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A Mathematical Model of a Rotational Frequency Translation Device

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CONTENTS

Abstract	ii
Problem Status	ii
Authorization	ii
INTRODUCTION	1
THE MATHEMATICAL MODEL	1
TORQUE ON DIPOLE	4
RADIATION	8
THE REFLECTION CASE	13
TORQUE ON DIPOLE (REFLECTION CASE)	14
PHOTONS DISCUSSION	16
SUMMARY	17
REFERENCES	18

ABSTRACT

An analysis was made which provides a theoretical explanation of a unique radiation torque experiment performed by Allen, in which circularly polarized (cp) microwave power was used to impart continuous rotary motion to an object supported within a circular waveguide. Under the action of the cp wave, an oscillating current was set up in a rod dipole resulting in an oscillating dipole moment. Part of the energy scattered by the rotating dipole is frequency shifted. Torques arise from the interactions between the E-field of the wave and the induced dipole moment, and between the oscillating current in the dipole and the longitudinal H-field in the waveguide. The magnitudes of these torques were derived for two cases of interest; the open waveguide and the shorted waveguide. The magnitudes of the spectral components of the radiation scattered by the dipole were also derived along with their characteristic polarizations. A cursory analysis shows that photons are conserved in such an experiment.

PROBLEM STATUS

This is an interim report; work on the problem is continuing.

AUTHORIZATION

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A MATHEMATICAL MODEL OF A ROTATIONAL FREQUENCY TRANSLATION DEVICE

INTRODUCTION

In a simple experiment to demonstrate the torque exerted by circularly polarized radiation [1], Allen suspended a thin conducting rod at its midpoint so that it was free to turn about the axis of a vertically oriented circular waveguide. With less than a watt of circularly polarized (CP) power at 9000 Mc, the torque was sufficient to cause continuous rotation of the rod about the waveguide axis. In each instance, the rotating rod would assume a terminal velocity (limited by atmospheric drag) which was proportional to the incident microwave power.

There are two versions of the experiment of particular interest: (a) one in which the circular waveguide is terminated in a matched load, and (b) one in which the waveguide is short-circuited one-quarter wavelength beyond the rod. In the first case, both the transmitted and the reflected components of the incident wave are found to contain two frequencies, one which is downshifted by twice the rotational frequency of the rod, and the other which is unshifted. In the second case, all of the reflected power is downshifted in frequency by twice the rotational frequency of the rod. Once rotating, the rod acts as a passive mechanism for frequency shifting the microwave radiation.

The present work is a theoretical explanation of the observed results, based on a simple model. This model has successfully explained the origin of the frequency shift, the distribution of power among shifted and unshifted spectral components, the characteristic polarizations for these components, and the dynamical origin of torques on the rod, as well as other aspects of the observed results.

THE MATHEMATICAL MODEL

Under the action of the CP wave, an oscillating current is set up in the rod and hence an oscillating dipole moment. The induced dipole moment and the oscillating current through interaction with the incident CP wave cause a torque on the rod. Eventually an equilibrium is reached in which the torque due to the incident wave is balanced by the drag torques of the air and pivot. When this occurs, the rod will rotate at a constant rate. The oscillating current in the rod will radiate producing radiation at the incident frequency and at a component which is downshifted by twice the rotation frequency. The rod is assumed to have a very high conductivity, vanishing thickness, and length $\lambda/2$, where λ is the free-space wavelength of the waveguide radiation. It is assumed to be free to rotate in a plane perpendicular to the waveguide axis (Fig. 1), subject only to friction arising from the pivot on which the rod is supported and to air drag.

A pure circularly-polarized wave cannot be propagated in a waveguide, but a field configuration which closely approximates a CP wave in a region close to the waveguide axis may be formed in a cylindrical waveguide by superimposing two TE_{11} modes with a 90° phase difference.

In cylindrical coordinates the electric field configuration can be written:

$$E_r = \frac{2E_0}{kr} \sin\phi\, J_1(kr) \cos(\omega_0 t - \beta z) + \frac{2E_0}{kr} \cos\phi\, J_1(kr) \sin(\omega_0 t - \beta z) \tag{1}$$

Note: This analysis was completed in July 1965 while Mr. Sachs was a summer employee at NRL.

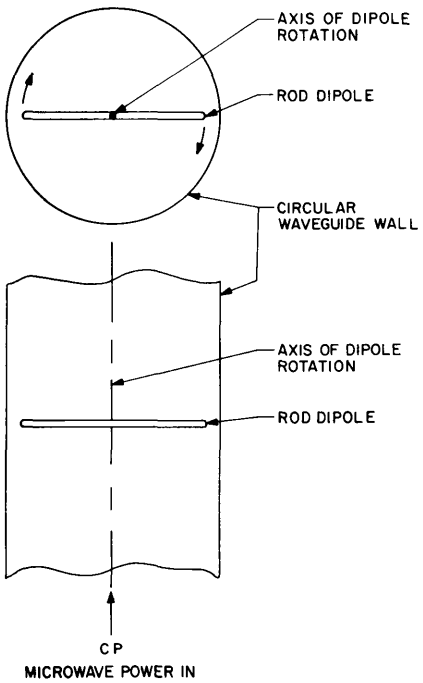

Fig. 1. Rotating rod dipole in a circular waveguide.

$$E_\phi = 2E_0 \cos\phi\, J_1'(kr) \cos(\omega_0 t - \beta z) - 2E_0 \sin\phi\, J_1'(kr) \sin(\omega_0 t - \beta z) \quad \text{(1 cont.)}$$

$$E_z = 0$$

and the magnetic field configuration can be written

$$\begin{aligned} B_r &= -\frac{2E_0\beta}{\omega_0} J_1'(kr) \cos\phi \cos(\omega_0 t - \beta z) + \frac{2E_0\beta}{\omega_0} J_1'(kr) \sin\phi \sin(\omega_0 t - \beta z) \\ B_\phi &= \frac{2E_0\beta}{\omega_0 kr} \sin\phi\, J_1(kr) \cos(\omega_0 t - \beta z) + \frac{2E_0\beta}{\omega_0 kr} \cos\phi\, J_1(kr) \sin(\omega_0 t - \beta z) \\ B_z &= \frac{2E_0 k}{\omega_0} J_1(kr) \cos\phi \sin(\omega_0 t - \beta z) + \frac{2E_0 k}{\omega_0} J_1(kr) \sin\phi \cos(\omega_0 t - \beta z) \end{aligned} \tag{2}$$

where

$k = 1.84/a$

a = waveguide radius

$\beta = \omega_0/c \sqrt{1 - (\omega_c/\omega_0)^2}$

ω_c = cutoff angular frequency = $c(1.84/a)$

ω_0 = angular frequency of wave.

The field propagates in the $+z$ direction (Fig. 2).

We can easily see how this configuration approximates a CP wave. Expressing the E-field in vector form, we have

$$\mathbf{E} = \hat{r} \frac{2E_0}{kr} J_1(kr) \sin(\omega_0 t - \beta z + \phi) + \hat{\phi}\, 2E_0\, J_1'(kr) \cos(\omega_0 t - \beta z + \phi) \tag{3}$$

where $\hat{r}$ and $\hat{\phi}$ are unit vectors in directions of increasing r and ϕ.

