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Potential and Magnetic Moment of a Superconducting Torus in a Nonaxial Uniform Magnetic Field

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CONTENTS

ABSTRACT

The magnetic constants of an ideal superconducting torus, located in a uniform magnetic field perpendicular to its axis of symmetry, are calculated from a scalar potential expansion in toroidal Legendre functions. An algorithm for finding the expansion coefficients, suitable for digital computation, is included. Graphs and approximate formulas are given for the induced magnetic moment of the torus and the maximum field values (with locations) on its surface, as functions of its radius ratio.

PROBLEM STATUS

This final report completes a supporting study on one phase of the problem involving the investigation of electromagnetic forces on superconducting toroids.

AUTHORIZATION

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POTENTIAL AND MAGNETIC MOMENT OF A SUPERCONDUCTING TORUS IN A NONAXIAL UNIFORM MAGNETIC FIELD

INTRODUCTION

The torus has long been of interest in the study of electromagnetism and in hydrodynamics. The toroidal body considered here is an ideal superconducting ring (no flux penetration) of circular cross section, placed in a uniform magnetic field.* The magnetostatic potential and magnetic moment of such a torus have previously been calculated (1) for the case where the applied magnetic field is parallel to the axis of the torus. The analogous calculations are made here for the case of an applied field perpendicular to the axis. To treat the problem of a torus at an arbitrary angle to the field, the axial and transverse solutions are superposed linearly. This allows the calculation of net Meissner effect torques on such a ring.

A scalar potential solution of Laplace's equation $\nabla^2 U = 0$ is appropriate for this problem. The methods used here are similar to those of Ref. 1 in solving the problem of a superconducting torus in an axisymmetric field. The analogous problem in fluid flow has also been treated by the method of conjugate functions (2), an approach which cannot be used here because of the lower degree of symmetry.

The axisymmetric problem is a Neumann potential problem for a doubly connected domain, with the potential dependent on only two coordinates. The transverse problem considered here is more complicated in that it is truly three-dimensional, but simpler in that the domain of the potential function is, in effect, simply connected. There will be no flux passing through the central opening of the torus, and no circulation effects (persistent currents), since these are inherently associated with axial symmetry. Thus the potential to be found will be strictly single-valued; no other case need be considered.

LAPLACE'S EQUATION IN TOROIDAL COORDINATES

The toroidal coordinate system used here will be described briefly; further details can be found in Refs. 1 and 3.

The transformations between toroidal coordinates (s, φ, θ) and cylindrical coordinates (ρ, θ, z) are:

$$\rho = \frac{a\,(s^2-1)^{1/2}}{s - \cos\varphi} \qquad \varphi = \cot^{-1}\left(\frac{\rho^2 + z^2 - a^2}{2az}\right)$$

$$z = \frac{a \sin\varphi}{s - \cos\varphi} \qquad s = \frac{\rho^2 + z^2 + a^2}{\left[(\rho^2 + z^2 + a^2)^2 - 4a^2\rho^2\right]^{1/2}} .$$

The azimuthal angle θ is the same in both systems, and a is a scale factor for the toroidal system. Some constant surfaces and unit vectors are shown in Fig. 1.

*Thus the boundary conditions correspond to those of ideal fluid flow past a solid torus.

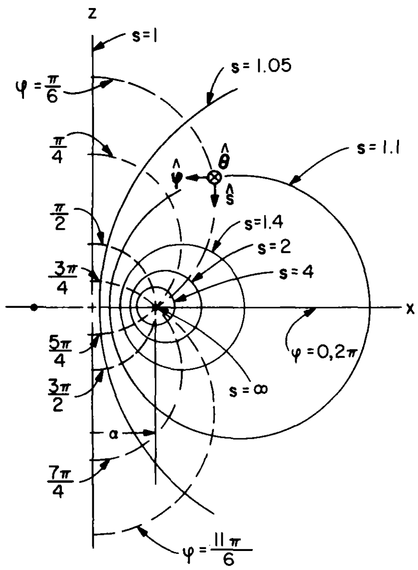


Fig. 1 - Toroidal Coordinate System. Some constant surfaces for s and φ are shown in a section through the half-plane $\theta = 0$. Each constant surface for s is a torus centered on the z axis. The constant surfaces for φ are spheres having the common rim $s = \infty$ (a circle of radius a in the x-y plane). The unit vectors $\hat{\mathbf{s}}$, $\hat{\boldsymbol{\varphi}}$, $\hat{\boldsymbol{\theta}}$ are shown at the point $s = 1.1$, $\varphi = \pi/6$, $\theta = 0$.

