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Discrete Finite Fourier Transforms: a Tutorial Approach

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ABSTRACT

A discrete, finite time series of N points

$$f_n, \quad n = 0, 1, 2, \ldots, N-1,$$

often arises, for example, in the sampling of continuous data. This time series can be expressed as the finite series

$$f_n = \sum_{k=0}^{N-1} F_k e^{2\pi i kn/N} \quad n = 0, 1, 2, \ldots, N-1,$$

where the coefficients F_k, to be interpreted as frequency values, are given by

$$F_k = \frac{1}{N} \sum_{n=0}^{N-1} f_n e^{-2\pi i kn/N} \quad k = 0, 1, 2, \ldots, N-1.$$

Experimental errors inherent in f_n will, of course, be reflected in F_k. Given the original time series f_n, however, it is exactly (i.e., without approximation) represented by F_k at the N points of interest.

These formulas are derived from basic principles, and their relationship to digital analysis of data sampled from continuous signals is discussed. In particular, the efficient use of the recently popular algorithms for rapid machine computation of Fourier transforms is explained. Although the results obtained are for the most part well known in the literature, it is felt that the importance of these transforms warrants a complete and notationally consistent exposition.

PROBLEM STATUS

An interim report; work is continuing.

AUTHORIZATION

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DISCRETE FINITE FOURIER TRANSFORMS: A TUTORIAL APPROACH

INTRODUCTION

Fourier transforms play an important role in the analysis of experimental data as, for example, in the computation of power spectra (1). If the computation is done digitally it is natural to consider the summation involved as an approximation to the integral of a continuous Fourier transform. This is not the case. The data are inevitably of finite extent. If they are given as discrete finite time series, as are those obtained by sampling continuous data, no further approximations need be made in the computation of a discrete finite Fourier transform. The relationship of this transform, considered as a discrete finite frequency series, to the frequency spectrum of the original continuous data is to be examined.

The derivation given here of the discrete transform pair follows that of Hamming (2). It may be considered as a guided tour or interpretation of the pertinent portions of Hamming's work, with complex exponentials replacing the sinusoidal functions used there. The transformation is based on the orthogonality of the N distinct Nth roots of unity over the set of N points. This orthogonality can be established in many ways. The discrete finite difference and summation calculus is used here in the belief that it provides a useful insight into the meaning of the equations. We shall need only the rudiments of this calculus; they are given here for convenience and completeness.

THE DISCRETE FINITE DIFFERENCE AND SUMMATION CALCULUS

The discrete finite difference calculus is analogous to the differential calculus but lacks the subtlety of the latter, since it does not require the concept of limits which plagues the student of elementary calculus. Indeed, the difference operator, to be defined below, is identical to the first steps in the application of the usual definition of the derivative.

In general the difference operator Δ and the basic operation of differencing applied to a function $f(x)$ is defined as

$$\Delta f(x) \stackrel{\Delta}{=} f(x+h) - f(x),$$

where the symbol $\stackrel{\Delta}{=}$ means "is equal by definition to" and h is the increment of the independent variable x. We will be concerned with discrete series and with an index (the independent variable) changing in increments of unity, and we will therefore let $h = 1$. Thus

$$\Delta f(x) = f(x+1) - f(x). \tag{1}$$

If we let $f(x) = \exp(\gamma x)$, where γ is any constant, and apply the difference operator, we get

$$\Delta e^{\gamma x} = e^{\gamma(x+1)} - e^{\gamma x} = e^{\gamma x}(e^{\gamma} - 1),$$

or, if $e^{\gamma} \neq 1$

$$e^{\gamma x} = \frac{\Delta e^{\gamma x}}{e^{\gamma} - 1} . \tag{2}$$

This result will be used presently.

The summation calculus, similarly, is analogous to integral calculus but lacks the subtlety. As with the difference calculus we need only the basic results.