But (Fig. 3)

$$\begin{aligned} \hat{r} &= \hat{i} \cos\phi + \hat{j} \sin\phi \\ \hat{\phi} &= -\hat{i} \sin\phi + \hat{j} \cos\phi. \end{aligned} \tag{4}$$

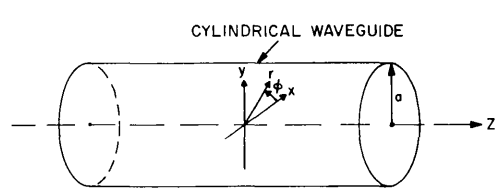

Fig. 2. Coordinate axis in cylindrical waveguide.

Thus, on substitution of Eq. 4, Eq. 3 becomes

$$\mathbf{E} = [\hat{i}\cos\phi + \hat{j}\sin\phi]\,\frac{2E_0}{kr}\,J_1(kr)\sin(\omega_0 t - \beta z + \phi) + [-\hat{i}\sin\phi + \hat{j}\cos\phi]\;2E_0 J_1'(kr)\cos(\omega_0 t - \beta z + \phi) \tag{5}$$

If for $r \ll a$ and $kr = (1.84/a)r \ll 1$, the approximations $J_1(kr) \approx kr/2$ and $J_1'(kr) \approx 1/2$ are used. Then the E-field becomes

$$\mathbf{E} = \hat{i}E_0\sin(\omega_0 t - \beta z) + \hat{j}E_0\cos(\omega_0 t - \beta z) \tag{6}$$

which represents a CP wave in cartesian coordinates.

Let us now choose our coordinate system such that the origin coincides with the pivot point of the metal rod (Fig. 4). Let the waveguide modes propagate in the direction of increasing z. Let the rod rotate at a constant angular rate ω.

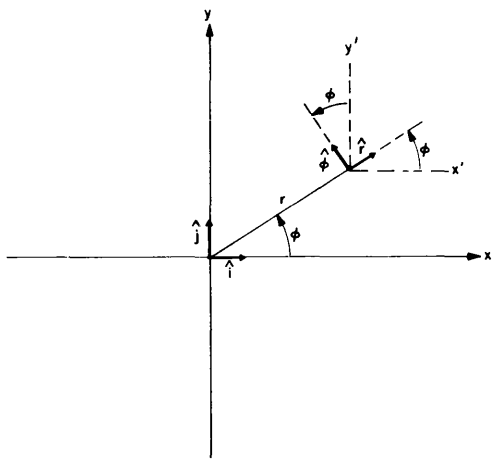

Fig. 3. Cylindrical coordinates and unit vector.

Let s be the distance along the rod from the center. For $s > 0$,

$$\phi_{rod} + \omega t = \frac{\pi}{2}$$

or

$$\phi_{rod} = \frac{\pi}{2} - \omega t$$

and for $s < 0$,

$$\phi_{rod} = -\frac{\pi}{2} - \omega t.$$

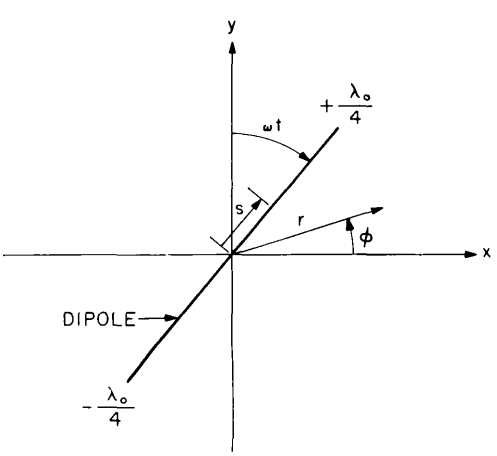

Fig. 4. Position of the rod dipole in the coordinate system.

Then the E-field along the rod becomes for $s > 0$,

$$\begin{aligned} E_r(s,\phi_{rod}) &= \frac{2E_0}{k|s|}J_1(k|s|)\sin\left(\frac{\pi}{2} - \omega t\right)\cos\omega_0 t \\ &\quad + \frac{2E_0}{k|s|}J_1(k|s|)\cos\left(\frac{\pi}{2} - \omega t\right)\sin\omega_0 t \\ &= \frac{2E_0}{k|s|}J_1(k|s|)\sin\left(\omega_0 t - \omega t + \frac{\pi}{2}\right) \\ &= \frac{2E_0}{k|s|}J_1(k|s|)\cos(\omega_0 t - \omega t) \end{aligned} \tag{7a}$$

and for $s < 0$,

$$E_r(s, \phi_{rod}) = -\frac{2E_0}{k|s|} J_1(k|s|) \cos(\omega_0 t - \omega t). \tag{7b}$$

Experimentally

$$\frac{\lambda_0}{4} = 0.82 \text{ cm} \qquad \text{and} \qquad a = 1.19 \text{ cm},$$

therefore

$$\frac{k\lambda_0}{4} = \frac{1.84}{1.19}(0.82) = 1.27$$

Thus, the field is not constant over the rod length. The length is

$$\frac{\pi c}{\omega_0} = \frac{\lambda_0}{2}.$$

Experimentally we know that

$$\frac{\omega}{2\pi} \approx 10 \text{ cps} \qquad \text{and} \qquad \frac{\omega_0}{2\pi} \approx 10^{10} \text{ cps}.$$

Hence the rod acts as a half-wave dipole. The current in the rod [2] will be

$$I = I_0 \cos\left(\frac{\omega_0 - \omega}{c}\right) s \cos(\omega_0 t - \omega t). \tag{8}$$

The charge density in the rod can be determined from the equation of continuity

$$\text{div } \mathbf{j} + \frac{\partial \rho}{\partial t} = 0. \tag{9}$$

Let the rod have a cross-sectional area A. The charge density can be assumed to be uniformly distributed over the cross-sectional area, i.e., $\rho = \rho(s, t)$ and $\mathbf{j} = I(s, t)/A$. The continuity equation becomes

$$\frac{d}{ds}\left[\frac{I_0}{A} \cos\left(\frac{\omega_0 - \omega}{c}\right) s \cos(\omega_0 t - \omega t)\right] = -\frac{\partial}{\partial t}[\rho(s, t)]. \tag{10}$$

Then, we obtain

$$\rho(s, t) = \frac{I_0}{Ac} \sin\left(\frac{\omega_0 - \omega}{c}\right) s \sin(\omega_0 t - \omega t). \tag{11}$$

We are now in a position to discuss the torques acting on the rod.

TORQUE ON DIPOLE

The force acting on an infinitesimal amount of charge at s is

$$d\mathbf{F} = \rho(s, t)\, \mathbf{E}(s, t) dV \qquad \text{and} \qquad dV = A\, ds. \tag{12}$$

The moment of this force is

$$d\tau = \mathbf{r} \times d\mathbf{F}. \tag{13}$$

Substituting Eq. 12 for $d\mathbf{F}$, then

$$d\tau = (|s|\,\hat{r}) \times \rho(s,t)\,\mathbf{E}(s,t)\,dV. \tag{14}$$

Since the E-field along the rod in vector form is for $s > 0$,

$$\mathbf{E}(s,t) = \hat{r}\,\frac{2E_0}{k|s|}J_1(k|s|)\sin\left(\omega_0 t - \omega t + \frac{\pi}{2}\right) + \hat{\phi}\,2E_0\,J_1'(k|s|)\cos\left(\omega_0 t - \omega t + \frac{\pi}{2}\right), \tag{15a}$$

and for $s < 0$,

$$\mathbf{E}(s,t) = \hat{r}\,\frac{2E_0}{k|s|}J_1(k|s|)\sin\left(\omega_0 t - \omega t - \frac{\pi}{2}\right) + \hat{\phi}\,2E_0\,J_1'(k|s|)\cos\left(\omega_0 t - \omega t - \frac{\pi}{2}\right). \tag{15b}$$

