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Techniques for Numerically Evaluating the Formulas Describing Monostatic Reflection of Acoustic Waves by Elastic Spheres

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ABSTRACT

Numerical methods are described by which the theoretical formulas specifying the acoustic field resulting from scattering of an acoustic wave by an elastic sphere may be evaluated. They are restricted to the case of monostatic reflection of a plane harmonic wave incident upon the elastic sphere — the sphere being immersed in a uniform homogeneous fluid, the acoustic properties of which are assumed to be the same at all frequencies. Methods of calculating reflection form-functions as a function of sphere size-parameter are shown in the range of size-parameters $0 < ka \leq 30.0$. The reflection form-functions may be calculated at any distance from the elastic sphere by one of two formulas, one describing reflection in the far-field case. A description is given of the means by which the spherical Bessel function values, required in computations made using the reflection formulas, are calculated. Examples are given of form-function computational results in the case of reflection by elastic spheres and, for purposes of comparison, also in the case of reflection by rigid, immovable spheres. Brief tables of computed values of the far-field reflection form-function as a function of sphere size-parameter are given in the cases of an aluminum and of a rigid sphere. Implementation of the numerical methods described requires access to a digital computer. The capabilities and limitations of the two different computing machines used in the treatment of the elastic sphere problem by the numerical methods described, are discussed.

PROBLEM STATUS

This is an interim report on the problem; work is continuing.

AUTHORIZATION

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TECHNIQUES FOR NUMERICALLY EVALUATING THE FORMULAS DESCRIBING MONOSTATIC REFLECTION OF ACOUSTIC WAVES BY ELASTIC SPHERES

INTRODUCTION

This report describes numerical methods by which the theoretical formulas specifying the acoustic field resulting from scattering of an acoustic wave by an elastic sphere may be evaluated. The treatment presented has as its aim the exposition and description of techniques which provide the experimental investigator, having access to automatic computing facilities, with reliable theoretical values with which he may compare his experimental data. The present treatment is restricted to the problem of monostatic reflection in the case of insonification with a plane, harmonic acoustic wave. The field point, however, need not be at a great distance from the reflector to enable one to calculate the scattered portion of the total field.

The problem of acoustic reflection by an elastic sphere, that is by a solid, homogenous sphere capable of supporting both dilatational and shear waves, was first solved by Faran (1,2). Faran gives a comprehensive and complete description (2) of the theoretical solution and numerically evaluates several scattering patterns (i.e., distributions, in angle, of the pressure amplitudes of the scattered acoustic wave) that were of interest to him experimentally. Faran expressed his results in terms of the phase-shift angles introduced by Morse (3) and effected his numerical calculations by means of tabulated values (4) of these quantities. His sphere reflection formulas are restricted to the case of harmonic plane-wave insonification of the reflector, and his numerical calculations treat the case of far-field scattering patterns.

Hickling (5,6) also examined the problem of reflection by an elastic sphere, extending the treatment in several directions. He gives expressions that describe the reflected field for an incident spherical harmonic wave and for the case when the amplitude of the incident wave, either spherical or plane, may be time-varying (i.e., the case of the reflection of a pulse from the sphere). Hickling used a digital computer to evaluate, in a considerable number of cases, the theoretical formulas he presents. The major portion of his numerical evaluations are concerned with finding, in the monostatic far-field case when the reflector is insonified by a plane harmonic wave, the variation in the modulus of the reflection form-function as a function of sphere size-parameter (ka) for spheres composed of particular elastic materials (brass, aluminum, Armco iron, etc.). Hickling also reevaluated, for incident plane harmonic waves, the modulus of the monostatic reflection form-function as a function of reflector size-parameter in the far-field case for ideally rigid and ideally soft spheres. He examined this form-function modulus versus size-parameter relationship, in the cases of an ideally rigid sphere and of a brass sphere, for several constant ratios of range-parameter to size-parameter (kr/ka). Finally, using a plane wave pulse, whose temporal waveform was a sinusoid modulated by a single rectangular pulse (i.e., a truncated sinusoid), Hickling numerically constructed the pulse reflected monostatically into the far-field, in the case of an Armco iron reflector, for several different values of pulse length and dominant frequency component.

FORMULAS SPECIFYING THE REFLECTED FIELD OF AN ELASTIC SPHERE

A solid sphere, homogeneous and isotropic, is surrounded by a uniform, inviscid, elastic fluid, considered to be initially undisturbed. At the center of the sphere the

origin of a Cartesian coordinate system is placed coincidentally with the origin of a spherical coordinate system. If the same geometry is maintained, with plane waves advancing in the direction of the Z axis (Fig. 1), the displacement u of an element of the sphere is due both to deformation of the sphere, assumed to obey the equations of linear elasticity theory, and to rigid body motion of the sphere under the influence of the impinging sound wave. The coordinate systems used to describe the incident and reflected fields are thus at rest with respect to the equilibrium position of the reflecting sphere. The reflected field is described in terms of the fixed polar coordinate system. In this system, the reflected field is independent of the azimuthal angle ϕ, and, in the case of monostatic reflection, the polar angle θ equals π.

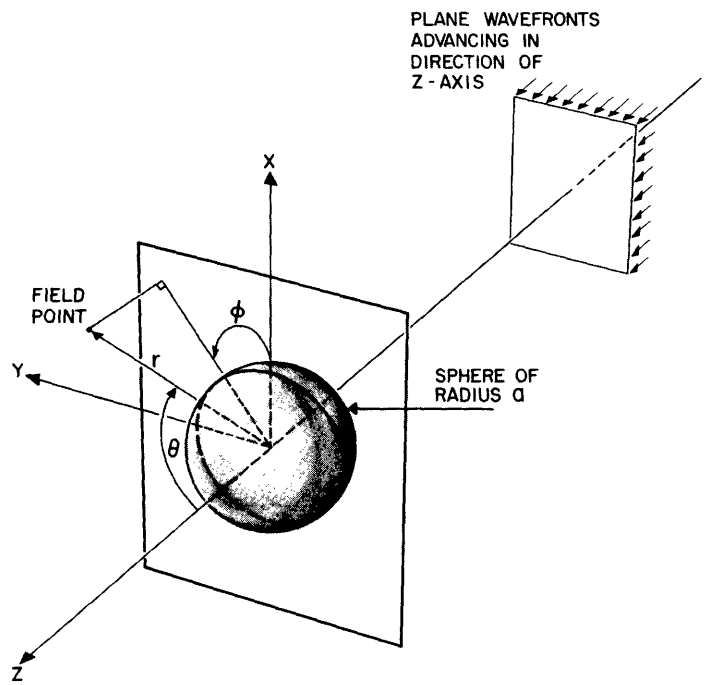


Fig. 1 - The geometry of monostatic reflection of an acoustic wave by an elastic sphere

If the pressure in the incident plane wave is

$$p_{(i)} = p_0 e^{i(\omega t - kz)}, \tag{1}$$

with k the wavenumber and ω the angular frequency, the pressure p at an arbitrary field point outside the sphere is the sum of the incident pressure field $p_{(i)}$ and a scattered pressure field $p_{(s)}$, that is,

$$p = p_{(i)} + p_{(s)}. \tag{2}$$

Let a be the sphere's radius and suppose the sphere to be composed of a material of density ρ_1 in which the speeds of dilatational and shear waves are c_1 and c_2 respectively. Let the density of the fluid in which the sphere is immersed be ρ and the sound speed in this fluid be c. The scattered portion of the total acoustic field at a point (r, θ) may be written (1,5):

$$p_{(s)} = \frac{p_0}{r}\,\frac{a}{2}\,e^{i(\omega t - kr)} f_r\,, \tag{3}$$

where f_r is the *reflection form-function* given as:

$$f_r = 2\left(\frac{kr}{ka}\right) e^{ikr} \sum_{n=0}^{\infty} (-i)^{n+1} h_n^{(2)}(kr) P_n(\cos\theta)(2n+1)\sin\eta_n\, e^{i\eta_n}. \tag{4}$$

In this expression $h_n^{(2)}(kr)$ is an nth order spherical Bessel function of the third kind [Eq. (A4b)] in Appendix A, $P_n(\cos\theta)$ is the nth order Legendre polynomial, and η_n is defined as

$$\tan\eta_n = -\left[\frac{\zeta_n j_n(ka) - (ka) j_n'(ka)}{\zeta_n y_n(ka) - (ka) y_n'(ka)}\right], \tag{5}$$

in which $j_n(ka)$ and $y_n(ka)$ are nth order spherical Bessel functions of the first and second kind respectively, with primes here denoting the derivatives of these functions with respect to the argument. In Eq. (5) the quantity ζ_n is defined as

$$\zeta_n = \frac{\rho}{\rho_1}\,\frac{(k_2 a)^2}{2}\left(\frac{A_n}{B_n} - \frac{C_n}{D_n}\right)\Big/\left(\frac{E_n}{B_n} - \frac{F_n}{D_n}\right), \tag{6}$$

where

$$A_n = (k_1 a)\, j_n'(k_1 a), \tag{6a}$$

$$B_n = (k_1 a)\, j_n'(k_1 a) - j_n(k_1 a), \tag{6b}$$

$$C_n = 2n(n+1) j_n(k_2 a), \tag{6c}$$

$$D_n = (k_2 a)^2 j_n''(k_2 a) + (n-1)(n+2) j_n(k_2 a), \tag{6d}$$

$$E_n = (k_1 a)^2 \left[\left(\frac{\sigma}{1-2\sigma}\right) j_n(k_1 a) - j_n''(k_1 a)\right], \tag{6e}$$

and

$$F_n = 2n(n+1)\left[j_n(k_2 a) - (k_2 a) j_n'(k_2 a)\right], \tag{6f}$$

in which primes and double primes denote, respectively, first and second derivatives of the spherical Bessel functions with respect to their arguments. In Eq. (6e), σ is Poisson's ratio of the reflector material and can be expressed in terms of the dilatational and shear wave speeds of the material:

$$\sigma = \frac{c_1^2 - 2c_2^2}{2(c_1^2 - c_2^2)}\,. \tag{7}$$

For a reflecting sphere composed of a given material, the values of $k_1 a$ and $k_2 a$ involved in the complete expression for ζ_n are determined from the value of ka (equal to $\omega a/c$) in question and from c_1 and c_2 for the material:

$$k_1 a = (c/c_1)\, ka\,, \tag{8a}$$

$$k_2 a = (c/c_2)\, ka\,. \tag{8b}$$

Certain differences will be noted between several of the formulas shown above and corresponding expressions given by Faran and Hickling. In Faran's journal article (1), the $\sigma/(1-2\sigma)$ term was misplaced in the expression which corresponds to Eq. (6e), although it is correctly placed in his original report (2). Hickling, in both his original report (5) and his journal article (6), omitted the factor ka which multiplies $j_n'(ka)$ in the numerator and $y_n'(ka)$ in the denominator of the expression which corresponds to Eq. (5). His computed results, however, are obtained from the correct expression for $\tan \eta_n$, and he points out this misprint in a later article (7). Corresponding expressions of Faran and Hickling also differ slightly because Hickling uses the convention of Morse (3) in describing the incident wave; that is, Hickling expresses the incident pressure wave advancing in the positive Z direction as $p_{(i)} = p_0 \exp[i(kz - \omega t)]$. The expressions in this report follow the other convention for expressing the incident wave, the one used by Faran. Since Faran used special tables to effect his numerical computations, many of his equations are expressed in terms of tabulated phase-shift angles — a procedure which, except in the case of the quantity η_n, is unnecessarily cumbersome when performing numerical evaluations using a digital computer.

For large distances from the reflector the asymptotic expressions for the $h_n^{(2)}(kr)$ terms in the form-function sum are used [Eq. (A13b)] with the result that f_r, which may now be written f_∞, is expressed

$$f_\infty = \frac{-2}{ka} \sum_{n=0}^{\infty} P_n(\cos\theta)(2n+1) \sin\eta_n \, e^{i\eta_n} . \tag{9}$$

Equation (9) describes the reflection characteristics in the far-field region of the sphere. For monostatic reflection, $\theta = \pi$, making $P_n(\cos\theta) = (-1)^n$, and in this case f_r and f_∞ are

$$f_r = -2\left(\frac{kr}{ka}\right) e^{ikr} \sum_{n=0}^{\infty} (-i)^{n+1} h_n^{(2)}(kr)(-1)^n(2n+1) \sin\eta_n \, e^{i\eta_n} \tag{10}$$

and

$$f_\infty = -\frac{2}{ka} \sum_{n=0}^{\infty} (-1)^n(2n+1) \sin\eta_n \, e^{i\eta_n}. \tag{11}$$

In order to numerically evaluate the formulas which describe the reflected field, it is necessary to transform several of the equations into another form. If the first and second derivatives of spherical Bessel functions are expressed in terms of spherical Bessel functions [Eqs. (A8), (A10), and (A11)] and the recursion relations [Eqs. (A7) and (A9)] are used, the quantities involved in the nth term of the reflection form-function sum may be expressed in terms of spherical Bessel functions of orders n and $n-1$. Thus Eq. (5) is transformed into

$$\tan\eta_n = -\left[\frac{(n+1+\zeta_n)j_n(ka) - (ka)j_{n-1}(ka)}{(n+1+\zeta_n)y_n(ka) - (ka)y_{n-1}(ka)}\right] , \tag{12}$$

and Eqs. (6) and (6a-f) are transformed into

$$\zeta_n = \frac{\rho}{\rho_1} \frac{(k_2 a)^2}{2} (A_n' - B_n')/(C_n' - D_n'), \tag{13}$$

with

$$A_n' = (k_1 a)\left\{\left[(k_2 a)^2 - 2n\right] j_n(k_2 a) + 2(k_2 a) j_{n-1}(k_2 a)\right\} j_{n-1}(k_1 a), \tag{13a}$$

$$B_n' = (k_2 a)(n+1)\left[(k_2 a) j_n(k_2 a) + 2 j_{n-1}(k_2 a)\right] j_n(k_1 a), \tag{13b}$$

$$C_n' = 2(k_1 a)\left\{\left[(k_2 a)^2 + n(n+2)(n-1)\right] j_n(k_2 a) - (k_2 a)(n+2)(n-1) j_{n-1}(k_2 a)\right\} j_{n-1}(k_1 a), \tag{13c}$$

and

$$D_n' = \left(\left\{(k_2 a)^2 (n+2)(n+1) + (k_1 a)^2 \left(\frac{1-\sigma}{1-2\sigma}\right)\left[2n(n+2) - (k_2 a)^2\right]\right\} j_n(k_2 a)\right.$$

$$\left. - 2(k_2 a)\left[(n+2)(n+1)(n-1) + (k_1 a)^2 \left(\frac{1-\sigma}{1-2\sigma}\right)\right] j_{n-1}(k_2 a)\right) j_n(k_1 a). \tag{13d}$$

FORMULAS SPECIFYING THE REFLECTED FIELD OF A RIGID, IMMOVABLE SPHERE

A reflecting sphere that is ideally rigid, rather than elastic, is placed with its center fixed at the origin of the coordinate system of Fig. 1. The boundary value problem, with Neumann boundary conditions at the surface of the sphere, is solved under conditions of insonification identical to those discussed in the case of the elastic sphere problem. The equation describing the scattered portion of the total field has the same form as Eq. (3), namely,

$$p_{(s)} = \frac{p_0}{r}\,\frac{a}{2}\, e^{i(\omega t - kr)} f_r, \tag{14}$$

where f_r is the reflection form-function for the rigid sphere. The expression giving the reflection form-function f_r for the rigid sphere is identical to Eq. (4).* In the rigid sphere case, however, the quantity η_n is given by the expression

$$\tan \eta_n = -\left[\frac{(n+1) j_n(ka) - (ka) j_{n-1}(ka)}{(n+1) y_n(ka) - (ka) y_{n-1}(ka)}\right], \tag{15}$$

rather than by Eq. (12), which applies to the elastic case. Since there is no term in Eq. (15) which corresponds to ζ_n in Eq. (12), there is no dependence on material constants, either of the reflecting sphere or of the surrounding fluid. For the rigid sphere the expressions giving (a) the general far-field reflection characteristics, (b) the monostatic reflection characteristics at arbitrary distances from the reflector, and (c) the monostatic reflection characteristics in the far-field are identical to Eqs. (9), (10), and (11) respectively, but with the quantity η_n being given in all cases by Eq. (15).

NUMERICAL EVALUATION OF MONOSTATIC REFLECTION FORMULAS

Initially all the computations were performed on the Naval Research Laboratory's NAREC digital computer.† Early in 1966, however, NRL's new digital computer – a

*See for example the treatment in Ref. 8; the expression for form-function there, however, differs slightly from that given in Eq. (4) because of a slightly different spherical coordinate system.

†The NAREC computer is a medium-speed machine, approximately equivalent to an IBM 701. Its core memory contains 16,384 words, each of which is a 48-digit message. Floating-point numbers have 10-decimal-place accuracy, and the range of computation employing floating-point numbers x is: $10^{-231} \leq |x| \leq 10^{307}$.

Control Data Corporation 3800 — became available, and, since the sphere problem computations could be handled much more effectively using the new machine, existing programs were adapted to it. Since programming was done in the NELIAC compiler language (9), which is, in general, 90 to 95 percent independent of the particular machine on which the NELIAC compiler is employed, the transition between the NAREC and CDC 3800 computers could be made without making any major changes in the existing programs. References will be made to the capabilities and limitations of both machines in handling the sphere reflection problem, since computations were made on both.

Reflection form-functions were computed in a number of cases. Generally it is of interest to compute the complex reflection form-function as a function of sphere size-parameter ka. The range of size-parameter values used was generally:

$$0.001 \leq ka \leq 30.0 ,$$

which was the range of interest in our experimental work. This range, moreover, coincided with that generally used by Hickling (5,6) in his numerical evaluations. When programming was undertaken, it was decided for expediency not to evaluate the case of $ka = 0$, but to treat it as an indeterminate case. In a few cases, notably for an ideally rigid and an ideally soft reflecting sphere (the latter case also being treated but not discussed in this report), the size-parameter range was increased in order to explore the limits to which calculations could be extended in the light of existing programs and available equipment. No attempt was made, however, to completely perfect the programs and utilize the computation machinery to its ultimate capability, since the numerical analysis described here was motivated by the practical goal of arriving at theoretical values which could be compared to the results of reflection experiments previously undertaken.

It was found that the evolution of machine computational procedures which give satisfactorily computed results involved trial-and-error testing at several intermediate stages in the programming, both in order to adapt them to the particular computing machine employed and in order to explore the way in which particular programming techniques affected the accuracy, range of validity, and computation time of the results. The computation problem may be divided into three parts: (a) calculation of spherical Bessel functions, (b) evaluation of the reflection form-function sums as a function of size-parameter, and (c) checking the results thus obtained by means of auxiliary computer programs. Practically, treatment of part (a) precedes treatment of part (b) of the problem, but a logical description of computation procedures is easier if the order in which these parts are discussed is reversed; therefore (b) will be discussed first. Discussion of (c), computational checks, will be deferred until computational results have been discussed.

Computation of Far-Field and Near-Range Reflection Form-Functions

The nth term of each of the form-function sums, Eqs. (10) and (11), which give respectively f_r and f_∞ in the monostatic case, is split into two parts, one involving the real part and the other involving the imaginary part of the complex factor $e^{i\eta_n}$. In the case of f_∞, given by Eq. (11), the form-function sum may then be directly expressed as two sums, one sum giving the real part of f_∞ and the other giving its imaginary part. In the case of f_r, given by Eq. (10), the real and imaginary parts of the nth term are obtained by multiplying together two complex numbers,

$$(-1)^n(2n+1)\sin\eta_n\, e^{i\eta_n} = (-1)^n(2n+1)\left[\sin\eta_n\cos\eta_n + i\sin^2\eta_n\right]$$

and

$$(-i)^{n+1} h_n^{(2)}(kr) = (-i)^{n+1} [j_n(kr) - i y_n(kr)],$$

and taking the real and imaginary parts of the result. Summing over n, the real parts and the imaginary parts of all the complex numbers thus obtained, and multiplying the complex number which results from these summings by the coefficient of the sum in Eq. (10), which involves the complex number e^{ikr}, yields the complex expression for f_r.

It is not known *a priori* to what index N form-function sums must be carried in order to contain enough terms to sufficiently approximate the infinite sums which actually express f_r and f_∞. In the case of f_∞, the number of terms required to make the absolute difference between the Nth term and the $(N-1)$th term less than some fixed small quantity ϵ depends upon the value of the size-parameter ka for which f_∞ is being calculated — more terms being required, the higher the value of ka in question. For f_r, the number of terms depends, in addition, on the particular range-parameter kr being considered. Moreover, since the index N to which sums must be carried is determined by the range of ka and kr values considered in the calculations, it is not known *a priori* to what order the various spherical Bessel functions required must be determined. But assuming for the moment that it is possible to produce spherical Bessel functions as needed, the summations may be carried out as described below.

Two loop statements are used in the program for computing the required sums for a given value of ka. First, suppose the form-function f_∞ is being computed for an elastic sphere according to Eq. (11). As has been stated, f_∞ is expressed as two sums, one giving $Re(f_\infty)$ and the other $Im(f_\infty)$, each of which may be computed in the same way. For either of these sums, a loop is used to sum the first ten terms ($n = 0, \ldots, 9$). A second loop is used to compute successive terms, one at a time, and cumulatively add them to the sum of the preceding terms. The absolute value of the sum involving the addition of N terms with this second loop is compared at each step to the absolute value of the sum involving $N-1$ terms, and when the difference between these two is less than some arbitrary but preassigned small number ϵ, the loop is terminated. Included in the subsequent printed output are the respective numbers of terms required to make the absolute difference between the Nth and $(N-1)$th partial sums less than ϵ for the real and the imaginary parts of f_∞. Fewer terms need be summed to obtain the imaginary part of f_∞ to within a given ϵ-interval of its true value (as expressed by an infinite sum) than to obtain the real part of f_∞ to the same tolerance, since the sum giving $Im(f_\infty)$ converges more rapidly than that giving $Re(f_\infty)$. Ten terms were summed with the initial loop before testing the absolute value of the difference between the Nth and $(N-1)$th terms of a sum, because the initial terms of any of the sums in question generally are all large and do not decrease monotonically as N is increased.

Computing the form-function f_r according to Eq. (10) is done in a manner similar to the computation of f_∞ according to Eq. (11). In fact, a simple modification may be made to the computer program which is used to evaluate f_∞ in order to calculate f_r, since the only substantial difference in the expressions for f_r and f_∞ is the presence of the complex term $(-i)^{n+1} h_n^{(2)}(kr)$ which multiplies the nth complex term of the sum in Eq. (10). This program modification for finding f_r was accomplished by summing the same number of terms when calculating $Im(f_r)$ as when calculating $Re(f_r)$. Therefore, after one loop is used to calculate the first ten terms of the modified sums expressing the real and imaginary parts of f_r, an additional loop is used to calculate and sum successive terms in these modified sums until a value of N is reached such that the Nth and $(N-1)$th partial sums in both cases differ in absolute value by an amount less than ϵ. That is, the sums expressing the real and imaginary parts of f_∞ [Eq. (11)] were found and checked for convergence, separately, using independent machine loops, while the corresponding sums in the case of f_r [Eq. (10)] were obtained simultaneously using a single loop.

The respective programs used to compute f_∞ and f_r for an elastic sphere may be modified to calculate f_∞ and f_r for an ideally rigid sphere. In the rigid case, the methods

used to sum the expressions for f_∞ and f_r exactly parallel the methods used for summing in the elastic case.

In Tables 1a and 1b, the convergence properties of the form-function sums are shown for a number of the sphere reflection cases considered. It will be noted that for a given tolerance ϵ the number of terms required to determine f_∞ is almost completely independent of the particular material of which the sphere is composed. In fact, the convergence does not depend on whether an elastic or an ideally rigid or soft reflecting sphere is being considered. The convergence of f_r is, in general, more rapid than that of f_∞ and is equally rapid, at the same kr value, for aluminum (elastic) spheres as for ideally rigid spheres. For a given ka value the number of terms required in summing is almost the same whether $\epsilon = 2 \times 10^{-4}$ or $\epsilon = 2 \times 10^{-6}$ is used as the tolerance value. Table 2 shows the N required in the case of an aluminum sphere in the far field for these two different tolerance values at various values of size-parameter. It was found that use of a tolerance value of 2×10^{-6} in computations had the advantage of reducing the number of what may be called *pathological values* of the form-function that were encountered. A pathological value results not as a mathematical consequence of the equations specifying sphere reflection characteristics but through the application of a given computational procedure to evaluate these equations. Occasionally an exceptional case is encountered where the procedure produces an erroneously computed value. In such a case, a pathological value appears as a value of the form-function, computed at some ka, that is significantly different from the values of the form-function computed at two neighboring values of ka — one of these neighboring form-function values occurring at a ka value slightly higher than, and the other at a ka value slightly lower than, the ka value of the point in question. If the form-function modulus is plotted as a function of ka, a pathological value gives a plotted point that lies away from a smooth curve drawn through its neighbors at higher and lower ka values. However, in the elastic case, a form-function can often change drastically with only a small change in ka; therefore, it is necessary to test points at which pathological behavior is suspected on the basis of discontinuities in a plotted curve. Two tests can be applied to a suspected pathological point. First, additional values for the form-function may be computed in a very small ka-neighborhood of the suspected pathological point. If all form-function values so computed lie on the previously determined smooth curve with the exception of the originally suspected point, this point may then be discarded. Second, the number N of terms required to evaluate a form-function's real and imaginary parts is an indicator of pathological behavior. For a pathological value of the form-function, it is found that abnormally few terms have been summed when obtaining either its real or its imaginary part — a fact which may be readily determined if the value of the N required in each case is a printed output when the form-function is determined. What occurs in such a case is that two successive partial sums are, by chance, almost equal, making the absolute values of the Nth and $(N-1)$th partial sums differ by less than the set tolerance ϵ without the absolute difference between the value of the Nth partial sum and the true value of the form-function being less than ϵ. Such behavior happens only at a few scattered ka values, and its occurrence depends upon the tolerance limit set. Choosing a smaller value for ϵ causes the summation to be properly made at a ka value at which the pathological behavior had been encountered when some lower tolerance was specified. Except for pathological points, N increased slowly and monotonically with increasing values of the size-parameter for both the real and the imaginary parts of the form-function. Also, in all cases when questionable points were encountered, both tests — computing additional values in the neighborhood of the questionable point and checking the values of N at which summations were terminated for that point — were in agreement concerning whether or not such points were pathological.

