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Brownian Motion as a Route to Statistical Mechanics and to a Tentative Quantum Communication Theory

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ABSTRACT

We consider a molecular bath in statistical equilibrium at temperature T. A heavy particle in this bath undergoes Brownian motion, acquiring a mean kinetic energy $3kT/2$ regardless of the nature of its encounters with the molecules of the bath. Specifying that these encounters are elastic collisions occurring only for molecules in a narrow band of relative velocities, we use Brownian-motion principles to show, in one-dimensional examples, that the number density of molecules in that band is determined by the $kT/2$ mean kinetic energy of the Brownian particle. We thus obtain the one-dimensional equilibrium energy distribution for classical and quantum molecules, at the price of assuming no more than the relation between the first two moments (the mean count and the mean-square count of particles per unit time) of the appropriate microstatistics.

Toward a quantum communication theory, we consider the Brownian particle to be a narrow-reflection-band filter freely sliding in an ideal transmission line terminated by matched loads at temperature T. We insert Planck's constant h into the mathematics so that the resulting one-dimensional electromagnetic spectrum is Planckian for temperature T. The result gives the following rule of thumb for quantum communication theoretic problems classically entailing no more than power spectrums $S(f)$ and their squares $S^2(f)$: add $hfS(f)$ to $S^2(f)$ to produce the quantum version. The merits of this approach, of the general method, and of some related problems are discussed.

PROBLEM STATUS

This is an interim report on one phase of the problem; work on this and other phases continues.

AUTHORIZATION

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BROWNIAN MOTION AS A ROUTE TO STATISTICAL MECHANICS AND TO A TENTATIVE QUANTUM COMMUNICATION THEORY*

INTRODUCTION

The developments on the subject of this report were started in 1956. The opening problem was to find the spectrum of electromagnetic radiation in an enclosure having perfectly conducting walls; a lossless "Brownian particle" was to be pushed around in the enclosure by the radiation pressure of the field and was to shuffle the frequencies in the field via Doppler-effect reflections. Since the Rayleigh-Jeans law cannot describe the frequency distribution of the electromagnetic part of the finite conserved energy, the solution of the problem might lead to something new, possibly to an understanding of how Planck's constant must be "invented" to give a spectrum of finite energy content.

The problem is basically not a Brownian-motion one, and we have not made headway with it even in the simplest one-dimensional version described herein. Instead, considerations have led to a somewhat backhanded way of getting from assumed microstatistics to thermal equilibrium macrostatistics. Applied to the one-dimensional radiation field, the method uses the mathematics of communication theory; one has to insert Planck's constant into this mathematics in a particular way to get the right spectrum for the radiation field. The result amounts to a prescription for converting classical communication theory to the quantum version.

These developments are published as a report so as to preserve mathematical detail, polemical loose ends, and so as not to have sacrifice either the physics or the communication theoretic aspects, as would be necessary in a paper publishable in a journal in either field.

In due course, we hope to consider a rotatable Brownian particle in a three-dimensional radiation field. The problem should reveal or confirm the way one describes the polarization statistics.

THERMAL EQUILIBRIUM STATISTICS AS DERIVED FROM THE BROWNIAN MOTION OF AN ARBITRARY TEST PARTICLE

According to classical statistical mechanics, a system in thermal equilibrium at temperature T contains mean energy $kT/2$ per degree of freedom. In particular, suppose that an observable Brownian test particle of mass M is placed in a bath of temperature T. If V is the velocity component of the Brownian particle along some x axis fixed with respect to the bath-containing vessel, the particle's observable kinetic energy $MV^2/2$ has the time average $kT/2$ regardless of the details of the interaction between the Brownian particle and the bath. (Here and henceforth, we choose M large enough so that $(V^2)_{\text{average}} \approx kT/M << c^2$; the Brownian particle moves at nonrelativistic velocities.)

There are a number of systems in physics for which the concept "degree of freedom" is hard to define. What does "degree of freedom" mean for electromagnetic waves?

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Even when "degree of freedom" is well specified, as with a classical billiard-ball gas, the value $kT/2$ of the mean energy is insufficient to specify the macroscopic statistics, i.e., the distribution of thermal kinetic energies among the billiard-ball molecules of the gas as a whole. Apparently one needs a further principle, such as an entropy definition, for deducing the macrostatistics.

With or without a defined "degree of freedom" it may be possible to find the thermal-equilibrium energy distribution of the system by the following alternative argument: One regards the Brownian test particle as losslessly coupled only with selected energy bands of the equilibrium system. By means of the theory of Brownian motion, the test particle's mean kinetic energy can be worked out from its mode of interaction with the energies in the selected band. Setting the particle's mean kinetic energy (for its one degree of freedom of motion along the x axis) equal to $kT/2$ should then lead to specifying the energy in the selected band. The arbitrary selection of the energy band and/or mode of interaction should then lead to the full energy distribution of the system.