Let x be a discrete variable assuming integral values only. Using Eq. (1) we have

$$\begin{aligned}\sum_{x=a}^{b-1} \Delta f(x) &= \sum_{x=a}^{b-1} [f(x+1) - f(x)] \\ &= f(a+1) - f(a) + f(a+2) - f(a+1) + \ldots + f(b) - f(b-1) \\ &= f(b) - f(a).\end{aligned}$$

Thus, by analogy to the fundamental theorem of calculus in the form

$$\int_a^b \frac{df(x)}{dx}\, dx = f(b) - f(a),$$

we have what may be termed the "fundamental theorem of discrete calculus":

$$\sum_{x=a}^{b-1} \Delta f(x) = f(b) - f(a). \tag{3}$$

If Eq. (2) is summed over N discrete values of the parameter, we have

$$\sum_{x=0}^{N-1} e^{\gamma x} = \sum_{x=0}^{N-1} \frac{\Delta e^{\gamma x}}{e^{\gamma} - 1} = \frac{1}{e^{\gamma} - 1} \sum_{x=0}^{N-1} \Delta e^{\gamma x} .$$

We now apply the "fundamental theorem," Eq. (3) with $a = 0$ and $b = N$ to get

$$\sum_{x=0}^{N-1} e^{\gamma x} = \frac{1}{e^{\gamma} - 1} \left(e^{\gamma N} - e^{\gamma 0} \right) = \frac{e^{\gamma N} - 1}{e^{\gamma} - 1} . \tag{4}$$

This is, of course, simply the sum of a geometric progression of N terms,

$$1 + r + r^2 + \ldots + r^{N-1} = \frac{1 - r^N}{1 - r} ,$$

with $r = e^{\gamma}$. This sum could have been used as a starting point. To have done so, however, would have partially obscured the relationship between the discrete transforms to be developed and continuous Fourier transforms.

ORTHOGONALITY OF THE NTH ROOTS OF UNITY

With Eq. (4) and the fact that

$$\exp(2n\pi i) = \cos 2n\pi + i \sin 2n\pi = 1$$

for any integer n, we wish to establish the orthogonality of the N distinct Nth roots of unity of the form $\exp(2\pi ki/N)$.†

In the sequel the indices k, n, m, etc. will be restricted to the N integral values 0, 1, 2, ..., $N-1$. However, to enable us to write the equations in complete generality (to include $k \geq N$) we need the concept of congruence modulo N (3). We say that $k \equiv m$ (mod N) (to be read k is congruent to m, modulo N) if and only if $k - m = jN$ for some integer $j = \pm 1, \pm 2, \ldots$. This is an extension of the equality concept, since if $j = 0$ we have $k = m$. It is useful because if $k \equiv m$ (mod N), $\exp[2\pi i(k-m)/N] = \exp(2j\pi i) = 1$ for $j = 0, 1, 2, \ldots$. Thus if either $k = m$ or, more generally, if $k \equiv m$ (mod N) we can write

$$\exp[2\pi i(k-m)x/N] = \exp(2\pi i j x) = 1^x = 1$$

for all integral values of x, and

$$\sum_{x=0}^{N-1} e^{2\pi i(k-m)x/N} = \sum_{x=0}^{N-1} 1^x = N, \quad k = m, \text{ or } k \equiv m \pmod{N}. \tag{5a}$$

If $k \neq m$ and $k \not\equiv m$ (mod N) we let $\gamma = 2\pi i(k-m)/N \neq 0$ in Eq. (4) to get

$$\sum_{x=0}^{N-1} e^{\gamma x} = \sum_{x=0}^{N-1} e^{2\pi i(k-m)x/N} = \frac{e^{2\pi i(k-m)} - 1}{e^{2\pi i(k-m)/N} - 1} = 0, \tag{5b}$$

since $k - m$ is an integer while $(k-m)/N$ is not.

Equations (5a) and (5b) can be combined to give the desired orthogonality of the N distinct Nth roots of unity over the set of N points in the succinct notation

$$\sum_{x=0}^{N-1} e^{2\pi i kx/N} e^{(2\pi i mx/N)^*} = N\delta_{km} \pmod{N}, \tag{5c}$$

where

$$\delta_{km} \pmod{N} = \begin{cases} 1, & \text{if } k - m = jN, \text{ for some } j = 0, \pm 1, \pm 2, \ldots \\ 0, & \text{if } k - m \neq jN, \text{ for any } j = 0, \pm 1, \pm 2, \ldots, \end{cases}$$

and the asterisk denotes the complex conjugate.