Also, since $\hat{r} \times \hat{r} = 0$ and $\hat{r} \times \hat{\phi} = \hat{K}$, $d\tau$ becomes for $s > 0$,

$$d\tau = \hat{K}\,|s|\frac{I_0}{c}\sin\left(\frac{\omega_0 - \omega}{c}\right)sJ_1'(k|s|)\,[-\sin^2(\omega_0 t - \omega t)]\;2E_0\,ds \tag{16a}$$

and for $s < 0$,

$$d\tau = \hat{K}\,|s|\frac{I_0}{c}\sin\left(\frac{\omega_0 - \omega}{c}\right)sJ_1'(k|s|)\,[\sin^2(\omega_0 t - \omega t)]\;2E_0\,ds. \tag{16b}$$

Integrating over the length of the rod, $d\tau$ becomes

$$\begin{aligned}\tau &= (-\hat{K})\frac{2E_0 I_0}{c}\int_0^{\lambda_0/4} ds\,|s|\sin\left(\frac{\omega_0 - \omega}{c}\right)s\,J_1'(k|s|)\sin^2(\omega_0 t - \omega t)\\ &\quad + (\hat{K})\frac{2E_0 I_0}{c}\int_{-\lambda_0/4}^{0} ds\,|s|\sin\left(\frac{\omega_0 - \omega}{c}\right)s\,J_1'(k|s|)\sin^2(\omega_0 t - \omega t)\\ &= (-\hat{K})\frac{4E_0 I_0}{c}\int_0^{\lambda_0/4} ds\,|s|\sin\left(\frac{\omega_0 - \omega}{c}\right)s\,J_1'(k|s|)\sin^2(\omega_0 t - \omega t).\end{aligned} \tag{17}$$

We next approximate $J_1(k|s|)$ by

$$J_1(k|s|) = \frac{1}{2}(k|s|)\left[1 - \frac{\left(\frac{1}{2}k|s|\right)^2}{2} + \frac{\left(\frac{1}{2}k|s|\right)^4}{12} + \ldots\right] \tag{18}$$

so that

$$J_1'(k|s|) = \frac{d}{d(k|s|)}\,[J_1(k|s|)] = \frac{1}{2} - \frac{3}{16}(k|s|)^2 + \frac{5}{384}(k|s|)^4 + \ldots.$$

Carrying out the integration and averaging over time, we find

$$\bar{\tau} = (-\hat{K}) \frac{E_0 I_0}{c\left(\frac{\omega_0 - \omega}{c}\right)^2} \left\{ 1 - \frac{9}{8} \frac{k^2}{\left(\frac{\omega_0 - \omega}{c}\right)^2} \left[\left(\frac{\pi}{2}\right)^2 - 2 \right] + \frac{25}{192} \frac{k^4}{\left(\frac{\omega_0 - \omega}{c}\right)^4} \left[\left(\frac{\pi}{2}\right)^4 - 12\left(\frac{\pi}{2}\right)^2 + 24 \right] + \ldots \right\}. \tag{19}$$

Physically, this torque is due to the tendency of the induced dipole moment of the rod to line up with the impressed E-field. Since the E-field is rotating, the rod rotates in the same direction. It is, so to speak, dragged along by the rotating E-field.

Similarly, there is a second torque mechanism which arises from the interaction of the induced current in the rod with the longitudinal B-field in the waveguide. The force acting on the induced current at s is

$$d\mathbf{F} = \mathbf{j}(s, t) \times \mathbf{B}(s, t)\, dV. \tag{20}$$

The moment of this force is

$$d\tau = |s|\, \hat{r} \times d\mathbf{F}. \tag{13}$$

For $s > 0$

$$d\tau = A ds \frac{I_0}{A} |s| \cos\left(\frac{\omega_0 - \omega}{c}\right) s \cos(\omega_0 t - \omega t)\, [\hat{r} \times (+\hat{r} \times \mathbf{B}(s, t))] \tag{21a}$$

and for $s < 0$

$$d\tau = A ds \frac{I_0}{A} |s| \cos\left(\frac{\omega_0 - \omega}{c}\right) s \cos(\omega_0 t - \omega t)\, [\hat{r} \times (-\hat{r} \times \mathbf{B}(s, t))]. \tag{21b}$$

Now for $s > 0$

$$\{\hat{r} \times [+\hat{r} \times \mathbf{B}(s, t)]\} = -\hat{\phi} B_\phi(s, t) - \hat{K} B_z(s, t) \tag{22a}$$

and for $s < 0$

$$\{\hat{r} \times [-\hat{r} \times \mathbf{B}(s, t)]\} = +\phi B_\phi(s, t) + \hat{K} B_z(s, t) \tag{22b}$$

Also, for $s > 0$

$$\begin{aligned} d\tau = ds\, I_0 |s| \cos\left(\frac{\omega_0 - \omega}{c}\right) s \cos(\omega_0 t - \omega t) \Bigg[& -\phi \frac{2E_0 \beta}{\omega_0 k |s|} J_1(k|s|) \sin\left(\omega_0 t - \omega t + \frac{\pi}{2}\right) \\ & - \hat{K} \frac{2E_0 k}{\omega_0} J_1(k|s|) \sin\left(\omega_0 t - \omega t + \frac{\pi}{2}\right) \Bigg] \end{aligned} \tag{23a}$$

and for $s < 0$

$$d\tau = ds\, I_0\, |s| \cos\left(\frac{\omega_0 - \omega}{c}\right) s \cos(\omega_0 t - \omega t) \left[+ \hat{\phi}\, \frac{2E_0\beta}{\omega_0 k |s|} J_1(k|s|) \sin\left(\omega_0 t - \omega t - \frac{\pi}{2}\right) \right.$$
$$\left. + \hat{K}\, \frac{2E_0 k}{\omega_0} J_1(k|s|) \sin\left(\omega_0 t - \omega t - \frac{\pi}{2}\right)\right] \tag{23b}$$

If we note that $\hat{\phi}$ for $s > 0$ is in the opposite direction to $\hat{\phi}$ for $s < 0$ we see that there is no net torque in the $\hat{\phi}$ direction. Finally by integrating over the rod, we obtain

$$\tau = \frac{2I_0E_0k}{\omega_0}(-\hat{K}) \int_0^{\lambda_0/4} ds\, |s| \cos\left(\frac{\omega_0 - \omega}{c}\right) s\, J_1(k|s|) \cos^2(\omega_0 t - \omega t)$$
$$+ \frac{2I_0E_0k}{\omega_0}(-\hat{K}) \int_{-\lambda_0/4}^{0} ds\, |s| \cos\left(\frac{\omega_0 - \omega}{c}\right) s\, J_1(k|s|) \cos^2(\omega_0 t - \omega t) \tag{24}$$
$$= (-\hat{K}) \frac{4I_0E_0k}{\omega_0} \int_0^{\lambda_0/4} ds\, |s| \cos\left(\frac{\omega_0 - \omega}{c}\right) s\, J_1(k|s|) \cos^2(\omega_0 t - \omega t).$$