For reasons of mathematical simplicity, we use the foregoing approach in a few one-dimensional cases only. As will be seen, the physically interesting and essential problem is the prescription of statistics to the energies in a selected narrow band. That is, one must correctly assume the microstatistics, the statistical distribution of the result of some finite counting process for bath molecules which have been sorted into some small energy (or velocity or momentum) range. For instance, counts of classical particles, arriving at the counter in some narrow velocity interval during some short time interval, are distributed according to the Poisson statistics, which are thus the classical molecular microstatistics.

Further, one may assume that Planck's quantum h was, and is, an ad hoc constant introduced in order to obtain the correct Planckian spectrum, or macrostatistics, of blackbody thermal radiation. To derive Planck's one-dimensional power spectrum

$$S(f) = hf/(\exp[hf/(kT)] - 1) \tag{1}$$

via the present Brownian-motion method thus requires the ad hoc introduction of h into the microstatistics of a one-dimensional radiation field. But this is essentially the central problem in adapting conventional continuous-wave communication theory to take account of the quantum granularity of the waves with which one communicates. We therefore anticipate having something to say concerning quantum communication theory.

The entropy definition permits one to argue from appropriate microstatistics to the macroscopic energy distributions. The present (one-dimensional) derivations will be interesting if one can obtain the macrostatistics with weaker requirements on the governing microstatistics, or with fewer ad hoc assumptions inserted more naturally in the microstatistics. Whether our method proves interesting in these senses will be discussed later in the report.

We shall first outline the necessary deductions from general Brownian-motion theory, and then illustrate how one is led from well-known microstatistics to the correct macrostatistics of (one-dimensional) thermal baths of classical and quantum molecules. (The microstatistics are set forth in the Appendix.) Next we consider a (one-dimensional) bath of electromagnetic waves on a transmission line containing a freely moving filter treated as a Brownian test particle. The objective is to determine how to insert h into the microstatistics so as to achieve both the correct Planckian macrostatistics and to find a plausible basis for constructing a quantum communication theory. We find a limited, ad hoc, answer here. In the final section of this report we discuss whether the present scheme of proceeding from micro- to macrostatistics has any advantages, whether the method is germane to a quantum communication theory, and raise a few related open problems which we hope will be of interest to the reader.

RELEVANT ASPECTS OF BROWNIAN-MOTION THEORY

First, we recall some facts from the theory of Brownian motion (1,2). Let t measure the time and let $V(t)$ be the instantaneous x component of velocity of the Brownian particle, of mass M, along the one dimension of interest. Owing to impacts from the molecules or waves in the time interval Δt, the velocity will have the average change $\langle \Delta V \rangle \equiv [V(t+\Delta t) - V(t)]_{\text{average}} = -A(V)\Delta t + 0(\Delta t)^2$ and the random mean-square change $\langle (\Delta V)^2 \rangle \equiv [V(t+\Delta t) - V(t)]^2_{\text{average}} = B(V)\Delta t + 0(\Delta t)^2$. Then the probability $W(V,t)\,dV$ that the Brownian particle has velocity between V and $V+dV$ at a given time t satisfies the Fokker-Planck equation

$$W_t = \Big(A(V)W\Big)_V + \frac{1}{2}\Big(B(V)W\Big)_{VV}, \tag{2}$$

with the velocity V_0 at time $t = t_0$ being given as initial condition. Here we are not concerned with the details of W as a solution of a diffusion equation, but need only the mean kinetic energy

$$\overline{MV^2}/2 = (M/2)\int_{-\infty}^{\infty} V^2 W(V,t)\,dV .$$

For this purpose we can assume that all traces of the initial conditions are lost so that the left side of Eq. (2) vanishes: the equation is solvable by quadratures and, if $W(\pm\infty, t) \equiv 0$, we get

$$W(V,t) \equiv W(V) = (K/B)\exp\left(-2\int_0^V (A/B)\,dV\right)$$

where the constant K is chosen so that

$$\int_{-\infty}^{\infty} W\,dV = 1 .$$

The bombarding molecules or waves are assumed to be in thermal equilibrium, so that if the origin of the x axis is at rest with respect to the walls of the containing vessel, then the statistics are assumed independent of reversal of the x direction. The mean velocity $\overline{V}$ of the Brownian particle, with respect to the origin, then vanishes, and A is an odd function, B an even function of V. Thus

$$A(V) = VA_1 + 0(V^3), \qquad B(V) = B_0 + 0(V^2), \tag{3}$$

and

$$\overline{V^2} = \int_{-\infty}^{\infty} W(V)V^2 dV = B_0/(2A_1),$$

with all relative errors decreasing inversely with M. Then the relation $\overline{MV^2} = kT$ yields

$$2A_1 kT = B_0 M . \tag{4}$$

Thus, one should be able to determine the full statistics of the particles or waves of the thermal bath through a calculation of only the leading terms A_1 and B_0 and their effects on the mean and mean-square incremental velocity of the massive Brownian particle.