A line element is given by

$$d\mathbf{r} = N\left[\frac{ds}{(s^2-1)^{1/2}}\,\hat{\mathbf{s}} + d\varphi\,\hat{\boldsymbol{\varphi}} + (s^2-1)^{1/2}\,d\theta\,\hat{\boldsymbol{\theta}}\right]$$

and the gradient by

$$\nabla U = \frac{(s^2-1)^{1/2}}{N}\,\frac{\partial U}{\partial s}\,\hat{\mathbf{s}} + \frac{1}{N}\frac{\partial U}{\partial \varphi}\,\hat{\boldsymbol{\varphi}} + \frac{1}{N(s^2-1)^{1/2}}\,\frac{\partial U}{\partial \theta}\,\hat{\boldsymbol{\theta}}$$

where

$$N = \frac{a}{s - \cos\varphi}\ .$$

Laplace's equation can be partially separated in toroidal coordinates (3), with the solutions:

$$U(s, \varphi, \theta) = (s - \cos\varphi)^{1/2}\, S(s)\, \Phi(\varphi)\, \Theta(\theta)$$

where

$$S(s) = c_1\, P^{m}_{n-1/2}(s) + c_2\, Q^{m}_{n-1/2}(s)$$

$$\Phi(\varphi) = c_3 \cos n\varphi + c_4 \sin n\varphi$$

$$\Theta(\theta) = c_5 \cos m\theta + c_6 \sin m\theta$$

and $\{c_1, \ldots, c_6\}$ are arbitrary expansion coefficients.

For $U(s, \varphi, \theta)$ to be single-valued, n and m in these expressions must be integers.

The Legendre functions of half-integral degree and integral order m (ring functions) have the following representations (4):

$$P^0_{n-1/2}(s) = \frac{1}{\pi}\int_0^\pi \left[s + (s^2-1)^{1/2}\cos\varphi\right]^{n-1/2} d\varphi \tag{1}$$

$$Q^0_{n-1/2}(s) = \frac{1}{\sqrt{2}}\int_0^\pi \frac{\cos n\varphi \, d\varphi}{(s-\cos\varphi)^{1/2}} \tag{2}$$

$$P^m_{n-1/2}(s) = (s^2-1)^{m/2}\frac{d^m}{ds^m}P^0_{n-1/2}(s) \qquad m = 1,2,\ldots \tag{3}$$

$$Q^m_{n-1/2}(s) = (s^2-1)^{m/2}\frac{d^m}{ds^m}Q^0_{n-1/2}(s) \qquad m = 1,2,\ldots \tag{4}$$

These functions satisfy the usual recursion formulas for Legendre functions:

$$2ns\,K^m_{n-1/2}(s) = (n-m+1/2)\,K^m_{n+1/2}(s) + (n+m-1/2)\,K^m_{n-3/2}(s)$$

$$K^m_{n+1/2}(s) - K^m_{n-3/2}(s) = 2n(s^2-1)^{1/2}\,K^{m-1}_{n-1/2}(s)$$

where K denotes either P or Q.

The ring functions of the first kind $\{P^m_{n-1/2}(s)\}$ diverge as $s \to \infty$, while those of the second kind $\{Q^m_{n-1/2}(s)\}$ diverge as $s \to 1$. For approximate formulas and graphs of some of these functions, see Ref. 5. (Note that many authors use $u = \cosh^{-1}s$ as the argument.)