THE DISCRETE FINITE FOURIER TRANSFORM

Equation (5c) enables us to represent any finite discrete function (e.g., time series) of N points in terms of the orthogonal Nth roots of unity as a basis. Thus, given an arbitrary discrete finite time series of N points

$$f_n, \quad n = 0, 1, 2, \ldots, N-1,$$

let us assume that f_n can be expressed as the finite series

† $\exp(2\pi ki/N)$ is an Nth root of unity for $k = 0, 1, 2, \ldots, N-1$, since $[\exp(2\pi ki/N)]^N = \exp(2\pi ki) = 1$. Of course, k can take on additional integral values, but these will not yield distinct roots.

$$f_n = \sum_{k=0}^{N-1} F_k e^{2\pi i k n/N}, \quad n = 0, 1, 2, \ldots, N-1. \tag{6}$$

For this to be the desired representation we must determine the N coefficients F_k such that this series is exactly equal to the original function at the N given points. (The value of Eq. (6) for n not equal to one of the N points is, as yet, undefined.)

Multiplying both sides of Eq. (6) by $\exp(-2\pi i m n/N)$ with $0 \leq m \leq N-1$ and summing over n, we get, with the aid of Eq. (5c),

$$\sum_{n=0}^{N-1} f_n e^{-2\pi i m n/N} = \sum_{k=0}^{N-1} F_k \sum_{n=0}^{N-1} e^{2\pi i (k-m) n/N}$$

$$= \sum_{k=0}^{N-1} F_k N \delta_{km} = N F_m,$$

where the fact that these sums are finite permits us to interchange the order of summation.

We have, therefore,

$$F_k = \frac{1}{N} \sum_{n=0}^{N-1} f_n e^{-2\pi i k n/N}, \quad k = 0, 1, 2, \ldots, N-1. \tag{7}$$

It remains to be shown that the assumed series, Eq. (6), with coefficients given by Eq. (7) does in fact exactly represent the original function f_n for $n = 0, 1, 2, \ldots, N-1$. We can, of course, rewrite Eq. (7) as

$$F_k = \frac{1}{N} \sum_{j=0}^{N-1} f_j e^{-2\pi i k j/N}, \quad k = 0, 1, 2, \ldots, N-1. \tag{8}$$

Substituting Eq. (3) into Eq. (6) we get

$$f_n = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} f_j e^{-2\pi i k j/N} e^{2\pi i k n/N}$$

$$= \frac{1}{N} \sum_{j=0}^{N-1} f_j \sum_{k=0}^{N-1} e^{2\pi i (n-j) k/N}$$

Making use of Eq. (5c) this becomes

$$f_n = \frac{1}{N} \sum_{j=0}^{N-1} f_j N \delta_{nj} \pmod{N} = f_n,$$

which was to be verified.

Equations (6) and (7) together constitute the discrete finite Fourier transform pair

$$f_n = \sum_{k=0}^{N-1} F_k e^{2\pi i k n/N}, \quad n = 0, 1, 2, \ldots, N-1$$

and (9)

$$F_k = \frac{1}{N} \sum_{n=0}^{N-1} f_n e^{-2\pi i k n/N}, \quad k = 0, 1, 2, \ldots, N-1.$$

Notice that this derivation is exact in the sense that these sums are not approximations to integrals. Given the original time series f_n it is exactly represented by the transform F_k at the N points of interest. Naturally, if there are experimental errors in the time series f_n, as will arise from sampling errors (4), these will not be eliminated but will be reflected in the F_k.

DIGITAL ANALYSIS OF CONTINUOUS SIGNALS

Let us assume that the discrete finite time series f_n has been obtained by sampling a continuous time function $f(t)$. We assume further that $f(t)$ is band-limited. In other words, if $F(\omega)$ is the Fourier transform of $f(t)$:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt ,$$

there exists $W > 0$ such that

$$F(\omega) = 0 \text{ for } |\omega| > \Omega = 2\pi W;$$

or, in words, the spectrum of $f(t)$ vanishes outside the frequency band $-W \leq f \leq W$, where $\omega = 2\pi f$.