ONE-DIMENSIONAL BILLIARD-BALL GASES

We suppose a typical length dx (light-seconds) of the x axis to contain an average number $N(v)\,dv\,dx$ of molecules, each of mass m, in the velocity range from v to $v + dv$. We assume that the Brownian particle of mass M, velocity V, collides elastically with all molecules reaching it (from either side) with relative speeds in the "collision range" from v_1 to v_2; the particle is transparent to molecules of other relative velocities. The Brownian mass M is taken so large compared with m that in a single elastic collision the molecule's momentum p, relative to the Brownian particle, is simply reversed. Thus (with error of negligible order m/M) a collision transfers the momentum $2p$ to the particle. We also regard M as sufficiently large so that V is always negligible compared with the velocity of light, which is the velocity unit here. (In any event, these assumptions are requirements for using the approximations of the previous section.)

The problem is to specify the velocity distribution $N(v)$ of the molecules in such a way that the relation $\overline{MV^2} = kT$ holds independent of the "collision interval" v_1, v_2 or its infinitesimal version $v,\ v + \Delta v$.

In a time interval Δt (seconds), just those molecules, with velocity v, initially at distances less than $(v - V)\Delta t$ to the left of the particle will overtake the particle; thus the mean number of molecules in the velocity range $v,\ v + \Delta v$ overtaking the particle from the left in time Δt is $N(v)\,\Delta v(v - V)\,\Delta t$. With $N(-v) = N(v)$ as appropriate to a thermal equilibrium, the average number $N(v)\ \Delta v(v + V)\ \Delta t$ of molecules in the speed range $v,\ v + \Delta v$ reaches the particle from the right in the time Δt.

Only those molecules moving to the right with velocities in the range

$$(v_1', v_2') = \left(\frac{v_1 + V}{1 + v_1 V},\ \frac{v_2 + V}{1 + v_2 V}\right) \approx \left[v_1 + V\left(1 - v_1^2\right),\ v_2 + V\left(1 - v_2^2\right)\right] \tag{5}$$

will hit the Brownian particle with relative velocities in the range v_1, v_2. Each such colliding molecule will be reflected, transferring momentum

$$\Delta p' = 2m(v - V)(1 - v^2)^{-1/2}[1 + 0(V^2)] \tag{6}$$

as measured with respect to the fixed container walls. For molecules moving toward the left, we have similarly the "collision range"

$$(v_1'', v_2'') = \left[v_1 - V(1 - v_1^2),\quad v_2 - V(1 - v_2^2)\right] \tag{7}$$

and the momentum transfer, per molecule,

$$\Delta p'' = -2m(v + V)(1 - v^2)^{-1/2}\,. \tag{8}$$

Thus, in time Δt, the mean momentum $M\langle\Delta V\rangle$ transferred to the Brownian particle is

$$M\langle\Delta V\rangle = \int_{v_1'}^{v_2'} \Delta p'(v - V)N(v)\,dv\,\Delta t + \int_{v_1''}^{v_2''} \Delta p''(v + V)N(v)\,dv\,\Delta t\,. \tag{9}$$

After substituting from Eqs. (5-8) to express Eq. (9) in terms of V, v_1, and v_2, we expand in powers of V and retain only the first power of V to obtain the A_1 of Eqs. (3) and (4):

$$-MA_1 = 4m\left\{(1-v_2^2)^{1/2} v_2^2 N(v_2) - (1-v_1^2)^{1/2} v_1^2 N(v_1) - 2\int_{v_1}^{v_2} vN(v)(1-v^2)^{-1/2} dv\right\} . \quad (10)$$

Finally, with $(v_1, v_2) \to (v, v+\Delta v)$ we get, to order $(\Delta v)^2$,

$$MA_1 = -4m\Delta v\left\{\frac{d}{dv}\left[v^2(1-v^2)^{1/2} N(v)\right] - 2vN(v)(1-v^2)^{-1/2}\right\}$$

$$= \frac{-4mv^2\Delta v}{1-v^2}\frac{d}{dv}\left[(1-v^2)^{3/2} N(v)\right] \quad (11)$$

As $B(V) = B_0 + 0(V^2)$, we may simplify the calculation of B_0 by assuming $V = 0$. If the particular numbers N_L and N_R of molecules collide with the Brownian particle from left and right, respectively, in any time interval Δt, and if the relative velocity range for collisions is the narrow one $(v, v+\Delta v)$, so that the individual momentum transfers have the common magnitude $\Delta p = 2mv(1-v^2)^{-1/2}$, then the net transferred momentum is $M\Delta v = \Delta p(N_L - N_R)$. We thus have $B_0 \Delta t \approx (\Delta p)^2 M^{-2} \langle (N_L - N_R)^2 \rangle$, where the angular bracket $\langle\rangle$ denotes the average. We can assume that the collisions from left and right are statistically independent and have the same statistical distribution by symmetry. Thus

$$\langle (N_L - N_R)^2 \rangle = \langle N_L^2 \rangle + \langle N_R^2 \rangle - 2\langle N_L N_R \rangle = 2\left[\langle N^2 \rangle - \langle N \rangle^2\right] ,$$

where $\langle N \rangle = vN(v)\Delta v \Delta t$, as previously assumed (but with $V = 0$). Hence

$$B_0 \Delta t = 8\left(\frac{mv}{M}\right)^2 \left[\langle N^2 \rangle - \langle N \rangle^2\right] (1-v^2)^{-1} .$$

