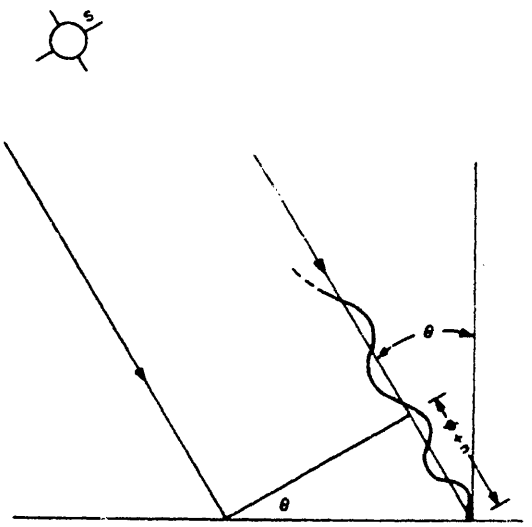


Fig. C1 - Geometry relating the phase angle ϕ plus the integer n to the zenith angle θ

The combiner is an electronic device for solving the ambiguity problem which arises in the process of finding the elevation angle of a satellite which has been observed in the Spasur fence. This piece of equipment was designed by William Alford of the Operational Research Branch, Applications Research Division. Its method of operation is best explained by the use of Fig. C2, which illustrates the gates or pulses which exist in the case of a three-baseline combiner. The center of each gate is given by

$$x_i = \frac{\lambda}{d_i}(\phi_i + n_i) .$$

where d_i is the baseline length, with $i = 1, 2, 3$ for the three baselines, λ is the wavelength of the radiated signal, ϕ_i is the observed phase angle, and n_i is one of the integers which produce values of x_i in the range -1.067 to 1.067. The proper value of $\sin\theta$,

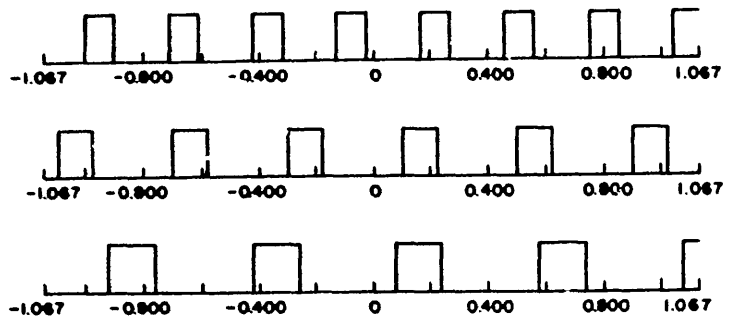


Fig. C2 - Combiner gates in the case of a three-baseline combiner

where θ is the zenith angle of the satellite, is obtained by scanning these pulses from right to left to find the first set of partial gates which cover the same numerical intervals. The middle of the interval on the longest baseline is taken to be $\sin\theta$.

The combiner simulator is a NAREC program which has the capability of simulating electronic combiners which possess a wide range of specifications. To illustrate simply the resolution process used in the combiner simulator consider what is done to simulate an electronic combiner which uses three baselines. Three phase readings ϕ_i, with $i = 1, 2, 3$, are used together with all integers n_i which produce sets of values of $\sin\theta_i$ in the range -1.067 to 1.067. By definition $\sin\theta_i = (\phi_i + n_i)\lambda/d_i$ where ϕ_i, λ, d_i, and n_i are as previously defined. A gate increment G and gate increment multipliers m_i, with $i = 1, 2, 3$, are used to form gates of pulses for the sets of $\sin\theta_i$ readings, a gate being an interval $[\sin\theta_i - m_i G\lambda/2d_i, \sin\theta_i + m_i G\lambda/2d_i]$. If a value of $\sin\theta_i + m_i G\lambda/2d_i$ in this set of gates exceeds 1.067, it is replaced by 1.067; likewise if $\sin\theta_i - m_i G\lambda/2d_i$ is less than -1.067, it is replaced by -1.067. Figure C2 illustrates these three sets of intervals on which simulated pulses have been drawn. A solution for $\sin\theta$ exists when a vertical straight line passes through a pulse or gate on each of the three rows. The resolved $\sin\theta$ is in general the center of the pulse, of this set of three, which came from the longest baseline, since if the right edge of this pulse is positive, the solution is taken to be the pulse right edge less $m_3 G\lambda/2d_3$, while if it is negative, the solution is taken to be the pulse left edge plus $m_3 G\lambda/2d_3$.

The resolution process used in the combiner simulator is the same for n phase readings for $n = 3$ through $n = 16$. The $\sin\theta$ resolution is achieved as follows: Starting with n phase readings consider n sets of gates formed as described above and examine the set of n gates formed by taking the gate of each of the n sets of gates which lies closest to 1.067. If this set of n gates has a common interval, then a resolution exists and this solution is stored and tallied. After a resolution has been found, examination is made of the set of gates formed from the last set by replacing the gate on the longest baseline by the gate which is one lobe to its left. This new set of gates is then checked for the existence of a resolution. If a resolution does not exist for this or any set of n gates, the new set of gates to be examined is chosen as follows: The gate of the last set examined which has its right edge farthest to the left, called the reference gate, is retained and the remaining gates, unless their left edges lie to the left of the right edge of the reference gate, are replaced by gates which are displaced by the least possible number of lobes to the left which do satisfy this condition. Each new set of gates is examined to see if it yields a solution. This examination and formation of new sets of gates continues until the right edge of some gate is less than -1.067, the electrical horizon.

To summarize the flexibility of the combiner simulator the following is noted: It will simulate any combiner whose input consists of interferometer phase readings of from three to 16 baselines. The lengths of the baselines as well as the wavelength of the radiation are completely variable subject to the parameter tape. The gates are individually specified in multiples of an arbitrary unit, with the arbitrary unit and the multiple indicators being given on the parameter tape.