Approximating $J_1(k|s|)$ by

$$J_1(k|s|) = \frac{1}{2}k|s| - \frac{1}{16}(k|s|)^3 + \frac{(k|s|)^5}{384} + \ldots \tag{16}$$

and integrating over the rod and averaging over time, we get

$$\bar{\tau} = (-\hat{K}) \frac{2E_0I_0k}{\omega_0} \left\{ \frac{k}{2\left(\frac{\omega_0 - \omega}{c}\right)^3} \left[\left(\frac{\pi}{2}\right)^2 - 2\right] - \frac{k^3}{16\left(\frac{\omega_0 - \omega}{c}\right)^5} \left(\frac{\pi}{2}\right)^4 + \frac{4}{16} \frac{k^3}{\left(\frac{\omega_0 - \omega}{c}\right)^5} \left[3\left(\frac{\pi}{2}\right)^2 - 6\right] \right.$$
$$\left. + \frac{k^5}{384\left(\frac{\omega_0 - \omega}{c}\right)^7} \left(\frac{\pi}{2}\right)^6 + \ldots \right\}. \tag{25}$$

Since $\omega << \omega_0$, $\omega_0 \approx \omega_0 - \omega$ and the total torque is

$$\bar{\tau}_t = (-\hat{K}) \frac{E_0I_0}{c\left(\frac{\omega_0 - \omega}{c}\right)^2} \left\{ 1 - \frac{1}{8}\left(\frac{k}{\frac{\omega_0 - \omega}{c}}\right)^2 \left[\left(\frac{\pi}{2}\right)^2 - 2\right] + \frac{1}{8(24)}\left(\frac{k}{\frac{\omega_0 - \omega}{c}}\right)^4 \left[\left(\frac{\pi}{2}\right)^4 - 12\left(\frac{\pi}{2}\right)^2 + 24\right] + \ldots \right\}$$
$$\equiv (-\hat{K}) \frac{E_0I_0}{c\left(\frac{\omega_0 - \omega}{c}\right)^2} [1 - \delta]. \tag{26}$$

Experimentally, $k/[(\omega_0 - \omega)/c] = \omega_c/(\omega_0 - \omega) \approx (4.65 \times 10^{10}/2\pi \times 10^{10}) \approx 0.8$, so that the δ is small. We note that as the radius of the waveguide increases, $\delta \to 0$ since $\omega_c \to 0$.

RADIATION

We have seen that there is an induced oscillating current and dipole moment established in the dipole under the influence of the impressed electromagnetic field. We expect that this oscillating dipole moment will radiate energy in both directions down the waveguide. To calculate this radiated field, let us now consider a somewhat ad hoc algorithm which is justified mainly by the fact that the end results agree with the experiments.

First we calculate the magnitude of the induced dipole moment $|\mathbf{p}|$ in the rod by integrating over the length of the rod:

$$|\mathbf{p}| = \int_{-\lambda_0/4}^{+\lambda_0/4} s\rho(s, t)\, A\, ds$$

Using Eq. 10,

$$\begin{aligned} |\mathbf{p}| &= \frac{I_0}{c} \sin(\omega_0 t - \omega t) \int_{-\lambda_0/4}^{+\lambda_0/4} s \sin\left(\frac{\omega_0 - \omega}{c}\right) s\, ds \\ &= \frac{I_0}{c} \frac{\sin(\omega_0 t - \omega t)}{\left(\frac{\omega_0 - \omega}{c}\right)^2} \left[\sin\left(\frac{\omega_0 - \omega}{c}\right)s - \left(\frac{\omega_0 - \omega}{c}\right)s \cos\left(\frac{\omega_0 - \omega}{c}\right)s\right]_{-\lambda_0/4}^{+\lambda_0/4}. \end{aligned} \tag{27}$$

Since $\omega << \omega_0$ and since $(\omega_0 - \omega)/c(+\lambda_0/4) = +\pi/2$ and $(\omega_0 - \omega)/c(-\lambda_0/4) = -\pi/2$, Eq. 27 reduces to

$$|\mathbf{p}| = \frac{2I_0 \sin(\omega_0 t - \omega t)}{c\left(\frac{\omega_0 - \omega}{c}\right)^2}. \tag{28}$$

We define a unit vector $\hat{n}$ directed along the dipole direction as

$$\hat{n} = \hat{i} \sin \omega t + \hat{j} \cos \omega t. \tag{29}$$

The dipole moment is then

$$\mathbf{p} = \hat{n}\, |\mathbf{p}| \tag{30}$$

And by substituting Eqs. 27 and 28, the dipole moment becomes

$$\mathbf{p} = (\hat{i} \sin \omega t + \hat{j} \cos \omega t) \left[\frac{2I_0 \sin(\omega_0 t - \omega t)}{c\left(\frac{\omega_0 - \omega}{c}\right)^2}\right]. \tag{31}$$

We now assert, without rigorous justification, that on the waveguide axis the E-field radiated by the dipole is given for $z > 0$

$$\mathbf{E}_{\text{rad}}(z, t) = a\left[\mathbf{p}'\left(t - \frac{|z|}{v}\right) \times (+\hat{K})\right] \times (+\hat{K}) \tag{32a}$$

and for $z < 0$

$$\mathbf{E}_{\text{rad}}(z, t) = a \left[\mathbf{p}' \left(t - \frac{|z|}{v} \right) \times (-\hat{K}) \right] \times (-\hat{K}) \tag{32b}$$

where

$$\mathbf{p}'\left(t - \frac{|z|}{v}\right) = \frac{\partial}{\partial t} \left[\mathbf{p} \left(t - \frac{|z|}{v} \right) \right]$$

and v is the phase velocity in the waveguide. Since $(\hat{j} \times \hat{K}) \times \hat{K} = -\hat{j}$ and $(\hat{i} \times \hat{K}) \times \hat{K} = -\hat{i}$, when $z = 0$

$$\begin{aligned} \mathbf{p}'(t) &= \frac{\partial}{\partial t} \left\{ (\hat{i} \sin \omega t + \hat{j} \cos \omega t) \left[\frac{2I_0 c}{(\omega_0 - \omega)^2} \sin (\omega_0 t - \omega t) \right] \right\} \\ &= \hat{i} \left[\frac{2cI_0}{(\omega_0 - \omega)} \sin \omega t \cos (\omega_0 t - \omega t) + \frac{2cI_0 \omega}{(\omega_0 - \omega)^2} \sin (\omega_0 t - \omega t) \cos \omega t \right] \\ &\quad + \hat{j} \left[\frac{2cI_0}{(\omega_0 - \omega)} \cos \omega t \cos (\omega_0 t - \omega t) - \frac{2cI_0 \omega}{(\omega_0 - \omega)^2} \sin (\omega_0 t - \omega t) \sin \omega t \right]. \end{aligned} \tag{33}$$

We neglect each term containing $\sin (\omega_0 t - \omega t)$ since $\omega \ll \omega_0$. Thus,

$$\mathbf{p}'(t) = \hat{i} \frac{2cI_0}{\omega_0 - \omega} \sin \omega t \cos (\omega_0 t - \omega t) + \hat{j} \frac{2cI_0}{\omega_0 - \omega} \cos \omega t \cos (\omega_0 t - \omega t). \tag{34}$$

Using the substitutions

$$\cos (\omega_0 t - \omega t) \cos \omega t = \frac{1}{2} [\cos \omega_0 t + \cos (\omega_0 t - 2\omega t)] \tag{35a}$$

$$\sin \omega t \cos (\omega_0 t - \omega t) = \frac{1}{2} [\sin \omega_0 t - \sin (\omega_0 t - 2\omega t)], \tag{35b}$$