If the sampling rate is greater than $2W$ samples per second (with W given in hertz) the continuous function $f(t)$ is completely determined for all t by the sequence $\{f(n/2W)\}$. In fact, Shannon's sampling theorem (4-7):

$$f(t) = \sum_{-\infty}^{\infty} f\left(\frac{n}{2W}\right) \frac{\sin 2\pi W[t - (n/2W)]}{2\pi W[t - (n/2W)]} , \tag{10}$$

yields $f(t)$ explicitly in terms of the $f(n/2W)$.

As Eq. (10) indicates, an infinite number of samples are required for an exact representation of $f(t)$. If $f(t)$ is small outside an interval of T seconds, however, and is sampled at a rate of $2W$ samples per second for these T seconds, there are $N = 2WT$ samples in the range in which the function has appreciable value.

The function $f(t)$ cannot be both time-limited and band-limited, but it is possible to keep the spectrum within the band W and to have the time function sufficiently small outside the interval T so that $f(t)$ can be satisfactorily approximated by the $N = 2WT$ samples. We will assume that this is the case. The rapid attenuation of the sampling function (6)

$$\operatorname{sinc} 2Wt = \frac{\sin 2\pi Wt}{2\pi Wt} \tag{11}$$

implies that the effect of any term in Eq. (10) will be of consequence only within a small number of intervals in the neighborhood of the corresponding sample point (7).

Thus we have the discrete time series

$$f_n = f(n/2W), \quad n = 0, 1, 2, \ldots, N-1 = 2WT-1,$$

where, approximately,

$$f(t) \sim \sum_{n=0}^{2WT-1} f_n \operatorname{sinc}(2Wt - n). \tag{12}$$

That the coefficients f_n are in fact equal to the sampled values may be seen from Eq. (12) by first changing the dummy index and then letting $t = n/2W$ as follows:

$$f\left(\frac{n}{2W}\right) = \sum_{j=0}^{N-1} f_j \frac{\sin(\pi)(n-j)}{\pi(n-j)} = \sum_{j=0}^{N-1} f_j \delta_{nj} = f_n, \tag{13}$$

where δ_{nj} is the Kronecker delta (unity when $n = j$ and zero when $n \neq j$).

We are interested, usually, not so much in $f(t)$ as in $F(\omega)$, its Fourier transform, or in its power spectrum, the Fourier transform of the autocorrelation function of $f(t)$ (1). We seek the relationship, if any, between the "spectral content" of f_n as indicated by its finite Fourier transform F_k and the "spectral content" of $f(t)$ as indicated by its (continuous) transform $F(\omega)$. In other words we want $F(\omega)$, but we can get F_k. How are they related?

Taking the (continuous) Fourier transform symbolized by $\mathfrak{F}\{\cdot\}$ of both sides of Eq. (12), we get

$$F(\omega) = \mathfrak{F}\{f(t)\} = \sum_{n=0}^{N-1} f_n \mathfrak{F}\{\operatorname{sinc}(2Wt - n)\}. \tag{14}$$

It is shown in Appendix A that

$$\mathfrak{F}\{\operatorname{sinc}(2Wt - n)\} = (1/2W)\, e^{-in\omega/2W} \operatorname{rect}(\omega/4\pi W), \tag{15a}$$

where (6)

$$\operatorname{rect}(\omega/4\pi W) \triangleq \begin{cases} 1, & \text{if } |\omega| < 2\pi W \\ 0, & \text{if } |\omega| > 2\pi W \end{cases} \tag{15b}$$

Using this result in Eq. (14) we get

$$F(\omega) = \sum_{n=0}^{N-1} f_n \left[\frac{1}{2W} e^{-in\omega/2W} \operatorname{rect}\left(\frac{\omega}{4\pi W}\right)\right]$$

or

$$F(\omega) = \begin{cases} \dfrac{1}{2W} \displaystyle\sum_{n=0}^{N-1} f_n e^{-in\omega/2W}, & \text{if } |\omega| < 2\pi W \\ 0, & \text{if } |\omega| > 2\pi W \end{cases}$$

In terms of the frequency $f = \omega/2\pi$ (not to be confused with the continuous function $f(t)$ or the discrete time series f_n) we have

$$F(f) = \frac{1}{2W} \sum_{n=0}^{N-1} f_n \, e^{-2\pi i f n/2W}, \quad \text{with } |f| < W. \tag{16}$$

Remembering that $N = 2WT$, if we let $f = k/T$ in Eq. (16) we have

$$F\left(\frac{k}{T}\right) = \frac{T}{N} \sum_{n=0}^{N-1} f_n \, e^{-2\pi i k n/N} = T F_k, \tag{17}$$

where F_k is given by Eq. (7).