APPENDIX D

PROPOSAL OF A JOINT WALKUP RESOLUTION AND RECTIFICATION PROCEDURE

Suppose we have a time sequence of phase readings $\phi_j(\tau_k)$ from a radio interferometer with baseline lengths $d_1, d_2, \ldots, d_N$. Suppose further that the shortest baseline d_1 is less than 1/2 wavelength long. Then there exists a one-to-one mapping of electrical phase to sine of the space angle given by the relationship

$$\sin \theta_1(\tau_k) = x_1(\tau_k) = \frac{\lambda}{d_1}\left[\phi_1(\tau_k) - \frac{1}{2}\right]$$

and it is possible to do a least mean square fit in the $x(\tau)$ domain of the form $x_i = A_i + B_i\tau$. Let us further suppose the phase readings on all baselines are taken at the same time sequence; we can say that $\sigma_{A_i} = c_A\sigma_{x_i}$ and $\sigma_{B_i} = c_B\sigma_{x_i}$, where now c_A and c_B are constants independent of the baseline. Now we may approximate $x_2(\tau)$ by $\hat{x}_2(\tau) = x_1(\tau)$; then

$$\frac{d_2 x_1(\tau)}{\lambda} + \frac{1}{2} = \hat{\phi}_2(\tau) .$$

Choose m such that $|\phi_2(\tau) + m - \hat{\phi}_2(\tau)| < 1/2$ and do a least mean square fit of

$$x_2(\tau) = A_2 + B_2\tau .$$

We can now form

$$\bar{x}_2(\tau) = \frac{\sum_{i=1}^{2} (d_i^2/s_i^2)\, x_i(\tau)}{\sum_{i=1}^{2} (d_i^2/s_i^2)} .$$

We now have three choices of s_i^2:

1. We can set all $s_i^2 = \infty$ except the one associated with the longest baseline, and we get $\bar{x}_2(\tau) = x_2(\tau)$.

2. We can set s_i^2 = constant, and we get

$$\bar{x}_2(\tau) = \frac{\sum_{i=1}^{2} d_i^2\, x_i(\tau)}{\sum_{i=1}^{2} d_i^2} .$$

3. We can take s_i^2 to be the mean squared phase residual on the respective baselines, which yields the equation

---

Note: This appendix was an internal NRL Code 5130 memorandum (5130:FAP:yi) dated November 2, 1962.

$$\bar{x}_2(\tau) = \frac{\sum_{i=1}^{2} \frac{x_i(\tau)}{\sigma^2_{x_i}}}{\sum_{i=1}^{2} \frac{1}{\sigma^2_{x_i}}} .$$

We now project $\bar{x}_2(\tau)$ to the third baseline and repeat the procedure until all baselines have been used.

APPENDIX E

TRAJECTORY GENERATING SUBROUTINE

1. Purpose: The purpose of this subroutine is to generate typical phase data for a set of baselines to be used in the Spasur simulator.

2. Language Used: This program is available in NAR 1B symbolic language on Tape S4331. Symbolic addresses used: all those beginning with tgs plus tgt1 to tgt8. The subroutine is written in fixed point and requires the fixed point subroutine Tape 4400-3d70.

3. Statement: This program assumes a satellite is initially at an angle θ_0 relative to the local station vertical (positive θ is east) and traveling at a velocity v along a path parallel to the tangent to the earth at the station in the fence. The path passes at a height h vertically above the station. The equation for the phase on a baseline of length d_i (wavelengths) at the jth sample is

$$\phi_{ij} = d_i \frac{(\sin\theta_0 + j(vh)\Delta\tau\cos\theta_0)}{\sqrt{\cos^2\theta_0 + (\sin\theta_0 + j(vh)\Delta\tau\cos\theta_0)^2}} + \frac{1}{2} + m$$

where m is chosen to be an integer such that $0 \leq \phi_{ij} < 1$; vh = the velocity-to-height ratio ($\sec^{-1}$), with positive velocity to the east; and $\Delta\tau$ = time between data samples (sec).

The sine of the space angle in the middle of the interval is given by

$$\sin\theta_{(n-1)/2} = \frac{\sin\theta_0 + \left(\frac{n-1}{2}\right) vh\,\Delta\tau\cos\theta_0}{\sqrt{\cos^2\theta_0 + \left[\sin\theta_0 + \left(\frac{n-1}{2}\right) vh\,\Delta\tau\cos\theta_0\right]^2}}.$$

The phase rate on a 1-wavelength baseline in the middle of the interval is given by

$$\dot{\phi}_{(n-1)/2} = \frac{1}{\Delta\tau}\left[\frac{\sin\theta_0 + (n/2)(vh)\Delta\tau\cos\theta_0}{\sqrt{\cos^2\theta_0 + [\sin\theta_0 + (n/2)(vh)\Delta\tau\cos\theta_0]^2}} - \frac{\sin\theta_0 + \left(\frac{n-1}{2}\right)(vh)\Delta\tau\cos\theta_0}{\sqrt{\cos^2\theta_0 + \left[\sin\theta_0 + \left(\frac{n-1}{2}\right)(vh)\Delta\tau\cos\theta_0\right]^2}}\right].$$

4. Instructions for Using the Subroutine

a) Additional Symbolic Addresses

1) stho sine of the space angle at $t = 0$ (times 10^{-3} converted)

Note: This appendix was an internal memorandum (5130-52:FAP:yi) from NRL Code 5134 to NRL Code 5130 dated October 18, 1962.

2) vh the velocity-to-height ratio (sec^{-1}) (times 10^{-3} converted)

3) delt time between data samples (sec) (times 10^{-3} converted)

4) one 1×10^{-3} converted

5) ndw left address count which equals the number of data phases desired per baseline (in hexadecimal)

6) blo symbolic address of the shortest baseline (first address of the table of 16 possible baselines). Baseline lengths are placed in the table in units of wavelengths times 10^{-3} converted.