The calculation of B_0 therefore reduces to specifying the mean-square number of discrete molecular collisions in a time-velocity interval $\Delta v \Delta t$, the mass of the molecule being m. This specification, depending on the microstatistics of the particular molecular species, is treated in the Appendix:

$$\langle N^2 \rangle - \langle N \rangle^2 = \langle N \rangle \begin{pmatrix} 1 \\ 1 \mp Ch\, vN(v)(mv)^{-1} (1-v^2)^{3/2} \end{pmatrix} \quad (12)$$

depending on whether the microstatistics are Maxwell-Boltzmann (MB), Fermi-Dirac (FD), or Bose-Einstein (BE). Thus

$$M^2 B_0 = \frac{8m^2 v^3 N(v)\Delta v}{1-v^2} \begin{pmatrix} 1 \\ 1 \mp (Ch/m) N(v)(1-v^2)^{3/2} \end{pmatrix} \quad (13)$$

Taking Eq. (4) in the form $2MA_1 kT = B_0 M^2$, we have, from Eqs. (11) and (13),

$$-\frac{d}{dv}\left[(1-v^2)^{3/2} N(v)\right] = \frac{mv\, N(v)}{kT} \cdot \begin{pmatrix} 1 \\ 1 \mp (Ch/m)(1-v^2)^{3/2} N(v) \end{pmatrix} . \quad (14)$$

Now if $N(v)\,dv\,dx$ is the mean number of molecules in an interval dx having velocities in an interval $(v, v+dv)$, then the mean number of molecules that would arrive (say from the left) at a fixed point x_1, per second, with energies $E = m/(1-v^2)^{1/2}$ in the range $(E, E+dE)$ is

$$vN(v)\left(\frac{dv}{dE}\right)dE = \left[(1-v^2)^{3/2}N(v)/m\right]dE \equiv \bar{N}(E)\,dE \tag{15}$$

Thus Eq. (14) becomes

$$-\frac{d\bar{N}}{dE} = \frac{\bar{N}}{kT}\left(\frac{1}{1 \mp Ch\bar{N}}\right).$$

The solutions, with E_0 as the integration constant, are

$$\bar{N} = \exp[(E_0 - E)/(kT)] \qquad [\mathrm{MB}] \tag{16}$$

and

$$\bar{N} = \frac{1}{Ch(\exp[(E-E_0)/(kT)] \pm 1)} \qquad \begin{bmatrix}\mathrm{FD}\\ \mathrm{BE}\end{bmatrix}. \tag{17}$$

The photon is a BE "molecule." The energy of a photon corresponding to a light wave of frequency f is hf. We consider some blackbody at temperature T, emitting photons down some one-dimensional tube or waveguide to a photon counter or radio receiver. From $E = hf$ and Eq. (17), the mean number of photons N counted per second in the frequency interval from f to $f + df$ is

$$h\bar{N}\,df = \left(C\exp[h(f-f_0)/(kT)] - C\right)^{-1}$$

so that the power $S(f)$ received per unit frequency interval is given through

$$S(f) = hf\left(C\exp[h(f-f_0)/(kT)] - C\right)^{-1} \tag{18}$$

If $S(f)$ is to have a finite positive value at $f = 0$ we must have $f_0 = 0$. Then it is only necessary to choose $C = 1$, instead of $C \approx 1$, to get Eq. (1). But this choice of C is ad hoc; any specific value of C is not within reach of the present considerations.

We have obtained the equilibrium energy distribution functions, Eqs. (16) and (17), for the bath molecules from the Brownian-motion principle. The sole assumption concerning the molecular microstatistics is the relation, introduced in Eq. (12), between the mean number and the mean-square number of molecules counted in a small time-energy interval. The derivation itself would have been somewhat simpler if one assumed nonrelativistic speeds for the molecules, but this would have ruled out the desired extrapolation to massless photons, Eq. (18).

AD HOC QUANTIZATION OF ELECTROMAGNETIC FIELDS

The foregoing examples suggest that a priori knowledge of the relation between the mean and mean-square momentum of molecules of the particular type in a small time-energy interval suffices to prescribe their entire thermal energy spectrum, to within an integration constant. In the following, we treat a converse problem: Dealing with classically described electromagnetic waves running along a transmission line, we do not know the relation between mean and mean-square energy transport, or momentum, for small time-frequency intervals, but we do know that the power spectrum $S(f)$ for waves from a matched load at temperature T should be given by Eq. (1). Thus we treat the Brownian motion of a particle arbitrarily coupled with a classically described one-dimensional electromagnetic field. We require that the mean kinetic energy of the particle be $kT/2$ and that the spectrum of the field be of the form given in Eq. (1) and force the microstatistics of the field to yield Eq. (1), regardless of the coupling of the particle with the field.