Computation of Spherical Bessel Function Values

Values of spherical Bessel functions of both the first and second kind must be found up to relatively high orders and over a wide range of arguments in order to evaluate

Table 1
Convergence Properties of Reflection Form-Function Sums

a. Far-Field Reflection Form Function f_∞

Reflecting Sphere	Tolerance (ϵ)	Size - Parameter (ka) Value for which Stated Value of N Last Suffices when Summing to Obtain f_∞ (ka increments = 0.1)									
		$N = 10$		$N = 20$		$N = 30$		$N = 40$		$N = 50$	
		$Re(f_\infty)$	$Im(f_\infty)$	$Re(f_\infty)$	$Im(f_\infty)$	$Re(f_\infty)$	$Im(f_\infty)$	$Re(f_\infty)$	$Im(f_\infty)$	$Re(f_\infty)$	$Im(f_\infty)$
Brass	2×10^{-4}	4.8	6.7	12.5	15.3	20.9	24.3	29.5	> 30.0		
Aluminum	2×10^{-4}	4.8	6.7	12.5	15.3	20.9	24.3	29.5	> 30.0		
Aluminum	2×10^{-6}	3.8	5.9	10.9	14.1	18.9	22.9	27.3	> 30.0		
Tungsten carbide	2×10^{-6}	3.8	5.9	10.9	14.2	18.9	23.0	27.3	> 30.0		
Lucite	2×10^{-6}	3.7	5.4	10.6	13.5	18.0	23.2	26.8	> 30.0		
Aluminum oxide	2×10^{-6}	3.8	5.9	10.9	14.2	18.9	23.0	27.3	> 30.0		
Ideally rigid	2×10^{-6}	3.8	5.9	10.9	14.2	18.9	23.0	27.3	32.0	≈ 35.7	> 40.0
Ideally soft	2×10^{-6}	3.8	5.9	10.9	14.2	18.9	23.0	27.3	32.0	≈ 35.7	> 36.3

b. Reflection Form-Function f_r at Various Ranges ($\epsilon = 2 \times 10^{-6}$ in all cases)

Reflecting Sphere	Range-Parameter (kr)	Size-Parameter (ka) Value for which Stated Value of N Last Suffices when Summing to Obtain f_r (ka increments = 0.1)			
		$N = 10$	$N = 20$	$N = 30$	$N = 40$
Aluminum	60	4.7	12.4	20.6	29.2
Aluminum	90	4.8	12.5	20.9	29.5
Aluminum	150	5.0	12.8	21.2	29.7
Aluminum	300	5.1	13.1	21.4	>30.0
Aluminum	450	5.3	13.1	21.6	>30.0
Ideally rigid	60	4.7	12.4	20.7	29.3
Ideally rigid	90	4.8	12.5	20.9	29.5
Ideally rigid	150	5.0	12.8	21.1	29.8
Ideally rigid	300	5.1	13.1	21.4	>30.0
Ideally rigid	450	5.3	13.2	21.5	>30.0

Table 2
Convergence of the Reflection Form-Function f_∞ of an Aluminum Sphere for Different Specified Tolerances ϵ

Size-Parameter (ka)	Number (N) of Terms Required to Make the Absolute Difference Between the Nth and $(N-1)$th Partial Form-Function Sums Less than ϵ at the Stated ka			
	$\epsilon = 2 \times 10^{-4}$		$\epsilon = 2 \times 10^{-6}$	
	$Re(f_\infty)$	$Im(f_\infty)$	$Re(f_\infty)$	$Im(f_\infty)$
10.0	17	14	19	16
15.0	23	20	26	21
20.0	29	26	32	27
25.0	35	31	38	33
30.0	41	37	44	38

reflection form-function sums in both the elastic and the rigid cases. One's ability to compute spherical Bessel functions fixes the limits within which the form-functions may be found. One's ability to compute the Bessel functions, in turn, depends upon the limitations of the computing machine available. Thus there are three interdependent factors that must be taken into account in the spherical Bessel function computation problem: (a) the range over which form-function values are desired, as determined by external considerations such as an existing experimental capability, (b) the convergence properties of form-function summations, and (c) the limitations of the available machine in any Bessel function computation process. It is not possible to determine the convergence properties of reflection form-functions until Bessel functions are available for computing them.

Therefore, one must initially guess, for a given ka-region in which form-function computations are to be made, the range of indices and the range of arguments of those spherical Bessel functions whose values will have to be computed in order to carry out the form-function computations in that region. Since it may not be possible to compute all the spherical Bessel function values needed to evaluate form-functions in a given region of interest, one might be required to restrict his form-function calculations to those regions where he can compute the required Bessel functions. It was found, however, that accurate values of spherical Bessel functions of both the first and the second kind of all orders from 0 to 50, having arguments between 0.001 and 50.0,* could be computed on the NAREC machine and that such a range of spherical Bessel functions is sufficient to compute accurate values of far-field reflection form-functions f_∞ for rigid spheres and spheres composed of various elastic materials. When the CDC 3800 machine became available, it was possible to compute spherical Bessel functions of orders 0 to 50 up to arguments of 1500.0, which is more than sufficient to calculate the form-function f_r at any range between the reflector's surface and the region where f_r becomes essentially equal to f_∞.

It was found that generating spherical Bessel functions as they are needed in a form-function summation by using the relations given in Eqs. (A14a,b) and (A15a-c), which give $j_n(x)$ and $y_n(x)$ in terms of finite sums and trigonometric functions, is not a practical

*It will be noted that even if $ka = 30.0$ is the highest value of the sphere size-parameter of interest, for certain materials $k_1 a$ or $k_2 a$ may be greater than ka owing to a wave speed in the sphere that is lower than the sound speed in water, and spherical Bessel functions with arguments greater than 30.0 will be encountered.

computational procedure. When the sums P and Q computed according to Eqs. (A15a-c) are multiplied by trigonometric functions and summed together in Eqs. (A14a, b), the value obtained for the resulting spherical Bessel function is usually seriously in error. The error arises because, as the number of terms in the sums P and Q [Eqs. (A15a, b)] increases, P and Q become large. Generally, then, the spherical Bessel function value, which is small compared to either P or Q, is computed as the difference between two large numbers. The digits needed to determine this difference are found to be partially or totally lost in rounding off the large values of P and Q in the computation process. Thus, unless large numbers of significant digits can be retained, not only in the terms of P and Q but also in the values of the trigonometric functions, the formulas of Eqs. (A14a, b) and (A15a-c) do not constitute a practical basis for a scheme to calculate the spherical Bessel functions in the sphere reflection problem.

A practical scheme of computing spherical Bessel functions is described at the end of Sections 9 and 10 in Ref. 10. This method is particularly well suited to the sphere reflection problem because it generates at one time an entire table of spherical Bessel function values of various orders, for a given argument, which may be stored in the computer memory and used as needed in a reflection form-function summation. The method is based on iterative use of the recursion relation for spherical Bessel functions given by Eq. (A7).

For spherical Bessel functions of the first kind, having an argument x determined by the value of ka in question, the recursion relation, Eq. (A7), is used in the form

$$j_{n-1}(x) = \frac{2n+1}{x} j_n(x) - j_{n+1}(x).$$

Starting with some integer M, sufficiently larger than the largest index N required in order to compute an accurate value of the form-function, one inserts two numbers $G_{M+1} = 0$ and $G_M = 1$ into the right-hand side of this recursion relation, in place of $j_{M+1}(x)$ and $j_M(x)$, and generates a sequence (G_K) in the direction of decreasing n:

$$(G_K) = G_{M-1}, G_{M-2}, \ldots, G_N, \ldots, G_1, G_0.$$

The terms of the sequence $G_0, G_1, \ldots, G_{N-1}, G_N$, are respectively proportional to the terms in the sequence $j_0(x), j_1(x), \ldots, j_{N-1}(x), j_N(x)$, to a certain number of significant figures, which depends on how much larger than N the value of M was chosen. Computing $j_0(x)$ according to Eq. (A16a) gives q, the constant of proportionality, as

$$q = \frac{\sin x}{xG_0}.$$

When the terms from $K = 0$ to $K = N$ of the sequence (G_K) are multiplied by this q, spherical Bessel functions of orders 0 to N have been generated, accurate to a certain number of significant figures; that is

$$\begin{aligned} j_0(x) &= qG_0, \\ j_1(x) &= qG_1, \\ &\vdots \\ j_{N-1}(x) &= qG_{N-1}, \end{aligned}$$

and

$$j_N(x) = qG_N.$$

The accuracy to which the Bessel function values are determined for a fixed N is increased by increasing the value of M used to generate the sequence (G_K).

Spherical Bessel functions of the second kind may be generated by iterative use of the recursion relation, Eq. (A7), in the form

$$y_{n+1}(x) = \frac{2n+1}{x} y_n(x) - y_{n-1}(x),$$

starting with $y_0(x)$ and $y_1(x)$ computed as

$$y_0(x) = -\frac{\cos x}{x}$$

and

$$y_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}.$$

In contrast to the functions of the first kind, $y_n(x)$ may be computed in the direction of increasing orders without introducing accumulative errors through loss of significant digits by rounding off in successive iterations.

Calculation of spherical Bessel functions of the second kind $y_n(x)$ engendered no difficulty. It was necessary, however, to define by trial the procedure for iteratively using the recursion relation in the direction of decreasing orders [this iterative technique may be called "Miller's device" — see Ref. 10, p. 452] in order to calculate values of spherical Bessel functions of the first kind $j_n(x)$ with a given computing machine. For a given value of the argument x, the Miller's device iteration may generate numbers either too large or too small for the computing machine to handle unless the value of M, the index at which the iteration to obtain (G_K) is begun, is carefully chosen. If M is chosen too large, the machine generates numbers outside its capabilities and cannot proceed, but if M is chosen too small, Bessel function values will not have sufficient accuracy. The maximum value that M can take and still not cause the machine's capabilities to be exceeded increases as the argument x, for which a table $j_0(x), j_1(x), \ldots, j_N(x)$ is to be computed, increases — N remaining fixed. This allows the range of arguments of the spherical Bessel functions considered in the sphere problem to be divided into intervals, in each of which a different value of M is used to start the iteration but in all of which accurate Bessel function values are found up to the same fixed maximum order N. The intervals into which the x range is divided are somewhat arbitrary, since the requirement is that the value of M be sufficiently small to compute the Bessel function value for the largest argument included in the preceding interval (which has a different, smaller M) and yet be sufficiently large to accurately determine Bessel function values for all arguments in the interval to which it applies. As an illustration, take the actual case of generating spherical Bessel function values with the NAREC machine. In order to generate accurate function values for $j_n(x)$ in the range $0.001 \leq x \leq 50.0$, $0 \leq n \leq 50$, the x-range was divided into two intervals for which the following M values were required:

$$0.001 \leq x \leq 10.0, \quad M = 60$$

$$10.0 < x \leq 50.0, \quad M = 100.$$

The value $M = 60$ was sufficient to allow the generation of accurate Bessel function values in the interval $x \leq 10.0$, and a value of $M = 100$ could be used for computing function values for $x = 10.0$ without exceeding the machine's capabilities. The entire range of arguments x of interest can thus be covered using the two x-intervals described. To take $M = 100$ in the entire range $0.001 \leq x \leq 50.0$ would not have worked with the NAREC machine,

because for some x value less than 10.0 this M would cause the NAREC's range limitation to be encountered. Calculating spherical Bessel functions $j_n(x)$ in the range $0.001 \leq x \leq 1500.0$, $0 \leq n \leq 50$ by Miller's device is within the capabilities of the CDC 3800 machine. The range of arguments was divided into the following intervals which have M values as given below:

$$0.001 \leq x \leq 10.0, \quad M = 60,$$

$$10.0 < x \leq 50.0, \quad M = 100,$$

$$50.0 < x \leq 100.0, \quad M = 200,$$

$$100.0 < x \leq 250.0, \quad M = 450,$$

$$250.0 < x \leq 400.0, \quad M = 600,$$

$$400.0 < x \leq 800.0, \quad M = 1000,$$

$$800.0 < x \leq 1200.0, \quad M = 1400,$$

and

$$1200.0 < x \leq 1500.0, \quad M = 1700.$$

For arguments $x \leq 100.0$, the accuracy with which spherical Bessel function values could be computed may be obtained by comparing computed values with published tables (10,11). It was found that even though identical values of M were used to start the Miller's device iteration, the CDC 3800 computer produced slightly more accurate Bessel function values than did the NAREC machine for a given x. The accuracy difference between corresponding computations performed on the two machines is negligible for $x < 10.0$ and amounts to a gain of about one significant figure for $10.0 \leq x \leq 50.0$ with the CDC 3800 machine. For representative arguments between $x = 0.01$ and $x = 50.0$, computed spherical Bessel function values for functions of the first kind are shown in Table B1a of Appendix B, with an indication of the errors in their computation. Table B1b is similar, showing values of functions of the first kind for representative arguments between $x = 100.0$ and $x = 1500.0$. Only for $x = 100.0$ in this range could comparison with tables be made.

The digits that were uncertain in computed values of $j_n(x)$ for values of x greater than 100.0 could be inferred by computing spherical Bessel function values of the various orders $0 \leq n \leq 50$ for a given argument x using various values of M in the iterative process. The uncertain digits in a computed value of $j_n(x)$ for a given x and a given n were taken to be those figures which changed when M was changed. For the computed values in Table B1b at arguments $x \geq 200.0$, the differences between the function values obtained for a given x, using the value of M appropriate to that interval of the total range of arguments in which x lies (the intervals being as previously described), and those obtained using other, higher values of M are indicated. Greater computational accuracy, especially at the higher values x, could no doubt have been obtained but would have involved considerably more exploratory computation than was necessary in order to provide spherical Bessel function values adequate in the sphere problem, would have entailed breaking the range of arguments up into more than the eight intervals used, and would have required that more computer memory be set aside for storage in Bessel function computation. As it is, for a sphere in the ka-range of from 0 to 30, it is not necessary to extend the range of spherical Bessel function arguments up to $x = 1500.0$, because the value of kr at which the formulas of Eqs. (10) and (11) give essentially the same result occurs much before $kr = 1500.0$, as will be indicated later.

Spherical Bessel function values, for functions of the first kind, are seen to be generally accurate to nine or ten places for $x \leq 50.0$, with accuracy decreasing somewhat as zeros of the functions are approached at the higher values of x in this range. At $x = 100.0$,

computed values have an accuracy of from eight to ten significant figures in the cases which can be compared, but, judging from the behavior at smaller arguments, for values of x near 100 some further loss of accuracy might be expected near zeros of the functions. There were, in general, seven to ten certain figures at x = 200.0 and six to eight certain figures at x = 1500.0, with the least accurate entries occurring at the higher values and near the zeros of the functions in all cases considered. Tables B1a and B1b were computed using the CDC 3800 machine.

Tables B2a and B2b, also computed using the CDC 3800 machine, show values of spherical Bessel functions of the second kind and correspond respectively to Tables B1a and B1b as to the values of the arguments considered. Those computed values which can be checked against tables, that is, those values between x = 1.0 and x = 100.0, can be seen to be accurate to nine or ten significant figures except very near zeros of the functions.

COMPUTATIONAL RESULTS

A value of the monostatic reflection form-function f_∞ in the far field may be calculated when six quantities are specified: ka, c_1, c_2, ρ_1, c, and ρ. If kr is also specified, f_r may be calculated. As an example of a far-field form-function determination, Fig. 2 shows a plot of $|f_\infty|$ vs ka in the case of an aluminum sphere in water for $0 < ka \leq 28.0$, and Appendix C gives numerical data relating to this case covering the range $0 < ka \leq 30.0$ in ka-increments of 0.1. The curve was computed using the following values of the pertinent quantities:

$$c_1 = 6296.0 \text{ m/sec},$$

$$c_2 = 3106.0 \text{ m/sec},$$

$$\rho_1 = 2.7118 \text{ g/cm}^3,$$

$$c = 1482.25 \text{ m/sec},$$

and

$$\rho = 0.998203 \text{ g/cm}^3.$$

The values used for wave speeds c_1 and c_2 were those measured by ultrasonic means using a sample of 2S aluminum; the material density ρ_1 was obtained by accurately weighing a precision-made 2S sphere; c is the sound speed, at 20.00°C in water, obtained from a table derived from recent experimental measurements (12), and the value of ρ, the density of water at 20°C, is that found in the Handbook of Chemistry and Physics (13). For these values of c_1 and c_2 the value of Poisson's ratio computed according to Eq. (7) is σ = 0.3391718830. Numerical data suitable for plotting such $|f_\infty|$ vs ka curves were obtained on both the NAREC and CDC 3800 machines. Since the rate of convergence of form-function sums is almost completely independent of the sphere's elastic properties, all similar computations took about the same length of time on a given computing machine. A table of values such as given in Appendix C, where ka ranged between 0.1 and 30.0 in ka-increments of 0.1, took approximately 3 hours to compute on the NAREC machine. On the CDC 3800, a computation of a table for the same range of ka-values, but with ka-increments of 0.01, could be accomplished in approximately 6-1/2 minutes, exclusive of compilation time, or about 277 times faster. A table of form-function values in ka-increments of 0.1 will allow a rough $|f_\infty|$ vs ka curve to be plotted while ka-increments of 0.01 permit all but the most rapid variations in a form-function curve to be described. An auxiliary program was used to compute a set of form-function values, in very small ka-increments, in a neighborhood of each point within the $0 < ka \leq 30.0$ range near which the form-function value changed rapidly or at which pathological behavior was suspected.

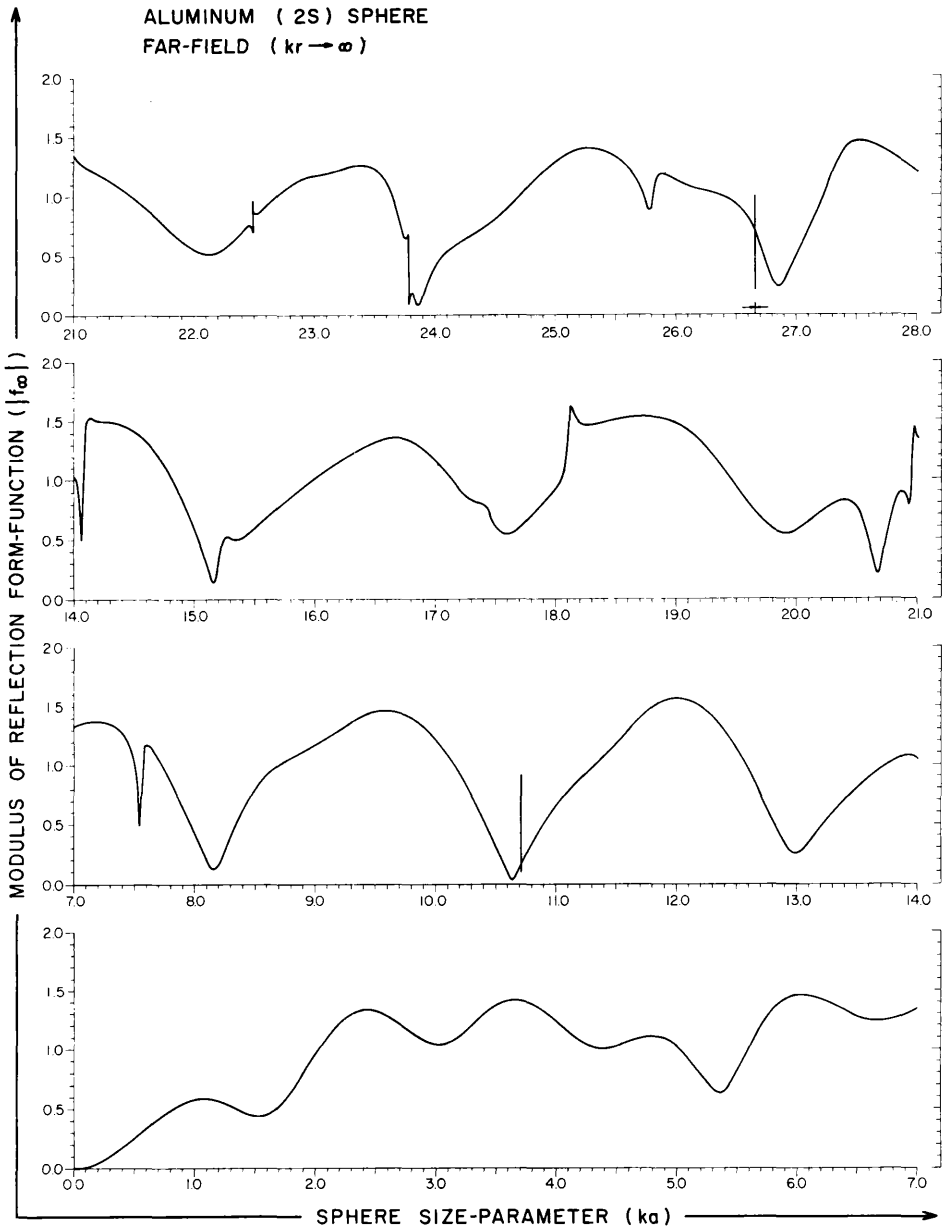


Fig. 2 - Monostatic far-field reflection form-function modulus vs sphere size-parameter for a sphere composed of 2S aluminum

Several types of rapid variation are encountered in the curves displaying the form-function modulus vs sphere size-parameter. First, variations occur such as the sharp dip near ka = 14.07 in Fig. 2. The existence of variations of this type are readily apparent when plotting the $|f_\infty|$ vs ka curve from tabulated values of f_∞ in ka-increments of 0.01. It may be necessary, however, to calculate form-function values in ka-increments of 0.001 in a neighborhood which contains the minimum of such a dip in order to determine the actual minimum value that the form-function attains in this region. Another type of rapid variation is that which might be called a "flection" in the form-function curve. Four such flections, each of which occurs within a very small ka-interval, were

found in the $|f_\infty|$ vs ka curve of Fig. 2 in the range $0 < ka \leq 28.0$; for these the major variations in $|f_\infty|$ occurred in the intervals given approximately as

$$10.7070 < ka < 10.7160,$$

$$22.460 < ka < 22.520,$$

$$23.760 < ka < 23.820,$$

and

$$26.65030 < ka < 26.65200.$$

The existence of flections is a mathematical consequence of the formulas which specify the reflection form-function and are not, as in the case of pathological points, the results of vagaries in the computational process which can be eliminated by requiring that a more accurate summation be performed. When the form-function curve passes through any flection, even one that occurs in an extremely narrow ka-interval, its variation is continuous and behaves in a manner generally characteristic of all variations of this type. Figures 3a and 3b show the detail of a particularly narrow but otherwise typical flection — in this case, the one which occurs near $ka = 26.65$ in Fig. 2. The region of the size-parameter axis between the arrowheads in Fig. 2 is expanded to cover the complete horizontal scale in Fig. 3a. Figure 3a shows the smooth way in which the form-function curve passes into and out of a flection. The portion of the graph of Fig. 3a which lies in the region indicated by the arrows is expanded in Fig. 3b to show the shape of the flection. This is the shape generally encountered in any flection, but the minimum in a flection may precede its maximum as ka is increased, rather than follow it as in the case shown.

For purposes of comparison a computation of f_∞ was made in the case of an aluminum sphere with the specifying constants identical to those used by Hickling (6). Hickling in his computations, however, specifies an additional elastic constant independently. This constant, Poisson's ratio σ, is actually completely determined according to Eq. (7) when the wave speeds c_1 and c_2 are specified, but the present programming was done in a way which allowed σ to be specified independently in the manner of Hickling, if desired. Figure 4 shows a $|f_\infty|$ vs ka curve for the range $0 < ka \leq 28.0$ when the set of pertinent constants is specified as follows:

$$c_1 = 6420.0 \text{ m/sec},$$

$$c_2 = 3040.0 \text{ m/sec},$$

$$\sigma = 0.355,$$

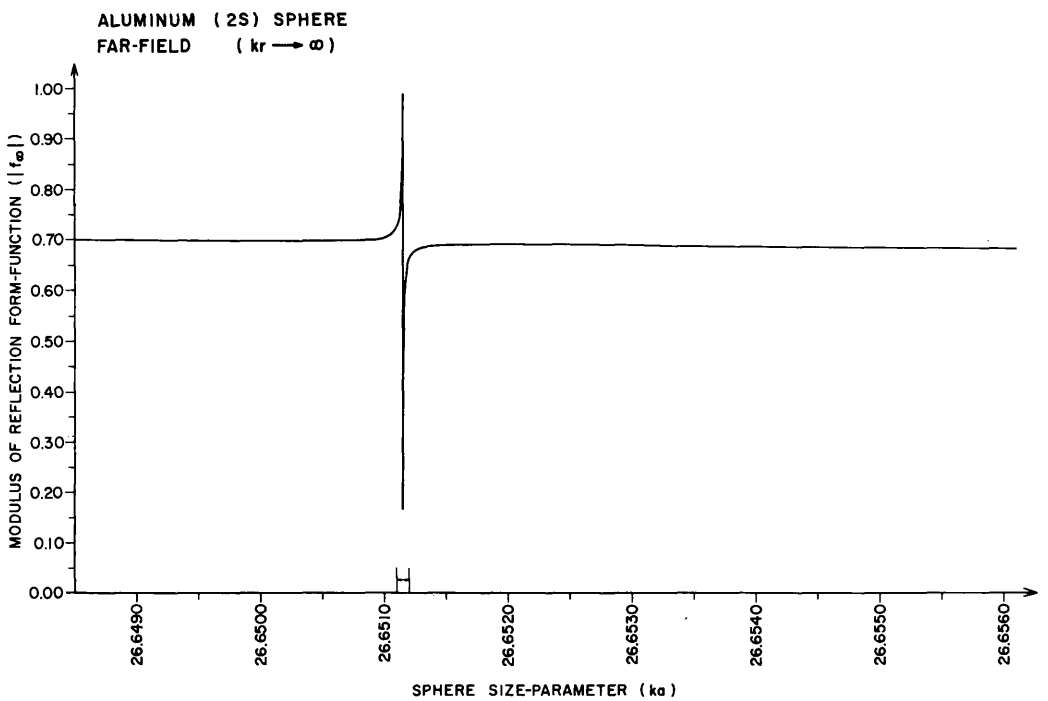
$$\rho_1 = 2.70 \text{ g/cm}^3,$$

$$c = 1410.0 \text{ m/sec},$$

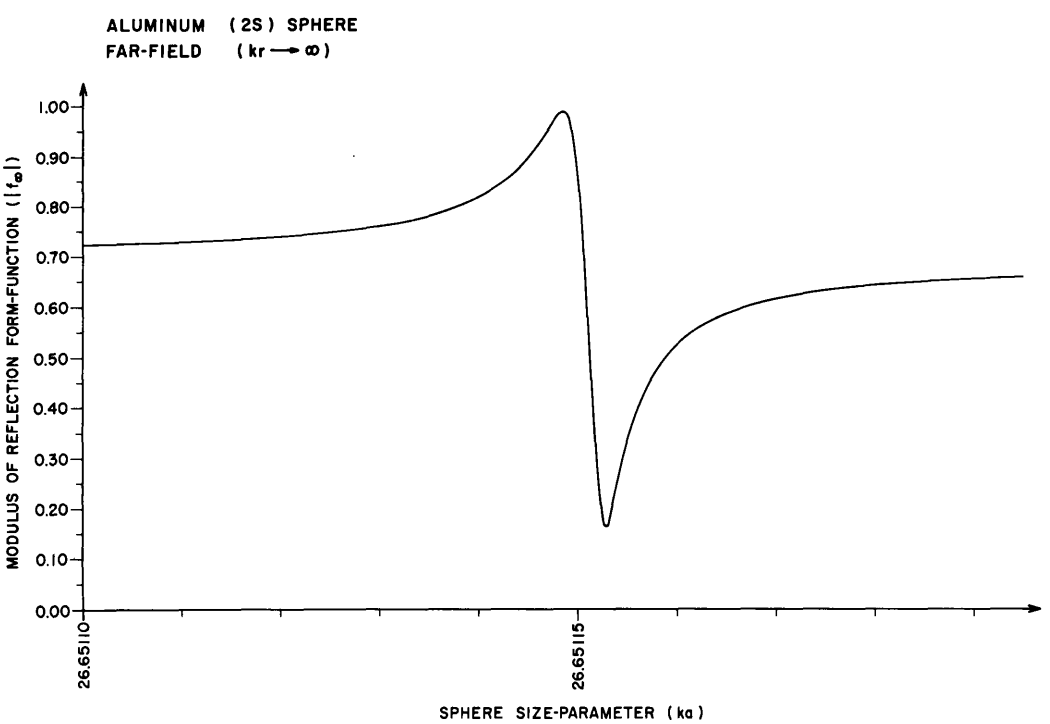
and

$$\rho = 1.0 \text{ g/cm}^3.$$

Comparing the curve of Fig. 4 to the corresponding curve obtained by Hickling (his Fig. 7), it is seen that the two curves are the same except in two minor respects. First, the scale of Hickling's abscissa should really be slightly compressed, as can be seen by comparing the values of $|f_\infty|$ at $ka = 28.0$ in the two figures. Figure 4 shows that, at $ka = 28.0$, the form-function modulus has not yet attained a value of 1.0, while Hickling's plot shows it to be at or near a local maximum value of approximately 1.43 at this value of size-parameter (actually, this maximum is not attained until $ka \approx 28.56$, with the "bump" in the curve which precedes it lying at $ka \approx 28.21$ at which point $|f_\infty| \approx 0.997$). Also, Hickling did not exhibit the existence of the several flections which occur in this instance in approximately the following size-parameter intervals:



(a) region $26.6485 \leq ka \leq 26.6561$, expanded



(b) region $26.651100 \leq ka \leq 26.651194$, expanded

Fig. 3 - Flection in the form-function modulus vs size-parameter curve of Fig. 2 that occurs near $ka = 26.65$

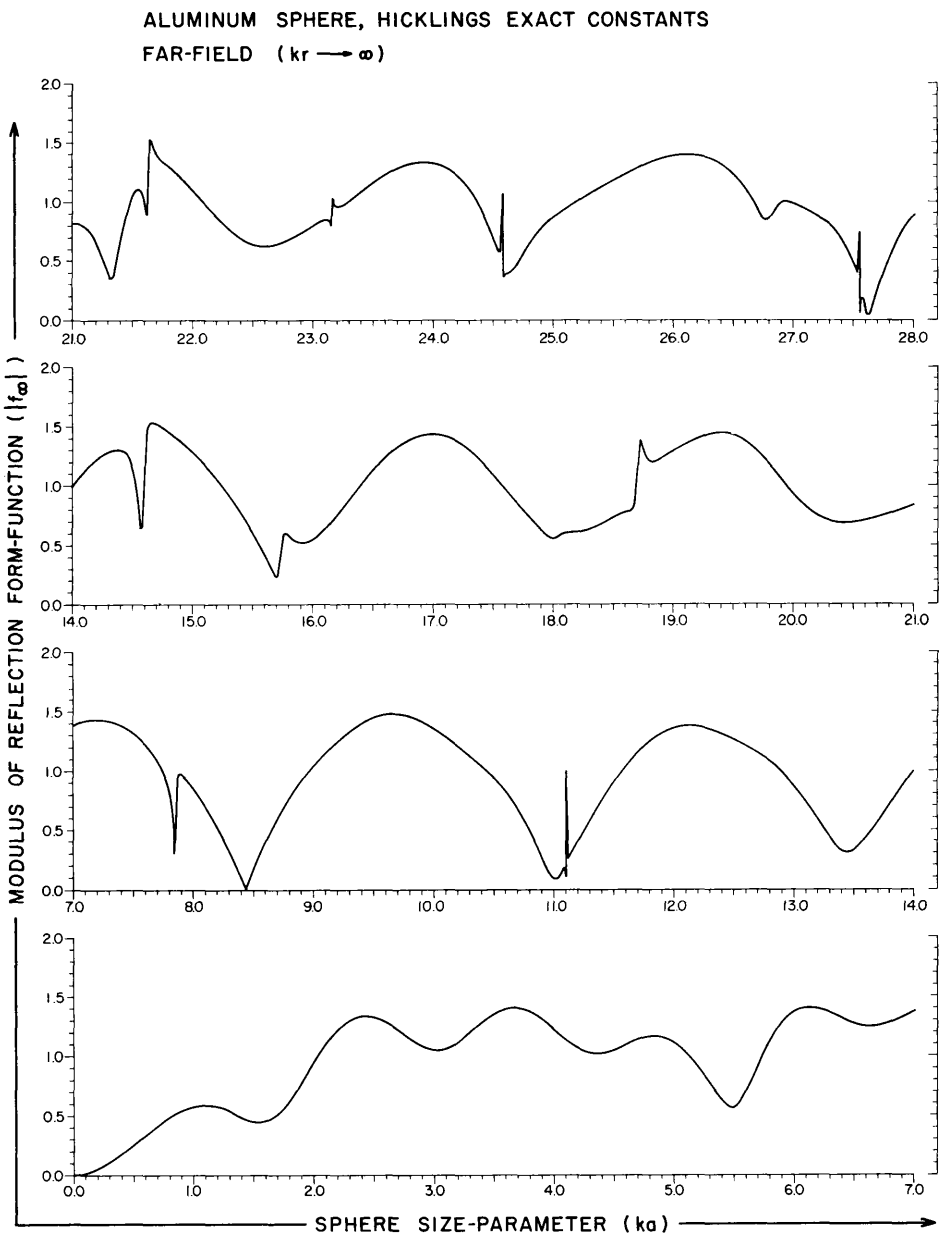


Fig. 4 - Monostatic far-field reflection form-function modulus vs sphere size-parameter for a sphere composed of aluminum when using the constants specified by Hickling

$$11.090 < ka < 11.130,$$

$$23.110 < ka < 23.220,$$

$$24.550 < ka < 24.600,$$

and

$$27.520 < ka < 27.560.$$

It will be noted that here the flections are generally somewhat broader than those exhibited in the previously considered 2S aluminum case. In fact, it was their greater

broadness in this case that led to the detection of such features in the first place. Their existence here showed which portions of the size-parameter (ka) axis should be examined, in the 2S aluminum case, in order to locate the flections, since the presence of flections was not always discernable in the tables of form-function values computed for 2S aluminum.

Some general remarks about the existence of flections are appropriate at this point. While a number of portions of the size-parameter axes of the various $|f_\infty|$ vs ka and $|f_r|$ vs ka plots in this report were searched by using an auxiliary program to compute f_∞ or f_r in fine increments, it is quite possible that flections exist in the curves which have not yet been resolved. Flections are not, however, very densely spaced along the ka-axis in the size-parameter range considered, as searches with the auxiliary program have shown, and probably only a few, if any, are missing from the plots. Now that is known that such rapid variations are to be expected in the form-function vs size-parameter curves, the most effective way to detect and resolve them would be to modify the main computing program in order to test, during the process of computation, any tabulation of form-function values for the presence of flections and to compute, at that time, enough additional form-function values to display these flections completely.

Figure 5 shows a plot of the modulus of the finite-range form-function f_r vs the size-parameter in the interval $0 < ka \leq 28.0$ for two different values of the range-parameter, $kr = 60.0$ and $kr = 450.0$, in the case of the 2S aluminum spheres previously considered. Since, for each curve, kr is held constant and ka is varied, the ratio r/a grows smaller as ka increases. Thus the differences between finite-range form-function values and corresponding far-field form-function values increase as ka increases. Figure 6 shows plots of $|f_r|$ vs ka at $kr = 450.0$ and $|f_\infty|$ vs ka, both in the range $21.0 \leq ka \leq 28.0$, illustrating the differences between the f_r, at large kr, and the f_∞ computations, in a portion of the ka range where these differences are most significant. Increasing kr would, of course, reduce the difference between $|f_r|$ and $|f_\infty|$ at a given value of ka. Figure 7 shows a plot of $|f_\infty|$ vs ka in the case of a rigid, immovable sphere, and Appendix D gives a table of values computed in the range $0 < ka \leq 30.0$ using ka-increments of 0.1 for this case. Curves of finite-range form-function modulus vs size-parameter in the case of a rigid, immovable sphere are shown for $0 < ka \leq 28.0$ in Fig. 8 for the two values of range-parameter used before, namely $kr = 60.0$ and $kr = 450.0$. Differences between the $|f_r|$ vs ka, at $kr = 450.0$, and the $|f_\infty|$ vs ka curves are shown for the rigid, immovable sphere in Fig. 9 for the size-parameter range $21.0 \leq ka \leq 28.0$. Note that no flections occur in the form-function modulus vs size-parameter curves for a rigid, immovable sphere, either in the far-field or in the near-range case.

COMPUTATIONAL CHECKS

An auxiliary program for checking the machine computations of f_∞ was used as follows. A sphere composed of a fictitious material was considered which had the properties

$$c_1 = 4690.0 \text{ m/sec},$$

$$c_2 = 2100.0 \text{ m/sec},$$

$$\rho_1 = 8.4 \text{ g/cm}^3,$$

and

$$\sigma = 0.375$$

and which was immersed in a fluid having

$$c = 1407.0 \text{ m/sec}$$

and

$$\rho = 1.0 \text{ g/cm}^3.$$

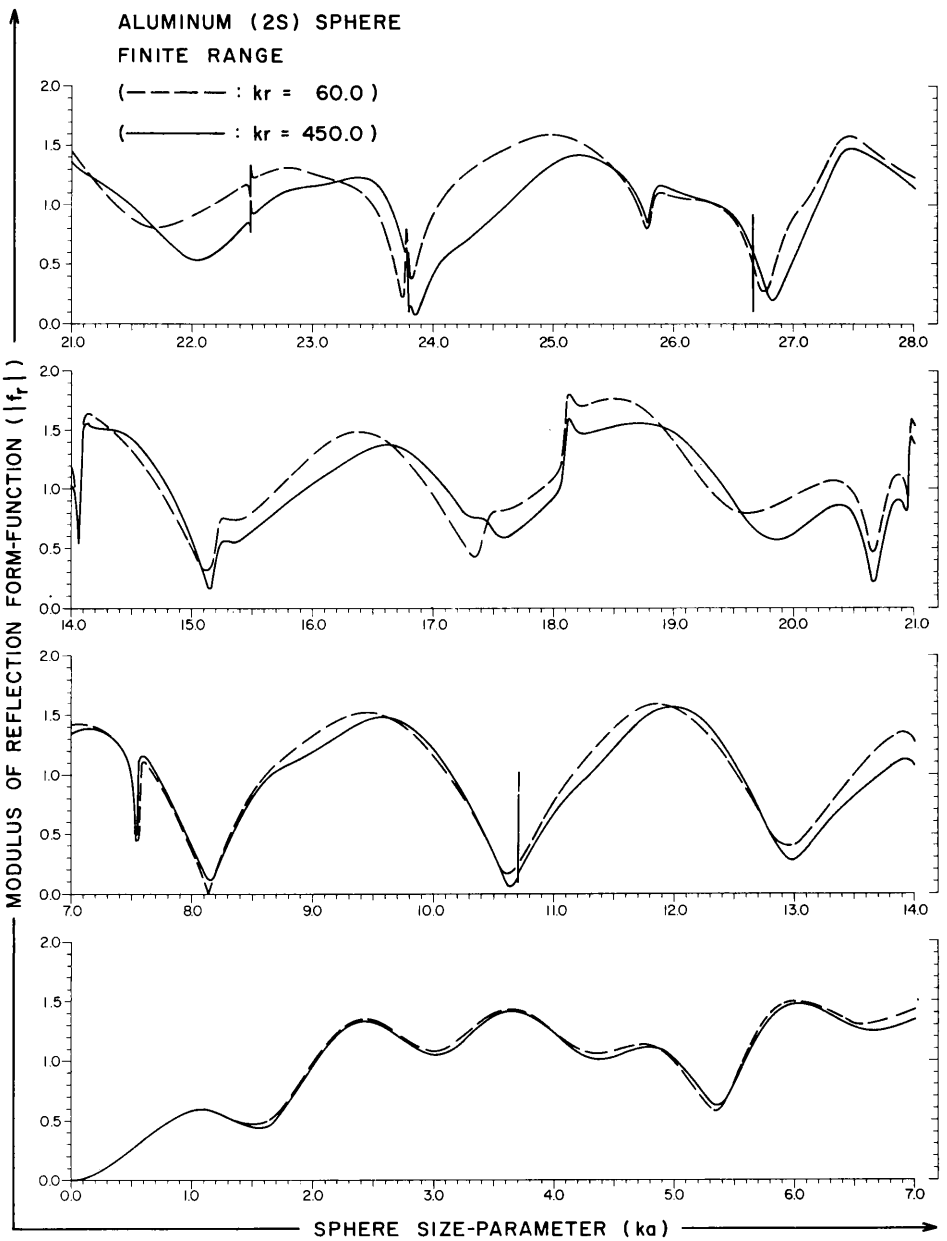


Fig. 5 - Monostatic finite-range reflection form-function moduli vs sphere size-parameter for a sphere composed of 2S aluminum at two values of the range-parameter (kr)

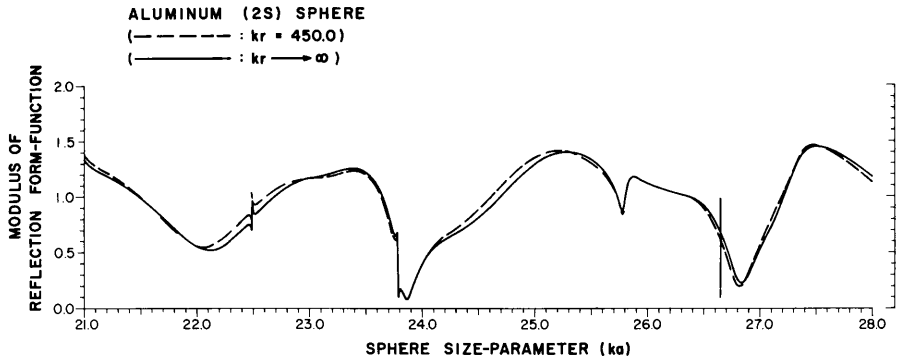


Fig. 6 - Comparison of the moduli of the far-field and the finite-range monostatic reflection form-functions as a function of sphere size-parameter for a sphere composed of 2S aluminum; the range-parameter in the finite-range case is 450.0

It will be noted that this case differs little from that of a brass sphere immersed in water at 0°C. A value of ka = 10 was chosen. The auxiliary program was used to print out tables which showed the following quantities for each value of n between 0 and 50 at this ka:

$$j_n(ka),\ y_n(ka),\ j_n(k_1a),\ j_n(k_2a),\ n,\ \zeta_n,\ \tan\eta_n,$$

$$\sin\eta_n,\ \cos\eta_n,\ 2n+1,\ \sin\eta_n\cos\eta_n,\ \sin^2\eta_n,$$

$$Re\ (n\text{th term in the form-function summation}),$$

and

$$Im\ (n\text{th term in the form-function summation}).$$

The auxiliary program also called upon the main program to compute f_∞ for the material in question and produced what would be equivalent to a single line in the table of Appendix C. Because of the choice of parameters, at ka = 10.0, the values of k_1a and k_2a are 3.0 and 6.7 respectively, and since spherical Bessel function values with arguments of 10.0, 3.0, and 6.7 are found in tables (11), an additional check was made on a number of the Bessel function values used in computing f_∞. Next, the tabulation of ζ_n was checked by calculating the value of this quantity by hand for several representative values of n. In the case in question, ζ_n was checked at n = 0 and n = 4, since these represented slightly different cases in the logic of the program used to compute f_∞. Using the appropriate recursion relations to eliminate derivatives of Bessel functions, the hand calculation of ζ_n was performed according to Eqs. (6) and (6a-f) and the results compared to the tabulated values which are found, using the auxiliary program, according to Eqs. (13) and (13a-d). This procedure afforded a check upon the equivalence of these two sets of expressions for ζ_n. After these two cases of ζ_n were checked, it could be inferred that the entire tabulation of ζ_n from n = 0 to n = 50 was correct, because all values were computed by a means identical to that used in one of the two cases checked. Then using the computer-tabulated values of ζ_n, $\tan\eta_n$ could be calculated by hand for representative values of n — again making hand calculations using Eq. (5) and comparing the result with the machine-computed result of this quantity according to Eq. (12) as an additional check on the equivalence of these different relations. It could then be inferred that all values of $\tan\eta_n$ for n between 0 and 50 had been correctly tabulated by the computer. From $\tan\eta_n$ the hand calculation proceeded in the same manner through the various pertinent quantities until the real and the imaginary parts of the nth term in the form-function sum

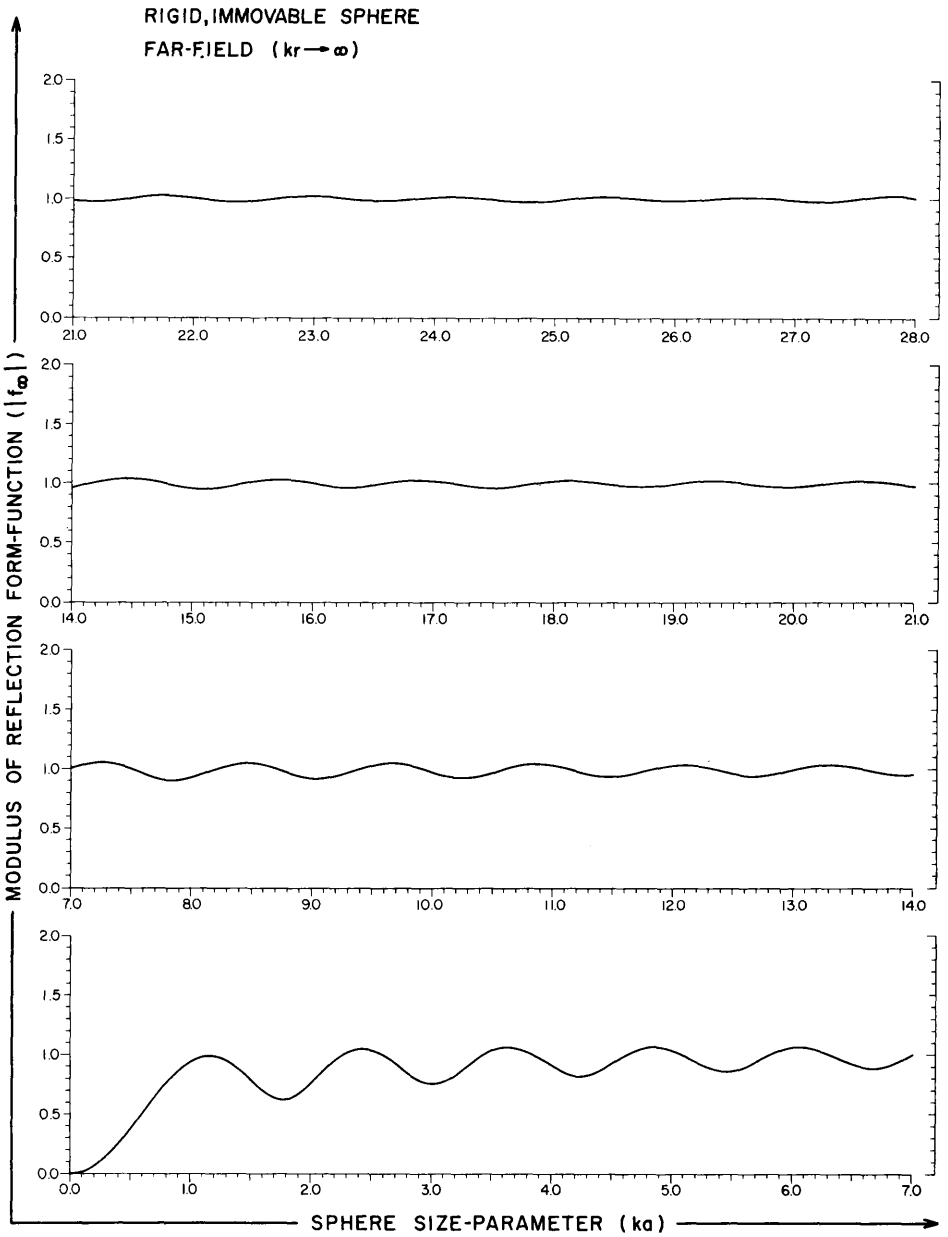


Fig. 7 - Monostatic far-field reflection form-function modulus vs sphere size-parameter for a rigid, immovable sphere

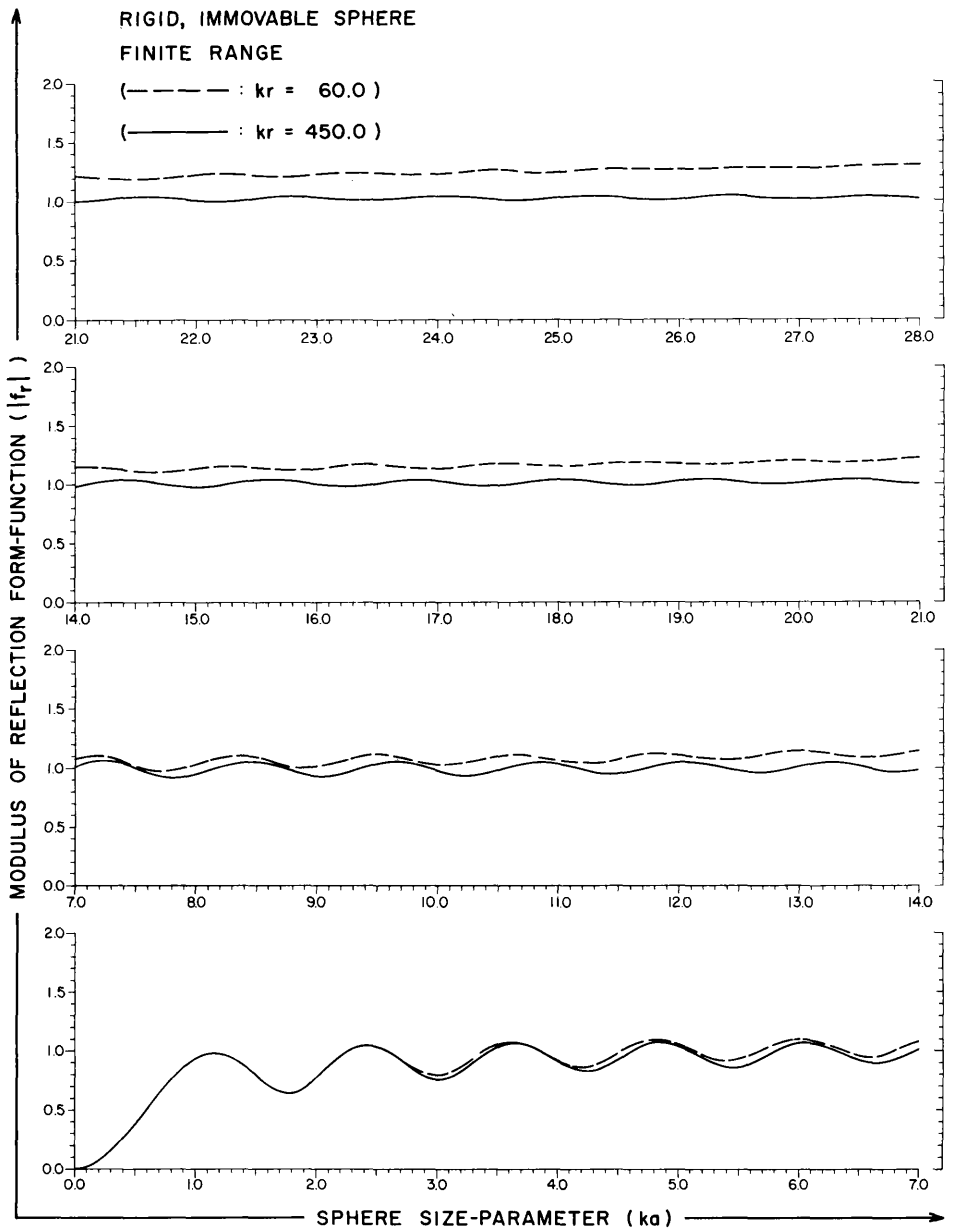


Fig. 8 - Monostatic finite-range reflection form-function moduli vs sphere size-parameter for a rigid, immovable sphere at two values of the range-parameter (kr)

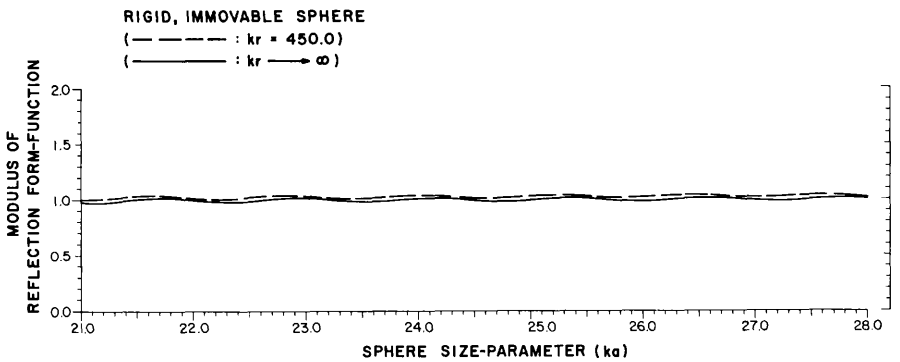


Fig. 9 - Comparison of the moduli of the far-field and the finite-range monostatic reflection form-functions as a function of the sphere size-parameter for a rigid, immovable sphere; the range-parameter in the finite-range case is 450.0

were calculated by hand for several representative n and compared to corresponding machine-tabulated results. It could then be inferred that all of the terms which contribute to the form-function summation for f_∞ at ka = 10.0 were correctly tabulated for the various values of n. If the respective real parts and imaginary parts of the form-function terms were then cumulatively summed by hand, starting at n = 0 and proceeding to some n at which adding a successive term made negligible difference in the result, $Re(f_\infty)$ and $Im(f_\infty)$ were found. These were compared to the corresponding values of $Re(f_\infty)$ and $Im(f_\infty)$ that were computed using the main program in the case in question. Thus a form-function sum could be checked by a hand calculation, with high certainty, in a reasonable length of time. Of course, such a check is not restricted to the material (with idealized constants) that was considered, nor to any particular value of ka, but may be applied to a sphere composed of any material at any ka. Checking the particular case considered merely reduced somewhat the amount of numerical manipulation in hand calculations, and the checking procedure was in no way unique in logical development because of the numerical values chosen.