7) blew A hexadecimal word which tells which of the baselines are to be used in the subroutine. A one in the 2^{-44} bit means that the shortest baseline in the table blo to blo + 15 is to be used. A one in the 2^{-43} bit means the second shortest baseline (location blo + 1) is to be used, etc. Any combination of baselines is valid.

b) Entrance

```
s:        sA    Lo    TGSR

s + 1:    return
```

c) Output

1) The appropriate phase readings ϕ are place location 1000 through [1000 + (ndw) (number of baselines) - 1] first in order of increasing baseline length and second in order of increasing time.

2) The sine of the space angle in the middle of the interval is left in symbolic location TGT2, scaled 10^{-3} in hexadecimal.

3) The phase rate on a 1-wavelength baseline in the middle of the interval is left in symbolic location TGT3, scaled 10^{-3} in hexadecimal.

APPENDIX F

SYSTEMATIC ERROR SUBROUTINE

1. Purpose: The purpose of this subroutine is to add to the Spasur simulator data a systematic time-independent error plus a systematic dependent error, to see how the simulator will perform with systematic errors.

2. Language Used: This program is available in NAR 1B symbolic language on Tape S4437. Symbolic addresses used: all those beginning with gs, g19, g20, g21, syex. In addition to internal working locations it uses locations 3CDO-3CEF. The subroutine is in fixed point and requires the fixed point subroutine Tape 4400-3d70.

3. Statement: This subroutine adds to the Spasur simulator phase data locations a systematic error of the form

$$E_{ij} = a_i + b_i \left[\frac{T_j - T_0}{D/2}\right]^2$$

where

t_0 = time of the center of the signal,

t_j = time of the jth sample,

D = duration of the signal,

a_i = dc offset of the signal, and

b_i = maximum time-dependent offset.

4. Instructions for Using the Subroutine

a) Additional Symbolic Addresses

1) one 1×10^{-3} converted

2) nbl left address counter which equals the number of baselines to be used (in hexadecimal)

3) ndw left address counter which equals the number of data phases desired per baseline

4) syet the systematic error table stored in a compact form. The first word of the table is $a_1 a_2 a_3$ where each a_i is contained in four hexadecimal characters, one for the sign and three for the phase error in revolutions, the second word in the table is $a_4 a_5 a_6$, etc. The sixth word in the table is

Note: This appendix was an internal memorandum (5130-53:FAP:yi) from NRL Code 5134 to NRL Code 5130 dated October 22, 1962.

$a_{16} b_1 b_2$; the seventh is $b_3 b_4 b_5$, etc. The eleventh (last) word in the table is $b_{15} b_{16} X$. The desired errors are to be put in the order of increasing length of the baselines used.

b) Entrance

s:	sA	Lo	SYER
s + 1:	return		

c) Output

The appropriate E_{ij} are added to the phase data locations (1000 to 1000 + (ndw) (nbl) - 1) in the Spasur simulator.

APPENDIX G

GAUSSIAN CORRELATED NOISE GENERATOR SUBROUTINE

1. Purpose: This subroutine adds a Gaussian correlated random variable of specified correlation and standard deviation to the phase data locations in the Spasur simulator.

2. Language Used: This program is available in NAR 1B symbolic language on Tape 4745. Symbolic addresses used: all those beginning with gcr, and gcsa, gcsb, and gcsc. It is written in fixed point and requires tape 4400-3d70.

3. Formal Statement: This program does what is called the "throw away method" to select Gaussian random numbers. First two flat distributed random numbers are generated by multiplication of a constant times the last random number, and the low order product is taken as a flat distributed random variable, and the process is repeated. Let us say two such numbers are x_1 and x_2. Then x_2 is said to be a Gaussian random variable, of mean zero and standard deviation σ, provided that

$$x_1 < e^{-x_2^2/2\sigma^2} .$$

Since x_1 is assumed to be flat distributed, the probability of selecting x_2 is $e^{-x_2^2/2\sigma^2}$. Then the relative frequency of a set of random numbers obeys the Gaussian law, provided of course that x_1 and x_2 are flat distributed.

This program also will generate Gaussian correlated random noise assuming that two consecutive numbers are joint Gaussian distributed, with correlation coefficient ρ, where by definition

$$\rho = \frac{\int_{-\infty}^{\infty} e^{j2\pi f \Delta\tau} |w(f)|^2 df}{\int_{-\infty}^{\infty} |w(f)|^2 df}$$

where

$$|w(f)|^2 = \frac{1}{1 + (f/f_0)^2} .$$

That is we assume noise appears to have come through an RC low-pass filter of corner frequency f_0, where $\Delta\tau$ is the time between sampled data. Then

$$\rho = e^{-2\pi f_0 \Delta\tau} .$$

Again if x_1 and x_2 are jointly Gaussian, then the conditional probability of x_2 given x_1 is

Note: This appendix was an internal memorandum (5130-51:FAP:yi) from NRL Code 5134 to NRL Code 5130 dated October 17, 1962.

$$P(x_2/x_1) = \frac{e^{-\left[(x_2-\rho x_1)/\sigma_x\sqrt{1-\rho^2}\right]^2/2}}{\sqrt{2\pi}\,\sigma_x\sqrt{1-\rho^2}} .$$

Assuming that x_1 and x_2 have mean zero and standard deviation σ_x, let

$$Y_{12} = \frac{x_2 - \rho x_1}{\sigma_x\sqrt{1-\rho^2}} .$$

Then Y_{12} has mean zero and standard deviation one and

$$x_2 = \sigma_x\sqrt{1-\rho^2}\; Y_{12} + \rho x_1 .$$

Suppose we let

$$x_1 = \sigma_x\sqrt{1-\rho^2}\; Y_{01}$$

$$x_i = \sigma_x\sqrt{1-\rho^2}\; Y_{i-1,i} + \rho x_{i-1} \text{ for } i \geq 2 .$$

This program discards the first eight numbers $x_1, x_2, \ldots, x_8$, so that x_9 is almost independent of the assumption that $x_0 = 0$. It then adds x_9 to storage location 1000, x_{10} to storage location (1000 plus number of baselines used in the program), etc. For the second baseline, we start another sequence $x_0 = 0, x_1, x_2, \ldots, x_8$ and add the new x_9 to location 1001, x_0 to location (1001 plus number of baselines used), etc.