The simplest problem entailing the interaction of classical electromagnetic waves with a Brownian object seems to be the following: In vacuum and with the velocity of light taken as unity, we assume plane waves normally incident on a lossless, partially reflecting thin slab, or filter, of mass M per unit area. The simplest filter, in turn, is one having a narrow rejection band, so that the filter totally reflects waves impinging on it with frequencies between f_1 and f_2, as measured at the filter, and is totally transparent at other frequencies. The waves arriving from both the left and the right are assumed to have their origins in independent, stationary, matched loads at temperature T. Hence, from Eq. (1), the average energy and the momentum $S(f)\,df$ arrive, per unit area per second, from each direction in a frequency interval of width df at f, at any point stationary with respect to the loads.

An electromagnetic wave $F(t-x)$ is seen by the filter, which moves through $x=0$ with velocity V, to have the time variation

$$qF(qt') \tag{19}$$

where $q = [(1-V)/(1+V)]^{1/2} \equiv 1/r$ and t' is the time according to a clock attached to the moving filter. Thus the average momentum transferred to the filter by the wave in time $\Delta t'$ is

$$2q^2 \int_{f_1 r}^{f_2 r} S(f)\,df. \tag{20}$$

A similar, oppositely directed momentum transfer, with $-V$ replacing V, takes place in the time $\Delta t'$ owing to the wave $G(t+x)$ moving from the right. Expanding in powers of V, we therefore have

$$\frac{1}{2} M\langle\Delta V\rangle = -4V\int_{f_1}^{f_2} S(f)\,df + 2V[f_2 S(f_2) - f_1 S(f_1)] + 0(V^2).$$

With $(f_1, f_2) \to (f, f+\Delta f)$ we have, for the A_1 of Eqs. (3) and (4),

$$MA_1 = 4\Delta f[2S(f) - 2fS_f] \quad (S_f \equiv \partial S(f)/\partial f). \tag{21}$$

For the B_0 of Eqs. (3) and (4), we cite a known result: Given a Gaussian noise uniform over a frequency band of width Δf, and zero elsewhere, the time integral of the noise power over time Δt has, say, an expected mean value W and an expected rms variation (among independent samples) $D = W/(\Delta f \Delta t)^{1/2}$ (Refs. 2 (p. 141), 3, and 4). For $V=0$, the mean momentum transfer imparted by the waves from the left in the time Δt is $2S(f)\Delta f\,\Delta t$; the mean-square fluctuation is then $4S^2(f)\Delta f\Delta t$ on the assumption that the waves formed a Gaussian process. Adding in the mean-square fluctuation in momentum independently transferred by waves from the right, we have

$$M^2\langle(\Delta V)^2\rangle = 8S^2(f)\Delta f\Delta t + 0(\Delta t^2),$$

or (22)

$$M^2 B_0 = 8S^2(f)\Delta f.$$

From $2A_1 kT = MB_0$, we now have

$$S(f) - fS_f = [S^2(f) + QM]/(kT) \tag{23}$$

where QM stands for hitherto ignored terms of quantum-mechanical origin, representing deviation of the statistics from the classical Gaussian ideal. Setting $QM = 0$ for the classical case, we have $S(f) = fkT/(f + KkT)$, where K is an integration constant. For the only plausible choice $K = 0$, we get the Rayleigh-Jeans law $S(f) = kT$. Toward obtaining Planck's law we set $QM = hfS(f)$, whence

$$S(f) = hf/(K \exp[hf/(kT)] - 1). \tag{24}$$

Here one must take $K = 1$ for $S(f) > 0$, at $f = 0$, or for the Rayleigh-Jeans law to be a valid low-frequency approximation. But for $K = 1$, Eq. (24) is identical with Planck's result, Eq. (1). (The form of QM was, of course, actually found by differentiating Eq. (1) to find the first-order differential equation, Eq. (23), satisfied by Planck's $S(f)$.)

While the foregoing calculation seems to be the simplest of its type, it may not be utterly convincing. First, the physics of waves in the transmission line and of the radiation pressure on the filter was given by fiat, and second, the narrow, square-reflection-band filter sorts out waves according to frequency on an all-or-nothing basis, and thus gives the appearance of treating waves as billiard-ball particles. To overcome both objections we have taken the linear, lossless, passive filter of mass M to be an element shunting the transmission line. The filter is described through a response function $F(t)$ such that if $E(t)$ is the transverse electric field in the line, or voltage drop across a line of unit cross section, then the current $j(t)$ in the transverse element is given through

$$j(t) = \int_0^\infty F(t')\, E(t - t')\, dt'.$$

Here the filter is already linear and passive, but for it to be lossless we must also have

$$F(t) = -2 \int_0^\infty F(t')\, F(t + t')\, dt',$$

according to our calculation. This last relation (or its Fourier transform) proves vital in factoring out the largely arbitrary $F(t)$ from the ensuing version of $2A_1kT = B_0M$. But calculation with this filter merely serves to confirm the physics already assumed and gives our final result, Eq. (23), with no further suggestions as to how the ad hoc terms QM are to be introduced to get Eq. (24). Therefore, we omit the very tedious details of this calculation.