For calculation of the reflection form-function f_r, only a slight modification of the program used for computing f_∞ had to be made, as has been previously stated. This involved, principally, multiplying terms like those in the sum which expressed f_∞ by complex quantities of the form

$$(-i)^{n+1} h_n^{(2)}(kr) = (-i)^{n+1} [j_n(kr) - i y_n(kr)] , \tag{16}$$

which could be readily done if the real and imaginary parts of this latter term are stored in the computer. Checking a form-function f_r was then merely a matter of verifying that the real and imaginary parts of the expression of Eq. (16) are correctly computed and tabulated. This could be done by means of another simple auxiliary program. As an additional check in this case it could be observed that at large values of kr the form-function f_r values, calculated according to Eq. (10), approached those of f_∞ calculated according to Eq. (11).

SUMMARY

Methods of numerically evaluating the expressions describing monostatic reflection of an acoustic wave by an elastic sphere have been discussed, and examples of the computed results have been shown. No attempt has been made, however, to describe how

particular features observed in the curves that exhibit a sphere's reflection characteristics are influenced, either by the behavior of particular terms or groups of terms in the formulas which describe these characteristics, or by the particular values of the elastic parameters specified in a given sphere-reflection computation problem. The study of both these important aspects of the elastic sphere reflection problem is greatly facilitated by the availability of the computing programs described here, but, as each in itself represents a large area of inquiry whose surface has only been scratched at the present time, it was decided to report here only the results which were complete at this time.

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Appendix A

BESSEL FUNCTION FORMULAS

In this section, all Bessel function formulas pertinent to the elastic sphere reflection problem are collected for reference. Most may be found in, or may be readily derived from expressions found in, Ref. 10. The spherical Bessel functions of the first kind $j_n(x)$ and of the second kind $y_n(x)$, of order n and argument x, are defined

$$j_n(x) = \sqrt{\frac{\pi}{2x}}\, J_{n+1/2}(x) \tag{A1a}$$

and

$$y_n(x) = \sqrt{\frac{\pi}{2x}}\, Y_{n+1/2}(x), \tag{A1b}$$

in which $J_{n+1/2}(x)$ and $Y_{n+1/2}(x)$ are the Bessel functions of the first and second kind having indices of odd half-integral values (i.e., $n = 0, 1, 2, 3, \ldots$). The functions $J_\nu(x)$ and $Y_\nu(x)$ are related through the expression

$$Y_\nu(x) = \begin{cases} \dfrac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}, & (\nu \neq \text{integer}) \quad \text{(A2a)} \\[2ex] \dfrac{1}{\pi}\left[\dfrac{\partial}{\partial\xi} J_\xi(x) - (-1)^\xi \dfrac{\partial}{\partial\xi} J_{-\xi}(x)\right]_{\xi=\nu}, & (\nu = \text{integer}). \quad \text{(A2b)} \end{cases}$$

The Bessel function $J_\nu(x)$ may be expressed by the series

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m}}{m!\,\Gamma(\nu+m+1)}, \tag{A3}$$

where $\Gamma(\tau)$ denotes the gamma function with some argument τ. The spherical Bessel functions of the third kind can be expressed by

$$h_n^{(1)}(x) = j_n(x) + i\,y_n(x) \tag{A4a}$$

and

$$h_n^{(2)}(x) = j_n(x) - i\,y_n(x), \tag{A4b}$$

and are analogous to the Hankel functions of the first and second kind $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$ given by

$$H_\nu^{(1)}(x) = J_\nu(x) + i\,Y_\nu(x), \tag{A5a}$$

and

$$H_\nu^{(2)}(x) = J_\nu(x) - i\,Y_\nu(x). \tag{A5b}$$

The spherical Bessel functions have the following properties. From Eq. (A2a), when $\nu = n + 1/2$,

$$y_n(x) = (-1)^{n+1} j_{-n-1}(x). \tag{A6}$$

If $z_n(x)$ is any one of the functions $j_n(x)$, $y_n(x)$, $h_n^{(1)}(x)$, or $h_n^{(2)}(x)$, the following recursion relations hold:

$$z_{n-1}(x) + z_{n+1}(x) = \frac{2n+1}{x} z_n(x) \tag{A7}$$

and

$$z_n'(x) = z_{n-1}(x) - \frac{n+1}{x} z_n(x), \tag{A8}$$

where the prime indicates the first derivative of $z_n(x)$, with respect to x. From Eqs. (A7) and (A8) it follows that

$$z_{n-2}(x) = \frac{2n-1}{x} z_{n-1}(x) - z_n(x), \tag{A9}$$

$$z_{n-1}'(x) = \frac{n-1}{x} z_{n-1}(x) - z_n(x), \tag{A10}$$

and

$$z_n''(x) = \frac{1}{x^2} \left[(n+1)(n+2) - x^2\right] z_n(x) - \frac{2}{x} z_{n-1}(x), \tag{A11}$$

where the double prime indicates the second derivative of $z_n(x)$ with respect to x.

The pairs $j_n(x)$, $y_n(x)$ and $h_n^{(1)}(x)$, $h_n^{(2)}(x)$ are linearly independent solutions of the differential equation

$$x^2 w'' + 2xw' + [x^2 - n(n+1)]\, w = 0. \tag{A12}$$

For large values of argument, $h_n^{(1)}(x)$, $h_n^{(2)}(x)$, $j_n(x)$, and $y_n(x)$ are often approximated by the first term in their asymptotic representations:

$$h_n^{(1)}(x) \approx \frac{1}{x} e^{i[x-(n+1)\pi/2]} = \frac{1}{x} (-i)^{n+1} e^{ix}, \tag{A13a}$$

$$h_n^{(2)}(x) \approx \frac{1}{x} e^{-i[x-(n+1)\pi/2]} = \frac{1}{x} (i)^{n+1} e^{-ix}, \tag{A13b}$$

$$j_n(x) \approx \frac{1}{x} \cos\left[x - \frac{n+1}{2}\pi\right], \tag{A13c}$$

and

$$y_n(x) \approx \frac{1}{x} \sin\left[x - \frac{n+1}{2}\pi\right]. \tag{A13d}$$

The spherical Bessel functions can be expressed in terms of trigonometric functions:

$$j_n(x) = \frac{1}{x}\left[P(n+1/2,x)\ \sin\left(x - \frac{n\pi}{2}\right) + Q(n+1/2,x)\ \cos\left(x - \frac{n\pi}{2}\right)\right] \tag{A14a}$$

and

$$y_n(x) = (-1)^{n+1}\ \frac{1}{x}\left[P(n+1/2,x)\ \cos\left(x + \frac{n\pi}{2}\right) - Q(n+1/2,x)\ \sin\left(x + \frac{n\pi}{2}\right)\right], \tag{A14b}$$

where

$$P(n+1/2,x) = \sum_{m=0}^{[n/2]} (-1)^m\ (n+1/2,2m)(2x)^{-2m} \tag{A15a}$$

and

$$Q(n+1/2,x) = \sum_{m=0}^{[(n-1)/2]} (-1)^m\ (n+1/2,2m+1)(2x)^{-2m-1}, \tag{A15b}$$

in which

$$(n+1/2,m) = \frac{(n+m)!}{m!\,(n-m)!}, \tag{A15c}$$

and for any real number τ, the notation $[\tau]$ in the upper limit of the summations means "the largest integer $\leq \tau$." For $n = 0$, $Q(n+1/2,x)$ is taken to be equal to zero. Then by Eqs. (A14a, b) and (A15a-c), and using Eq. (A6),

$$j_0(x) = y_{-1}(x) = \frac{\sin x}{x} \tag{A16a}$$

and

$$y_0(x) = -\ j_{-1}(x) = -\ \frac{\cos x}{x}. \tag{A16b}$$

Appendix B

EXAMPLES OF COMPUTED VALUES OF SPHERICAL BESSEL FUNCTIONS OF THE FIRST AND SECOND KIND FOR VARIOUS ARGUMENTS

Digits uncertain in the computed entries, when comparing them either to published tables or to entries computed using higher values of the index M, are underscored. A computed entry checked against tables in Ref. 10 is indicated by $\circ$, and a computed entry checked against tables of Ref. 11 is indicated by $\bullet$.

Table B1a

Spherical Bessel Functions of the First Kind, $0.01 \leq x \leq 50.0$

ORDER	X = 0.01	X = 0.10	X = 1.00	X = 10.0	X = 25.0	X = 50.0
0	.9999833334 +000	.9983341665 +000	.8414709848 +000	-.5440211109 -001	-.5294070004 -002	-.5247497070 -002
1	.3333300000 -002	.3330001190 -001	.3011686790 +000	.7846694180 -001	-.3985987527 -001	-.1940427051 -001
2	.6666619048 -005	.6661906085 -003	.6203505201 -001	.7794219363 -001	.5108849709 -003	.4083240839 -002
3	.9523756615 -008	.9518519722 -005	.9006581118 -002	-.3949584499 -001	.3996205227 -001	.1981259460 -001
4	.1058196248 -010	.1057720150 -006	.1011015808 -002	-.1055892851 +000	.1067848966 -001	-.1309477596 -002
5	.9619972621 -014	.9616310235 -009	.9256115862 -004	-.5553451162 -001	-.3611779599 -001	-.2004830056 -001
6	.7399982735 -017	.7397541095 -011	.7156936311 -005	.4450132234 -001	-.2657031990 -001	-.3101148528 -002
7	.4933323758 -020	.4931887477 -013	.4790134199 -006	.1133862307 +000	.2230122964 -001	.1924200194 -001
8	.2901956050 -023	.2901200103 -015	.2826498803 -007	.1255780237 +000	.3995105769 -001	.8873749112 -002
9	.1527345672 -026	.1526985694 -017	.1491376503 -008	.1000964095 +000	.4865489584 -002	-.1622492725 -001
10	.7273076136 -030	.7271510998 -020	.7116552641 -010	.6460515450 -001	-.3625328560 -001	-.1503922147 -001
11	.3162207566 -033	.3161581506 -022	.3099551855 -011	.3557441489 -001	-.3531824949 -001	.9908454232 -002
12	.1264883213 -036	.1264651338 -024	.1241662597 -012	.1721599975 -001	.3760496072 -002	.1959711041 -001
13	.4684753242 -040	.4683953666 -027	.4604637678 -014	.7465584477 -002	.3907874556 -001	-.1098990255 -003
14	.1615432332 -043	.1615174403 -029	.1589575988 -015	.2941078342 -002	.3844454914 -001	-.1965645589 -001
15	.5211072548 -047	.5210290942 -032	.5132686116 -017	.1063542715 -002	.5516931435 -002	-.1129084539 -001
16	.1579113030 -050	.1578889713 -034	.1556708271 -018	.3559040735 -003	-.3160355415 -001	.1265613175 -001
17	.4511751863 -054	.4511148302 -037	.4451177505 -020	.1109407280 -003	-.4723362292 -001	.1964389234 -001
18	.1219392480 -057	.1219237720 -039	.1203855742 -021	.3238847439 -004	-.3452351793 -001	.1094592893 -002
19	.3126647580 -061	.3126270116 -042	.3088742364 -023	.8896627270 -005	-.3861183620 -002	-.1883389360 -001
20	.7625970141 -065	.7625092315 -045	.7537795724 -025	.2308371961 -005	.2850007148 -001	-.1578502990 -001
21	.1773481519 -068	.1773286447 -047	.1753882578 -026	.5676977728 -006	.5060130086 -001	.5890169082 -002
22	.3941070230 -072	.3940655181 -050	.3899361311 -028	.1327284582 -006	.5853416599 -001	.2085057531 -001
23	.8385256172 -076	.8384409132 -053	.8300118917 -030	.2958028995 -007	.5476019792 -001	.1287534870 -001
24	.1711276839 -079	.1711110751 -055	.1694580174 -031	.6298904527 -008	.4441500610 -001	-.8747747535 -002
25	.3355444906 -083	.3355131533 -058	.3323936367 -033	.1284342236 -008	.3229321405 -001	-.2144814128 -001
26	.6331028342 -087	.6330458573 -061	.6273730961 -035	.2512408773 -009	.2146315055 -001	-.1312935657 -001
27	.1151096098 -090	.1150996139 -063	.1141042434 -036	.4723441381 -010	.1320866511 -001	.7531023315 -002
28	.2019466900 -094	.2019297477 -066	.2002424425 -038	.8548398628 -011	.7595912700 -002	.2141348222 -001
29	.3422825349 -098	.3422547606 -069	.3394883234 -040	.1491458367 -011	.4110015842 -002	.1688034642 -001
30	.5611189244 -102	.5610748381 -072	.5566831269 -042	.2512057385 -012	.2103724689 -002	-.1494673449 -002
31	.8906649810 -106	.8905971559 -075	.8838398944 -044	.4089663774 -013	.1023072399 -002	-.1870384802 -001
32	.1370253849 -109	.1370152617 -077	.1360066051 -045	.6443079246 -014	.4744177559 -003	-.2207217506 -001
33	.2045155042 -113	.2045008329 -080	.2030388613 -047	.9833773547 -015	.2104137664 -003	-.9989979555 -002
34	.2963992875 -117	.2963786237 -083	.2943193082 -049	.1455490311 -015	.8949113806 -004	.8685602458 -002
35	.4174637932 -121	.4174354867 -086	.4146142466 -051	.2091095955 -016	.3658177465 -004	.2197611095 -001
36	.5718682204 -125	.5718304783 -089	.5680684961 -053	.2918781736 -017	.1440110196 -004	.2252047509 -001
37	.7624909739 -129	.7624419580 -092	.7575558616 -055	.3961471214 -018	.5469443082 -005	.1090378268 -001
38	.9902480343 -133	.9901859888 -095	.9840005689 -057	.5232167519 -019	.2007227281 -005	-.6164801068 -002
39	.1253478544 -136	.1253401945 -097	.1245765051 -058	.6729777513 -020	.7128169442 -006	-.2039757632 -001
40	.1547504398 -140	.1547412110 -100	.1538210375 -060	.8435671635 -021	.2452742625 -006	-.2606336952 -001
41	.1864463157 -144	.1864354582 -103	.1853528273 -062	.1031165119 -021	.8187166620 -007	-.2182508231 -001
42	.2193486097 -148	.2193361298 -106	.2180916453 -064	.1229988550 -022	.2653966929 -007	-.1016626710 -001
43	.2521248420 -152	.2521108196 -109	.2507124354 -066	.1432514838 -023	.8363209395 -008	.4542428232 -002
44	.2832863427 -156	.2832709337 -112	.2817341560 -068	.1629935848 -024	.2564299404 -008	.1807009223 -001
45	.3113036771 -160	.3112871081 -115	.3096345646 -070	.1812806648 -025	.7656964845 -009	.2762233593 -001
46	.3347351403 -164	.3347176994 -118	.3329780847 -072	.1971820227 -026	.2228357988 -009	.3220255917 -001
47	.3523527833 -168	.3523348027 -121	.3505412913 -074	.2098616274 -027	.6325268724 -010	.3227442412 -001
48	.3632502957 -172	.3632321337 -124	.3614204144 -076	.2186523394 -028	.1752441267 -010	.2911884666 -001
49	.3669194944 -176	.3669015121 -127	.3651076437 -078	.2231141772 -029	.4742033933 -011	.2421613840 -001
50	.3632866315 -180	.3632691731 -130	.3615274719 -080	.2230696024 -030	.1254041698 -011	.1882910737 -001

Table B1b
Spherical Bessel Functions of the First Kind, $100.0 \leq x \leq 1500.0$

ORDER	X = 100.00	X = 200.00	X = 400.00	X = 800.00	X = 1200.0	X = 1500.0
0	-.5063656408 -002	-.4366486481 -002	-.2127298406 -002	.1137462060 -002	-.7356549906 -004	-.6626013389 -003
1	-.8673825287 -002	-.2457770808 -002	.1307922601 -002	.5615562227 -003	-.8301411567 -003	.7306986747 -004
2	.4803441650 -002	.4329619919 -002	.2137107825 -002	-.1115356225 -002	.7149014616 -004	.6627474786 -003
3	.8913997370 -002	.2566011306 -002	-.1281208753 -002	-.5685271991 -003	.8304390323 -003	-.7086070922 -004
4	-.4179461834 -002	-.4239809524 -002	-.2159528978 -002	.1110381612 -002	-.6664591848 -004	-.6630781619 -003
5	-.9290148935 -002	-.2756802734 -002	.1232619351 -002	.5810189922 -003	-.8309388778 -003	.6688224025 -004
6	.3157545451 -002	.4088185373 -002	.2193426011 -002	-.1102392600 -002	.5902897878 -004	.6635686317 -003
7	.9700629844 -002	.3022534783 -002	-.1161333005 -002	-.5989328720 -003	.8315783573 -003	-.6113131211 -004
8	-.1702450974 -002	-.3861495265 -002	-.2236975998 -002	.1091162609 -002	-.4563424931 -004	-.6641799448 -003
9	-.9990046509 -002	-.3350761881 -002	.1066261526 -002	.6221200774 -003	-.8322673425 -003	.5360393939 -004
10	-.1956578625 -003	.3543172886 -002	.2287623421 -002	-.1076387257 -002	.3545668305 -004	.6648589281 -003
11	.9948958358 -002	.3722795034 -002	-.9461612959 -003	-.6503752430 -003	.8328876345 -003	-.4429591440 -004
12	.2483918285 -002	-.3115051457 -002	-.2342027695 -002	.1057688969 -002	-.1949299956 -004	-.6655381320 -003
13	-.9327978787 -002	-.4112176466 -002	.7997845650 -003	.6834280232 -003	-.8332939386 -003	.3320361220 -004
14	-.5002472557 -002	.2559907634 -002	.2396013153 -002	-.1034623273 -002	.7438859374 -006	.6661357970 -003
15	.7877261745 -002	.4483363073 -002	-.6266736114 -003	-.7209331168 -003	.8333119159 -003	-.2032498679 -004
16	.7444423698 -002	-.1864986358 -002	-.2444533858 -002	.1006687115 -002	.2078333856 -004	-.6665558468 -003
17	-.5420601925 -002	-.4791085822 -002	.4243995681 -003	.7624589604 -003	-.8327403741 -003	.5660758162 -005
18	-.9341634372 -002	.1026546339 -002	.2481668821 -002	-.9733295354 -003	-.4507159947 -004	.6666879312 -003
19	.1964197207 -002	.4980996894 -002	-.1948452022 -003	-.8074754514 -003	.8313506665 -003	.1078421081 -004
20	.1010767128 -001	-.5525194439 -004	-.2500666228 -002	.9339651071 -003	.7209049614 -004	-.6664075417 -003
21	.2179948019 -002	-.4992323543 -002	-.6147308614 -004	.8553411631 -003	-.8288875745 -003	-.2899935028 -004
22	-.9170293635 -002	-.1018097617 -002	.2494057871 -002	-.8879905196 -003	-.1017923009 -003	.6655762270 -003
23	-.6306580155 -002	.4763251579 -002	.3420545966 -003	-.9052906299 -003	.8250703633 -003	.4896663709 -004
24	.6206200962 -002	.2137461739 -002	-.2453866456 -002	.8348046951 -003	.1341075566 -003	-.6640419390 -003
25	.9347618626 -002	-.4239573453 -002	-.6426532375 -003	.9564224174 -003	-.8195943946 -003	-.7065867376 -004
26	-.1438915463 -002	-.3218552969 -002	.2371928168 -002	-.7738327660 -003	-.1689403147 -003	.6616395441 -003
27	-.1011024382 -001	.3386656917 -002	.9569337197 -003	-.1007688838 -002	.6121327740 -003	.9403660431 -004
28	-.4121718639 -002	.4149883621 -002	-.2240349782 -002	.7045541583 -003	.2061630669 -003	-.6581915353 -003
29	.7760864197 -002	-.2203940084 -002	-.1276183564 -002	.1057888322 -002	-.6023400254 -003	-.1190478827 -003
30	.8700628515 -002	-.4800045947 -002	.2052112706 -002	-.6265348946 -003	-.2456114516 -003	.6535089852 -003
31	-.2453480803 -002	.7399260709 -003	.1589130751 -002	-.1105661608 -002	.7898547796 -003	.1456239147 -003
32	-.1024632142 -001	.5033122659 -002	-.1801824613 -002	.5394640430 -003	.2870788272 -003	-.6473927808 -003
33	-.4206628121 -002	.8958387931 -003	-.1881927251 -002	.1149493061 -002	-.7743046765 -003	-.1736776019 -003
34	.7427880580 -002	-.4733016663 -002	.1486601798 -002	-.4431939991 -003	-.3303108386 -003	.6396351813 -003
35	.9331865722 -002	-.2528729542 -002	.2138366061 -002	-.1167718544 -002	.7553116032 -003	.2031008202 -003
36	-.8022559181 -003	.3835317676 -002	-.1107041822 -002	.3377839784 -003	.3750001203 -003	-.6300217424 -003
37	-.9917512542 -002	.3928620493 -002	-.2340401194 -002	.1218541332 -002	-.7324992959 -003	-.2337618784 -003
38	-.6635878488 -002	-.2362084991 -002	.6682165985 -003	-.2235457286 -003	-.4207813263 -003	.6183336485 -003
39	.4807886106 -002	-.4838023215 -002	.2469032889 -002	-.1240057608 -002	.7054991608 -003	.2655030057 -003
40	.1043410851 -001	.4510658208 -003	-.1805826029 -003	.1010900398 -003	.4672266878 -003	-.6043504902 -003
41	.3643741789 -002	.5020704872 -002	-.2505600866 -002	.1250292975 -002	-.6739613594 -003	-.2981379321 -003
42	-.7409802827 -002	.1632526701 -002	-.3393295767 -003	.2862785632 -004	-.5138423484 -003	.5878535246 -003
43	-.9942074192 -002	-.4326881024 -002	.2433493331 -002	-.1227251265 -002	.6375641930 -003	.3314496319 -003
44	-.1239801719 -002	-.3514719946 -002	.8686143762 -003	-.1642664314 -003	.5500657524 -003	-.5686294460 -003
45	.8838650661 -002	.2762830648 -002	-.2240226632 -002	.1228976624 -002	-.5960259831 -003	-.3651883123 -003
46	.9282973821 -002	.4771807892 -002	-.1378265935 -002	.3040625224 -003	-.6052643895 -003	.5464746884 -003
47	-.2054850077 -003	-.5439399783 -003	.1919779802 -002	-.1193629356 -002	.5491179929 -003	.3990697430 -003
48	-.9478184579 -002	-.5030179381 -002	.1834213638 -002	-.4458060084 -003	.6487362386 -003	-.5212002713 -003
49	-.8988354033 -002	-.1895697022 -002	-.1474982995 -002	.3189575378 -002	-.4966784809 -003	-.4327740272 -003
50	.5797140853 -003	.4091809355 -002	-.2199271929 -002	.5868284614 -003	-.6897122053 -003	.4926371855 -003