$\mathbf{p}'(t)$ becomes

$$\mathbf{p}'(t) = \frac{cI_0}{(\omega_0 - \omega)} (\hat{j} \cos \omega_0 t + \hat{i} \sin \omega_0 t) + \frac{cI_0}{(\omega_0 - \omega)} [\hat{j} \cos (\omega_0 t - 2\omega t) - \hat{i} \sin (\omega_0 t - 2\omega t)]. \tag{36}$$

Substituting the value for $\mathbf{p}'(t)$ into Eq. 32 to find the E-field radiated, we find for $z = 0$

$$\mathbf{E}_{\text{rad}}(z, t) = -\frac{cI_0 a}{(\omega_0 - \omega)} [\hat{i} \sin \omega_0 t + \hat{j} \cos \omega_0 t] - \frac{cI_0 a}{(\omega_0 - \omega)} [-\hat{i} \sin (\omega_0 t - 2\omega t) + \hat{j} \cos (\omega_0 t - 2\omega t)] \tag{37}$$

and for $z \neq 0$

$$\mathbf{E}_{\text{rad}}(z,t) = -\frac{cI_0 a}{\omega_0 - \omega}\left[\hat{i}\sin\omega_0\left(t-\frac{|z|}{v}\right)+\hat{j}\cos\omega_0\left(t-\frac{|z|}{v}\right)\right]$$
$$-\frac{cI_0 a}{\omega_0-\omega}\left\{-\hat{i}\sin\left[(\omega_0-2\omega)\left(t-\frac{|z|}{v}\right)\right]+\hat{j}\cos\left[(\omega_0-2\omega)\left(t-\frac{|z|}{v}\right)\right]\right\}. \tag{38}$$

The total E-field for $z > 0$ is the sum of the incident plus radiated E-fields,

$$\mathbf{E}_{\text{tot}} = \mathbf{E}_{\text{inc}} + \mathbf{E}_{\text{rad}}.$$

Then, by substituting Eq. 38

$$\mathbf{E}_{\text{tot}} = \left(E_0 - \frac{cI_0 a}{\omega_0-\omega}\right)\left[\hat{i}\sin\omega_0\left(t-\frac{z}{v}\right)+\hat{j}\cos\omega_0\left(t-\frac{z}{v}\right)\right]$$
$$-\frac{cI_0 a}{\omega_0-\omega}\left\{-\hat{i}\sin\left[(\omega_0-2\omega)\left(t-\frac{z}{v}\right)\right]+\hat{j}\cos\left[(\omega_0-2\omega)\left(t-\frac{z}{v}\right)\right]\right\}, \tag{39a}$$

and for $z < 0$,

$$\mathbf{E}_{\text{rad}}(z,t) = -\frac{cI_0 a}{\omega_0-\omega}\left[\hat{i}\sin\omega_0\left(t+\frac{z}{v}\right)+\hat{j}\cos\omega_0\left(t+\frac{z}{v}\right)\right]$$
$$-\frac{cI_0 a}{\omega_0-\omega}\left\{-\hat{i}\sin\left[(\omega_0-2\omega)\left(t+\frac{z}{v}\right)\right]+\hat{j}\cos\left[(\omega_0-2\omega)\left(t+\frac{z}{v}\right)\right]\right\}. \tag{39b}$$

We have two unknown quantities in the equations, a and I_0. To determine these, we can apply the conservation of energy and angular momentum. Losses in the waveguide walls will be neglected.

Let us consider a cylindrical volume V defined by the waveguide walls and two cross-sectional areas, A_1 and A_2, A_1 in front of the dipole and A_2 behind the dipole (Fig. 5). No energy or angular momentum flows through the cylindrical walls. Let the incident field carry power $K(E_0)^2$ across A_1 into V. Flowing out of V across A_1 is the radiated power from the dipole, $2K\,[cI_0 a/(\omega_0-\omega)]^2$. At A_2 the power transported out of V is

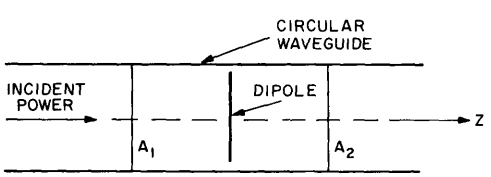

Fig. 5. Lengthwise cross section of the circular waveguide.

$$K\left(E_0 - \frac{cI_0 a}{\omega_0-\omega}\right)^2 + K\left(\frac{cI_0 a}{\omega_0-\omega}\right)^2. \tag{40}$$

In V, the mechanical power developed is

$$|\tau_t|\omega = \frac{E_0 I_0\,[1-\delta]}{c\left(\dfrac{\omega_0-\omega}{c}\right)^2}. \tag{41}$$

The conservation of energy states that the power flowing into V is equal to the power flowing out plus the mechanical power developed in V, i.e.,

$$K(E_0)^2 = 3K\left(\frac{cI_0 a}{\omega_0-\omega}\right)^2 + K\left(E_0 - \frac{cI_0 a}{\omega_0-\omega}\right)^2 + \frac{E_0 I_0\,[1-\delta]}{c\left(\dfrac{\omega_0-\omega}{c}\right)^2}. \tag{42}$$

Solving for I_0, we find

$$I_0 = -\frac{E_0}{4Kca^2}\,[\omega(1-\delta) - 2Ka(\omega_0 - \omega)]\,. \tag{43}$$

Let us now consider the conservation of angular momentum. A pure circularly polarized wave of frequency ω carries across a unit area per second an amount of angular momentum $|\mathbf{L}| = |\mathbf{S}|/\omega$ where $\mathbf{S}$ is the Poynting vector. This angular momentum is a vector which is directed either parallel or antiparallel to the direction of propagation, depending on the sense of polarization. If the wave propagates toward the observer, and the E vector rotates in the clockwise direction, the angular momentum vector is directed antiparallel to the propagation direction. For counterclockwise rotation, the angular momentum vector is parallel to the propagation direction.

In a waveguide the situation is the same in all respects except that

$$|\mathbf{L}| = \frac{|\mathbf{S}|}{\omega}\gamma$$

where γ is a numerical factor which depends on the modes involved. For our problem [3] $\gamma = 0.994$. We will let $\gamma = 1$ in the following.

Consider now the waveguide situation. At A_1, the angular momentum flow into V is

$$(\mathbf{L})_{\text{in}} = -\hat{K}\,\frac{K(E_0)^2}{\omega_0}. \tag{44}$$

Flowing out of V at A_1 is

$$(\mathbf{L})_{A_1} = -\hat{K}\,K\left(\frac{cI_0a}{\omega_0-\omega}\right)^2\frac{1}{\omega_0} + \hat{K}K\left(\frac{cI_0a}{\omega_0-\omega}\right)^2\left(\frac{1}{\omega_0-2\omega}\right). \tag{45}$$

Flowing out of V at A_2 is

$$(\mathbf{L})_{A_2} = -\hat{K}\,K\left(E_0 - \frac{cI_0a}{\omega_0-\omega}\right)^2\frac{1}{\omega_0} + \hat{K}K\left(\frac{cI_0a}{\omega_0-\omega}\right)^2\left(\frac{1}{\omega_0-2\omega}\right). \tag{46}$$

As derived earlier, the torque on the dipole is

$$\bar{\tau}_t = -\hat{K}\,\frac{E_0I_0(1-\delta)}{c\left(\dfrac{\omega_0-\omega}{c}\right)^2}. \tag{26}$$