4. Instructions for Using the Subroutine

a) Additional Symbolic Addresses

1) delt — time between samples of data in seconds times 10^{-3} (converted)

2) BNDW — bandwidth of the correlating low-pass filter in cycles per second times 10^{-3} (converted). A negative bandwidth will yield a zero correlation coefficient.

3) sig — standard deviation of the phase errors in revolutions times 10^{-3} (converted)

4) nbl — a left address counter which equals the number of baselines to be used (in hexadecimal)

5) ndw — a left address counter which equals the number of data words to be used per baseline (in hexadecimal)

6) one — 1×10^{-3} converted

b) Entrance

s:	sA	Lo	GCRN
s + 1:	return		

c) Output: A Gaussian correlated random number is added appropriately to locations 1000 to [1000 + (nbl) (ndw) - 1]. There is no printout directly from this subroutine.

d) Special Features

1) Since this program generates pseudorandom numbers, unless the program is reset after each time the program is read into the machine, it will generate the same sequence of random numbers (or multiples thereof). In order to obtain an essentially independent sample on a repeated trial, read out the contents of symbolic location gcr0 at the end of one trial and put that number back into gcr0 before starting the next trial.

2) A Gaussian random number of mean zero and standard deviation 1/4 by the entrance

```
s:        sA    Lo    gcrc
s + 1:    return
```

The resulting random number is left in the A register.

3) A flat distributed random variable $0 \leq x < 1$ can be gotten by entrance

```
s:        sA    Lo    gcri
s + 1:    return
```

The resulting random number is left in the A register.

APPENDIX H

SUBROUTINE FOR THE MAXIMUM LIKELIHOOD METHOD OF AMBIGUITY RESOLUTION

1. Purpose: The purpose of this subroutine is to perform a maximum likelihood resolution of the phase data in the Spasur simulator.

2. Language Used: This program is available in NAR 1B symbolic language on Tape S4476. Symbolic addresses used: max1, m1x, m1ai, hr0-hr70, hy0-hy45, hn0-hn21, and h1y. In addition to the location within Tape S4476, the program uses locations 3C60-3CCF.

3. Formal Statement: The basic hypothesis of the maximum likelihood method of ambiguity resolution is that the "best" estimate of the sine of the space angle $\bar{x}$ is that which maximizes the Gaussian likelihood function

$$L = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\,\sigma_{x_i}} e^{-\left[(x_i-\bar{x})/\sigma_{x_i}\right]^2/2}$$

$$= \left(\prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\,\sigma_{x_i}}\right) e^{-\left[\sum_{i=1}^{N}(x_i-\bar{x})/\sigma_{x_i}\right]^2 \Big/ 2}$$

where x_i is a possible choice of the sine of the space angle observed on the ith base and $\sigma_{x_i} = (\lambda/d_i)\,\sigma_{\phi_i}$ is the standard deviation of the sine of the space angle measured on the ith baseline (of length d_i); σ_{ϕ_i} is the standard deviation of the phase error on the ith baseline.

Since σ_{x_i} is just a number, maximization of the likelihood function is equivalent to the minimization of the sum

$$A' = \frac{1}{\lambda^2} \sum_{i=1}^{N} \left(\frac{x_i - \bar{x}}{\sigma_{\phi_i}}\right)^2 d_i^2$$

or

$$A = \sum_{i=1}^{N} \left(\frac{x_i - \bar{x}}{\sigma_{\phi_i}}\right)^2 d_i^2 \ .$$

Setting $\partial A/\partial \bar{x} = 0$ we get

Note: This appendix was an internal memorandum (5130-54:FAP:yi) from NRL Code 5134 to NRL Code 5130 dated October 23, 1962.

$$\bar{x} = \frac{\sum_{i=1}^{N} d_i^2 x_i / \sigma_{\phi_i}^2}{\sum_{i=1}^{N} d_i^2 / \sigma_{\phi_i}^2} .$$

The program selects one choice of the sine of the space angle on the longest baseline x_ℓ (say the smallest possible value of x_ℓ); it then finds the two closest values of x_i for the baselines $i = 1$ to $N-1$. It then computes the maximum likelihood estimator for 2^{N-1} possible combinations of x_i's and computes the appropriate value of A. The program continues to choose each possible value x_ℓ in succession until the largest permissible value of x_ℓ has been used in this computation, at which time it chooses that sequence of $x_1, x_2, \ldots, x_n$, which has the minimum value of A and declares that value of the maximum likelihood estimator $\bar{x}$ to be the "best" solution.

4. Instructions for Using the Subroutine

a) Additional Symbolic Addresses

1) mla — the smallest allowable value of the sine of the space angle as measured on the longest baseline (scaled 10^{-3} converted)

2) mlb — the largest allowable value of the sine of the space angle as measured on the longest baseline (scaled 10^{-3} converted)

3) half — $(1/2)\times 10^{-3}$ converted

4) one — 1×10^{-3} converted

5) hun — 100×10^{-3} converted

6) blo — symbolic address of the first element in a table of baseline lengths. The baselines are put in order of increasing length, in terms of number of wavelengths, scaled 10^{-3} converted.

7) blew — A hexadecimal word which tells which of the baselines are to be used in the subroutine. A one in the 2^{-44} bit means that the shortest baseline in the table blo to blo + 15 is to be used. A one in the 2^{-43} bit means that the second shortest baseline (location blo + 1) is to be used, etc.

b) Additional Information Needed

1) The phases (in revolutions) which are to be resolved are to be placed in order of increasing baseline length starting in location 3DBO. They must be scaled 10^{-3} and converted.

2) Corresponding to the above phases, a table of standard deviations of phase σ_{ϕ_i} are to be placed in a table beginning with location 3CCO, with the condition $0 \le \sigma_{\phi_i} < (1/2) \times 10^{-3}$. They must be converted.

c) Entrance

```
s:       sA      Lo      MAXL

s + 1:   return
```

d) Output

1) There is no direct printout from this program

2) The maximum likelihood estimator $\bar{x}$ which minimizes the function A is left in symbolic location mix and is scaled 10^{-3}.