Table B2a

Spherical Bessel Functions of the Second Kind, $0.01 \leq x \leq 50.0$

ORDER	X = 0.01	X = 0.10	X = 1.00	X = 10.0	X = 25.0	X = 50.0
0	-.9999500005 +002	-.9950041653 +001	-.5403023059 +000°	.8390715292 -001°	-.3964811248 -001	-.1929932057 -001°
1	-.1000049999 +005	-.1004987507 +003	-.1381773291 +001	.6279282638 -001	.3708145505 -002	.4861510662 -002
2	-.3000050001 +007	-.3005012479 +004	-.3605017566 +001	-.6506930499 -001	.4009308994 -001	.1959101121 -001
3	-.1500015000 +010	-.1501501252 +006	-.1664331454 +002	-.9532747888 -001	.4310472483 -002	-.2902409541 -002
4	-.1050007500 +013	-.1050750375 +008	-.1128981842 +003	-.1659930220 -002	-.3888615764 -001	-.1999734855 -001
5	-.9450052500 +015	-.9455251876 +009	-.9994403434 +003	.9383354169 -001	-.1830948923 -001	-.6971131969 -003
6	-.1039504725 +019	-.1039972631 +012	-.1088094559 +005	.1048768261 +000	.3082998238 -001	.1984398364 -001
7	-.1351355197 +022	-.1351869868 +014	-.1404528524 +006	.4250633221 -001	.3434108007 -001	.5856548944 -002
8	-.2027031757 +025	-.2027700805 +016	-.2095911840 +007	-.4111732776 -001	-.1022533434 -001	-.1808701896 -001
9	-.3445952635 +028	-.3446956182 +018	-.3549004843 +008	-.1124057894 +000	-.4129430742 -001	-.1200613539 -001
10	-.6547307979 +031	-.6549013975 +020	-.6722150083 +009	-.1724536721 +000	-.2115833930 -001	.1352468751 -001
11	-.1374934331 +035	-.1375258465 +023	-.1408102513 +011	-.2497469220 +000	.2352130241 -001	.1768650415 -001
12	-.3162348306 +038	-.3163028980 +025	-.3231913629 +012	-.4019642485 +000	.4279793751 -001	-.5388895605 -002
13	-.7905869391 +041	-.7907434923 +027	-.8065703047 +013	-.7551636992 +000	.1927663511 -001	-.2038095195 -001
14	-.2134584419 +045	-.2134975799 +030	-.2174507909 +015	-.1636977739 +001	-.2197917160 -001	-.5616818447 -002
15	-.6190294025 +048	-.6191350742 +032	-.6298007233 +016	-.3992071745 +001	-.4477247416 -001	.1712319725 -001
16	-.1918990934 +052	-.1919297380 +035	-.1950207734 +018	-.1073844467 +002	-.3353869636 -001	.1623320074 -001
17	-.6332669465 +055	-.6333619442 +037	-.6429387516 +019	-.3144479566 +002	.5013949613 -003	-.6409284759 -002
18	-.2216434121 +059	-.2216747611 +040	-.2248335423 +021	-.9931834016 +002	.3424064931 -001	-.2071970007 -001
19	-.8200805613 +062	-.8201902826 +042	-.8312411677 +022	-.3360330629 +003	.5017476601 -001	-.8923293294 -002
20	-.3198313967 +066	-.3198719935 +045	-.3239592219 +024°	-.1211210605 +004°	.4403198567 -001	.1375953130 -001°
21	-.1311308645 +070	-.1311466971 +048	-.1327401568 +026	-.4629930419 +004	.2203769049 -001	.2020610896 -001
22	-.5638626852 +073	-.5639275989 +050	-.5704587152 +027	-.1869749019 +005	-.6127158031 -002	.3617722405 -002
23	-.2537381952 +077	-.2537661081 +053	-.2565736817 +029	-.7950877546 +005	-.3306657495 -001	-.1695015880 -001
24	-.1192569461 +081	-.1192695069 +056	-.1205325845 +031	-.3549937544 +006	-.5603800287 -001	-.1955087167 -001
25	-.5843590106 +084	-.5844180459 +058	-.5903530905 +032	-.1659960621 +007	-.7676791067 -001	-.2209695444 -002
26	-.2980230835 +088	-.2980520107 +061	-.3009595435 +034	-.8110805414 +007	-.1005685349 +000	.1729698232 -001
27	-.1579522284 +092	-.1579669813 +064	-.1594495228 +036	-.4132730807 +008	-.1364373833 +000	.2054449671 -001
28	-.8687372263 +095	-.8688154164 +066	-.8766714157 +037	-.2191893890 +009	-.1995937084 +000	.5301964054 -002
29	-.4951802032 +099	-.4952232077 +069	-.4995432574 +039	-.1208052209 +010	-.3186362719 +000	-.1450025768 -001
30	-.2921563112 +103	-.2921808237 +072	-.2946428547 +041°	-.6908318645 +010°	-.5523878932 +000	-.2241226812 -001°
31	-.1782153449 +107	-.1782298072 +075	-.1796821870 +043	-.4093269152 +011	-.1029190188 +001	-.1284270942 -001
32	-.1122756643 +111	-.1122844864 +078	-.1131703136 +045	-.2509676380 +012	-.2041171379 +001	.6230454247 -002
33	-.7297918004 +114	-.7298473791 +080	-.7354273559 +046	-.1590356955 +013	-.4277855399 +001	.2094229994 -001
34	-.4889604951 +118	-.4889966211 +083	-.4926231582 +048	-.1040442396 +014	-.9423481089 +001	.2183222768 -001
35	-.3373827343 +122	-.3374069388 +086	-.3398364364 +050	-.7020016838 +014	-.2173095241 +002	.9186174253 -002
36	-.2395417365 +126	-.2395584375 +089	-.2412346075 +052	-.4880167715 +015	-.5229242374 +002	-.8787860241 -002
37	-.1748654642 +130	-.1748773220 +092	-.1760672798 +054	-.3492322264 +016	-.1309629249 +003	-.2201645020 -001
38	-.1311490958 +134	-.1311577519 +095	-.1320263364 +056	-.2570440021 +017	-.3405963510 +003	-.2423681507 -001
39	-.1009848020 +138	-.1009912941 +098	-.1016426723 +058	-.1944315593 +018	-.9180738362 +003	-.1530824500 -001
40	-.7977799226 +141	-.7978299119 +100	-.8028450850 +059°	-.1510304918 +019°	-.2560516972 +004	.4978797142 -004°
41	-.6462017273 +145	-.6462412187 +103	-.6502028761 +061	-.1203903828 +020	-.7378001151 +004	.1538890151 -001
42	-.5363474256 +149	-.5363794136 +106	-.5395881027 +063	-.9841371280 +020	-.2193444685 +005	.2549578853 -001
43	-.4559953053 +153	-.4559218554 +109	-.4585848670 +065	-.8244775205 +021	-.6719911814 +005	.2795393900 -001
44	-.3966289103 +157	-.3966514778 +112	-.3989148755 +067	-.7074540716 +022	-.2119184843 +006	.2314406532 -001
45	-.3529997256 +161	-.3530193593 +115	-.3549883807 +069	-.6213893485 +023	-.6872306859 +006	.1324249727 -001
46	-.3212297462 +165	-.3212472203 +118	-.3229995349 +071	-.5583897664 +024	-.2289601212 +007	.9572797190 -003
47	-.2987436605 +169	-.2987595618 +121	-.3003540686 +073	-.5130885893 +025	-.7830085824 +007	-.1146195700 -001
48	-.2838064743 +173	-.2838212625 +124	-.2853040653 +075	-.4818502621 +026	-.2746472492 +008	-.2273499802 -001
49	-.2752922771 +177	-.2753063259 +127	-.2767149079 +077	-.4622638684 +027	-.9873304685 +008	-.3264393915 -001
50	-.2725393514 +181	-.2725529788 +130	-.2739192284 +079°	-.4528227270 +028°	-.3635181406 +009	-.4190000150 -001°

Table B2b
Spherical Bessel Functions of the Second Kind, $100.0 \leq x \leq 1500.0$

ORDER	X = 100.00	X = 200.00	X = 400.00	X = 800.00	X = 1200.0	X = 1500.0
0	-.8623188724 -002	-.2435938375 -002	.1313240847 -002	.5601593916 -003	-.8300798522 -003	.7351160168 -004
1	.4977424524 -002	.4354306795 -002	.2130581501 -002	-.1116761861 -002	.7287377218 -004	.6626503124 -003
2	.8772511459 -002	.2501252977 -002	-.1297261485 -002	-.5643472485 -003	.8302620366 -003	-.7218630105 -004
3	-.4538798951 -002	-.4291775470 -002	-.2146797270 -002	.1113234691 -002	-.6941434703 -004	-.6628909334 -003
4	-.9090227385 -002	-.2651465118 -002	.1259692533 -002	.5740880521 -003	-.8306669536 -003	.6909281003 -004
5	.3720678486 -002	.4172459540 -002	.2175140352 -002	-.1106776200 -002	.6318434487 -004	.6633054903 -003
6	.9499502019 -002	.2880950393 -002	-.1199876173 -002	-.5893062248 -003	.8312461435 -003	-.6422856977 -004
7	-.2485743224 -002	-.3985197764 -002	-.2214136327 -002	.1097199974 -002	-.5417917832 -004	-.6638621378 -003
8	-.9872363503 -002	-.3179840225 -002	.1116846061 -002	.6098787243 -003	-.8319233832 -003	.5758994839 -004
9	.8074414282 -003	.3714911345 -002	.2261602285 -002	-.1084240051 -002	.4239359706 -004	.6645148239 -003
10	.1002577737 -001	.3532756803 -002	-.1009419953 -002	-.6356294256 -003	.8325946151 -003	-.4917276062 -004
11	.1297971820 -002	-.3343971881 -002	-.2314596833 -002	.1067554779 -002	-.2782319130 -004	-.6652032426 -003
12	-.9727243855 -002	-.3917313569 -002	.8763306347 -003	.6663216255 -003	-.8331278929 -003	.3897297757 -004
13	-.3729782785 -002	.2854307684 -002	.2369367497 -002	-.1046732228 -002	.1046636019 -004	.6658527923 -003
14	.8720202504 -002	.4302645107 -002	-.7163983287 -003	-.7016488382 -003	.8333633861 -003	-.2698762731 -004
15	.6258641510 -002	-.2230424144 -002	-.2421306376 -002	.1021297458 -002	.9673254970 -005	-.6663745530 -003
16	-.6780023636 -002	-.4648360849 -002	.5287470845 -003	.7412241146 -003	-.8331134936 -003	.1321588655 -004
17	-.8496049310 -002	.1463444604 -002	.2464928011 -002	-.9907219628 -003	-.3258387604 -004	.6666653025 -003
18	.3806406377 -002	.4904463655 -002	-.3130658836 -003	-.7845682005 -003	.8321631305 -003	.2339637180 -005
19	.9904419670 -002	-.5561188276 -003	-.2493886605 -002	.9544356836 -003	.5824223924 -004	-.6666075914 -003
20	.5631729408 -004	-.5012906826 -002	.6991193965 -004	.8310969400 -003	-.8302702577 -003	-.1967143456 -004
21	-.9881329579 -002	-.4715270719 -003	.2501052579 -002	-.9118419654 -003	-.8660980638 -004	.6660699056 -003
22	-.4305289013 -002	.4911528506 -002	.1989512126 -003	-.8801084457 -003	.8271667397 -003	.3876543852 -004
23	.7943949524 -002	.1576620986 -002	-.2478670567 -002	.8623358653 -003	.1176285591 -003	-.6649069424 -003
24	.8038945289 -002	-.4541022574 -002	-.4901950042 -003	.9307706778 -003	-.8225596212 -003	-.5959918938 -004
25	-.4004866332 -002	-.2689171516 -002	.2418621679 -002	-.8053261613 -003	-.1512164103 -003	.6629600355 -003
26	-.1008142712 -001	.3855283838 -002	.7985692683 -003	-.9821102205 -003	.8161329237 -003	.8213983059 -004
27	-.1338290041 -002	.3710821733 -002	-.2312811251 -002	.7402613592 -003	.1872622811 -003	-.6600577615 -003
28	.9345367595 -002	-.2834807861 -002	-.1116580815 -002	.1033003189 -002	-.8075500692 -003	-.1063419485 -003
29	.6665149571 -002	-.4518741974 -002	.2153698485 -002	-.6666598820 -003	-.2256209094 -003	.6560167676 -003
30	-.5412929349 -002	.1501778979 -002	.1434251342 -002	-.1082169355 -002	.7964570411 -003	.1321452747 -003
31	-.9967036474 -002	.4976784562 -002	-.1934975155 -002	.5841444686 -003	.2661074757 -003	-.6506428597 -003
32	-.8663036295 -003	.6590815835 -004	-.1739009929 -002	.1128170732 -002	-.7824863986 -003	-.1594722748 -003
33	.9403939115 -002	-.4955364411 -002	.1652386042 -002	-.4924805966 -003	-.3084921556 -003	.6437323944 -003
34	.7166942836 -002	-.1725955236 -002	.2015784591 -002	-.1169415982 -002	.7652622533 -003	.1882256551 -003
35	-.4458748557 -002	.4359909854 -002	-.1304663200 -002	.3916184682 -003	.3524947351 -003	-.6350740143 -003
36	-.1033265431 -001	.3273723234 -002	-.2247362309 -002	.1204172121 -002	-.7444063148 -003	-.2182858251 -003
37	-.3084089090 -002	-.3165000874 -002	.8945195785 -003	-.2817377621 -003	-.3977794526 -003	.6244507708 -003
38	.8019587494 -002	-.4460598562 -002	.2415084730 -002	-.1230585036 -002	.7195450990 -003	.2495083636 -003
39	.9259171460 -002	.1447670428 -002	-.4296157681 -003	.1632939524 -003	.4439502631 -003	-.6116426748 -003
40	-.7048420403 -003	.5032428380 -002	-.2499933844 -002	.1246710314 -002	-.6903183733 -003	-.2817215445 -003
41	-.9830093514 -002	.5904630666 -003	-.7662083537 -004	-.3706453304 -004	-.4905467533 -003	.5964297114 -003
42	-.7454135575 -002	-.4787386208 -002	.2484035021 -002	-.1250555760 -002	.6563888895 -003	.3147239885 -003
43	.3494078274 -002	-.2625102205 -002	.6044782773 -003	-.9580701640 -004	.5370409664 -003	-.5785953520 -003
44	.1049398367 -001	.3645466749 -002	-.2352560996 -002	.1240136747 -002	-.6174534195 -003	-.3482825190 -003
45	.5845567196 -002	.4247334908 -002	-.1127923099 -002	.2337722295 -003	-.5828354283 -003	.5579305892 -003
46	-.5174517526 -002	-.1712929366 -002	.2095958490 -002	-.1213545155 -002	.5732550662 -003	.3821303080 -003
47	-.1065786849 -001	-.5043847063 -002	.1615233448 -002	-.3748468538 -003	.6272626959 -003	-.5342385101 -003
48	-.4950457544 -002	-.6828979895 -003	-.1712340547 -002	.1169032091 -002	-.5235967694 -003	-.4159654137 -003
49	.5855924677 -002	.4712641539 -002	-.2030476030 -002	.5165919949 -003	-.6695867682 -003	.5073394134 -003
50	.1074782297 -001	.3015655551 -002	.1209797729 -002	-.1105103832 -002	.4683558610 -003	.4494498150 -003

Appendix C

TABLE OF VALUES OF THE FAR-FIELD MONOSTATIC REFLECTION FORM-FUNCTION FOR $0 < ka \leq 30.0$ IN SIZE-PARAMETER INCREMENTS OF 0.1 FOR A SPHERE COMPOSED OF 2S ALUMINUM

[Note: $0 \leq$ arg (radians) $< 2\pi$, $0^\circ \leq$ arg (degrees) $< 360^\circ$, tolerance $\epsilon = 2 \times 10^{-6}$]

SIZE PARAMETER (KA)	MODULUS OF FORM FUNCTION	ARGUMENT OF FORM FUNCTION (DEG MIN SEC)			ARGUMENT OF FORM FUNCTION (RADIANS)	REAL PART OF FORM FUNCTION	NO. TERMS USED IN REAL PART	IMAG PART OF FORM FUNCTION	NO. TERMS USED IN IMAG PART
.0000000000 +000	INDETERMINATE POINT								
.1000000000 +000	.1172384511 -001	180	0	28.1	.3141728749 +001	-.1172384500 -001	10	-.1595555015 -005	10
.2000000000 +000	.4596929147 -001	180	3	41.0	.3142663952 +001	-.4596926510 -001	10	-.4924682022 -004	10
.3000000000 +000	.1000512254 +000	180	12	7.9	.3145121403 +001	-.1000506025 +000	10	-.3530549745 -003	10
.4000000000 +000	.1697723886 +000	180	27	56.3	.3149719720 +001	-.1697667820 +000	10	-.1379736268 -002	10
.5000000000 +000	.2497514897 +000	180	53	1.9	.3157019119 +001	-.2497217729 +000	10	-.3852629759 -002	10
.6000000000 +000	.3337727299 +000	181	29	33.5	.3167644101 +001	-.3336594742 +000	10	-.8694279318 -002	10
.7000000000 +000	.4151440294 +000	182	20	31.8	.3182471237 +001	-.4147972127 +000	10	-.1696577388 -001	10
.8000000000 +000	.4870938665 +000	183	30	41.0	.3202878133 +001	-.4861794124 +000	10	-.2983309792 -001	10
.9000000000 +000	.5432498046 +000	185	7	38.3	.3231081154 +001	-.5410760316 +000	10	-.4854975049 -001	10
.1000000000 +001	.5782309237 +000	187	23	40.2	.3270650873 +001	-.5734220898 +000	10	-.7441846547 -001	10
.1100000000 +001	.5883867695 +000	190	38	45.9	.3327402003 +001	-.5782588985 +000	10	-.1066997556 +000	10
.1200000000 +001	.5727851900 +000	195	26	13.8	.3411022217 +001	-.5521207574 +000	10	-.1524648915 +000	10
.1300000000 +001	.5347706224 +000	202	42	20.9	.3537883515 +001	-.4933253974 +000	10	-.2064211010 +000	10
.1400000000 +001	.4848024454 +000	213	56	57.3	.3734118774 +001	-.4021594304 +000	10	-.2707419503 +000	10
.1500000000 +001	.4448697332 +000	230	49	58.6	.4028794921 +001	-.2809724274 +000	10	-.3449109662 +000	10
.1600000000 +001	.4481147256 +000	252	34	34.5	.4408287064 +001	-.1341818693 +000	10	-.4275535443 +000	10
.1700000000 +001	.5172075762 +000	273	31	50.2	.4774009867 +001	.3185062359 -001	10	-.5162259338 +000	10
.1800000000 +001	.6421860291 +000	289	0	25.4	.5044124808 +001	.2091501678 +000	10	-.6071730423 +000	10
.1900000000 +001	.7964570131 +000	299	12	52.1	.5222277655 +001	.3887348069 +000	10	-.6951467640 +000	10
.2000000000 +001	.9557564358 +000	305	58	40.1	.5340320248 +001	.5614800550 +000	10	-.7734406973 +000	10
.2100000000 +001	.1101415823 +001	310	45	33.1	.5423771156 +001	.7190938350 +000	10	-.8342786532 +000	10
.2200000000 +001	.1219473679 +001	314	30	52.9	.5489316746 +001	.8549632538 +000	10	-.8695710947 +000	10
.2300000000 +001	.1300296431 +001	317	53	30.6	.5548259305 +001	.9646644022 +000	10	-.8718907041 +000	10
.2400000000 +001	.1338731234 +001	321	23	16.2	.5609275949 +001	.1046068556 +001	10	-.8354291675 +000	10
.2500000000 +001	.1334445196 +001	325	27	10.7	.5680225717 +001	.1099130320 +001	10	-.7567407215 +000	10
.2600000000 +001	.1292376894 +001	330	33	39.0	.5769374945 +001	.1125502662 +001	10	-.6352021676 +000	10
.2700000000 +001	.1223313703 +001	337	14	32.7	.5885990747 +001	.1128078558 +001	10	-.4732179041 +000	10
.2800000000 +001	.1144318251 +001	346	1	55.9	.6039401314 +001	.1110482543 +001	10	-.2762114795 +000	10
.2900000000 +001	.1077782399 +001	357	12	48.3	.6234550350 +001	.1076507979 +001	10	-.5239723940 -001	10
.3000000000 +001	.1046510150 +001	10	19	58.6	.1803439692 +000	.1029537905 +001	10	.1877104062 +000	10
.3100000000 +001	.1063749788 +001	23	57	52.2	.4182592805 +000	.9720517406 +000	10	.4320636815 +000	10
.3200000000 +001	.1125137766 +001	36	25	25.7	.6357154505 +000	.9053388418 +000	10	.6680543192 +000	10
.3300000000 +001	.1211899966 +001	46	48	27.6	.8169478268 +000	.8294844587 +000	10	.8835479950 +000	10
.3400000000 +001	.1301228801 +001	55	8	52.2	.9625112111 +000	.7436015339 +000	10	.1067826367 +001	10
.3500000000 +001	.1373813934 +001	61	56	27.1	.1081071848 +001	.6462184017 +000	10	.1212339268 +001	10
.3600000000 +001	.1416460447 +001	67	46	35.8	.1182925207 +001	.5357315948 +000	10	.1311240579 +001	10
.3700000000 +001	.1422392396 +001	73	12	35.2	.1277751613 +001	.4108727653 +000	10	.1361715796 +001	10
.3800000000 +001	.1390790944 +001	78	45	24.2	.1374564096 +001	.2711698322 +000	10	.1364099106 +001	10
.3900000000 +001	.1326957266 +001	84	55	31.1	.1482225987 +001	.1173754523 +000	11	.1321755874 +001	10

SIZE PARAMETER (KA)	MODULUS OF FORM FUNCTION	ARGUMENT OF FORM FUNCTION (DEG MIN SEC)			ARGUMENT OF FORM FUNCTION (RADIANS)	REAL PART OF FORM FUNCTION	NO. TERMS USED IN REAL PART	IMAG PART OF FORM FUNCTION	NO. TERMS USED IN IMAG PART
.4000000000 +001	.1241649068 +001	92	13	28.7	.1609623647 +001	-.4819779351 -001	11	.1240713255 +001	10
.4100000000 +001	.1150593352 +001	101	6	0.7	.1764531442 +001	-.2215185234 +000	11	.1129068026 +001	10
.4200000000 +001	.1072415237 +001	111	43	21.2	.1949926252 +001	-.3969140929 +000	11	.9962598270 +000	10
.4300000000 +001	.1023910590 +001	123	39	4.6	.2158121995 +001	-.5673865633 +000	11	.8523293875 +000	10
.4400000000 +001	.1012919476 +001	135	42	48.8	.2368648480 +001	-.7251065290 +000	11	.7072668427 +000	10
.4500000000 +001	.1033690392 +001	146	30	4.4	.2556928568 +001	-.8619918650 +000	12	.5705136727 +000	10
.4600000000 +001	.1069821288 +001	155	5	14.5	.2706785258 +001	-.9702756909 +000	12	.4506469476 +000	10
.4700000000 +001	.1101808335 +001	161	11	22.9	.2813290937 +001	-.1042962179 +001	12	.3552625769 +000	10
.4800000000 +001	.1112764524 +001	164	50	13.0	.2876947316 +001	-.1074023994 +001	12	.2910624445 +000	10
.4900000000 +001	.1090540403 +001	165	59	12.4	.2897015856 +001	-.1058085799 +001	12	.2640697111 +000	10
.5000000000 +001	.1027957484 +001	164	12	59.2	.2866117404 +001	-.9891994122 +000	12	.2796088545 +000	10
.5100000000 +001	.9236104399 +000	158	20	18.1	.2763525558 +001	-.8583849358 +000	12	.3409274801 +000	10
.5200000000 +001	.7878787679 +000	145	42	21.4	.2543048242 +001	-.6509112657 +000	13	.4439228278 +000	10
.5300000000 +001	.6645925899 +000	121	42	35.3	.2124236608 +001	-.3493211994 +000	13	.5653830650 +000	10
.5400000000 +001	.6547430604 +000	86	4	25.3	.1502269459 +001	.4483238352 -001	13	.6532063476 +000	10
.5500000000 +001	.8075798150 +000	53	47	27.9	.9388314462 +000	.4770626717 +000	13	.6516105930 +000	10
.5600000000 +001	.1024178043 +001	33	15	54.4	.5805857469 +000	.8563577681 +000	13	.5617757881 +000	10
.5700000000 +001	.1213356064 +001	21	28	25.1	.3747855130 +000	.1129132361 +001	13	.4441768201 +000	10
.5800000000 +001	.1347567831 +001	15	14	45.6	.2660927472 +000	.1300141157 +001	13	.3543614413 +000	10
.5900000000 +001	.1428806154 +001	12	40	15.2	.2211488147 +000	.1394009152 +001	14	.3134094930 +000	10
.6000000000 +001	.1464896768 +001	12	38	28.1	.2206293733 +000	.1429387536 +001	14	.3205835514 +000	11
.6100000000 +001	.1463456842 +001	14	34	23.8	.2543516706 +000	.1416372504 +001	14	.3682320706 +000	11
.6200000000 +001	.1432555463 +001	18	13	27.0	.3180716925 +000	.1360698816 +001	14	.4480110338 +000	11
.6300000000 +001	.1382196644 +001	23	32	49.7	.4109752548 +000	.1267103398 +001	14	.5521924865 +000	11
.6400000000 +001	.1324875337 +001	30	33	3.8	.5332165359 +000	.1140951563 +001	14	.6734420447 +000	11
.6500000000 +001	.1274616209 +001	39	7	52.4	.6829686866 +000	.9887231147 +000	14	.8044085300 +000	11
.6600000000 +001	.1244130293 +001	48	53	52.1	.8534276016 +000	.8178964601 +000	15	.9374996351 +000	11
.6700000000 +001	.1240649452 +001	59	7	51.2	.1032028670 +001	.6365506969 +000	15	.1064901063 +001	11
.6800000000 +001	.1262811617 +001	68	59	3.4	.1204002975 +001	.4528744674 +000	15	.1178812070 +001	12
.6900000000 +001	.1301173702 +001	77	48	41.4	.1358066829 +001	.2747150511 +000	15	.1271843011 +001	12
.7000000000 +001	.1341954934 +001	85	19	44.5	.1489272484 +001	.1092801806 +000	15	.1337497995 +001	12
.7100000000 +001	.1371132016 +001	91	32	28.6	.1597696616 +001	-.3687939909 -001	15	.1370635953 +001	12
.7200000000 +001	.1376751947 +001	96	33	45.4	.1685335510 +001	-.1573474700 +000	15	.1367730857 +001	12
.7300000000 +001	.1348611176 +001	100	25	0.0	.1752601587 +001	-.2438361485 +000	16	.1326384573 +001	12
.7400000000 +001	.1271018216 +001	102	36	23.5	.1790821722 +001	-.2774053264 +000	16	.1240376391 +001	12
.7500000000 +001	.1026554091 +001	99	15	2.2	.1732250141 +001	-.1650219406 +000	16	.1013203366 +001	12
.7600000000 +001	.1184799154 +001	132	42	26.3	.2316179241 +001	-.8035938610 +000	16	.8706238810 +000	13
.7700000000 +001	.1055766240 +001	123	23	48.2	.2153678879 +001	-.5811283715 +000	16	.8814375584 +000	13
.7800000000 +001	.8753761436 +000	122	36	7.7	.2139811004 +001	-.4716546291 +000	16	.7374451191 +000	13
.7900000000 +001	.6724043715 +000	121	24	0.8	.2118833799 +001	-.3503314996 +000	16	.5739298556 +000	13