Conservation of angular momentum says that

$$(\mathbf{L})_{\text{in}} - (\mathbf{L})_{A_1} - (\mathbf{L})_{A_2} = \tau. \tag{47}$$

and substituting Eqs. 26, 44, 45, and 46, we have

$$(-\hat{K})\frac{K(E_0)^2}{\omega_0} + \hat{K}\,K\left(\frac{cI_0a}{\omega_0-\omega}\right)^2\frac{1}{\omega_0} - (2\hat{K})\,K\left(\frac{cI_0a}{\omega_0-\omega}\right)^2\frac{1}{(\omega_0-2\omega)}$$

$$+ (\hat{K})\,K\left(E_0 - \frac{cI_0a}{\omega_0-\omega}\right)^2\frac{1}{\omega_0} = (-\hat{K})\,\frac{cI_0E_0}{(\omega_0-\omega)^2}\,(1-\delta).$$

Solving for a we find

$$a = \frac{(1-\delta)}{2K}. \tag{48}$$

Then from eq. 42 we have

$$I_0 = \frac{E_0 K}{c(1-\delta)}(\omega_0 - 2\omega).$$

Finally, substituting for a and I_0 our results are for $z > 0$ and $kr << 1$

$$\begin{aligned}\mathbf{E}_{\text{tot}} = &\frac{E_0}{2}\left(\frac{\omega_0}{\omega_0-\omega}\right)\left[\hat{i}\sin\omega_0\left(t-\frac{z}{v}\right)+\hat{j}\cos\omega_0\left(t-\frac{z}{v}\right)\right]\\ &-\frac{E_0}{2}\left(\frac{\omega_0-2\omega}{\omega_0-\omega}\right)\left\{-\hat{i}\sin\left[(\omega_0-2\omega)\left(t-\frac{z}{v}\right)\right]+\hat{j}\cos\left[(\omega_0-2\omega)\left(t-\frac{z}{v}\right)\right]\right\},\end{aligned} \tag{49a}$$

and for $z < 0$, $kr << 1$

$$\begin{aligned}\mathbf{E}_{\text{rad}} = &-\frac{E_0}{2}\left(\frac{\omega_0-2\omega}{\omega_0-\omega}\right)\left[\hat{i}\sin\omega_0\left(t+\frac{z}{v}\right)+\hat{j}\cos\omega_0\left(t+\frac{z}{v}\right)\right]\\ &-\frac{E_0}{2}\left(\frac{\omega_0-2\omega}{\omega_0-\omega}\right)\left\{-\hat{i}\sin\left[(\omega_0-2\omega)\left(t+\frac{z}{v}\right)\right]+\hat{j}\cos\left[(\omega_0-2\omega)\left(t+\frac{z}{v}\right)\right]\right\}.\end{aligned} \tag{49b}$$

Physically, the transmitted radiation, i.e., for $z > 0$, consists of two components of equal amplitude (recall $\omega << \omega_0$), one at frequency ω_0 which is circularly polarized in the same sense as that of the incident wave, and one at a frequency $\omega_0 - 2\omega$ which is circularly polarized in the opposite sense to that of the incident wave. The reflected radiation, i.e., for $z < 0$, also consists of two components of equal amplitude. The component at ω_0 is polarized in the opposite sense to that of the incident wave, and the component at $\omega_0 - 2\omega$ is polarized in the same sense as the incident wave.

Finally, the time averaged torque on the dipole is

$$\bar{\tau}_t = (-\hat{K})\frac{K(\omega_0-2\omega)(E_0)^2}{(\omega_0-\omega)^2} \approx (-\hat{K})\frac{K(E_0)}{\omega_0} = (-\hat{K})\frac{(\text{power})_{\text{in}}}{\omega_0}. \tag{50}$$

If the dipole is turned (for example by a motor) in the sense opposite to that of the incident wave, ω is replaced by $-\omega$ in the above equations. In this case, one gets frequency-shifted radiation at $\omega_0 + 2\omega$.

If we set ω equal to zero, which corresponds physically to fixing the dipole along the y-axis, we see that in Eq. 48a and 48b for $z > 0$

$$\mathbf{E}_{\text{tot}} = \hat{i}E_0\sin\omega_0\left(t-\frac{z}{v}\right), \tag{51a}$$

and for $z < 0$

$$\mathbf{E}_{\text{rad}} = -E_0\,\hat{j}\cos\omega_0\left(t+\frac{z}{v}\right). \tag{51b}$$

That is, only the radiation perpendicular to the dipole is transmitted, while the radiation parallel to the dipole is completely reflected.

THE REFLECTION CASE

Let us now consider the same problem as before, except that the waveguide is terminated by a perfectly conducting plate at a distance $\lambda_g/4$ behind the dipole, where λ_g is the waveguide wavelength. In this case the field acting on the dipole will be the incident field plus a component which has been reflected by the termination. Hence we might expect the current to be different; although it turns out that this is not the case.

We may analyze the situation in the following way. Suppose the waveguide is not shorted. By Eq. 38a the transmitted wave for $z > 0$ and $kr << 1$ is

$$\mathbf{E}_{\text{tot}} = \left(E_0 - \frac{cI_0 a}{\omega_0 - \omega}\right)\left[\hat{i}\sin\omega_0\left(t - \frac{z}{v}\right) + \hat{j}\cos\omega_0\left(t - \frac{z}{v}\right)\right]$$
$$- \frac{cI_0 a}{\omega_0 - \omega}\left\{-\hat{i}\sin\left[(\omega_0 - 2\omega)\left(t - \frac{z}{v}\right)\right] + \hat{j}\cos\left[(\omega_0 - 2\omega)\left(t - \frac{z}{v}\right)\right]\right\},$$

where I_0 and a are now to be considered as unknown. Suppose that the waveguide is shorted, then the transmitted wave will undergo a 180° phase change at the short and propagate back toward the dipole. When this reflected wave reaches the dipole, the total field acting on the dipole for $kr << 1$ will be

$$\mathbf{E}(z = 0, t) = \mathbf{E}_{\text{inc}}(z = 0, t) - \mathbf{E}_{\text{tot}}\left(z = 0, t - \frac{\lambda_g}{2v}\right) \tag{52}$$

or

$$\mathbf{E}(z = 0, t) = E_0(\hat{i}\sin\omega_0 t + \hat{j}\cos\omega_0 t)$$
$$- \left(E_0 - \frac{cI_0 a}{\omega_0 - \omega}\right)\left[\hat{i}\sin\omega_0\left(t - \frac{\lambda_g}{2v}\right) + \hat{j}\cos\omega_0\left(t - \frac{\lambda_g}{2v}\right)\right]$$
$$+ \frac{cI_0 a}{\omega_0 - \omega}\left\{-\hat{i}\sin\left[(\omega_0 - 2\omega)\left(t - \frac{\lambda_g}{2v}\right)\right] + \hat{j}\cos\left[(\omega_0 - 2\omega)\left(t - \frac{\lambda_g}{2v}\right)\right]\right\}.$$

We know $-\omega_0\lambda_g/2v = -\pi$ and $(\omega_0 - 2\omega)[-(\lambda_g/2v)] = -\pi + 2\pi\omega/\omega_0 \approx -\pi$. Hence,

$$\mathbf{E}(z = 0, t) = \left(2E_0 - \frac{cI_0 a}{\omega_0 - \omega}\right)(\hat{i}\sin\omega_0 t + \hat{j}\cos\omega_0 t)$$
$$- \frac{cI_0 a}{\omega_0 - \omega}\left[-\hat{i}\sin(\omega_0 - 2\omega)t + \hat{j}\cos(\omega_0 - 2\omega)t\right]. \tag{53}$$