DISCUSSION

At least in simple, one-dimensional circumstances, we have used Brownian-motion principles, together with the relation between the first two moments $\langle n \rangle$ and $\langle n^2 \rangle$ of the appropriate particle microstatistics, to obtain the full thermal-equilibrium energy distribution of the particles. We claim this derivation to be conceptually simpler than the usual one, which we understand to be roughly the following:

1. Write down the appropriate particle microstatistics, as done in the Appendix here;

2. Define an additive entropy function measuring the randomness attributable to finding various numbers of particles in various discrete energy or momentum intervals;

3. Find the thermal-equilibrium distribution which maximizes the entropy when all possible energy or momentum intervals are considered, subject to a total energy or temperature constraint.

Our layman's impression is that Brownian-motion principles are a direct development from elementary classical physics, whereas the entropy definition often appears to be given by mystical fiat. Of course, the entropy concept is so useful in physics and communication theory that it would have to be invented anyway; perhaps the entropy definition is already implied in the a priori microstatistics. But our Brownian-motion derivation uses only the relation between the first two moments of the microstatistics, and we claim that this is conceptually simpler than the derivation of macrostatistics via the entropy concept, unless it is shown that both the entropy definition and the maximum-entropy calculation (under item 3 above) require no more than the relation between the first two moments of the microstatistics.

The relation between the first two moments of the Bose-Einstein and Fermi-Dirac microstatistics being adequate for the present purposes, one wonders whether this one relation is not adequate for all theoretical and experimental purposes. Is there a theoretical deduction, or an experimental result, which demonstrably requires more information from the microstatistics?

In communication theory, the situation is analogous; in limited experience with optimum-filtering and signal-processing problems, no case is recalled where more than the spectral density $S(f)$ and its square were required. But these are the entities encountered in the Brownian-motion result, Eq. (24). We are therefore led to propose a rule of thumb, governing problems in quantum communication theory, as follows: In any classical communication theoretic problem, it is assumed that solutions are obtainable in terms of the signal, noise, jamming, or other spectrums $S(f)$ and their squares $S^2(f)$. One obtains the quantum version, taking account of the quantum granularity of the physical processes, by replacing each $S^2(f)$ by $S^2(f) + hfS(f)$.

This rule of thumb would certainly be inadequate when the classical problem requires knowledge of, say, S^3. Further, this prescription merely generalizes what was done to get the right answer in the known case of the blackbody radiation formula, and one seeks other known problems for confirming that it really works. One such problem is the following. Messages are transmitted electromagnetically from A to B down an ideal transmission line. An enemy J has access to a directional coupler attached to the line and has a fixed power P available for jamming purposes. What kind of jamming modulation should J use and how should his available power P be distributed in frequency so that A's best possible choice of signaling modulation and frequency gives minimum communication rate, in the presence of J's fixed jamming? For a finite communication band, the classical answer is that J should distribute the power P uniformly across the band, in the form of Gaussian white noise. For an infinite band, this prescription would give zero jamming power in any one-cycle-wide communication band — but we have ignored quantum granularity. The classical finite-band answer results if J uses all his power to heat a matched load, which is connected to the transmission line via a filter whose passband is the finite band of the classical problem. This filter has infinite band, i.e., could be removed for the present problem, and the tentative conclusion is that J's power is then in the form of Planckian blackbody radiation from the matched load. The fact that the jamming power density tapers off at high frequencies is compensated for by the quantum granularity deteriorating the quality of the signals A might choose to use at such frequencies.

We have described this jamming problem in some detail because, to our knowledge, it is the only theoretical problem, other than that discussed in the previous section, for which the answer is known (or plausibly inferred) a priori and which might therefore test the theoretical method. As written, the jamming problem appears soluble in terms of Gabor's (5) information-cell quantization. However, it is not seen how Gabor's cells can be used for the Brownian-motion problem of the previous section, which uses a coherent filter that is missing in the above jamming problem.

Our rule-of-thumb prescription is believed to apply to the physical signals, or waves, which form the substantial aspect of communication. Therefore, one might seek an appropriate experiment, rather than a theoretical argument, for testing the prescription. The famous Brown-Twiss experiment (6) is apparently relevant here, but the experiment entails not only the boson statistics of the lightwaves but also the Fermi-Dirac statistics of the photoelectronic currents by which the waves are recorded. H. Gamo's ingenious triple correlator (7) seems to have the same complication. Is it conceptually possible to measure the Bose-Einstein statistics of fluctuations of a "photon stream" without having to allow for Fermi-Dirac statistics in the measuring process? (Here we note that the velocities of our Brownian filters are not directly observable, but must be inferred from observations of the filter's time-varying position. The observables in the problem are the filter's mass M, its bandpass characteristic, its drag coefficient A_1, and its time-varying position $x(t)$ in the transmission line. The drag coefficient A_1 is essentially the same as Chandrasekhar's coefficient β [Ref. (1), Eq. (132)] and is measured by slowly and steadily dragging the filter down a long length L of transmission line in time t: (Measured mean force required for dragging) = $A_1 L/t$. With this auxiliary measurement, the theory permits inference of $\overline{V^2} = kT/M$ from observations of the statistical "diffusion" of $x(t)$, or permits prediction of the latter when T is known from independent measurement.)