SIZE PARAMETER (KA)	MODULUS OF FORM FUNCTION	ARGUMENT OF FORM FUNCTION (DEG MIN SEC)			ARGUMENT OF FORM FUNCTION (RADIANS)	REAL PART OF FORM FUNCTION	NO. TERMS USED IN REAL PART	IMAG PART OF FORM FUNCTION	NO. TERMS USED IN IMAG PART
.8000000000 +001	.4435576964 +000	116	36	12.2	.2035113293 +001	-.1986305418 +000	17	.3965971985 +000	13
.8100000000 +001	.2039213015 +000	95	52	20.5	.1673288267 +001	-.2086371744 -001	17	.2028511831 +000	13
.8200000000 +001	.1649123592 +000	358	7	36.6	.6250492236 +001	.1648242349 +000	17	-.5390531121 -002	13
.8300000000 +001	.3980328906 +000	327	38	42.9	.5718488218 +001	.3362385808 +000	17	-.2130112644 +000	13
.8400000000 +001	.6218798769 +000	320	40	8.1	.5596728637 +001	.4810219889 +000	17	-.3941477229 +000	14
.8500000000 +001	.7987559159 +000	318	53	31.8	.5565718497 +001	.6018415449 +000	17	-.5251645151 +000	14
.8600000000 +001	.9254657826 +000	320	2	22.9	.5585746612 +001	.7093600036 +000	17	-.5943864906 +000	14
.8700000000 +001	.1011806931 +001	323	29	15.8	.5645925615 +001	.8132188192 +000	17	-.6020202797 +000	14
.8800000000 +001	.1072988205 +001	328	49	37.8	.5739116537 +001	.9180591010 +000	18	-.5554018133 +000	14
.8900000000 +001	.1124285501 +001	335	34	29.5	.5856886276 +001	.1023664587 +001	18	-.4648964411 +000	14
.9000000000 +001	.1177151160 +001	343	7	38.0	.5988699982 +001	.1126476747 +001	18	-.3416650281 +000	14
.9100000000 +001	.1236812081 +001	350	51	4.3	.6123508695 +001	.1221078290 +001	18	-.1966518104 +000	14
.9200000000 +001	.1301952394 +001	358	14	3.9	.6252370206 +001	.1301334295 +001	18	-.4011344539 -001	15
.9300000000 +001	.1366410838 +001	4	58	53.3	.8694318014 -001	.1361249661 +001	18	.1186504896 +000	15
.9400000000 +001	.1421798513 +001	11	0	27.5	.1921194408 +000	.1395639916 +001	18	.2714778799 +000	15
.9500000000 +001	.1459850857 +001	16	22	12.8	.2857141162 +000	.1400669422 +001	19	.4114482918 +000	15
.9600000000 +001	.1473968866 +001	21	11	40.4	.3699145660 +000	.1374267012 +001	19	.5328924810 +000	15
.9700000000 +001	.1459939233 +001	25	37	13.6	.4471608808 +000	.1316395658 +001	19	.6312883932 +000	15
.9800000000 +001	.1415990222 +001	29	46	7.6	.5195632266 +000	.1229130660 +001	19	.7030406322 +000	15
.9900000000 +001	.1342317878 +001	33	43	12.6	.5885280240 +000	.1116484611 +001	19	.7451707168 +000	15
.1000000000 +002	.1240201538 +001	37	30	1.8	.6545070630 +000	.9839115448 +000	19	.7549953157 +000	16
.1010000000 +002	.1110919869 +001	41	4	36.8	.7169268597 +000	.8374431220 +000	19	.7299534038 +000	16
.1020000000 +002	.9549181053 +000	44	22	42.2	.7745490844 +000	.6825148000 +000	19	.6678638602 +000	16
.1030000000 +002	.7719725176 +000	47	22	9.4	.8267499961 +000	.5228341910 +000	20	.5679665278 +000	16
.1040000000 +002	.5630566948 +000	50	14	14.0	.8768051386 +000	.3601368571 +000	20	.4328213093 +000	16
.1050000000 +002	.3336718897 +000	54	3	23.4	.9434639882 +000	.1958611017 +000	20	.2701395173 +000	16
.1060000000 +002	.9868763335 -001	69	30	20.2	.1213101681 +001	.3455209230 -001	20	.9244134299 -001	16
.1070000000 +002	.1415896899 +000	217	33	35.3	.3797135072 +001	-.1122406097 +000	20	-.8631156246 -001	16
.1080000000 +002	.3500940959 +000	227	32	16.8	.3971287191 +001	-.2363489230 +000	20	-.2582732323 +000	17
.1090000000 +002	.5257135886 +000	232	41	59.6	.4061379007 +001	-.3185772408 +000	20	-.4181905294 +000	17
.1100000000 +002	.6687336989 +000	238	5	36.6	.4155515660 +001	-.3534488383 +000	21	-.5676959387 +000	17
.1110000000 +002	.7857831696 +000	244	26	24.6	.4266285691 +001	-.3390288663 +000	21	-.7088826542 +000	17
.1120000000 +002	.8877498664 +000	251	43	3.8	.4393303221 +001	-.2784858898 +000	21	-.8429385710 +000	17
.1130000000 +002	.9857499993 +000	259	34	1.7	.4530301344 +001	-.1785026556 +000	21	-.9694533836 +000	17
.1140000000 +002	.1087631024 +001	267	27	54.6	.4668147984 +001	-.4810218467 -001	21	-.1086566806 +001	17
.1150000000 +002	.1195559709 +001	274	53	47.7	.4797850470 +001	.1020499848 +000	21	-.1191196381 +001	17
.1160000000 +002	.1305132137 +001	281	30	25.3	.4913224608 +001	.2603585020 +000	21	-.1278899271 +001	17
.1170000000 +002	.1406590904 +001	287	11	28.4	.5012432573 +001	.4157346122 +000	21	-.1343749495 +001	18
.1180000000 +002	.1488537004 +001	292	4	32.5	.5097682697 +001	.5594388514 +000	22	-.1379409505 +001	18
.1190000000 +002	.1542124588 +001	296	23	44.0	.5173078170 +001	.6855754682 +000	22	-.1381352425 +001	18

SIZE PARAMETER (KA)	MODULUS OF FORM FUNCTION	ARGUMENT OF FORM FUNCTION (DEG MIN SEC)			ARGUMENT OF FORM FUNCTION (RADIANS)	REAL PART OF FORM FUNCTION	NO. TERMS USED IN REAL PART	IMAG PART OF FORM FUNCTION	NO. TERM USED IN IMAG PAR
.1200000000 +002	.1562502331 +001	300	21	45.5	.5242317212 +001	.7898002685 +000	22	-.1348194745 +001	18
.1210000000 +002	.1547617084 +001	304	7	1.2	.5307842788 +001	.8680344603 +000	22	-.1281262976 +001	18
.1220000000 +002	.1496733786 +001	307	44	41.8	.5371162667 +001	.9162220148 +000	22	-.1183532529 +001	18
.1230000000 +002	.1409700147 +001	311	18	59.2	.5433497057 +001	.9307084609 +000	22	-.1058790001 +001	18
.1240000000 +002	.1286823699 +001	314	55	49.2	.5496571121 +001	.9088146067 +000	22	-.9110275757 +000	18
.1250000000 +002	.1129337110 +001	318	47	25.0	.5563940009 +001	.8496038693 +000	22	-.7440265941 +000	19
.1260000000 +002	.9407228552 +000	323	21	38.1	.5643706712 +001	.7548426317 +000	23	-.5614018986 +000	19
.1270000000 +002	.7290924651 +000	329	44	8.0	.5754970987 +001	.6297232977 +000	23	-.3674566519 +000	19
.1280000000 +002	.5111352315 +000	340	43	59.7	.5946916941 +001	.4825078508 +000	23	-.1686576379 +000	19
.1290000000 +002	.3236333124 +000	4	29	46.6	.7847482368 -001	.3226373085 +000	23	.2537100804 -001	19
.1300000000 +002	.2569892914 +000	52	1	6.7	.9078944247 +000	.1581529439 +000	23	.2025614529 +000	19
.1310000000 +002	.3507982171 +000	90	57	54.1	.1587639081 +001	-.5908128745 -002	23	.3507484613 +000	19
.1320000000 +002	.4896988110 +000	109	58	48.3	.1919514712 +001	-.1673269558 +000	23	.4602245271 +000	19
.1330000000 +002	.6178392027 +000	121	46	36.3	.2125405297 +001	-.3253608733 +000	24	.5252290762 +000	19
.1340000000 +002	.7251954018 +000	131	22	25.6	.2292904735 +001	-.4793311854 +000	24	.5441966424 +000	20
.1350000000 +002	.8146712533 +000	140	23	58.1	.2450432922 +001	-.6277101347 +000	24	.5192968686 +000	20
.1360000000 +002	.8928129542 +000	149	17	57.4	.2605763924 +001	-.7676814809 +000	24	.4558290414 +000	20
.1370000000 +002	.9652161933 +000	157	58	52.0	.2757290670 +001	-.8948136652 +000	24	.3618712539 +000	20
.1380000000 +002	.1032414161 +001	166	3	26.5	.2898247836 +001	-.1001996629 +001	24	.2487604394 +000	20
.1390000000 +002	.1082089479 +001	172	50	52.5	.3016765040 +001	-.1073669897 +001	24	.1347241328 +000	20
.1400000000 +002	.1032355951 +001	176	25	21.8	.3079157256 +001	-.1030344451 +001	24	.6441368621 -001	20
.1410000000 +002	.1407770107 +001	223	51	27.3	.3907051867 +001	-.1015092709 +001	25	-.9753991315 +000	20
.1420000000 +002	.1504343034 +001	203	7	19.0	.3545146911 +001	-.1383500792 +001	25	-.5907398099 +000	21
.1430000000 +002	.1495435573 +001	204	46	29.4	.3573995103 +001	-.1357798101 +001	25	-.6266672732 +000	21
.1440000000 +002	.1478833073 +001	207	49	44.3	.3627299781 +001	-.1307798722 +001	25	-.6903692938 +000	21
.1450000000 +002	.1432851556 +001	210	56	58.4	.3681764365 +001	-.1228842704 +001	25	-.7368915722 +000	21
.1460000000 +002	.1348756940 +001	213	57	18.3	.3734220512 +001	-.1118761206 +001	25	-.7533383353 +000	21
.1470000000 +002	.1223082055 +001	216	55	29.5	.3786053119 +001	-.9777611713 +000	25	-.7347875925 +000	21
.1480000000 +002	.1056335862 +001	220	5	8.4	.3841219568 +001	-.8081993656 +000	25	-.6802216504 +000	21
.1490000000 +002	.8522827494 +000	223	49	34.4	.3906504517 +001	-.6148737580 +000	26	-.5901831467 +000	21
.1500000000 +002	.6134541206 +000	228	49	43.6	.3993815825 +001	-.4038437743 +000	26	-.4617750145 +000	21
.1510000000 +002	.3195230552 +000	238	42	40.7	.4166298171 +001	-.1659444784 +000	26	-.2730520333 +000	22
.1520000000 +002	.2993708394 +000	345	2	24.7	.6022087352 +001	.2892243032 +000	26	-.7728002261 -001	22
.1530000000 +002	.5217898359 +000	330	50	21.0	.5774232572 +001	.4556557294 +000	26	-.2542488726 +000	22
.1540000000 +002	.5226914288 +000	347	3	23.1	.6057277110 +001	.5094104043 +000	26	-.1170784766 +000	22
.1550000000 +002	.6007159840 +000	4	24	33.9	.7695886365 -001	.5989379415 +000	26	.4618479845 -001	22
.1560000000 +002	.6991335297 +000	17	23	39.1	.3035862053 +000	.6671625394 +000	26	.2090020058 +000	22
.1570000000 +002	.7914719592 +000	27	55	25.9	.4873633269 +000	.6993214147 +000	27	.3706446023 +000	22
.1580000000 +002	.8716634654 +000	37	27	1.4	.6536323677 +000	.6919964465 +000	27	.5300359563 +000	22
.1590000000 +002	.9419155983 +000	46	39	6.8	.8142289680 +000	.6465584546 +000	27	.6849577790 +000	22

SIZE PARAMETER (KA)	MODULUS OF FORM FUNCTION	ARGUMENT OF FORM FUNCTION (DEG MIN SEC)			ARGUMENT OF FORM FUNCTION (RADIANS)	REAL PART OF FORM FUNCTION	NO. TERMS USED IN REAL PART	IMAG PART OF FORM FUNCTION	NO. TERMS USED IN IMAG PART
.1600000000 +002	.1007293116 +001	55	43	24.1	.9725560178 +000	.5672966688 +000	27	.8323544381 +000	23
.1610000000 +002	.1072534378 +001	64	34	8.2	.1126940917 +001	.4605725165 +000	27	.9686087703 +000	23
.1620000000 +002	.1139868460 +001	72	56	56.9	.1273202486 +001	.3342329841 +000	27	.1089765304 +001	23
.1630000000 +002	.1207910670 +001	80	36	35.4	.1406907121 +001	.1970785039 +000	27	.1191724906 +001	23
.1640000000 +002	.1271767581 +001	87	22	18.0	.1524923458 +001	.5831916817 -001	27	.1270429713 +001	23
.1650000000 +002	.1324076966 +001	93	9	38.6	.1625961455 +001	-.7300583379 -001	28	.1322062767 +001	23
.1660000000 +002	.1356492241 +001	97	59	38.6	.1710319092 +001	-.1886480994 +000	28	.1343310498 +001	23
.1670000000 +002	.1361253624 +001	101	56	46.1	.1779295891 +001	-.2817688729 +000	28	.1331772402 +001	23
.1680000000 +002	.1332713739 +001	105	6	40.6	.1834537673 +001	-.3474309337 +000	28	.1286630349 +001	24
.1690000000 +002	.1268773966 +001	107	33	20.2	.1877199499 +001	-.3827019291 +000	28	.1209680376 +001	24
.1700000000 +002	.1172174926 +001	109	13	20.0	.1906287426 +001	-.3859185842 +000	28	.1106824694 +001	24
.1710000000 +002	.1051787104 +001	109	40	24.7	.1914164352 +001	-.3540950844 +000	28	.9903902180 +000	24
.1720000000 +002	.9251913530 +000	107	13	45.0	.1871502140 +001	-.2740365183 +000	28	.8836758604 +000	24
.1730000000 +002	.8277926298 +000	95	24	33.7	.1665207623 +001	-.7803692421 -001	28	.8241061074 +000	24
.1740000000 +002	.8042967353 +000	48	31	45.6	.8469967656 +000	.5326346046 +000	29	.6026554707 +000	24
.1750000000 +002	.6106749663 +000	325	49	19.1	.5686666187 +001	.5052089955 +000	29	-.3430565338 +000	24
.1760000000 +002	.5446129745 +000	298	56	54.6	.5217635803 +001	.2636054445 +000	29	-.4765663245 +000	24
.1770000000 +002	.6045348852 +000	299	19	6.5	.5224092854 +001	.2960187187 +000	29	-.5271008875 +000	25
.1780000000 +002	.7063858276 +000	305	35	43.0	.5333643752 +001	.4111560629 +000	29	-.5743967526 +000	25
.1790000000 +002	.8191767743 +000	313	18	9.4	.5468161894 +001	.5618335004 +000	29	-.5961490631 +000	25
.1800000000 +002	.9228510364 +000	323	27	16.8	.5645349066 +001	.7414056637 +000	29	-.5495194966 +000	25
.1810000000 +002	.1435358114 +001	340	44	34.3	.5947084806 +001	.1355046852 +001	29	-.4733930135 +000	25
.1820000000 +002	.1479389839 +001	324	24	55.3	.5662116275 +001	.1203123915 +001	30	-.8608641820 +000	25
.1830000000 +002	.1463485761 +001	328	46	13.2	.5738124590 +001	.1251420545 +001	30	-.7587734782 +000	25
.1840000000 +002	.1484114160 +001	334	11	27.1	.5832730638 +001	.1336072651 +001	30	-.6461460459 +000	25
.1850000000 +002	.1504254128 +001	340	6	15.3	.5935939183 +001	.1414470386 +001	30	-.5119121122 +000	25
.1860000000 +002	.1520516664 +001	346	16	51.4	.6043742758 +001	.1477136827 +001	30	-.3606074296 +000	26
.1870000000 +002	.1530593211 +001	352	28	16.5	.6151784023 +001	.1517398374 +001	30	-.2005436411 +000	26
.1880000000 +002	.1529872145 +001	358	30	6.8	.6257038168 +001	.1529349209 +001	30	-.3999722192 -001	26
.1890000000 +002	.1512537790 +001	4	18	58.2	.7533156133 -001	.1508248111 +001	30	.1138340969 +000	26
.1900000000 +002	.1473336297 +001	9	58	17.7	.1740370540 +000	.1451079694 +001	31	.2551226457 +000	26
.1910000000 +002	.1409029660 +001	15	37	8.4	.2726027486 +000	.1356998982 +001	31	.3793657137 +000	26
.1920000000 +002	.1319166266 +001	21	29	2.0	.3749644797 +000	.1227511427 +001	31	.4831307634 +000	26
.1930000000 +002	.1206167995 +001	27	51	56.1	.4863462977 +000	.1066308585 +001	31	.5637616816 +000	26
.1940000000 +002	.1075071594 +001	35	10	7.2	.6138090051 +000	.8788280489 +000	31	.6192254778 +000	26
.1950000000 +002	.9334484935 +000	43	58	41.0	.7675618000 +000	.6717149975 +000	31	.6481705425 +000	27
.1960000000 +002	.7920133313 +000	55	10	14.5	.9629101703 +000	.4523454044 +000	31	.6501297962 +000	27
.1970000000 +002	.6661496894 +000	69	56	31.5	.1220719713 +001	.2284692093 +000	31	.6257453388 +000	27
.1980000000 +002	.5769850327 +000	89	12	39.9	.1557026951 +001	.7944472876 -002	32	.5769303366 +000	27
.1990000000 +002	.5455374650 +000	111	40	29.2	.1949092691 +001	-.2014876212 +000	32	.5069653481 +000	27

SIZE PARAMETER (KA)	MODULUS OF FORM FUNCTION	ARGUMENT OF FORM FUNCTION (DEG MIN SEC)			ARGUMENT OF FORM FUNCTION (RADIANS)	REAL PART OF FORM FUNCTION	NO. TERMS USED IN REAL PART	IMAG PART OF FORM FUNCTION	NO. TERMS USED IN IMAG PART
.2000000000 +002	.5752214281 +000	133	0	47.0	.2321515956 +001	-.3923959991 +000	32	.4206008455 +000	27
.2010000000 +002	.6451719146 +000	149	49	24.3	.2614911687 +001	-.5577383440 +000	32	.3243065510 +000	27
.2020000000 +002	.7268553497 +000	161	48	35.1	.2824112692 +001	-.6905308430 +000	32	.2269049457 +000	27
.2030000000 +002	.7947580546 +000	169	46	59.4	.2963275220 +001	-.7821559968 +000	32	.1409693659 +000	27
.2040000000 +002	.8202934624 +000	174	3	6.1	.3037775280 +001	-.8158768506 +000	32	.8500781763 -001	28
.2050000000 +002	.7474304168 +000	173	56	60.0	.3036000207 +001	-.7432674503 +000	32	.7877642556 -001	28
.2060000000 +002	.4417729066 +000	172	43	48.2	.3014708028 +001	-.4382214677 +000	33	.5590390162 -001	28
.2070000000 +002	.3441240778 +000	266	51	14.5	.4657481538 +001	-.1888548034 -001	33	-.3436054708 +000	28
.2080000000 +002	.8097317771 +000	267	47	49.8	.4673942150 +001	-.3112395130 -001	33	-.8091333948 +000	28
.2090000000 +002	.8599670413 +000	266	6	19.9	.4644417587 +001	-.5840815862 -001	33	-.8579812347 +000	28
.2100000000 +002	.1357056433 +001	268	49	57.5	.4692014850 +001	-.2764693231 -001	33	-.1356774783 +001	28
.2110000000 +002	.1249502442 +001	268	1	10.5	.4677824147 +001	-.4318024449 -001	33	-.1248756109 +001	28
.2120000000 +002	.1197740980 +001	272	10	10.6	.4750256013 +001	.4534405878 -001	33	-.1196882355 +001	29
.2130000000 +002	.1141532541 +001	277	48	18.1	.4848612516 +001	.1550231020 +000	33	-.1130957285 +001	29
.2140000000 +002	.1075374208 +001	284	25	48.4	.4964241891 +001	.2679820074 +000	33	-.1041448669 +001	29
.2150000000 +002	.9991707725 +000	291	55	30.8	.5095056268 +001	.3730865323 +000	34	-.9269027306 +000	29
.2160000000 +002	.9138608047 +000	300	21	52.7	.5242351681 +001	.4619576955 +000	34	-.7885027951 +000	29
.2170000000 +002	.8213041156 +000	310	1	43.9	.5411024557 +001	.5282410585 +000	34	-.6288893657 +000	29
.2180000000 +002	.7253617095 +000	321	28	31.6	.5610805145 +001	.5674804949 +000	34	-.4517914315 +000	29
.2190000000 +002	.6337287904 +000	335	35	32.3	.5857190695 +001	.5770912969 +000	34	-.2618736620 +000	29
.2200000000 +002	.5600310616 +000	353	22	26.5	.6167540466 +001	.5562903815 +000	34	-.6462044199 -001	29
.2210000000 +002	.5233478046 +000	14	48	50.1	.2585517141 +000	.5059523991 +000	34	.1338099193 +000	30
.2220000000 +002	.5387949419 +000	37	20	6.9	.6516229293 +000	.4283961685 +000	34	.3267670610 +000	30
.2230000000 +002	.6035582954 +000	57	11	2.3	.9980484824 +000	.3270944476 +000	35	.5072394289 +000	30
.2240000000 +002	.6982179119 +000	72	50	8.5	.1271222790 +001	.2060530176 +000	35	.6671209833 +000	30
.2250000000 +002	.8650142438 +000	84	51	29.1	.1481052996 +001	.7752509875 -001	35	.8615332268 +000	30
.2260000000 +002	.9249619629 +000	94	13	4.1	.1644410986 +001	-.6802927729 -001	35	.9224568555 +000	30
.2270000000 +002	.1015968456 +001	102	11	44.6	.1783651666 +001	-.2146250190 +000	35	.9930397799 +000	30
.2280000000 +002	.1088125471 +001	109	18	5.5	.1907671446 +001	-.3596684199 +000	35	.1026964298 +001	30
.2290000000 +002	.1137845259 +001	116	5	14.5	.2026106550 +001	-.5003571596 +000	35	.1021926879 +001	30
.2300000000 +002	.1168094463 +001	122	58	59.3	.2146460684 +001	-.6359015083 +000	35	.9798336324 +000	31
.2310000000 +002	.1189397588 +001	130	9	15.2	.2271619829 +001	-.7669798447 +000	36	.9090701511 +000	31
.2320000000 +002	.1215438216 +001	137	7	14.9	.2393209542 +001	-.8906610265 +000	36	.8270507805 +000	31
.2330000000 +002	.1246600957 +001	142	42	35.3	.2490756182 +001	-.9917674216 +000	36	.7552558023 +000	31
.2340000000 +002	.1260306243 +001	145	55	38.2	.2546911312 +001	-.1043945775 +001	36	.7060800567 +000	31
.2350000000 +002	.1226762474 +001	146	14	30.6	.2552401337 +001	-.1019918455 +001	36	.6816982570 +000	31
.2360000000 +002	.1111083398 +001	142	27	51.3	.2486470022 +001	-.8810593803 +000	36	.6769347730 +000	31
.2370000000 +002	.8535535202 +000	131	56	57.6	.2302950372 +001	-.5705776755 +000	36	.6348186576 +000	31
.2380000000 +002	.1274616312 +000	184	45	32.1	.3224651645 +001	-.1270222180 +000	36	-.1057466595 -001	31
.2390000000 +002	.1395613619 +000	257	25	35.5	.4492940308 +001	-.3038132929 -001	36	-.1362143479 +000	32

SIZE PARAMETER (KA)	MODULUS OF FORM FUNCTION	ARGUMENT OF FORM FUNCTION (DEG MIN SEC)	ARGUMENT OF FORM FUNCTION (RADIANS)	REAL PART OF FORM FUNCTION	NO. TERMS USED IN REAL PART	IMAG PART OF FORM FUNCTION	NO. TERMS USED IN IMAG PART
.2400000000 +002	.3934102273 +000	273 26 40.3	.4772507489 +001	.2363699197 -001	37	-.3926995029 +000	32
.2410000000 +002	.5356158920 +000	279 7 19.4	.4871598876 +001	.8491554982 -001	37	-.5288418791 +000	32
.2420000000 +002	.6165894712 +000	287 33 50.1	.5018937202 +001	.1860679463 +000	37	-.5878446184 +000	32
.2430000000 +002	.6758216594 +000	298 16 14.0	.5205803328 +001	.3200932852 +000	37	-.5952102184 +000	32
.2440000000 +002	.7360915351 +000	310 9 44.4	.5413354104 +001	.4747463241 +000	37	-.5625359329 +000	32
.2450000000 +002	.8089639695 +000	322 8 40.1	.5622481764 +001	.6387261432 +000	37	-.4964389368 +000	32
.2460000000 +002	.8972699659 +000	333 21 35.2	.5818225855 +001	.8020155596 +000	37	-.4023237923 +000	32
.2470000000 +002	.9973586332 +000	343 21 25.7	.5992712511 +001	.9555778550 +000	37	-.2856487461 +000	32
.2480000000 +002	.1101783698 +001	352 3 33.6	.6144594748 +001	.1091219455 +001	38	-.1522084712 +000	33
.2490000000 +002	.1201682165 +001	359 37 0.7	.6276498081 +001	.1201655296 +001	38	-.8035860554 -002	33
.2500000000 +002	.1288434582 +001	6 16 30.0	.1095191699 +000	.1280715272 +001	38	.1408263690 +000	33
.2510000000 +002	.1354714489 +001	12 17 53.7	.2146449117 +000	.1323626623 +001	38	.2885548662 +000	33
.2520000000 +002	.1395124343 +001	17 56 18.2	.3130838451 +000	.1327304962 +001	38	.4296899694 +000	33
.2530000000 +002	.1406486384 +001	23 25 26.0	.4088239697 +000	.1290576386 +001	38	.5591212224 +000	33
.2540000000 +002	.1387686789 +001	28 57 15.8	.5053493090 +000	.1214233466 +001	38	.6717973769 +000	33
.2550000000 +002	.1338598860 +001	34 41 8.5	.6053797239 +000	.1100711190 +001	38	.7617622879 +000	33
.2560000000 +002	.1255648309 +001	40 41 20.2	.7101562634 +000	.9521078983 +000	38	.8186227621 +000	33
.2570000000 +002	.1105226645 +001	47 13 18.1	.8241738532 +000	.7506296833 +000	39	.8112219278 +000	34
.2580000000 +002	.9366550194 +000	74 39 48.7	.1303124255 +001	.2477331946 +000	39	.9032999998 +000	34
.2590000000 +002	.1181438878 +001	76 47 14.2	.1340190648 +001	.2700381891 +000	39	.1150163987 +001	34
.2600000000 +002	.1127283156 +001	82 43 23.9	.1443794060 +001	.1427829549 +000	39	.1118204070 +001	34
.2610000000 +002	.1084109248 +001	90 52 17.1	.1586005547 +001	-.1648782065 -001	39	.1083983862 +001	34
.2620000000 +002	.1052889792 +001	98 51 47.6	.1725488952 +001	-.1622254715 +000	39	.1040317168 +001	34
.2630000000 +002	.1025744259 +001	105 41 41.2	.1844722100 +001	-.2774770719 +000	39	.9875007645 +000	34
.2640000000 +002	.9920258435 +000	110 18 54.7	.1925363571 +001	-.3444160570 +000	39	.9303186841 +000	34
.2650000000 +002	.9308265793 +000	111 9 1.0	.1939938374 +001	-.3358565703 +000	40	.8681235424 +000	34
.2660000000 +002	.8020297974 +000	105 46 48.4	.1846211257 +001	-.2181089978 +000	40	.7718032527 +000	35
.2670000000 +002	.5620570387 +000	90 19 0.7	.1576326489 +001	-.3108250956 -002	40	.5620484440 +000	35
.2680000000 +002	.2767057387 +000	48 57 39.2	.8545289352 +000	.1816777629 +000	40	.2087085439 +000	35
.2690000000 +002	.2975104664 +000	328 33 20.9	.5734380606 +001	.2538206618 +000	40	-.1552016407 +000	35
.2700000000 +002	.5178265910 +000	304 6 34.1	.5307711529 +001	.2903846433 +000	40	-.4287436732 +000	35
.2710000000 +002	.7274596790 +000	300 37 9.5	.5246796784 +001	.3705181308 +000	40	-.6260302702 +000	35
.2720000000 +002	.9588696042 +000	302 35 11.5	.5281131337 +001	.5164210979 +000	40	-.8079233674 +000	35
.2730000000 +002	.1229392583 +001	302 23 18.4	.5277674106 +001	.6585322613 +000	40	-.1038143239 +001	35
.2740000000 +002	.1410277334 +001	299 12 22.0	.5222131772 +001	.6881487090 +000	41	-.1230988836 +001	35
.2750000000 +002	.1454007168 +001	298 11 30.9	.5204430927 +001	.6869116378 +000	41	-.1281518336 +001	36
.2760000000 +002	.1435458042 +001	300 15 47.5	.5240581231 +001	.7234317763 +000	41	-.1239833156 +001	36
.2770000000 +002	.1392262271 +001	304 20 36.1	.5311793929 +001	.7854465084 +000	41	-.1149551222 +001	36
.2780000000 +002	.1333367295 +001	309 46 38.1	.5406632953 +001	.8530945113 +000	41	-.1024742943 +001	36
.2790000000 +002	.1262079948 +001	316 21 50.2	.5521592319 +001	.9134147730 +000	41	-.8709301055 +000	36