For $kr << 1$, the E field along the dipole is

$$\mathbf{E}(z = 0, t)\cdot\hat{n} = \mathbf{E}(z = 0, t)\cdot(\hat{i}\sin\omega t + \hat{j}\cos\omega t)$$
$$= \left(2E_0 - \frac{cI_0 a}{\omega_0 - \omega}\right)(\sin\omega_0 t\sin\omega t + \cos\omega_0 t\cos\omega t)$$
$$- \frac{cI_0 a}{\omega_0 - \omega}\left[-\sin\omega t\sin(\omega_0 - 2\omega)t + \cos\omega t\cos(\omega_0 - 2\omega)t\right] \tag{54}$$
$$= 2\left(E_0 - \frac{cI_0 a}{\omega_0 - \omega}\right)\left[\cos(\omega_0 t - \omega t)\right].$$

Hence again the current in the dipole is of the form

$$I = I_0 \cos\left(\frac{\omega_0 - \omega}{c}\right) S \cos(\omega_0 t - \omega t).$$

Note that although a CP wave at $\omega_0 - 2\omega$ is acting on the dipole, the current in the dipole oscillates at $\omega_0 - \omega$.

Torque on Dipole (Reflection Case)

We now consider the torques acting on the dipole in the reflection case. The E field in cylindrical coordinates for $kr \ll 1$ is

$$\begin{aligned}\mathbf{E}(z=0,t) = &\left(2E_0 - \frac{cI_0 a}{\omega_0 - \omega}\right)[\hat{r}\sin(\omega_0 t + \phi) + \hat{\phi}\cos(\omega_0 t + \phi)] \\ &- \frac{cI_0 a}{\omega_0 - \omega}\{-\hat{r}\sin[(\omega_0 - 2\omega)t - \phi] + \hat{\phi}\cos[(\omega_0 - 2\omega)t - \phi]\}.\end{aligned} \tag{55}$$

At the dipole, i.e., when $\phi = \pi/2 - \omega t$ for $kr \ll 1$, for $s > 0$

$$\mathbf{E}(z=0,t) = \left(2E_0 - \frac{cI_0 a}{\omega_0 - \omega}\right)[\hat{r}\cos(\omega_0 t - \omega t)] - 2E_0\,\hat{\phi}\sin(\omega_0 t - \omega t) \tag{56a}$$

and for $s < 0$

$$\mathbf{E}(z=0,t) = -\left(2E_0 - \frac{cI_0 a}{\omega_0 - \omega}\right)\hat{r}\cos(\omega_0 t - \omega t) + 2E_0\,\hat{\phi}\sin(\omega_0 t - \omega t). \tag{56b}$$

Referring back to our derivation of the torque due to the induced dipole moment, Eq. 19, we can see that

$$\bar{\tau} = (-\hat{K})\frac{2E_0 I_0}{c\left(\frac{\omega_0 - \omega}{c}\right)^2}\left\{1 - \frac{9}{8}\left(\frac{k}{\frac{\omega_0 - \omega}{c}}\right)^2\left[\left(\frac{\pi}{2}\right)^2 - 2\right] + \frac{25}{192}\left(\frac{k}{\frac{\omega_0 - \omega}{c}}\right)^4\left[\left(\frac{\pi}{2}\right)^4 - 12\left(\frac{\pi}{2}\right)^2 + 24\right] + \ldots\right\}. \tag{57}$$

This is the torque due to the induced dipole moment in the reflection case.

Similarly, the torque due to the interaction between the induced current and the longitudinal B-field is

$$\bar{\tau} = (-\hat{K})\frac{2E_0 I_0}{c\left(\frac{\omega_0 - \omega}{c}\right)^2}\left\{\left(\frac{k}{\frac{\omega_0 - \omega}{c}}\right)^2\left[\left(\frac{\pi}{2}\right)^2 - 2\right] - \frac{1}{8}\left(\frac{k}{\frac{\omega_0 - \omega}{c}}\right)^4\left[\left(\frac{\pi}{2}\right)^4 - 12\left(\frac{\pi}{2}\right)^2 + 24\right] + \ldots\right\}. \tag{58}$$

The total torque will be

$$\bar{\tau}_t = (-\hat{K})\frac{2E_0 I_0}{c\left(\frac{\omega_0 - \omega}{c}\right)^2}\left\{1 - \frac{1}{8}\left(\frac{k}{\frac{\omega_0 - \omega}{c}}\right)^2\left[\left(\frac{\pi}{2}\right)^2 - 2\right] + \frac{1}{8(24)}\left(\frac{k}{\frac{\omega_0 - \omega}{c}}\right)^4\left[\left(\frac{\pi}{2}\right)^4 - 12\left(\frac{\pi}{2}\right)^2 + 24\right] + \ldots\right\}. \tag{59}$$

We now consider the radiation field which propagates in the $-z$ direction. For $z < 0$, this field will consist of the field radiated by the dipole plus the field reflected by the short. For $z < 0$ and $kr << 1$,

$$\begin{aligned}\mathbf{E}_{\text{refl}} = &\left(E_0 - \frac{cI_0 a}{\omega_0 - \omega}\right)\left[\hat{i}\sin\omega_0\left(t+\frac{z}{v}\right)+\hat{j}\cos\omega_0\left(t+\frac{z}{v}\right)\right]\\ &-\frac{cI_0 a}{\omega_0-\omega}\left[-\hat{i}\sin(\omega_0-2\omega)\left(t+\frac{z}{v}\right)+\hat{j}\cos(\omega_0-2\omega)\left(t+\frac{z}{v}\right)\right].\end{aligned} \tag{60}$$

The radiated field is

$$\begin{aligned}\mathbf{E}_{\text{rad}} = &-\frac{cI_0 a}{\omega_0-\omega}\left[\hat{i}\sin\omega_0\left(t+\frac{z}{v}\right)+\hat{j}\cos\omega_0\left(t+\frac{z}{v}\right)\right]\\ &-\frac{cI_0 a}{\omega_0-\omega}\left\{-\hat{i}\sin\left[(\omega_0-2\omega)\left(t+\frac{z}{v}\right)\right]+\hat{j}\cos\left[(\omega_0-2\omega)\left(t+\frac{z}{v}\right)\right]\right\}.\end{aligned} \tag{61}$$

The total field is

$$\begin{aligned}\mathbf{E}_{\text{tot}} &= \mathbf{E}_{\text{refl}} + \mathbf{E}_{\text{rad}}\\ &= \left(E_0 - \frac{2cI_0 a}{\omega_0-\omega}\right)\left[\hat{i}\sin\omega_0\left(t+\frac{z}{v}\right)+\hat{j}\cos\omega_0\left(t+\frac{z}{v}\right)\right]\\ &\quad -\frac{2cI_0 a}{\omega_0-\omega}\left[-\hat{i}\sin(\omega_0-2\omega)\left(t+\frac{z}{v}\right)+\hat{j}\cos(\omega_0-2\omega)\left(t+\frac{z}{v}\right)\right].\end{aligned} \tag{62}$$

We can now determine I_0 and a by applying conservation of energy and angular momentum. Consider a volume defined by a cross section A of the waveguide at some point $z < 0$, the waveguide walls, and the terminating wall. No energy or angular momentum passes through the waveguide walls or the short. Energy and angular momentum is transported only through the cross section A.