Equations (13) and (17) are subject to simple interpretations. From Eq. (13)

$$f_n = f(n/2W) = f(nT/N).$$

That is, if t is an integral multiple of T/N ($t = nT/N$), then $f(t) = f_{(Nt/T)}$, which means that $f(t)$ is equal to the sample if t is the time at which the sample was obtained — a not too surprising result.

From Eq. (17) $F_k = (1/T)\, F(k/T)$, or

$$F(f) = T F_{Tf}$$

if f is an integral multiple of $1/T$ ($f = k/T$). That is, the continuous spectrum $F(f)$ is equal to T times the discrete finite Fourier transform of f_n if f is a harmonic of a fundamental frequency f_0 defined as the reciprocal of the total sampling duration:

$$f_0 = 1/T = 2W/N.$$

For example, if 1000 samples of a continuous time function having no frequency component greater than 1000 Hz are collected at a rate of 2000 samples per second, $T = 1/2$ sec. A digitally computed finite Fourier transform will then represent the "true" Fourier transform at multiples of 2 Hz. Intermediate values could be computed using Eq. (16). Resolution of 0.1 Hz without interpolation would require a data length of $T = 10$ sec.

The fact that $|f| < W = N/2T$ imposes a restriction on the discrete values F_k, namely,

$$k/T < N/2T, \text{ or } k < N/2.$$

Only the first $N/2$ coefficients obtained from Eq. (7) are distinct; the remaining ones can be obtained by symmetry. (It imposes no burden in practice to require that N be even, and we shall assume this henceforth.) If in Eq. (7) we replace k by $N-k$, we have

$$F_{N-k} = \frac{1}{N} \sum_{n=0}^{N-1} f_n \, e^{-2\pi i (N-k) n/N}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} f_n \, e^{2\pi i k n/N} = F_k^*. \tag{18}$$

if f_n is real. We have again made use of $\exp(\pm 2n\pi i) = 1$. Notice that though f_n is real, F_k is, in general, complex; $F_k = \mathrm{Re}\{F_k\} + i\,\mathrm{Im}\{F_k\}$.

In a conventional application of Eq. (7) one need compute only those coefficients F_k that are desired. Equation (18) may be used to obtain others. Efficient algorithms for the machine computation of Eq. (7) nicknamed "Fast Fourier Transforms" have recently become popular (8-11). The popularity of these algorithms is due to the appreciable saving in computer time that they afford. Straightforward application results in computation of all N coefficients, however. For real time series this is wasteful, as can be seen in Eq. (18). Bingham et al. (10) present two schemes for the efficient use of the fast algorithms. These will be explained in Appendix B.

APPENDIX A

FOURIER TRANSFORM OF THE SINC FUNCTION

The inverse Fourier transform, symbolized by $\mathcal{F}^{-1}\{\cdot\}$, is given by

$$\mathcal{F}^{-1}\{F(\omega)\} = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega .$$

In general,

$$\mathcal{F}^{-1}\{F(\omega)\, e^{-i\alpha\omega}\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i(t-\alpha)\omega}\, d\omega = f(t-\alpha) , \tag{A1}$$

when α is any constant.

Following Woodward (6) we have defined $\mathrm{rect}\,(\omega/4\pi W)$ in Eq. (15b). If we take the inverse Fourier transform of this function, we have

$$\mathcal{F}^{-1}\left\{\mathrm{rect}\ \frac{\omega}{4\pi W}\right\} = \frac{1}{2\pi}\int_{-2\pi W}^{2\pi W} e^{i\omega t}\, d\omega = \frac{\sin 2\pi W t}{\pi t} = 2W\ \mathrm{sinc}\ 2Wt , \tag{A2}$$

using Eq. (11).