SIZE PARAMETER (KA)	MODULUS OF FORM FUNCTION	ARGUMENT OF FORM FUNCTION (DEG MIN SEC)			ARGUMENT OF FORM FUNCTION (RADIANS)	REAL PART OF FORM FUNCTION	NO. TERMS USED IN REAL PART	IMAG PART OF FORM FUNCTION	NO. TERMS USED IN IMAG PART
.2800000000 +002	.1182521704 +001	324	8	20.9	.5657295097 +001	.9583651822 +000	41	-.6927436447 +000	36
.2810000000 +002	.1100736203 +001	333	14	22.3	.5816127120 +001	.9828437619 +000	41	-.4956191372 +000	36
.2820000000 +002	.1024423239 +001	343	47	38.0	.6000335498 +001	.9837167326 +000	42	-.2859097108 +000	36
.2830000000 +002	.9619744575 +000	355	47	30.1	.6209736697 +001	.9593808429 +000	42	-.7059217631 -001	36
.2840000000 +002	.9207642845 +000	8	56	35.1	.1560860780 +000	.9095708102 +000	42	.1431356307 +000	37
.2850000000 +002	.9049848295 +000	22	38	2.7	.3950390847 +000	.8352842768 +000	42	.3482782202 +000	37
.2860000000 +002	.9141728623 +000	36	5	9.4	.6298187721 +000	.7387744137 +000	42	.5384462720 +000	37
.2870000000 +002	.9436253986 +000	48	38	6.1	.8488415288 +000	.6235977129 +000	42	.7082053272 +000	37
.2880000000 +002	.9863544967 +000	59	53	44.3	.1045375984 +001	.4947324267 +000	42	.8533082791 +000	37
.2890000000 +002	.1034800804 +001	69	42	42.4	.1216699819 +001	.3588100043 +000	42	.9706019185 +000	37
.2900000000 +002	.1079716939 +001	78	0	19.7	.1361452418 +001	.2243848062 +000	42	.1056143990 +001	37
.2910000000 +002	.1035500128 +001	87	44	49.6	.1531475908 +001	.4070580819 -001	43	.1034699741 +001	37
.2920000000 +002	.1182850267 +001	89	22	34.8	.1559911118 +001	.1287531813 -001	43	.1182780191 +001	37
.2930000000 +002	.1171512055 +001	91	7	55.5	.1590554922 +001	-.2314592683 -001	43	.1171283382 +001	38
.2940000000 +002	.1102880378 +001	89	15	14.0	.1557774019 +001	.1436164219 -001	43	.1102786865 +001	38
.2950000000 +002	.9702162416 +000	81	7	28.8	.1415892432 +001	.1496899542 +000	43	.9585992245 +000	38
.2960000000 +002	.4198674515 +000	82	33	21.2	.1440872194 +001	.5439757159 -001	43	.4163286935 +000	38
.2970000000 +002	.8720649463 -001	91	45	7.2	.1601374260 +001	-.2666178835 -002	43	.8716572833 -001	38
.2980000000 +002	.2892322163 +000	229	38	14.7	.4007929168 +001	-.1873132395 +000	43	-.2203838134 +000	38
.2990000000 +002	.5391868491 +000	226	45	35.6	.3957706840 +001	-.3693738041 +000	44	-.3927918674 +000	38
.3000000000 +002	.6584225164 +000	225	52	28.0	.3942252811 +001	-.4584154770 +000	44	-.4726261319 +000	38

Appendix D

TABLE OF VALUES OF THE FAR-FIELD MONOSTATIC REFLECTION FORM-FUNCTION FOR $0 < ka \leq 30.0$ IN SIZE-PARAMETER INCREMENTS OF 0.1 FOR A RIGID, IMMOVABLE SPHERE

[Note: $0 \leq \arg$ (radians) $< 2\pi$, $0^\circ \leq \arg$ (degrees) $< 360^\circ$, tolerance $\epsilon = 2 \times 10^{-6}$]

SIZE PARAMETER (KA)	MODULUS OF FORM FUNCTION	ARGUMENT OF FORM FUNCTION (DEG	MIN	SEC)	ARGUMENT OF FORM FUNCTION (RADIANS)	REAL PART OF FORM FUNCTION	NO. TERMS USED IN REAL PART	IMAG PART OF FORM FUNCTION	NO. TERMS USED IN IMAG PART
.0000000000 +000	INDETERMINATE POINT								
.1000000000 +000	.1658206400 -001	180	0	6.7	.3141625169 +001	-.1658206399 -001	10	-.5391793674 -006	10
.2000000000 +000	.6532274770 -001	180	0	49.9	.3141834359 +001	-.6532274579 -001	10	-.1578883350 -004	10
.3000000000 +000	.1432715039 +000	180	2	29.5	.3142317443 +001	-.1432714662 +000	10	-.1038417018 -003	10
.4000000000 +000	.2456949926 +000	180	5	3.8	.3143065673 +001	-.2456947260 +000	10	-.3619134190 -003	10
.5000000000 +000	.3662635069 +000	180	8	20.8	.3144020406 +001	-.3662624275 +000	10	-.8891964226 -003	10
.6000000000 +000	.4972267114 +000	180	12	26.0	.3145209527 +001	-.4972234591 +000	10	-.1798402234 -002	10
.7000000000 +000	.6296313530 +000	180	18	22.4	.3146937065 +001	-.6296223610 +000	10	-.3364992940 -002	10
.8000000000 +000	.7537079648 +000	180	28	51.7	.3149988297 +001	-.7536814018 +000	10	-.6327788875 -002	10
.9000000000 +000	.8595520244 +000	180	48	54.2	.3155818113 +001	-.8594650548 +000	10	-.1222711036 -001	10
.1000000000 +001	.9381223076 +000	181	26	22.0	.3166715595 +001	-.9378262695 +000	10	-.2356591291 -001	10
.1100000000 +001	.9824200281 +000	182	32	37.1	.3185987300 +001	-.9814520689 +000	10	-.4359986458 -001	10
.1200000000 +001	.9886113898 +000	184	23	36.5	.3218273205 +001	-.9857063420 +000	10	-.7573299868 -001	10
.1300000000 +001	.9569261332 +000	187	22	4.0	.3270184687 +001	-.9490252069 +000	10	-.1227142254 +000	10
.1400000000 +001	.8924125161 +000	192	1	30.8	.3351472252 +001	-.8728294217 +000	10	-.1859271353 +000	10
.1500000000 +001	.8059195015 +000	199	11	41.3	.3476605200 +001	-.7611153615 +000	10	-.2649710348 +000	10
.1600000000 +001	.7156480660 +000	209	58	27.2	.3664741470 +001	-.6199303487 +000	10	-.3575451260 +000	10
.1700000000 +001	.6479720542 +000	225	10	4.9	.3929923447 +001	-.4568397771 +000	10	-.4595271496 +000	10
.1800000000 +001	.6308742523 +000	243	36	49.9	.4251864176 +001	-.2803720612 +000	10	-.5651493870 +000	10
.1900000000 +001	.6746277375 +000	261	31	41.5	.4564528227 +001	-.9938788965 -001	10	-.6672665372 +000	10
.2000000000 +001	.7617464316 +000	275	50	54.6	.4814464356 +001	.7762059645 -001	10	-.7577814125 +000	10
.2100000000 +001	.8632453380 +000	286	21	48.2	.4997983789 +001	.2432005694 +000	10	-.8282789365 +000	10
.2200000000 +001	.9547575519 +000	294	12	13.9	.5134826054 +001	.3914358019 +000	10	-.8708271906 +000	10
.2300000000 +001	.1020202953 +001	300	31	29.9	.5245150435 +001	.5181753765 +000	10	-.8788107562 +000	10
.2400000000 +001	.1050778336 +001	306	13	38.6	.5344676241 +001	.6210009440 +000	10	-.8476396275 +000	10
.2500000000 +001	.1043846579 +001	312	2	25.8	.5446134187 +001	.6990178985 +000	10	-.7752352270 +000	10
.2600000000 +001	.1002479251 +001	318	39	4.6	.5561514088 +001	.7525638837 +000	10	-.6622780752 +000	10
.2700000000 +001	.9355778072 +000	326	48	10.6	.5703787286 +001	.7828843962 +000	10	-.5122478459 +000	10
.2800000000 +001	.8582932100 +000	337	17	44.6	.5886920953 +001	.7917834623 +000	10	-.3312796149 +000	10
.2900000000 +001	.7916440077 +000	350	42	26.8	.6120999690 +001	.7812550395 +000	10	-.1278311317 +000	10
.3000000000 +001	.7582396475 +000	6	39	12.2	.1161237020 +000	.7531330673 +000	10	.8785184093 -001	10
.3100000000 +001	.7714468580 +000	23	14	39.5	.4056894748 +000	.7088289121 +000	10	.3044533267 +000	10
.3200000000 +001	.8258406878 +000	38	10	28.0	.6662698745 +000	.6492202567 +000	10	.5104173782 +000	10
.3300000000 +001	.9016773036 +000	50	24	11.4	.8797010429 +000	.5747124626 +000	10	.6947859707 +000	10
.3400000000 +001	.9770702500 +000	60	12	33.5	.1050850574 +001	.4854408021 +000	10	.8479466381 +000	10
.3500000000 +001	.1035117840 +001	68	22	12.7	.1193284937 +001	.3815529641 +000	10	.9622298470 +000	10
.3600000000 +001	.1065437027 +001	75	40	49.3	.1320871411 +001	.2635158275 +000	10	.1032334959 +001	10
.3700000000 +001	.1063853545 +001	82	51	0.7	.1446008533 +001	.1324116591 +000	10	.1055581128 +001	10
.3800000000 +001	.1032023533 +001	90	32	38.2	.1580290166 +001	-.9797718556 -002	10	.1031977024 +001	10
.3900000000 +001	.9772967945 +000	99	25	35.7	.1735321075 +001	-.1600651049 +000	11	.9640996768 +000	10

SIZE PARAMETER (KA)	MODULUS OF FORM FUNCTION	ARGUMENT OF FORM FUNCTION (DEG MIN SEC)			ARGUMENT OF FORM FUNCTION (RADIANS)	REAL PART OF FORM FUNCTION	NO. TERMS USED IN REAL PART	IMAG PART OF FORM FUNCTION	NO. TERMS USED IN IMAG PART
.4000000000 +001	.9125625730 +000	110	8	9.7	.1922236131 +001	-.3141496331 +000	11	.8567849541 +000	10
.4100000000 +001	.8551979144 +000	123	4	11.6	.2147974699 +001	-.4666486384 +000	11	.7166606735 +000	10
.4200000000 +001	.8232854477 +000	137	56	5.0	.2407414941 +001	-.6111922298 +000	11	.5515822573 +000	10
.4300000000 +001	.8280717643 +000	153	27	24.5	.2678326551 +001	-.7407911296 +000	11	.3700423613 +000	10
.4400000000 +001	.8673125680 +000	167	58	49.3	.2931810421 +001	-.8482978564 +000	11	.1806151638 +000	10
.4500000000 +001	.9269678717 +000	180	31	37.3	.3150790929 +001	-.9269286574 +000	12	-.8526385524 -002	10
.4600000000 +001	.9892158753 +000	191	4	25.4	.3334865620 +001	-.9707975142 +000	12	-.1900006163 +000	10
.4700000000 +001	.1038872130 +001	200	7	42.3	.3492899735 +001	-.9754216800 +000	12	-.3575022360 +000	10
.4800000000 +001	.1065906287 +001	208	20	20.1	.3636200248 +001	-.9381627988 +000	12	-.5059711223 +000	10
.4900000000 +001	.1065879331 +001	216	20	29.9	.3775873683 +001	-.8585636325 +000	12	-.6316385334 +000	10
.5000000000 +001	.1039809523 +001	224	44	36.9	.3922515420 +001	-.7385395548 +000	12	-.7319584486 +000	10
.5100000000 +001	.9939481689 +000	234	7	50.4	.4086350960 +001	-.5823927512 +000	12	-.8054512064 +000	10
.5200000000 +001	.9393483893 +000	245	1	24.2	.4276464989 +001	-.3966381302 +000	13	-.8515007870 +000	10
.5300000000 +001	.8905771472 +000	257	42	15.4	.4497787915 +001	-.1896552267 +000	13	-.8701485794 +000	10
.5400000000 +001	.8623971725 +000	271	54	49.3	.4745789302 +001	.2879898749 -001	13	-.8619161800 +000	10
.5500000000 +001	.8640816542 +000	286	41	27.8	.5003702927 +001	.2481738070 +000	13	-.8276755806 +000	10
.5600000000 +001	.8945679797 +000	300	46	40.4	.5249564648 +001	.4577607136 +000	13	-.7685746544 +000	10
.5700000000 +001	.9432245972 +000	313	20	17.8	.5468784445 +001	.6473390765 +000	13	-.6860209624 +000	10
.5800000000 +001	.9954514685 +000	324	14	26.5	.5659067532 +001	.8077879531 +000	13	-.5817235159 +000	10
.5900000000 +001	.1038017748 +001	333	49	52.0	.5826451835 +001	.9316187319 +000	14	-.4577853037 +000	10
.6000000000 +001	.1061758595 +001	342	38	18.6	.5980169913 +001	.1013385975 +001	14	-.3168283130 +000	11
.6100000000 +001	.1062388757 +001	351	13	20.3	.6129985438 +001	.1049945884 +001	14	-.1621219055 +000	11
.6200000000 +001	.1040550953 +001	0	7	39.8	.2229120734 -002	.1040548368 +001	14	.2319511784 -002	11
.6300000000 +001	.1001595363 +001	9	52	14.6	.1722766813 +000	.9867687853 +000	14	.1716992574 +000	11
.6400000000 +001	.9550518437 +000	20	53	8.6	.3645246719 +000	.8922985614 +000	14	.3404809857 +000	11
.6500000000 +001	.9133001765 +000	33	22	56.3	.5826313717 +000	.7626218138 +000	14	.5025188371 +000	11
.6600000000 +001	.8886550444 +000	47	7	40.8	.8225388352 +000	.6046077915 +000	15	.6512735267 +000	11
.6700000000 +001	.8888683747 +000	61	21	32.4	.1070916696 +001	.4260523019 +000	15	.7801066744 +000	11
.6800000000 +001	.9134397467 +000	75	4	43.4	.1310371037 +001	.2352029930 +000	15	.8826390672 +000	12
.6900000000 +001	.9540023094 +000	87	34	42.8	.1528534230 +001	.4030613765 -001	15	.9531504717 +000	12
.7000000000 +001	.9984419277 +000	98	41	11.3	.1722403811 +001	-.1507920612 +000	15	.9869893806 +000	12
.7100000000 +001	.1035282684 +001	108	38	38.1	.1896194098 +001	-.3309650787 +000	15	.9809548170 +000	12
.7200000000 +001	.1056267560 +001	117	53	10.5	.2057503044 +001	-.4940348966 +000	15	.9336116318 +000	12
.7300000000 +001	.1057351917 +001	126	54	13.4	.2214887804 +001	-.6349104213 +000	16	.8455068503 +000	12
.7400000000 +001	.1038897896 +001	136	11	5.4	.2376873543 +001	-.7496446771 +000	16	.7192646921 +000	12
.7500000000 +001	.1005498753 +001	146	11	12.1	.2551438992 +001	-.8354238454 +000	16	.5595486945 +000	12
.7600000000 +001	.9654248877 +000	157	16	44.6	.2745037202 +001	-.8905049211 +000	16	.3728889906 +000	13
.7700000000 +001	.9293176998 +000	169	37	25.3	.2960492135 +001	-.9141197048 +000	16	.1673814581 +000	13
.7800000000 +001	.9076207967 +000	183	0	51.0	.3194199719 +001	-.9063651645 +000	16	-.4772524591 -001	13
.7900000000 +001	.9069280110 +000	196	49	34.6	.3435266431 +001	-.8680995993 +000	16	-.2625290514 +000	13

SIZE PARAMETER (KA)	MODULUS OF FORM FUNCTION	ARGUMENT OF FORM FUNCTION (DEG MIN SEC)	ARGUMENT OF FORM FUNCTION (RADIANS)	REAL PART OF FORM FUNCTION	NO. TERMS USED IN REAL PART	IMAG PART OF FORM FUNCTION	NO. TERMS USED IN IMAG PART
.8000000000 +001	.9270865582 +000	210 14 55.7	.3669533688 +001	-.8008601230 +000	17	-.4670252132 +000	13
.8100000000 +001	.9613793719 +000	222 40 32.2	.3886422421 +001	-.7068092915 +000	17	-.6516678006 +000	13
.8200000000 +001	.9996418605 +000	233 55 10.1	.4082665203 +001	-.5887106237 +000	17	-.8079007680 +000	13
.8300000000 +001	.1031877177 +001	244 8 58.1	.4261212011 +001	-.4499242787 +000	17	-.9286219098 +000	13
.8400000000 +001	.1050643486 +001	253 43 36.1	.4428366279 +001	-.2944107303 +000	17	-.1008550473 +001	14
.8500000000 +001	.1052137540 +001	263 4 55.3	.4591647634 +001	-.1267280617 +000	17	-.1044477574 +001	14
.8600000000 +001	.1036498782 +001	272 39 13.8	.4758707037 +001	.4799144557 -001	17	-.1035387148 +001	14
.8700000000 +001	.1007678414 +001	282 50 58.3	.4936655715 +001	.2240991340 +000	17	-.9824435676 +000	14
.8800000000 +001	.9728848410 +000	293 59 19.7	.5131072665 +001	.3955342971 +000	18	-.8888518063 +000	14
.8900000000 +001	.9413383817 +000	306 12 6.3	.5344228856 +001	.5559830930 +000	18	-.7596056537 +000	14
.9000000000 +001	.9220154788 +000	319 18 28.5	.5572974520 +001	.6990946896 +000	18	-.6011481997 +000	14
.9100000000 +001	.9206342195 +000	332 47 21.1	.5808267242 +001	.8187478059 +000	18	-.4209743419 +000	14
.9200000000 +001	.9373086361 +000	345 58 17.7	.6038343068 +001	.9093539486 +000	18	-.2272066800 +000	15
.9300000000 +001	.9665873885 +000	358 19 46.9	.6254032997 +001	.9661766869 +000	18	-.2817426444 -001	15
.9400000000 +001	.9998547854 +000	9 40 18.6	.1688054505 +000	.9856430098 +000	18	.1679805021 +000	15
.9500000000 +001	.1028343847 +001	20 6 52.5	.3510659077 +000	.9656217432 +000	19	.3536463147 +000	15
.9600000000 +001	.1045323885 +001	29 57 34.6	.5228940103 +000	.9056451687 +000	19	.5220238049 +000	15
.9700000000 +001	.1047218262 +001	39 35 11.9	.6909172393 +000	.8070511458 +000	19	.6673339009 +000	15
.9800000000 +001	.1033980838 +001	49 23 23.1	.8620137323 +000	.6730285237 +000	19	.7849515785 +000	15
.9900000000 +001	.1009002656 +001	59 44 2.2	.1042554118 +001	.5085534081 +000	19	.8714699028 +000	15
.1000000000 +002	.9785784126 +000	70 53 59.0	.1237433436 +001	.3202129989 +000	19	.9247050043 +000	16
.1010000000 +002	.9507497247 +000	82 59 47.8	.1448564157 +001	.1159230360 +000	19	.9436561285 +000	16
.1020000000 +002	.9333286383 +000	95 52 10.4	.1673239375 +001	-.9544588172 -001	19	.9284354747 +000	16
.1030000000 +002	.9313675845 +000	109 5 1.5	.1903870479 +001	-.3045104186 +000	20	.8801812214 +000	16
.1040000000 +002	.9452008934 +000	122 4 11.6	.2130521501 +001	-.5018575622 +000	20	.8009642415 +000	16
.1050000000 +002	.9703748322 +000	134 22 1.6	.2345148421 +001	-.6785375622 +000	20	.6936959648 +000	16
.1060000000 +002	.9995260945 +000	145 47 4.0	.2544418531 +001	-.8265360422 +000	20	.5620414439 +000	16
.1070000000 +002	.1024921176 +001	156 23 57.4	.2729682345 +001	-.9391943943 +000	20	.4103380399 +000	16
.1080000000 +002	.1040450225 +001	166 27 51.9	.2905352395 +001	-.1011551520 +001	20	.2435163096 +000	16
.1090000000 +002	.1042741156 +001	176 18 54.4	.3077278973 +001	-.1040585381 +001	20	.6701629989 -001	17
.1100000000 +002	.1031582397 +001	186 18 22.0	.3251655148 +001	-.1025340535 +001	21	-.1133094413 +000	17
.1110000000 +002	.1009867398 +001	196 45 58.7	.3434219912 +001	-.9669372263 +000	21	-.2913152260 +000	17
.1120000000 +002	.9831002037 +000	207 56 43.4	.3629331619 +001	-.8684655131 +000	21	-.4607099553 +000	17
.1130000000 +002	.9583386438 +000	219 56 23.4	.3838674229 +001	-.7347764740 +000	21	-.6152369376 +000	17
.1140000000 +002	.9424577472 +000	232 37 5.6	.4059953811 +001	-.5721879711 +000	21	-.7488841906 +000	17
.1150000000 +002	.9399913924 +000	245 36 45.2	.4286747731 +001	-.3881270516 +000	21	-.8561198570 +000	17
.1160000000 +002	.9514568766 +000	258 26 9.0	.4510556383 +001	-.1907338631 +000	21	-.9321431122 +000	17
.1170000000 +002	.9732055939 +000	270 40 47.4	.4724254115 +001	.1154694488 -001	21	-.9731370900 +000	18
.1180000000 +002	.9989085901 +000	282 9 25.4	.4924569723 +001	.2103623942 +000	22	-.9765070581 +000	18
.1190000000 +002	.1021712371 +001	292 54 52.6	.5112324434 +001	.3978127947 +000	22	-.9410850915 +000	18

SIZE PARAMETER (KA)	MODULUS OF FORM FUNCTION	ARGUMENT OF FORM FUNCTION (DEG MIN SEC)			ARGUMENT OF FORM FUNCTION (RADIANS)	REAL PART OF FORM FUNCTION	NO. TERMS USED IN REAL PART	IMAG PART OF FORM FUNCTION	NO. TERMS USED IN IMAG PART
.1200000000 +002	.1036049067 +001	303	9	51.5	.5291215422 +001	.5667622730 +000	22	-.8672820734 +000	18
.1210000000 +002	.1038725359 +001	313	12	8.2	.5466410851 +001	.7110864488 +000	22	-.7571700170 +000	18
.1220000000 +002	.1029378420 +001	323	20	55.1	.5643498286 +001	.8258524244 +000	22	-.6144816560 +000	18
.1230000000 +002	.1010459651 +001	333	54	4.8	.5827677527 +001	.9074309241 +000	22	-.4445197685 +000	18
.1240000000 +002	.9867883283 +000	345	5	8.2	.6022880280 +001	.9535449262 +000	22	-.2539749564 +000	18
.1250000000 +002	.9645935291 +000	356	59	22.7	.6230644654 +001	.9632624455 +000	22	-.5065705993 -001	18
.1260000000 +002	.9499812526 +000	9	30	12.4	.1658662069 +000	.9369434333 +000	23	.1568482808 +000	19
.1270000000 +002	.9470694904 +000	22	18	49.2	.3894470558 +000	.8761521138 +000	23	.3595804403 +000	19
.1280000000 +002	.9565294058 +000	34	59	59.2	.6108611836 +000	.7835452428 +000	23	.5486395510 +000	19
.1290000000 +002	.9753769818 +000	47	11	50.0	.8237469929 +000	.6627460561 +000	23	.7156311353 +000	19
.1300000000 +002	.9981522989 +000	58	43	24.4	.1024917384 +001	.5182102280 +000	23	.8530921236 +000	19
.1310000000 +002	.1018757662 +001	69	36	5.0	.1214773176 +001	.3550875109 +000	23	.9548717367 +000	19
.1320000000 +002	.1032098570 +001	80	0	28.8	.1396403235 +001	.1790799053 +000	23	.1016443725 +001	19
.1330000000 +002	.1035137864 +001	90	12	18.2	.1574375412 +001	-.3704838532 -002	23	.1035131234 +001	19
.1340000000 +002	.1027376165 +001	100	28	55.1	.1753741426 +001	-.1869067551 +000	24	.1010231483 +001	20
.1350000000 +002	.1010873592 +001	111	6	36.3	.1939236767 +001	-.3640772165 +000	24	.9430340397 +000	20
.1360000000 +002	.9898532554 +000	122	17	41.2	.2134446715 +001	-.5288543054 +000	24	.8367332855 +000	20
.1370000000 +002	.9698369539 +000	134	7	8.1	.2340816448 +001	-.6751516886 +000	24	.6962427123 +000	20
.1380000000 +002	.9562878267 +000	146	29	30.5	.2556764214 +001	-.7973593013 +000	24	.5279247599 +000	20
.1390000000 +002	.9529830180 +000	159	8	44.5	.2777616420 +001	-.8905518657 +000	24	.3392550768 +000	20
.1400000000 +002	.9607252078 +000	171	42	52.8	.2996986229 +001	-.9506978285 +000	24	.1384433592 +000	20
.1410000000 +002	.9770876366 +000	183	52	16.1	.3209156594 +001	-.9748583380 +000	24	-.6596567629 -001	20
.1420000000 +002	.9973486692 +000	195	26	16.5	.3411034978 +001	-.9613638265 +000	25	-.2654881558 +000	20
.1430000000 +002	.1016064443 +001	206	25	6.8	.3602683436 +001	-.9099545100 +000	25	-.4520727171 +000	21
.1440000000 +002	.1028557092 +001	216	57	36.7	.3786669712 +001	-.8218720793 +000	25	-.6184302516 +000	21
.1450000000 +002	.1031928843 +001	227	17	38.9	.3967030962 +001	-.6998901992 +000	25	-.7583078838 +000	21
.1460000000 +002	.1025559153 +001	237	40	56.9	.4148341703 +001	-.5482751165 +000	25	-.8666983170 +000	21
.1470000000 +002	.1011161015 +001	248	22	24.0	.4334932388 +001	-.3726707731 +000	25	-.9399803680 +000	21
.1480000000 +002	.9924343635 +000	259	33	20.6	.4530101918 +001	-.1799077229 +000	25	-.9759913816 +000	21
.1490000000 +002	.9742905406 +000	271	18	32.9	.4735237888 +001	.2225953777 -001	26	-.9740362265 +000	21
.1500000000 +002	.9616478078 +000	283	33	37.2	.4949061238 +001	.2254765550 +000	26	-.9348405369 +000	21
.1510000000 +002	.9579992356 +000	296	4	47.6	.5167568951 +001	.4211594245 +000	26	-.8604575960 +000	22
.1520000000 +002	.9642596648 +000	308	32	51.3	.5385171106 +001	.6008921712 +000	26	-.7541387801 +000	22
.1530000000 +002	.9784736425 +000	320	40	2.6	.5596701885 +001	.7568294737 +000	26	-.6201772463 +000	22
.1540000000 +002	.9965522776 +000	332	16	4.5	.5799169246 +001	.8820815479 +000	26	-.4637333123 +000	22
.1550000000 +002	.1013622853 +001	343	20	13.1	.5992360636 +001	.9710585322 +000	26	-.2906486128 +000	22
.1560000000 +002	.1025378407 +001	353	59	45.2	.6178393974 +001	.1019753602 +001	26	-.1072542211 +000	22
.1570000000 +002	.1029048200 +001	4	26	56.2	.7764866135 -001	.1025947530 +001	26	.7982394475 -001	22
.1580000000 +002	.1023904676 +001	14	56	1.2	.2606417595 +000	.9893220859 +000	27	.2638609403 +000	22
.1590000000 +002	.1011351713 +001	25	40	40.0	.4481617144 +000	.9114758483 +000	27	.4382283266 +000	22