3) The actual minimum value of A is left in symbolic location mlai and is scaled 10^{-3}.

APPENDIX I

SUBROUTINE FOR THE GENERALIZED WALKUP METHOD OF AMBIGUITY RESOLUTION

1. Purpose: The purpose of this subroutine is to perform one of the walkup methods of ambiguity resolution of the phase data in the Spasur simulator.

2. Language Used: This program is available in NAR 1B symbolic language on Tape S4427. Symbolic addresses used: walk, hlbs, hg1-hg11, hm0-hm19, hs1-hs27, ht0-ht15, hw1-hw10, hx0-hx30, werr, and wux. This program is in fixed point. In addition to the locations within Tape S4427 the program uses locations 3CFO through 3DCF.

3. Formal Statement: The resolution program starts out by assuming the reduced phase on the shortest base (d_1) equals the unreduced phase (there is no loss of generality here, provided that the shortest baseline is less than 1/2 wavelength). The sine of the space angle measured on this baseline is then

$$x_1 = \sin \theta_1 = \frac{\lambda}{d_1}\left(\phi_1 - \frac{1}{2}\right).$$

An approximation to the phase on the next longest baseline (d_2) is

$$\hat{\phi}_2 = \frac{d_2}{\lambda} x_1 + \frac{1}{2}$$

and the phase on the second baseline is taken to be that which is closest to $\hat{\phi}_2$. That is, m is chosen such that

$$|\phi_2 + m - \hat{\phi}_2| < \frac{1}{2}$$

where $\phi_{r_2} = \phi_2 + m - \hat{\phi}_2$ is defined to be the phase residual on the second baseline. If this is greater than a present tolerance, an error indication occurs and the resolution continues. The program now gets an estimate of the true sine of the space angle by

$$\bar{x}_2 = \frac{c\,\dfrac{d_1^2 x_1}{s_1^2} + \dfrac{d_2^2 x_2}{s_2^2}}{c\,\dfrac{d_1^2}{s_1^2} + \dfrac{d_1^2}{s_2^2}}$$

where the terms s_1^2 and s_2^2 are preset constants equal to the variance of the phase noise on the respective baselines and c is either 1 or 0. If $c = 0$, the walkup technique is called unweighted walkup and $\bar{x}_2 = x_2$. If $c = 1$, the walkup technique is called weighted walkup and the corresponding value of $\bar{x}_2$ is the maximum likelihood estimator of the sine of the space angle if x_1 and x_2 are independently distributed. To generalize after $k + 1$ projections:

$$\bar{x}_k = c \sum_{j=1}^{k-1} \frac{d_j^2 x_j}{s_j^2} + \frac{d_k^2 x_k}{s_k^2} \Big/ c \sum_{j=1}^{k-1} \frac{d_j^2}{s_j^2} + \frac{d_k^2}{s_k^2}.$$

After walking up to the nth (last) baseline $\bar{x}_n$ is formed and this is taken to be the resolved angle.

4. Instructions for Using the Subroutine

a) Additional Symbolic Addresses (externally supplied)

1) one 1×10^{-3} converted

2) half $(1/2) \times 10^{-3}$ converted

3) wub a constant which equals the maximum (and negative of the minimum) value that the sine of space may have on the longest baseline. If a resolution on any baseline goes beyond either bound, the value of the sine of the space angle for that baseline is set equal to the bound.

4) wucw walkup code word. If the first hexadecimal character is a zero, $c = 0$ is to be used in the resolution (unweighted walkup). If the first hexadecimal character is a one, $c = 1$ is to be used (weighted walkup). The right address of this word contains a hexadecimal counter which equals the number of consecutive baselines to be used.

b) Externally Supplied Constants

1) Phase data in revolutions, scaled 10^{-3}. The phase for the shortest baseline to be used is placed in location 3CF0; the remaining phases are put in the table in order of increasing baseline length.

2) Baseline lengths in wavelengths scaled 10^{-3} in order of increasing baseline length in a table starting with location 3d10; their number must be less than or equal to 32.

3) A table of standard deviations s_i of phase noise in revolutions, converted and scaled 10^{-3}. The table may contain up to 32 elements starting with location 3d30.

4) A table of maximum allowable absolute phase residuals converted and scaled 10^{-3} and in order of increasing baseline length.

c) Entrance

```
s:        sA    Lo    WALK

s + 1:    return
```

d) Output

1) The maximum likelihood estimate of the sine of the space angle $\bar{x}_n$ is left in symbolic address WUX.

2) The error indicator is left in symbolic address WERR. Its left address is a hexadecimal counter which tells the number of times in the resolution that a phase residual is above the prescribed tolerances. The right address is a hexadecimal counter which tells the number of times the closest value of the sine of the space angle is in absolute value greater than the maximum allowable value.

APPENDIX J

SUBROUTINE FOR THE GENERALIZED PATTERN METHOD OF AMBIGUITY RESOLUTION

1. Purpose: The purpose of this program is to perform one of the pattern methods of ambiguity resolution on the Spasur phase data.

2. Language Used: This program is available in NAR 1B symbolic language on Tapes S4411 and S4411-A. Symbolic addresses used: pat, pmak, pmxk, pmxb, hb0-hb9, hba1, hbf, hc0-hcf, hcd1, hd0-hde, and h1-h111.