Applying the results of Eq. (A1) with $\alpha = n/2W$ to Eq. (A2), we get

$$\mathcal{F}^{-1}\{\mathrm{rect}\,(\omega/4\pi W)\, e^{-in\omega/2W}\} = 2W\ \mathrm{sinc}\ (2Wt-n) .$$

Since $\mathcal{F}\{\mathcal{F}^{-1}[F(\omega)]\} = F(\omega)$, it follows that

$$\mathcal{F}\{\mathrm{sinc}(2Wt-n)\} = (1/2W)\ \exp(-in\omega/2W)\ \mathrm{rect}\ (\omega/4\pi W) ,$$

which is Eq. (15a).

APPENDIX B

EFFICIENT USE OF "FAST FOURIER TRANSFORMS"

Let $\{x_n\}$ and $\{y_n\}$ be real time series of N points each. From Eq. (9) the finite discrete Fourier transforms of these series are

$$X_k = \frac{1}{N}\sum_{n=0}^{N-1} x_n \exp(-2\pi i kn/N), \quad k = 0, 1, 2, \ldots, N-1 \tag{B1a}$$

and

$$Y_k = \frac{1}{N}\sum_{n=0}^{N-1} y_n \exp(-2\pi i kn/N), \quad k = 0, 1, 2, \ldots, N-1. \tag{B1b}$$

As in Eq. (18) we have

$$X_{n-k} = X_k^* \text{ and } Y_{n-k} = Y_k^*. \tag{B2}$$

As pointed out by Bingham et al. (10), efficient use of the fast computational algorithms requires computation of only the $N/2$ distinct coefficients for each series. They suggest combination of the two real series into one complex series, transforming the latter, and then separating the transform.

To do this we let $z_n = x_n + i y_n$ be a complex time series, where x_n and y_n are the original real series. Equation (9) is still applicable, but Eq. (18) is not. From Eqs. (9) and (B1) we have

$$\begin{aligned} Z_k &= \frac{1}{N}\sum_{n=0}^{N-1} z_n \exp(-2\pi i kn/N) \\ &= \frac{1}{N}\sum_{n=0}^{N-1} (x_n + i y_n) \exp(-2\pi i kn/N) \\ &= X_k + i Y_k, \quad k = 0, 1, 2, \ldots, N-1. \end{aligned} \tag{B3}$$

If in Eq. (B3) we replace k by $N-k$, we have

$$Z_{N-k} = X_{N-k} + i Y_{N-k} = X_k^* + i Y_k^*, \tag{B4}$$

where we have made use of Eq. (B2). Taking the complex conjugate of Eq. (B3), on the other hand, gives us

$$Z_k^* = X_k^* - i Y_k^*.$$

which shows that $Z_{N-k} \neq Z_k^*$.

By separating the real and imaginary parts of Eqs. (B3) and (B4), we obtain four equations in the four unknowns: $\mathrm{Re}\{X_k\}$, $\mathrm{Im}\{X_k\}$, $\mathrm{Re}\{Y_k\}$, $\mathrm{Im}\{Y_k\}$. From Eq. (B3) we have

$$\begin{aligned}\mathrm{Re}\{Z_k\} &= \mathrm{Re}\{X_k + i\,Y_k\}\\ &= \mathrm{Re}\{\mathrm{Re}[X_k] + i\,\mathrm{Im}[X_k] + i\,\mathrm{Re}[Y_k] - \mathrm{Im}[Y_k]\}\\ &= \mathrm{Re}\{X_k\} - \mathrm{Im}\{Y_k\}\end{aligned} \tag{B5}$$

and

$$\begin{aligned}\mathrm{Im}\{Z_k\} &= \mathrm{Im}\{X_k + i\,Y_k\}\\ &= \mathrm{Im}\{\mathrm{Re}[X_k] + i\,\mathrm{Im}[X_k] + i\,\mathrm{Re}[Y_k] - \mathrm{Im}[Y_k]\}\\ &= \mathrm{Im}\{X_k\} + \mathrm{Re}\{Y_k\}.\end{aligned} \tag{B6}$$