SIZE PARAMETER (KA)	MODULUS OF FORM FUNCTION	ARGUMENT OF FORM FUNCTION (DEG MIN SEC)			ARGUMENT OF FORM FUNCTION (RADIANS)	REAL PART OF FORM FUNCTION	NO. TERMS USED IN REAL PART	IMAG PART OF FORM FUNCTION	NO. TERMS USED IN IMAG PART
.1600000000 +002	.9946273633 +000	36	51	23.4	.6432673015 +000	.7958413996 +000	27	.5965903607 +000	23
.1610000000 +002	.9781118347 +000	48	32	50.6	.8473118282 +000	.6475103570 +000	27	.7330982872 +000	23
.1620000000 +002	.9662561175 +000	60	41	33.5	.1059286339 +001	.4729770778 +000	27	.8425814908 +000	23
.1630000000 +002	.9623116172 +000	73	5	44.1	.1275758521 +001	.2798171445 +000	27	.9207312389 +000	23
.1640000000 +002	.9672872203 +000	85	28	28.6	.1491813248 +001	.7631991339 -001	27	.9642716617 +000	23
.1650000000 +002	.9796286762 +000	97	33	40.1	.1702763121 +001	-.1289035465 +000	27	.9711108171 +000	23
.1660000000 +002	.9957945489 +000	109	11	23.1	.1905720467 +001	-.3273152045 +000	28	.9404634711 +000	23
.1670000000 +002	.1011415846 +001	120	20	7.5	.2100249062 +001	-.5108267818 +000	28	.8729364315 +000	23
.1680000000 +002	.1022519146 +001	131	5	49.5	.2288075538 +001	-.6721393908 +000	28	.7705673517 +000	23
.1690000000 +002	.1026450911 +001	141	39	17.1	.2472341592 +001	-.8050316194 +000	28	.6368088915 +000	24
.1700000000 +002	.1022389567 +001	152	13	25.9	.2656807433 +001	-.9045847909 +000	28	.4764522870 +000	24
.1710000000 +002	.1011463236 +001	163	0	50.3	.2845130622 +001	-.9673392135 +000	28	.2954872654 +000	24
.1720000000 +002	.9965008855 +000	174	11	19.1	.3040165309 +001	-.9913795238 +000	28	.1008992304 +000	24
.1730000000 +002	.9814172702 +000	185	49	27.3	.3243244917 +001	-.9763510529 +000	28	-.9959156406 -001	24
.1740000000 +002	.9702576560 +000	197	52	36.1	.3453599932 +001	-.9234129509 +000	29	-.2978396232 +000	24
.1750000000 +002	.9660631597 +000	210	10	38.7	.3668287926 +001	-.8351355237 +000	29	-.4856198983 +000	24
.1760000000 +002	.9699200614 +000	222	28	40.6	.3882972571 +001	-.7153523146 +000	29	-.6549931233 +000	24
.1770000000 +002	.9806179822 +000	234	32	1.7	.4093387239 +001	-.5689765288 +000	29	-.7986722335 +000	24
.1780000000 +002	.9950932452 +000	246	11	8.5	.4296750846 +001	-.4017925966 +000	29	-.9103698566 +000	25
.1790000000 +002	.1009424683 +001	257	23	53.1	.4492444164 +001	-.2202320065 +000	29	-.9851071277 +000	25
.1800000000 +002	.1019940935 +001	268	15	1.2	.4681851624 +001	-.3114145894 -001	29	-.1019465409 +001	25
.1810000000 +002	.1024097869 +001	278	54	2.4	.4867734744 +001	.1584501723 +000	29	-.1011765778 +001	25
.1820000000 +002	.1020992240 +001	289	32	40.1	.5053504252 +001	.3415610208 +000	30	-.9621648627 +000	25
.1830000000 +002	.1011506688 +001	300	22	30.4	.5242534612 +001	.5114774484 +000	30	-.8726606440 +000	25
.1840000000 +002	.9981065708 +000	311	32	45.4	.5437502451 +001	.6619646102 +000	30	-.7470070827 +000	25
.1850000000 +002	.9842955740 +000	323	7	57.7	.5639729307 +001	.7874630987 +000	30	-.5905418234 +000	25
.1860000000 +002	.9737620699 +000	335	6	12.2	.5848657453 +001	.8832692978 +000	30	-.4099364781 +000	26
.1870000000 +002	.9693608210 +000	347	18	49.5	.6061768473 +001	.9456960816 +000	30	-.2128833542 +000	26
.1880000000 +002	.9722407244 +000	359	32	38.7	.6275228048 +001	.9722099443 +000	30	-.7736289683 -002	26
.1890000000 +002	.9814878211 +000	11	34	16.9	.2019581224 +000	.9615397489 +000	30	.1968747174 +000	26
.1900000000 +002	.9944580680 +000	23	14	32.5	.4056556659 +000	.9137516847 +000	31	.3924343353 +000	26
.1910000000 +002	.1007630803 +001	34	30	47.0	.6023666764 +000	.8302847373 +000	31	.5709177612 +000	26
.1920000000 +002	.1017610075 +001	45	26	44.2	.7931755086 +000	.7139410207 +000	31	.7251334251 +000	26
.1930000000 +002	.1021955402 +001	56	10	42.9	.9805011568 +000	.5688267817 +000	31	.8490164520 +000	26
.1940000000 +002	.1019693973 +001	66	53	21.1	.1167436454 +001	.4002408829 +000	31	.9378608819 +000	26
.1950000000 +002	.1011490096 +001	77	45	22.0	.1357100186 +001	.2145101520 +000	31	.9884824774 +000	27
.1960000000 +002	.9994846179 +000	88	55	25.4	.1552011933 +001	.1877360817 -001	31	.9993082873 +000	27
.1970000000 +002	.9868153164 +000	100	28	2.3	.1753485501 +001	-.1792793269 +000	31	.9703934211 +000	27
.1980000000 +002	.9768533941 +000	112	21	56.3	.1961150319 +001	-.3717081524 +000	32	.9033690292 +000	27
.1990000000 +002	.9722858210 +000	124	29	44.1	.2172857605 +001	-.5506468011 +000	32	.8013287829 +000	27

SIZE PARAMETER (KA)	MODULUS OF FORM FUNCTION	ARGUMENT OF FORM FUNCTION (DEG MIN SEC)			ARGUMENT OF FORM FUNCTION (RADIANS)	REAL PART OF FORM FUNCTION	NO. TERMS USED IN REAL PART	IMAG PART OF FORM FUNCTION	NO. TERMS USED IN IMAG PART
.2000000000 +002	.9743111033 +000	136	39	45.3	.2385212258 +001	-.7086407065 +000	32	.6686632000 +000	27
.2010000000 +002	.9822716167 +000	148	39	46.3	.2594656585 +001	-.8389797935 +000	32	.5108526549 +000	27
.2020000000 +002	.9938935999 +000	160	20	57.7	.2798624539 +001	-.9360099189 +000	32	.3342303392 +000	27
.2030000000 +002	.1006016477 +001	171	40	16.1	.2996226397 +001	-.9954059391 +000	32	.1457263486 +000	27
.2040000000 +002	.1015497305 +001	182	40	30.6	.3188283039 +001	-.1014390618 +001	32	-.4739673493 -001	28
.2050000000 +002	.1019995237 +001	193	28	56.4	.3376903963 +001	-.9918860121 +000	32	-.2378075307 +000	28
.2060000000 +002	.1018479424 +001	204	15	11.6	.3564891430 +001	-.9285872730 +000	32	-.4183611036 +000	28
.2070000000 +002	.1011419959 +001	215	9	11.4	.3755131079 +001	-.8269529811 +000	33	-.5823393340 +000	28
.2080000000 +002	.1000666919 +001	226	19	6.3	.3950001453 +001	-.6911106532 +000	33	-.7236714360 +000	28
.2090000000 +002	.9890303350 +000	237	49	26.1	.4150810285 +001	-.5266814933 +000	33	-.8371305801 +000	28
.2100000000 +002	.9795970813 +000	249	39	28.4	.4357352272 +001	-.3405322243 +000	33	-.9185032640 +000	28
.2110000000 +002	.9749010103 +000	261	42	56.9	.4567802397 +001	-.1404669925 +000	33	-.9647284612 +000	28
.2120000000 +002	.9761784528 +000	273	49	31.0	.4779152689 +001	.6512488786 -001	33	-.9740036554 +000	28
.2130000000 +002	.9829936534 +000	285	47	59.4	.4988148084 +001	.2676469944 +000	33	-.9458549619 +000	29
.2140000000 +002	.9934009129 +000	297	29	55.0	.5192330260 +001	.4586801130 +000	33	-.8811685013 +000	29
.2150000000 +002	.1004565222 +001	308	51	54.3	.5390712687 +001	.6303533260 +000	34	-.7821802668 +000	29
.2160000000 +002	.1013577630 +001	319	55	58.9	.5583884640 +001	.7756833817 +000	34	-.6524223355 +000	29
.2170000000 +002	.1018194052 +001	330	48	26.0	.5773675218 +001	.8888666747 +000	34	-.4966237634 +000	29
.2180000000 +002	.1017336262 +001	341	37	58.6	.5962619894 +001	.9655105139 +000	34	-.3205659326 +000	29
.2190000000 +002	.1011301625 +001	352	33	47.9	.6153390274 +001	.1002795004 +001	34	-.1308936818 +000	29
.2200000000 +002	.1005413204 +001	6	11	6.9	.1079528556 +000	.9995604397 +000	34	.1083265370 +000	11
.2210000000 +002	.9909835185 +000	15	11	56.8	.2652744591 +000	.9563194910 +000	34	.2598102484 +000	29
.2220000000 +002	.9820449263 +000	26	58	32.6	.4708153142 +000	.8751972082 +000	34	.4454683874 +000	30
.2230000000 +002	.9772556148 +000	38	58	7.6	.6801334955 +000	.7598052667 +000	35	.6145929493 +000	30
.2240000000 +002	.9778789812 +000	51	1	32.1	.8905644698 +000	.6150597628 +000	35	.7602294325 +000	30
.2250000000 +002	.9836713906 +000	62	58	31.5	.1099128550 +001	.4469533190 +000	35	.8762660185 +000	30
.2260000000 +002	.9929788648 +000	74	41	1.0	.1303475063 +001	.2622941702 +000	35	.9577101828 +000	30
.2270000000 +002	.1003262118 +001	86	5	21.4	.1502541376 +001	.6842444954 -001	35	.1000926058 +001	30
.2280000000 +002	.1011829854 +001	97	12	52.2	.1696713210 +001	-.1270700560 +000	35	.1003819135 +001	30
.2290000000 +002	.1016532527 +001	108	8	58.2	.1887565050 +001	-.3166475256 +000	35	.9659568954 +000	30
.2300000000 +002	.1016254423 +001	119	1	31.8	.2077386661 +001	-.4930852723 +000	35	.8886168846 +000	30
.2310000000 +002	.1011139648 +001	129	59	3.0	.2268651872 +001	-.6497341093 +000	35	.7747573659 +000	31
.2320000000 +002	.1002542009 +001	141	8	51.7	.2463492018 +001	-.7807452513 +000	36	.6289096370 +000	31
.2330000000 +002	.9927096167 +000	152	35	24.7	.2663201228 +001	-.8812646411 +000	36	.4569956407 +000	31
.2340000000 +002	.9842384653 +000	164	18	55.9	.2867847208 +001	-.9475904549 +000	36	.2660783464 +000	31
.2350000000 +002	.9793885982 +000	176	14	59.7	.3076141370 +001	-.9772915600 +000	36	.6405648283 -001	31
.2360000000 +002	.9794406554 +000	188	15	29.6	.3285725826 +001	-.9692846239 +000	36	-.1406816101 +000	31
.2370000000 +002	.9843173908 +000	200	11	3.0	.3493872951 +001	-.9238688479 +000	36	-.3396278519 +000	31
.2380000000 +002	.9926250422 +000	211	53	57.3	.3698339366 +001	-.8427175433 +000	36	-.5245299007 +000	31
.2390000000 +002	.1002093942 +001	223	20	21.1	.3898004214 +001	-.7288264237 +000	36	-.6877530905 +000	31

SIZE PARAMETER (KA)	MODULUS OF FORM FUNCTION	ARGUMENT OF FORM FUNCTION (DEG MIN SEC)	ARGUMENT OF FORM FUNCTION (RADIANS)	REAL PART OF FORM FUNCTION	NO. TERMS USED IN REAL PART	IMAG PART OF FORM FUNCTION	NO. TERMS USED IN IMAG PART
.2400000000 +002	.1010236056 +001	234 30 57.2	.4093074366 +001	-.5864190010 +000	37	-.8226114784 +000	32
.2410000000 +002	.1014994669 +001	245 30 22.6	.4284893002 +001	-.4208101188 +000	37	-.9236520023 +000	32
.2420000000 +002	.1015225849 +001	256 25 42.9	.4475522955 +001	-.2382301727 +000	37	-.9868788722 +000	32
.2430000000 +002	.1010938100 +001	267 24 50.3	.4667254249 +001	-.4561292917 -001	37	-.1009908562 +001	32
.2440000000 +002	.1003273360 +001	278 34 41.6	.4862107273 +001	.1496478357 +000	37	-.9920498779 +000	32
.2450000000 +002	.9942372667 +000	289 59 41.8	.5061366375 +001	.3399665289 +000	37	-.9343074985 +000	32
.2460000000 +002	.9862113830 +000	301 40 27.7	.5265210875 +001	.5178506018 +000	37	-.8393114119 +000	32
.2470000000 +002	.9813309587 +000	313 33 19.6	.5472575052 +001	.6761922844 +000	37	-.7111782092 +000	32
.2480000000 +002	.9808852086 +000	325 31 7.9	.5681376048 +001	.8085561149 +000	37	-.5553132464 +000	32
.2490000000 +002	.9849407797 +000	337 25 17.8	.5889118255 +001	.9094501597 +000	38	-.3781649728 +000	33
.2500000000 +002	.9923364299 +000	349 8 28.6	.6093664837 +001	.9745683341 +000	38	-.1869442487 +000	33
.2510000000 +002	.1001048986 +001	0 36 40.2	.1066694694 -001	.1000992035 +001	38	.1067793391 -001	33
.2520000000 +002	.1008780883 +001	11 50 3.1	.2065454229 +000	.9873394657 +000	38	.2068807628 +000	33
.2530000000 +002	.1013567111 +001	22 52 30.8	.3992478267 +000	.9338537504 +000	38	.3939993158 +000	33
.2540000000 +002	.1014244120 +001	33 50 25.5	.5906269227 +000	.8424227690 +000	38	.5648141413 +000	33
.2550000000 +002	.1010700880 +001	44 51 4.3	.7828009332 +000	.7165272051 +000	38	.7128148663 +000	33
.2560000000 +002	.1003888030 +001	56 1 2.1	.9776853777 +000	.5611165292 +000	38	.8324298272 +000	33
.2570000000 +002	.9955901245 +000	67 24 41.3	.1176552214 +001	.3824165183 +000	39	.9192155910 +000	33
.2580000000 +002	.9879910767 +000	79 2 59.0	.1379677947 +001	.1876758505 +000	39	.9700021355 +000	34
.2590000000 +002	.9831076114 +000	90 52 56.2	.1586194836 +001	-.1513779291 -001	39	.9829910594 +000	34
.2600000000 +002	.9822297290 +000	102 48 14.3	.1794267735 +001	-.2176778625 +000	39	.9578056112 +000	34
.2610000000 +002	.9855480806 +000	114 41 2.8	.2001615264 +001	-.4115797097 +000	39	.8954926923 +000	34
.2620000000 +002	.9921095047 +000	126 24 22.7	.2206206375 +001	-.5888245142 +000	39	.7984779025 +000	34
.2630000000 +002	.1000116677 +001	137 54 8.1	.2406848437 +001	-.7420888285 +000	39	.6704756060 +000	34
.2640000000 +002	.1007451194 +001	149 10 1.1	.2603454908 +001	-.8650627225 +000	39	.5163568477 +000	34
.2650000000 +002	.1012238753 +001	160 15 16.0	.2796967482 +001	-.9527212933 +000	39	.3419786999 +000	34
.2660000000 +002	.1013304205 +001	171 15 34.6	.2989044006 +001	-.1001536705 +001	40	.1539793501 +000	34
.2670000000 +002	.1010431607 +001	182 17 40.6	.3181641383 +001	-.1009621399 +001	40	-.4045568521 -001	35
.2680000000 +002	.1004398723 +001	193 27 48.6	.3376575199 +001	-.9767962446 +000	40	-.2338501473 +000	35
.2690000000 +002	.9967879384 +000	204 50 17.7	.3575102083 +001	-.9045822397 +000	40	-.4187089272 +000	35
.2700000000 +002	.9896001199 +000	216 26 22.4	.3777582852 +001	-.7961174017 +000	40	-.5877971419 +000	35
.2710000000 +002	.9847387824 +000	228 13 39.9	.3983325802 +001	-.6560046627 +000	40	-.7344170150 +000	35
.2720000000 +002	.9834876928 +000	240 6 38.0	.4190719630 +001	-.4900995917 +000	40	-.8526725234 +000	35
.2730000000 +002	.9861439280 +000	251 58 6.9	.4397681530 +001	-.3052493179 +000	40	-.9377114165 +000	35
.2740000000 +002	.9919405471 +000	263 41 29.5	.4602285196 +001	-.1089958730 +000	41	-.9859340489 +000	35
.2750000000 +002	.9992876039 +000	275 12 36.1	.4803320909 +001	.9074197633 -001	41	-.9951590874 +000	35
.2760000000 +002	.1006235661 +001	286 30 44.3	.5000583258 +001	.2859937462 +000	41	-.9647371576 +000	36
.2770000000 +002	.1011000329 +001	297 38 32.7	.5194840180 +001	.4690556122 +000	41	-.8956051014 +000	36
.2780000000 +002	.1012402256 +001	308 41 5.8	.5387568399 +001	.6327892540 +000	41	-.7912759571 +000	36
.2790000000 +002	.1010133766 +001	319 44 35.8	.5580572826 +001	.7708902223 +000	41	-.6527621998 +000	36

SIZE PARAMETER (KA)	MODULUS OF FORM FUNCTION	ARGUMENT OF FORM FUNCTION (DEG MIN SEC)	ARGUMENT OF FORM FUNCTION (RADIANS)	REAL PART OF FORM FUNCTION	NO. TERMS USED IN REAL PART	IMAG PART OF FORM FUNCTION	NO. TERMS USED IN IMAG PART
.2800000000 +002	.1004816470 +001	330 54 57.5	.5775573124 +001	.8781168465 +000	41	-.4884331499 +000	36
.2810000000 +002	.9978474167 +000	342 16 26.3	.5973807616 +001	.9504728405 +000	41	-.3038105439 +000	36
.2820000000 +002	.9910573317 +000	353 50 31.6	.6175709648 +001	.9853389791 +000	41	-.1063095998 +000	36
.2830000000 +002	.9862410885 +000	5 35 23.0	.9755890392 -001	.9815514169 +000	42	.9606404479 -001	36
.2840000000 +002	.9846698628 +000	17 26 10.0	.3043175247 +000	.9394259300 +000	42	.2950485734 +000	36
.2850000000 +002	.9867314954 +000	29 16 21.1	.5109020180 +000	.8607296410 +000	42	.4824764546 +000	37
.2860000000 +002	.9918257535 +000	40 59 40.3	.7154896689 +000	.7486024220 +000	42	.6506248836 +000	37
.2870000000 +002	.9985534102 +000	52 31 56.8	.9168640728 +000	.6074321444 +000	42	.7925497480 +000	37
.2880000000 +002	.1005124621 +001	63 52 6.9	.1114717146 +001	.4426884881 +000	42	.9023870604 +000	37
.2890000000 +002	.1009844187 +001	75 2 16.6	.1309659052 +001	.2607210029 +000	42	.9756074217 +000	37
.2900000000 +002	.1011535246 +001	86 6 55.6	.1502998070 +001	.6852779901 -001	42	.1009211324 +001	37
.2910000000 +002	.1009810614 +001	97 11 46.7	.1696395508 +001	-.1264981853 +000	43	.1001856120 +001	37
.2920000000 +002	.1005150864 +001	108 22 25.4	.1891478424 +001	-.3168375831 +000	43	.9539089085 +000	37
.2930000000 +002	.9987828373 +000	119 43 2.8	.2089463797 +001	-.4951199260 +000	43	.8674234346 +000	37
.2940000000 +002	.9923785531 +000	131 15 21.1	.2290846835 +001	-.6543968886 +000	43	.7460428305 +000	38
.2950000000 +002	.9876281761 +000	142 57 58.6	.2495232033 +001	-.7884048320 +000	43	.5948337878 +000	38
.2960000000 +002	.9857847205 +000	154 46 42.7	.2701395134 +001	-.8918074126 +000	43	.4200607741 +000	38
.2970000000 +002	.9873127826 +000	166 35 37.6	.2907609974 +001	-.9604092142 +000	43	.2289119308 +000	38
.2980000000 +002	.9917612329 +000	178 18 48.3	.3112155997 +001	-.9913315750 +000	43	.2918991909 -001	38
.2990000000 +002	.9979065459 +000	189 52 4.3	.3313819422 +001	-.9831431110 +000	43	-.1710178282 +000	38
.3000000000 +002	.1004109631 +001	201 14 4.1	.3512204034 +001	-.9359366982 +000	44	-.3636738195 +000	38

Appendix E

DEFINITIONS OF SYMBOLS

English Small Letters

a radius of the reflecting sphere

c sound speed in the fluid surrounding the reflector

c_1, c_2 respective speeds of dilatational and shear waves within the reflecting sphere

f_r, f_∞ respective reflection form-function at arbitrary range r and reflection form-function in the far-field (where $kr \to \infty$)

$h_n^{(2)}(x)$ nth order spherical Bessel function of the third kind with argument x — see Eq. (A4b)

i one of the square roots of -1

$j_n(x)$ nth order spherical Bessel function of the first kind with argument x

k wavenumber of the incident acoustic wave ($k = 2\pi/\lambda$)

k_1, k_2 respective wavenumbers of the dilatational and shear waves within the reflector

ka sphere size-parameter

kr range-parameter

n integer ($n = 0, 1, 2, \ldots$)

p instantaneous acoustic pressure at a field point outside the reflecting sphere

$p_{(i)}$ instantaneous pressure in the incident acoustic wave

$p_{(s)}$ scattered component of the instantaneous acoustic pressure

p_0 pressure amplitude of the incident acoustic wave

q proportionality factor relating the sequence (G_K) to a sequence of the spherical Bessel functions of the first kind produced by Miller's device

r radial distance from the sphere center to the field point (range)

t time

υ displacement of an element of the sphere

x arbitrary real number

$y_n(x)$ nth order spherical Bessel function of the second kind with argument x

z Z-coordinate in XYZ-coordinate system

English Capital Letters

$A_n, B_n, C_n, D_n, E_n, F_n$ parameters - see Eqs. (6a-f)

A'_n, B'_n, C'_n, D'_n parameters - see Eqs. (13a-d)

G_K Kth term of sequence generated by a Miller's device iteration

$Im(\tau)$ imaginary part of a complex number τ

K integer $K = (0, 1, 2, \ldots, M-1)$

M integer at which the Miller's device iteration begins

N value of index n to which the infinite summations needed for determining a reflection form-function must be carried in order to express the form-function to within a defined accuracy ϵ

P, Q respectively, $P(n+1/2, x)$ and $Q(n+1/2, x)$ — See Eqs. (A15a, b)

$P_n(x)$ nth order Legendre polynomial with argument x

$Re(\tau)$ real part of a complex number τ

X, Y, Z designations of the three axes of a Cartesian coordinate system

Greek Small Letters

ϵ arbitrary but fixed small quantity. The difference between the form-function values, that are calculated by summing N and $N-1$ terms respectively of the appropriate infinite sum, is less than ϵ.

ζ_n a parameter - see Eq. (6) for definition

η_n a parameter - see Eq. (5) for definition in the elastic case and Eq. (15) for definition in the rigid case

θ polar angle in the spherical coordinate system in which the reflecting sphere is located

λ wavelength of the incident acoustic wave

ρ density of the fluid surrounding the reflector

ρ_1 density of the reflecting sphere

σ Poisson's ratio of the reflector material

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13. ABSTRACT

Numerical methods are described by which the theoretical formulas specifying the acoustic field resulting from scattering of an acoustic wave by an elastic sphere may be evaluated. They are restricted to the case of monostatic reflection of a plane harmonic wave incident upon the elastic sphere — the sphere being immersed in a uniform homogeneous fluid, the acoustic properties of which are assumed to be the same at all frequencies. Methods of calculating reflection form-functions as a function of sphere size-parameter are shown in the range of size-parameters $0 < ka \leq 30.0$. The reflection form-functions may be calculated at any distance from the elastic sphere by one of two formulas, one describing reflection in the far-field case. A description is given of the means by which the spherical Bessel function values, required in computations made using the reflection formulas, are calculated. Examples are given of form-function computational results in the case of reflection by elastic spheres and, for purposes of comparison, also in the case of reflection by rigid, immovable spheres. Brief tables of computed values of the far-field reflection form-function as a function of sphere size-parameter are given in the cases of an aluminum and of a rigid sphere. Implementation of the numerical methods described requires access to a digital computer. The capabilities and limitations of the two different computing machines used in the treatment of the elastic sphere problem by the numerical methods described, are discussed.

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