Finally, the molecules of our Brownian-motion problem were assumed to arrive at the filter at random times and with random momenta, subject to an overall thermal molecular-velocity distribution. Correspondingly, the waves in the transmission line were supposed to be random because they originated in, and were eventually absorbed by, hot matched-load terminations. These somewhat mystical assumptions should really be clarified or eliminated.

Consider, for instance, a rigid-walled, massive, cubical box containing vacuum and N identical classical, elastic, billiard-ball molecules, for which one assigns positions and momenta at some initial time t_0. In principle, what ensues thereafter is completely determined by the assumed physics and geometry. In practice, the problem is insoluble; one asserts some ergodic principle and thereby infers that the contents of the box behave like a gas having temperature $T = E_0/(Nk)$, where E_0 is the total kinetic energy of the molecules according to their initial assignment.

Consider the same box, now equipped with perfectly conducting walls, and filled with nothing but one elastic dielectric or charged classical particle plus classical electromagnetic radiation. Presumably one can assign initial conditions for the field and the molecule such that the box contains total initial energy E_0, and presumably because the coupling between the particle and the field is complicated, we can again appeal from the too-complex deterministic problem to ergodic principles and deduce that the particle has some ergodic-average kinetic energy $3kT/2$, the remainder of the initial energy E_0 being distributed according to some prescription among the infinite number of standing electromagnetic-wave modes that the box supports.

We do not know what this prescription might be, but it is <u>not</u> a Rayleigh-Jeans prescription, as only a finite energy is available. Yet the problem is supposed classically posed. What, then, are the assumptions underlying statements that the Rayleigh-Jeans law ".... is the only possible radiation law that can be found from classical theory." (Ref. 8, p. 314)?

For a one-dimensional version of the waves-plus-one-particle-in-a-box problem, we assume that conditions are always periodic, with period L, along an infinite x axis. With a particle of mass M, velocity $V = V(t)$, position

$$X = X(t) = \int_0^t V(t')\,dt',$$

and with waves $f(t-x)$ and $g(t+x)$ periodically arrayed, we have for the initial energy $E_0 > M$ per period and the initial total momentum $p = 0$

$$E_0 = \frac{M}{(1-V^2)^{1/2}} + \int_0^L \left\{ f^2[t - X - y] + g^2[t + X + y] \right\} dy,$$

and

$$p = 0 = \frac{MV}{(1-V^2)^{1/2}} + \int_0^L \left\{ f^2[t - X - y] - g^2[t + X + y] \right\} dy,$$

expressing the conservation of energy and momentum with respect to the fixed origin on the x axis. Differentiating these equations with respect to t gives the relativistically correct formula for elastic reflection of the waves f and g at the moving particle. The problem is to find, say, the correlation function

$$R(t) = \lim_{T\to\infty} \int_0^T f^2(t')f^2(t+t')\,dt'$$

for "almost all" initial conditions, ruling out time-periodic solutions, and satisfying the foregoing space periodicity and conditions of initial energy and momentum.

The problem is a nonlinear one with delayed feedback, for which the generally unstable solutions require some statistical-ergodic expression like the foregoing correlation function $R(t)$. The problem being already too difficult, we ignore the added complexity of the case that the particle is provided with a lossless, phase-scrambling passive filter, as well as the ultimate question of how Planck's constant might be forced to appear in any resulting statistics. Neither do we attempt to pose, or diagram, the analogous problem in which billiard-ball molecules replace the foregoing waves.

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APPENDIX

SOME MOLECULAR MICROSTATISTICS

Repeated counts are made of the number of unidirectionally traveling molecules of uniform mass m arriving at the counter in the narrow velocity interval $(v, v + \Delta v)$ in time intervals of length Δt. If the mean count $\langle n \rangle$ per time interval is found to be $\langle n \rangle \equiv N = N_0 \Delta v \Delta t$, what should the mean-square count $\langle n^2 \rangle$ be, under the assumptions that the molecules satisfy Maxwell-Boltzmann, Fermi-Dirac, or Bose-Einstein (MB, FD, BE) statistics?