Conservation of energy implies

$$(\text{power})_{\text{in}} - (\text{power})_{\text{out}} = |\bar{\tau}|\omega \tag{63a}$$

$$(\text{power})_{\text{in}} = K(E_0)^2 \tag{63b}$$

$$(\text{power})_{\text{out}} = K\left(E_0 - \frac{2cI_0 a}{\omega_0-\omega}\right)^2 + 4K\left(\frac{cI_0 a}{\omega_0-\omega}\right)^2 \tag{63c}$$

Substituting the value for $|\bar{\tau}_t|$ from Eq. 26 we find

$$|\bar{\tau}_t|\omega = \frac{2\omega c E_0 I_0}{(\omega_0-\omega)^2}\,(1-\delta). \tag{64}$$

Conservation of angular momentum implies

$$(\mathbf{L})_{\text{in}} - (\mathbf{L})_{\text{out}} = \tau$$

and since

$$(\mathbf{L})_{\text{in}} = (-\hat{K})\frac{K(E_0)^2}{\omega_0}$$

and

$$(\mathbf{L})_{\text{out}} = (-\hat{K})\frac{K}{\omega_0}\left(E_0 - \frac{2cI_0a}{\omega_0-\omega}\right)^2 + \hat{K}K\left(\frac{2cI_0a}{\omega_0-\omega}\right)^2\frac{1}{\omega_0-2\omega} \tag{65}$$

and

$$\bar{\tau}_t = (-\hat{K})\frac{2cI_0E_0}{(\omega_0-\omega)^2}(1-\delta).$$

Solving the conservation equations for a and I_0 we find

$$a = \frac{(1-\delta)}{2K} \tag{66}$$

and

$$I_0 = \frac{KE_0(\omega_0-2\omega)}{c(1-\delta)}. \tag{67}$$

Our results for $z < 0$ and for $kr << 1$ are then

$$\mathbf{E}_{\text{tot}} = \frac{\omega}{\omega_0-\omega}E_0\left[\hat{i}\sin\omega_0\left(t+\frac{z}{v}\right)+\hat{j}\cos\omega_0\left(t+\frac{z}{v}\right)\right]$$
$$-E_0\frac{(\omega_0-2\omega)}{(\omega_0-\omega)}\left[-\hat{i}\sin(\omega_0-2\omega)\left(t+\frac{z}{v}\right)+\hat{j}\cos(\omega_0-2\omega)\left(t+\frac{z}{v}\right)\right]. \tag{68}$$

Since $\omega/(\omega_0-\omega) \approx 10^{-9}$, practically all the radiation is frequency shifted. The frequency shifted component is polarized in the same sense as the incident field. The torque is

$$\bar{\tau}_t = (-\hat{K})\,2K(E_0)^2\frac{(\omega_0-2\omega)}{(\omega_0-\omega)^2} \approx (-\hat{K})2\frac{(\text{power})_{\text{in}}}{\omega_0}. \tag{69}$$

We note that this torque is twice the torque for the open-ended waveguide case.

PHOTONS DISCUSSION

Using these results, it will now be shown that the number of photons going into the waveguide is the same as the number coming out (within the validity of the approximations which have been made). The number of photons entering the waveguide is

$$n^{\text{in}} = \frac{(\text{power})_{\text{in}}}{v\hbar\omega_0 A} = \frac{K(E_0)^2}{v\hbar\omega_0 A} \tag{70}$$

where v is the velocity of the photons in the waveguide, $\hbar$ is the Planck's constant, and A is the cross-sectional area of waveguide. The number of photons of frequency ω_0 leaving the waveguide is

$$n_{\omega_0}^{\text{out}} = \frac{K(E_0)^2}{\nu\hbar\omega_0 A} \frac{\omega^2}{(\omega_0 - \omega)^2} . \tag{71}$$

At $\omega_0 - 2\omega$,

$$n_{\omega_0 - 2\omega}^{\text{out}} = \frac{K(E_0)^2}{(\omega_0 - 2\omega)\nu\hbar A} \frac{(\omega_0 - 2\omega)^2}{(\omega_0 - \omega)^2} \tag{72}$$

and since

$$n^{\text{out}} = n_{\omega_0}^{\text{out}} + n_{\omega_0 - 2\omega}^{\text{out}} = \frac{K(E_0)^2}{\nu\hbar A} \left[\frac{(\omega_0 - 2\omega)^2}{(\omega_0 - 2\omega)(\omega_0 - \omega)^2} + \frac{\omega^2}{\omega_0(\omega_0 - \omega)^2} \right] = \frac{K(E_0)^2}{\nu\hbar A} \frac{1}{\omega_0}, \tag{73}$$

we find

$$n^{\text{out}} = n^{\text{in}}. \tag{74}$$

SUMMARY

We have considered a mathematical model which describes the effects of a circularly-polarized wave incident upon a freely rotating half-wave dipole in a waveguide. We have found that the dipole will be subjected to a torque which arises from two distinct effects. One part of the torque is due to the interaction of the induced electric dipole moment with the transverse electric field; and the remainder arises from the interaction between the induced current and the longitudinal magnetic field in the waveguide. This torque turns the dipole in the direction of rotation of the incident electric vector. If the waveguide is terminated in a matched load, the torque is

$$|\bar{\tau}_t| = \frac{(\text{power})_{\text{in}}}{\omega_0}$$

where ω_0 is the frequency of the incident wave. If the waveguide is terminated with a perfectly conducting metal plate, the torque is just twice that of the open-ended case.*

In the open-ended waveguide case, one half of the transmitted power is downshifted in frequency by twice the rotation rate and is polarized in the opposite sense to that of the incident CP wave. The remainder is unshifted and is polarized in the same sense as the incident wave. One half of the reflected power is downshifted in frequency by twice the rotation rate and is circularly polarized in the same sense as the incident wave. The other half of the reflected power is unshifted and polarized in the opposite sense to that of the input power.

If the waveguide is terminated with a conducting plate, all but a very small amount of the reflected power is downshifted in frequency by twice the rotation rate. The unshifted power is probably experimentally undetectable, since it is down by a factor of 10^{-18} as compared to the shifted component, for dipole rotation rates on the order of 10 cps.

We have shown also that the number of photons entering the waveguide is the same as the number leaving.

Finally, we have shown that energy and angular momentum are conserved in all cases considered.

*The descrepency between this conclusion and experimental results [1], in which a ratio of 2.8:1 was obtained, has not yet been satisfactorily explained.

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A MATHEMATICAL MODEL OF A ROTATIONAL FREQUENCY TRANSLATION DEVICE

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This is an interim report; work on the problem is continuing.

5. AUTHOR(S) *(First name, middle initial, last name)*

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13. ABSTRACT

An analysis was made which provides a theoretical explanation of a unique radiation torque experiment performed by Allen, in which circularly polarized (cp) microwave power was used to impart continuous rotary motion to an object supported within a circular waveguide. Under the action of the cp wave, an oscillating current was set up in a rod dipole resulting in an oscillating dipole moment. Part of the energy scattered by the rotating dipole is frequency shifted. Torques arise from the interactions between the E-field of the wave and the induced dipole moment, and between the oscillating current in the dipole and the longitudinal H-field in the waveguide. The magnitudes of these torques were derived for two cases of interest; the open waveguide and the shorted waveguide. The magnitudes of the spectral components of the radiation scattered by the dipole were also derived along with their characteristic polarizations. A cursory analysis shows that photons are conserved in such an experiment.

DD FORM 1 NOV 65 1473 (PAGE 1)

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DD FORM 1 NOV 65 1473 (BACK)
(PAGE 2)

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