3. Formal Statement: The pattern method of ambiguity resolution tries to imitate the maximum likelihood method along with having a substantial time savings. First it assumes that the phase noise on all baselines is Gaussian, of mean zero, and constant standard deviation σ_ϕ. The generalized pattern method looks for the sequence of possible values of the sine of the space angle on each baseline $x_1, x_2, \ldots, x_\ell$ which minimizes the expression

$$A = \sum_{i=1}^{N} |x_i - \hat{x}|^p \, d_i^q$$

where p and q are 0, 1, or 2 and d_i is the length of the ith baseline in units of wavelengths. The value of $\hat{x}$ is obtained in one of two ways. Tape S4411 assumes that the possible sine of the space angle measured on the longest baseline (x_ℓ) is a "good enough" estimate of the true sine of the space angle ($\hat{x}$). Tape S4411-A, however, insists that the maximum likelihood estimate

$$\bar{x} = \frac{\sum_{i=1}^{N} d_i^2 \, x_i}{\sum_{i=1}^{N} d_i^2}$$

is a "better" estimate of $\hat{x}$. The program selects a possible value of the sine of the space angle on the longest baseline x_ℓ (say the smallest allowable value of x_ℓ). It then chooses the nearest values of the sine of the space angle on all other baselines and calculates the appropriate value of $\hat{x}$ (that is x_ℓ on Tape S4411 or $\bar{x}$ on Tape S4411-A). The results are substituted into the equation for A. The next larger value of x_ℓ is chosen in turn, and its value of A is calculated, and the process is continued until after the largest allowable value of x_ℓ has been used. The sequence $x_1, x_2, \ldots, x_\ell$ which minimizes the value of A is used to compute the maximum likelihood estimator of the sine of the space angle $\bar{x}$, and this is taken to be the resolved angle.

Note: This appendix was an internal memorandum (5130-57:FAP:yi) from NRL Code 5134 to NRL Code 5130 dated October 25, 1962.

4. Instructions for Using the Subroutine

a) Additional Symbolic Addresses (externally supplied)

1) pma	the negative ot the smallest allowable value of x_ℓ, scaled 10^{-3} converted.
2) pmb	the largest allowable value of x_ℓ scaled 10^{-3} converted
3) half	$(1/2)\times 10^{-3}$ converted
4) one	1×10^{-3} converted
5) hun	100×10^{-3} converted
6) pmcw	the pattern method code word. The left address of this word contains the address of the first element in the phase data table. The eighth hexadecimal character is the value of p to be used, and the sixteenth hexadecimal character is the choice of q to be used.
7) blo	the symbolic address of the first element of the baseline table (i.e., the address of the shortest baseline)
8) blcw	A hexadecimal word which tells which of the baselines are to be used in the subroutine. A one in the 2^{-44} bit means that the shortest baseline in the table blo to blo + 15 is to be used. A one in the 2^{-43} bit means that the second shortest baseline (location blo + 1) is to be used, etc.

b) Additional Information Needed

The phases (in revolutions) which are to be resolved are to be placed in order of increasing baseline length, and the first address of this table is placed in the left address of pmcw. They are to be scaled 10^{-3} and converted.

c) Entrance

```
s:        sA     Lo     PAT

s + 1:    return
```

d) Output

1) There is no direct printout from this program.

2) The value of the sine of the space angle on the longest baseline x_ℓ, associated with the minimum value of A, is left in symbolic location PMXK. It is scaled 10^{-3}.

3) The associated maximum likelihood estimate of the sine of the angle $\bar{x}$ is left in symbolic location PMXB.

4) The minimum value of A is left in symbolic location PMAK and is scaled $10^{-3} \times 2^{-p}$.

APPENDIX K

EVALUATION OF POINT-BY-POINT AMBIGUITY RESOLUTION METHODS BY NAREC SIMULATION

PROBLEM DESCRIPTION

In Spasur the east-west angle between the local zenith and the vector from a Spasur station to a satellite as it passes through the Spasur beam is measured by the use of a radio interferometer operating as follows: Phasemeters are attached to the outputs of several antenna pairs having various spacings, and the phase difference of the received signals between two antennas is measured (modulo one cycle). Long baselines provide fine readings and short baselines provide coarse readings. The phase angle ϕ_k in cycles measured on the kth baseline is related to the zenith angle θ by the equation

$$0 \leq \phi_k = \frac{d_k}{\lambda} \sin \theta - n_k + \frac{1}{2} < 1$$

where n_k is the modulus, d_k is the baseline length, and λ is the wavelength (9.107 feet). The data from all the antenna pairs are combined, but due to both sky noise and errors in the receivers the various phasemeters will not in general agree exactly. Because the various readings do not have exactly the correct relative values, it is difficult to infer the precise value which should be assigned to the combined output of the system. This problem is known as the "ambiguity resolution problem," that is, the problem of the proper number of integral revolutions to assign to each phasemeter associated with a long baseline such that the resultant phase angles derived from the various phasemeters most nearly agree.

This brings up the question, by what criterion does one measure how well the phase angles agree? Three different criteria have been programmed for NAREC evaluation by simulation techniques, and a sample of 36 sets of phasemeter readings was tested using each program.

COMPUTER SIMULATION

The data for this experiment were generated by assuming a true target angle of 0 degree (target at zenith). To the corresponding phase angle for each baseline, an uncorrelated Gaussian random variable of zero mean and specified standard deviation was added. The same value of standard deviation was used for each baseline in the set. Separate runs were made for standard deviations corresponding to 30, 45, and 60 degree phase noise on all baselines. Here, 30-degree phase noise means that, assuming Gaussian noise with mean zero, 68% of the noise measurements were in the range -30 to +30 degrees. These data were used in the three programs, and the number of correct resolutions was counted. A resolution was defined to be correct if the unambiguous phase on the longest baseline was estimated correctly within 1/2 cycle. In all the programs, baselines of 16, 20, 52.8, 30.8 and 129.6 feet were used.

Note: This appendix was an internal Code 5130 memorandum (5130-1:FAP:ywi) dated March 20, 1962.