For MB molecules, the probability $P(n)$ that n molecules will be counted in a discrete Δt interval is given by the Poisson distribution $P(n) = N^n e^{-N}/n!$ Now let R represent the operator $N(\partial/\partial N)$ and R^r represent its r-fold repeated application, so that, for instance,

$$R^0 f(N) = f(N); \quad R^2 f(N) = N\partial/\partial N \{N\partial f/\partial N\};$$

etc. Then

$$\langle n^r \rangle = \sum_{n=0}^{\infty} n^r P(n) = e^{-N} \sum_{n=0}^{\infty} R^r N^n/n! = e^{-N} R^r e^N .$$

Thus

$$\langle n^0 \rangle = 1; \quad \langle n^1 \rangle = N; \quad \langle n^2 \rangle = N^2 + N; \quad \text{etc.}$$

For FD and BE molecules, it is necessary to specify the number S of distinguishable states to any of which a given counted molecule could, in principle, be assigned. According to the uncertainty principle, if a time interval δt is used to best theoretical advantage for the observation of the energy E of an arriving molecule, the probable error δE of the observation will be $\delta E = Ch/\delta t$, where C is a numerical constant of order unity and h is Planck's constant. Thus the number of distinguishable states in an interval $\Delta E \Delta t$ is given by $S = \Delta E \Delta t/(Ch)$. For molecules of mass m, velocity v, the (relativistic) energy is $m(1 - v^2)^{-1/2} = E$, so that $\Delta E = mv\Delta v/(1 - v^2)^{3/2}$ gives

$$S = \frac{mv\Delta v\Delta t}{Ch(1 - v^2)^{3/2}}$$

and

$$N/S = N_0(1 - v^2)^{3/2} Ch/(mv).$$

The probability that n molecules are found in any given state of the S states in our narrow energy range can be assumed to be independent of the precise energy level of the state. For FD molecules, no more than one molecule can be found in any state; let p be the probability that a given state contains a molecule, and $q = 1 - p$ the probability that the state is empty. The probability $P(n)$ that exactly n of the S distinguishable states contain molecules (or that n molecules arrive in Δt) is then $P(n) = S!\, p^n q^{S-n}/n!\,(S - n)!$.

Then taking $R = p\partial/\partial p$ and regarding q as independent of p until the derivatives have been taken, we find

$$<n^r> = \sum_{n=0}^{\infty} n^r P(n) = \sum_{n=0}^{\infty} \frac{q^{S-n} S! R^r p^n}{n!(S-n)!} = R^r (p+q)^S .$$

Thus,

$$<n^0> = 1$$

$$<n^1> = pS = N$$

$$<n^2> = pS + p^2 S^2 - p^2 S = N + N^2 - N^2/S$$

$$= N^2 + N[1 - N_0 Ch(1 - N^2)^{3/2}/(mv)] .$$

For BE molecules, more than one molecule can occupy a given state and the probabilities are less elementary. Mandel* gives the formula

$$P(n) = \frac{(S+n-1)!}{(1+N/S)^S (1+S/N)^N n!(S-1)!}$$

for the probability $P(n)$ that the S similar states contain in total exactly n BE molecules when the average number contained is N. Mandel shows that

$$\sum_{n=0}^{\infty} P(n) = 1;$$

we now observe that if $R = [N(N+S)/S]\partial/\partial N$,

$$<n^r> = \sum_{n=0}^{\infty} n^r P(n) = S^S (N+S)^{-S} \sum_{n=0}^{\infty} R^r \{(N+S)^S S^{-S} P(n)\}$$

$$= (N+S)^{-S} R^r (N+S)^S ,$$

whence

$$<n^0> = 1, \quad <n^1> = N,$$

and

$$<n^2> = N + N^2 + N^2/S = N^2 + N \left\{1 + N_0 Ch(1 - v^2)^{3/2}/(mv)\right\} .$$

Ideally, it should be possible to give all three derivations as special cases of one general method, but the only unity here is that both FD and BE statistics become MB when $h \to 0$ and $S \to \infty$.

While we are in a position to generate whatever moments are needed from the three types of microstatistics, we find use in the text only for the one quantity $<n^2> - <n^1>^2$ = a function of $<n^1>$, depending on the type of molecule. The first moment $<n^1>$ itself is then derived by the procedure of the text.

*L. Mandel, Proc. Phys. Soc. 74:233 (1959)

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We consider a molecular bath in statistical equilibrium at temperature T. A heavy particle in this bath undergoes Brownian motion, acquiring a mean kinetic energy $3kT/2$ regardless of the nature of its encounters with the molecules of the bath. Specifying that these encounters are elastic collisions occurring only for molecules in a narrow band of relative velocities, we use Brownian-motion principles to show, in one-dimensional examples, that the number density of molecules in that band is determined by the $kT/2$ mean kinetic energy of the Brownian particle. We thus obtain the one-dimensional equilibrium energy distribution for classical and quantum molecules, at the price of assuming no more than the relation between the first two moments (the mean count and the mean-square count of particles per unit time) of the appropriate microstatistics.

Toward a quantum communication theory, we consider the Brownian particle to be a narrow-reflection-band filter freely sliding in an ideal transmission line terminated by matched loads at temperature T. We insert Planck's constant h into the mathematics so that the resulting one-dimensional electromagnetic spectrum is Planckian for temperature T. The result gives the following rule of thumb for quantum communication theoretic problems classically entailing no more than power spectrums $S(f)$ and their squares $S^2(f)$: add $hfS(f)$ to $S^2(f)$ to produce the quantum version. The merits of this approach, of the general method, and of some related problems are discussed.

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