Similarly, from Eq. (B4) we have

$$\begin{aligned}\mathrm{Re}\{Z_{N-k}\} &= \mathrm{Re}\{X_k^* + i\,Y_k^*\}\\ &= \mathrm{Re}\{\mathrm{Re}[X_k] - i\,\mathrm{Im}[X_k] + i\,\mathrm{Re}[Y_k] + \mathrm{Im}[Y_k]\}\\ &= \mathrm{Re}\{X_k\} + \mathrm{Im}\{Y_k\}\end{aligned} \tag{B7}$$

and

$$\begin{aligned}\mathrm{Im}\{Z_{N-k}\} &= \mathrm{Im}\{X_k^* + i\,Y_k^*\}\\ &= \mathrm{Im}\{\mathrm{Re}[X_k] - i\,\mathrm{Im}[X_k] + i\,\mathrm{Re}[Y_k] + \mathrm{Im}[Y_k]\}\\ &= -\mathrm{Im}\{X_k\} + \mathrm{Re}\{Y_k\}.\end{aligned} \tag{B8}$$

Adding Eqs. (B5) and (B7) we get

$$\mathrm{Re}\{X_k\} = \mathrm{Re}\{(Z_k + Z_{N-k})/2\}. \tag{B9a}$$

Subtracting Eq. (B8) from Eq. (B6) gives

$$\mathrm{Im}\{X_k\} = \mathrm{Im}\{(Z_k - Z_{N-k})/2\}. \tag{B9b}$$

Similarly we have

$$\mathrm{Re}\{Y_k\} = \mathrm{Im}\{(Z_k + Z_{N-k})/2\} \tag{B10a}$$

and

$$\mathrm{Im}\{Y_k\} = \mathrm{Re}\{(Z_{N-k} - Z_k)/2\}. \tag{B10b}$$

Thus X_k and Y_k are completely determined by Z_k and Z_{N-k}.

The preceding technique amounts to what might be termed a "one-cent sale" — two computations for (almost) the price of one. Bingham et al. (10), following Danielson and Lanczos (8), describe an additional technique which permits efficient computation for a single series. Only half as many Fourier coefficients as there are data points are computed, but more pre- and posttransform analysis is required. To use this method the given real series of length N is split into two series of length $N/2$ each. These two shorter

series are then combined into a single complex series of length $N/2$; the analysis is then similar to that of the previous technique.

The original series cannot simply be divided in half, however. If this is attempted, the resulting summations would extend from 0 to $(N/2) - 1$ and thus would not be in the form of Eq. (7). The subsequent analysis would not be applicable. Instead, we divide the given series $\{f_n\}$ into its odd and even terms, getting the two series

$$f_0, f_2, f_4, \ldots, f_{N-2} \tag{B11a}$$

and

$$f_1, f_3, f_5, \ldots, f_{N-1}. \tag{B11b}$$

For the even terms we can let $n = 2m$, and for the odd terms we can let $n = 2m + 1$. Then, if m ranges over the $N/2$ points 0, 1, 2, ..., $(N/2) - 1$, all the terms in the original series are present and accounted for in one of the two new series of Eq. (B11). Equation (7) can therefore be written in the form

$$\begin{aligned} F_k &= \frac{1}{N}\sum_{n=0}^{N-1} f_n e^{-2\pi i k n/N} \\ &= \frac{1}{N}\sum_{m=0}^{(N/2)-1} f_{2m} e^{-4\pi i k m/N} + \frac{1}{N}\sum_{m=0}^{(N/2)-1} f_{2m+1} e^{-2\pi i k(2m+1)/N} \\ &= \frac{1}{N}\sum_{m=0}^{(N/2)-1} f_{2m} e^{\frac{-2\pi i k m}{N/2}} + e^{-2\pi i k/N}\frac{1}{N}\sum_{m=0}^{(N/2)-1} f_{2m+1} e^{\frac{-2\pi i k m}{N/2}} . \end{aligned} \tag{B12}$$