RESOLUTION METHODS

The most successful resolution method was the maximum likelihood method, derived by Dr. J. J. Freeman. A variable x_k is defined related to the measured phase x_k by the relation

$$x_k = \frac{\lambda}{d_k}\left(\phi_k + n_k - \frac{1}{2}\right).$$

This method states that, for each possible set i of n_k's the best estimate of resolved phase angle x_i is given by

$$\bar{x}_i = \sum_k d_k^2 x_k / \sum_k d_k^2$$

where d_k is the length of the kth baseline. It is assumed in these experiments that the standard deviation of the noise (which equals the rms value) is the same on all baselines. The desired set of n_k's is found by evaluating the relation

$$A_i = \frac{\sum_i (\bar{x}_i - x_k)^2 d_k^2}{S_\phi^2}$$

for each of the sets of n_k's and choosing the set which minimizes A_i. S is the standard deviation of noise on all baselines. $\bar{x}_i$ is then the maximum likelihood estimate of the resolved phase angle. In determining the minimum value of A_i, a large number of permutations in the n_k should be tried. However, this would require an excessively long computation time on the NAREC; therefore, approximate methods are used which produce adequate results.

The pattern method, which is a simplication of the maximum likelihood method, assumes that the phase of the longest baseline ϕ_ℓ is exactly correct. Each of the possible resolutions is tested by the criterion

$$A_i = \sum_k |x_\ell - x_k| \, d_k$$

and the value of x_ℓ is chosen such that A_i is a minimum. In this case for each possible value of x_ℓ, only the x_k are chosen which lie closest to x_ℓ. The appropriate value of x_ℓ is said to be the estimate of the resolved phase angle.

In the walkup method of ambiguity resolution it is necessary to first have an unambiguous estimate of the phase angle. This is achieved by differencing phasemeter readings from two baselines whose lengths differ by less than $\lambda/2$ feet (where λ is the wavelength of 108 Mc, i.e., 9.107 feet). Such a baseline was obtained by differencing the phase angle from the 16- and 20-foot baselines to obtain a synthetic 4-foot baseline. Similarly, an 8-foot baseline was obtained from the 52.8 and 60.8 foot baselines. Using as the initial estimate the resolved phase angle on an unambiguous baseline, the estimate for the next longer baseline is taken to be that possible value which is closest to the initial estimate. The estimate for the third baseline is taken to be that one which is closest to the value selected for the second, etc. There were two options in this program; one uses 4- and 8-foot baselines, obtained by differencing, in addition to the five physical baselines previously mentioned. The other option used the five physical baselines and the ten possible baselines which can be synthesized through forming positive first differences of the physical baselines (i.e., 4, 8, 32.8, 36.8, 40.8, 44.8, 68.8, 76.8, 109.6, and 113.6 feet).

RESULTS

The results obtained are plotted in Fig. K1. They are considered preliminary. The results are limited in that they apply only to point-by-point resolution methods without smoothing. It should be noted that the percent of correct solutions obtained in Fig. K1 is only indicative of the probability of a correct resolution from a single set of data points having the level of rms error indicated. If many resolutions are to be combined into a single result, it is to be expected that the probability of a correct final result will be increased above that indicated in the figure.

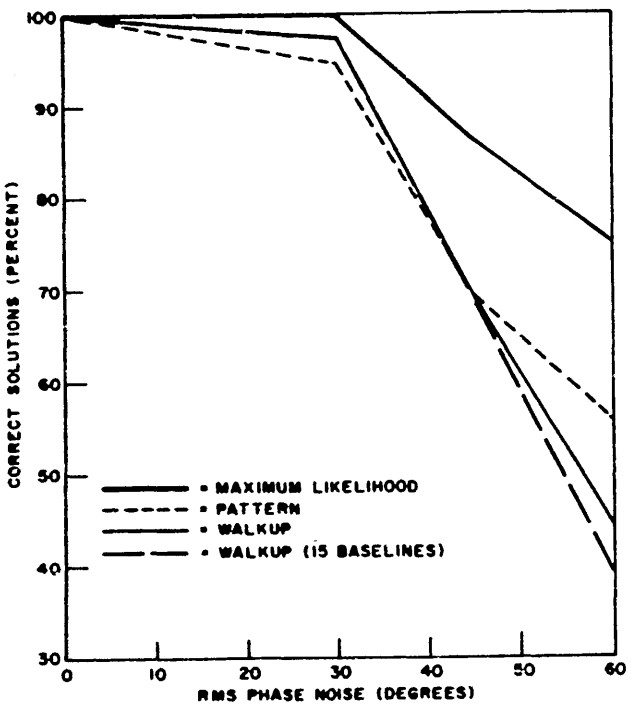


Fig. K1 - Comparison of point-by-point methods of ambiguity resolution

The error distributions occurring with each resolution method for the 36 sets of phasemeter readings are shown in Figs. K2, K3, and K4. In each figure, three separate histograms are plotted side by side, corresponding to the three levels of noise simulated. In all cases, no errors were made when there was no noise. It is interesting to note that certain values of error seem to be preferred by all three methods. A similar effect can sometimes be seen in plots of point-by-point resolutions produced by the NORC using the pattern method on actual Spasur data. It is also interesting to note that the pattern method shows no superiority over the walkup method except at high noise levels.

FUTURE PLANS

The present computer simulation programs have the following major program limitations: (a) they do not permit smoothing of the data, either before or after ambiguity resolution, (b) they do not provide simulation of a changing phase angle or of a phase rate,

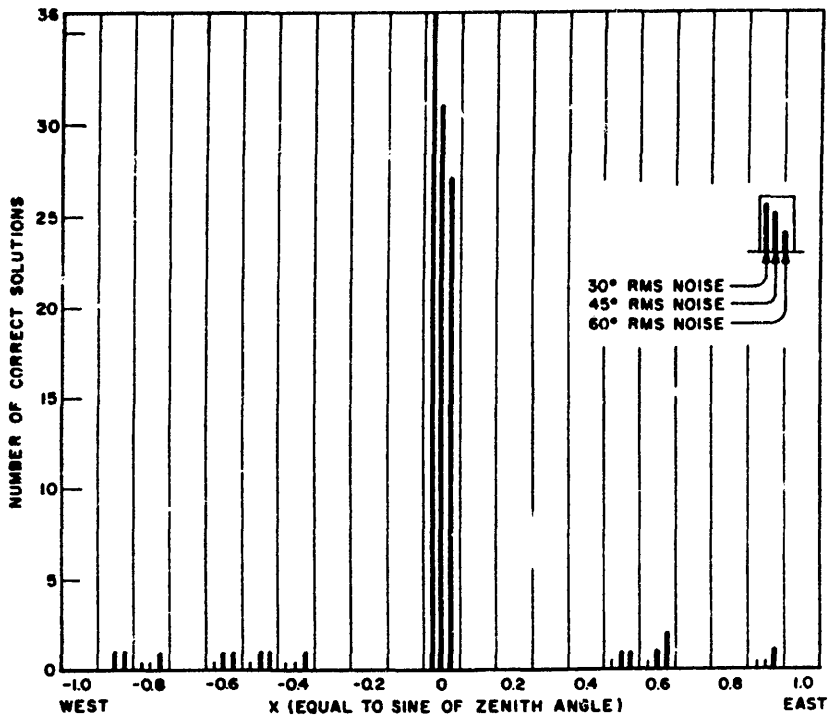


Fig- K2 - Distribution of resolutions for the maximum likelihood method

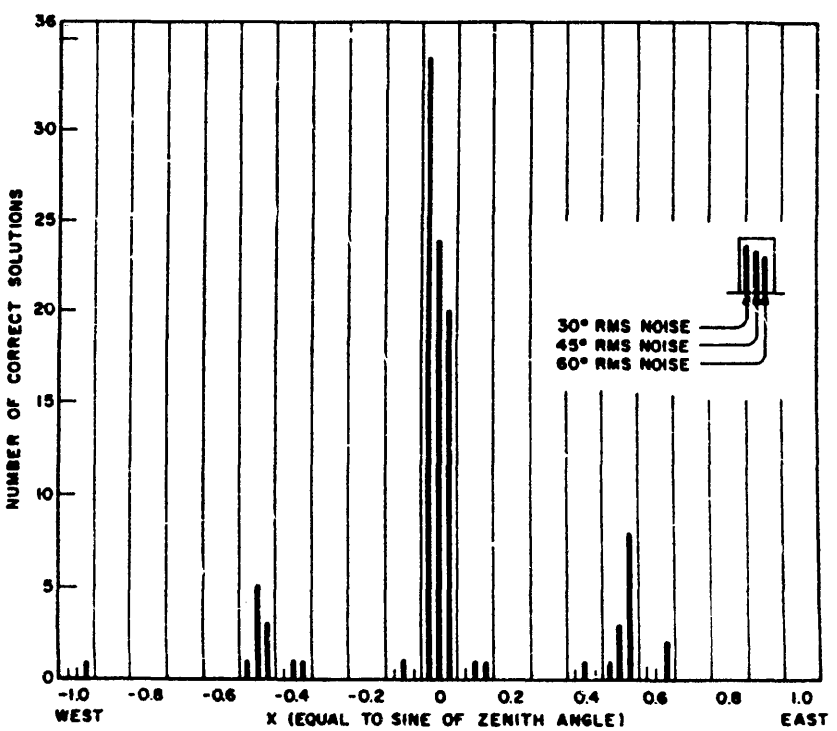


Fig. K3 - Distribution of resolutions for the pattern method

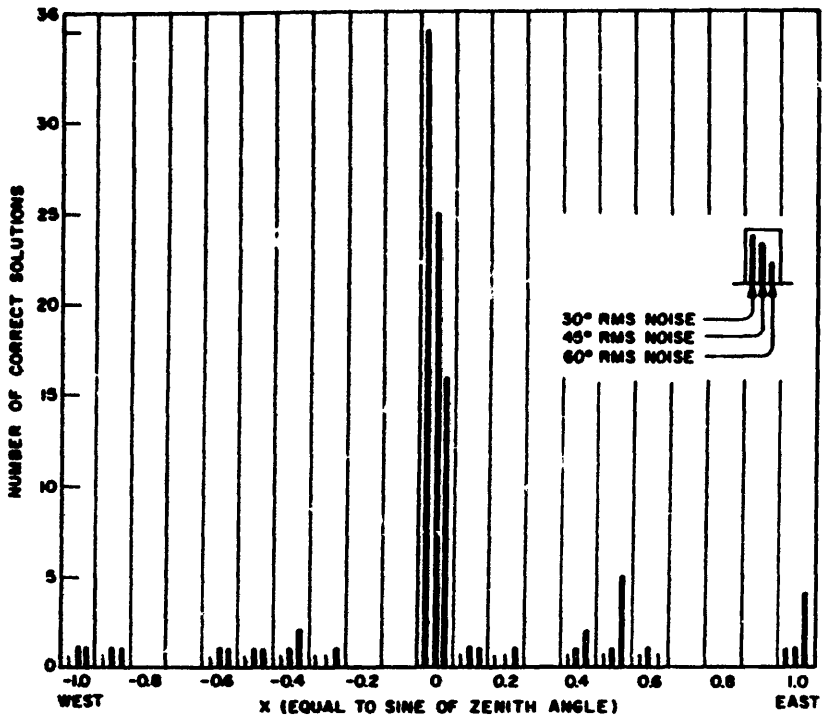


Fig. K4 - Distribution of resolutions for the walkup method

(c) they do not simulate noise having as narrow a bandwidth as that usually experienced in the present Spasur system, (d) they do not simulate systematic as well as random errors between individual phasemeter readings, (e) they have very limited flexibility in choice of baselines, etc., and (f) they are not sufficiently fast.

A set of NAREC programs is nearing completion which remove these limitations. They will be used to investigate the effects of noise level, bandwidth and choice of baselines on the determination of phase angle and phase rate. The ultimate purposes of this work are to evaluate present ambiguity resolution methods and to provide algorithms for future developments.

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13. ABSTRACT

A number of methods for ambiguity resolution for use in the Spasur radio interferometer direction-finding system ~~have been~~ derived and ~~have been~~ studied using both Monte Carlo and operational data. Implementations of the methods in both hardware and software were analyzed. The conclusion was drawn that the optimum resolution results can be obtained only by the joint optimization of the antenna configuration and the resolution process to form an optimum processing system.

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