We now relabel the terms in Eq. (B12) letting $f_{2m} = x_m$, $f_{2m+1} = y_m$, $M = N/2$, with $m = 0, 1, 2, \ldots, M - 1$. Equation (B12) becomes

$$\begin{aligned} F_k &= \frac{1}{2M}\sum_{m=0}^{M-1} x_m e^{-2\pi i k m/M} + \frac{1}{2} e^{-k\pi i/M}\frac{1}{M}\sum_{m=0}^{M-1} y_m e^{-2\pi i k m/M} \\ &= \frac{1}{2}\left[X_k + \exp(-k\pi i/M)\, Y_k\right], \end{aligned} \tag{B13}$$

where

$$X_k = \frac{1}{M}\sum_{m=0}^{M-1} x_m e^{-2\pi i k m/M} \qquad k = 0, 1, 2, \ldots, 2M - 1, \tag{B14a}$$

and

$$Y_k = \frac{1}{M}\sum_{m=0}^{M-1} y_m e^{-2\pi i k m/M} \qquad k = 0, 1, 2, \ldots, 2M - 1. \tag{B14b}$$

Equations (B14) are in the form of Eqs. (B1), but although N values of k are still required, only $M = N/2$ computational operations are required for each k. This saving is reflected in the computation of Z_k, Eq. (B3), and affects the analysis leading to Eqs. (B9) and (B10) only in the replacement of Z_{N-k} by $Z_{[(N/2)-k]}$. After computing Z_k, Eqs. (B9) and (B10) can be used to obtain X_k and Y_k; F_k is then obtained from Eq. (B13) as follows:

$$\mathrm{Re}\{F_k\} = \frac{1}{2}\left[\mathrm{Re}\{X_k\} + \cos\frac{\pi k}{M}\,\mathrm{Re}\{Y_k\} + \sin\frac{\pi k}{M}\,\mathrm{Im}\{Y_k\}\right] \tag{B15a}$$

and

$$\mathrm{Im}\{F_k\} = \frac{1}{2}\left[\mathrm{Im}\{X_k\} + \cos\frac{\pi k}{M}\,\mathrm{Im}\{Y_k\} - \sin\frac{\pi k}{M}\,\mathrm{Re}\{Y_k\}\right] \tag{B15b}$$

Finally, substitution of Eqs. (B9) and (B10) with N replaced by $N/2$ in Eq. (B15) yields, in terms of the N computed values of Z_k,

$$\mathrm{Re}\{F_k\} = \mathrm{Re}\{(Z_k + Z_{[(N/2)-k]})/2\} + \cos\frac{2\pi k}{N}\,\mathrm{Im}\{(Z_k - Z_{[(N/2)-k]})/2\}$$
$$+ \sin\frac{2\pi k}{N}\,\mathrm{Re}\{(Z_{[(N/2)-k]} - Z_k)/2\}$$

and

$$\mathrm{Im}\{F_k\} = \mathrm{Im}\{(Z_k + Z_{[(N/2)-k]})/2\} + \cos\frac{2\pi k}{N}\,\mathrm{Re}\{(Z_{[(N/2)-k]} - Z_k)/2\}$$
$$- \sin\frac{2\pi k}{N}\,\mathrm{Im}\{(Z_k + Z_{[(N/2)-k]})/2\}.$$

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13. ABSTRACT

A discrete, finite time series of N points

$$f_n \qquad n = 0, 1, 2, \ldots, N-1,$$

often arises, for example, in the sampling of continuous data. This time series can be expressed as the finite series

$$f_n = \sum_{k=0}^{N-1} F_k e^{2\pi i k n / N} \qquad n = 0, 1, 2, \ldots, N-1,$$

where the coefficients F_k, to be interpreted as frequency values, are given by

$$F_k = \frac{1}{N} \sum_{n=0}^{N-1} f_n e^{-2\pi i k n / N} \qquad k = 0, 1, 2, \ldots, N-1.$$

Experimental errors inherent in f_n will, of course, be reflected in F_k. Given the original time series f_n, however, it is exactly (i.e., without approximation) represented by F_k at the N points of interest.

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Continuous signals						
Power spectrum						
Autocorrelation function						
Algorithm						

These formulas are derived from basic principles, and their relationship to digital analysis of data sampled from continuous signals is discussed. In particular, the efficient use of the recently popular algorithms for rapid machine computation of Fourier transforms is explained. Although the results obtained are for the most part well known in the literature, it is felt that the importance of these transforms warrants a complete and notationally consistent exposition.

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