BOUNDARY CONDITIONS AND SYMMETRY

Consider a superconducting torus centered on the z axis, with large and small radii R and r, respectively (see Fig. 2). Then the surface of the torus is the constant coordinate surface $s = s_0$, where $s_0 = R/r$, and $R = a s_0/(s_0^2 - 1)^{1/2}$. If a uniform external magnetic field $\mathbf{H}_0 = H_0\hat{\mathbf{x}}$ is applied along the x axis, then the boundary conditions on the scalar magnetic potential $U(s,\varphi,\theta)$ are:

$$U \xrightarrow[\sqrt{\rho^2+z^2}\to\infty]{} U_0 = -H_0 x$$

$$\nabla U \cdot \hat{\mathbf{s}}\Big|_{s=s_0} = 0$$

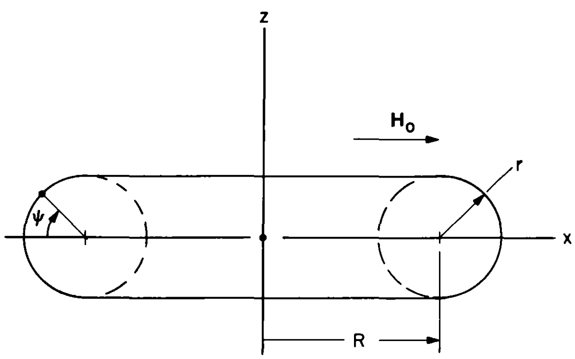


Fig. 2 - A torus defined by the constant coordinate surface $s = R/r$. The angle ψ, which can be used instead of φ to locate a point on the surface, is related to φ by $\cos\psi = (s_0\cos\varphi - 1)/(s_0 - \cos\varphi)$.

or

$$\left.\frac{\partial U}{\partial s}\right|_{s=s_0} = 0.$$

To apply the second boundary condition, which expresses the fact that the magnetic flux does not penetrate the torus, it is convenient to have an expansion in ring functions for

$$\begin{aligned} U_0 &= -H_0 x \\ &= -a\ H_0 \cos\theta \frac{(s^2-1)^{1/2}}{s - \cos\varphi}. \end{aligned}$$

Such an expansion (valid for $s > 1$) can be obtained by a Fourier analysis of $(s - \cos\varphi)^{-3/2}$, and by using Eqs. (2) and (4) above. The result is:

$$U_0 = \frac{2\sqrt{2}}{\pi}\ a\ H_0 \cos\theta\ (s - \cos\varphi)^{1/2} \left[Q^1_{-1/2}(s) + 2 \sum_{n=1}^{\infty} Q^1_{n-1/2}(s) \cos n\varphi \right].$$

The total potential can be expressed by $U = U_0 + U_1$, where U_0 is that component of the total potential corresponding to the external field, and U_1 is the potential due to induced currents on the surface of the torus. Since U_1 may not diverge outside the torus $(1 \leq s \leq s_0)$, U_1 must be expanded in the $\{P^m_{n-1/2}(s)\}$. Considering symmetry conditions about the plane $\varphi = 0$, the $\cos n\varphi$ solutions, and not the $\sin n\varphi$ solutions, must be used. Then U_1 can be written:*

$$U_1 = \frac{2\sqrt{2}}{\pi}\ aH_0 \cos\theta\ (s - \cos\varphi)^{1/2} \sum_{n=0}^{\infty} A_n P^1_{n-1/2}(s) \cos n\varphi$$

where $\{A_n\}$ are coefficients to be determined. (For later convenience, $(2\sqrt{2}/\pi)\,aH_0$ is factored out.)

The total potential can then be written:

$$U(s,\varphi,\theta) = \frac{2\sqrt{2}}{\pi}\ aH_0 \cos\theta\ (s - \cos\varphi)^{1/2} \sum_{n=0}^{\infty} \left[A_n P^1_{n-1/2}(s) + (2 - \delta_{n0})\ Q^1_{n-1/2}(s) \right] \cos n\varphi \qquad (5)$$

where δ_{n0} is the Kronecker delta.

The components of the gradient are:

*Explicit symmetry conditions on the θ-dependence require an expansion in

$$A^m_n P^m_{n-1/2}(s) \cos n\varphi \sin m\theta,$$

where m is an odd integer. That A^m_n can be set equal to zero for $m \neq 1$ is inferred from the form of the expansion for U_0. In any case, if such a solution can be found, the uniqueness theorem for the Neumann problem guarantees that any other solution differs, at most, by a constant.

$$\nabla U \cdot \hat{\mathbf{s}} = \frac{\sqrt{2}}{\pi} H_0 (s - \cos\varphi)^{1/2} (s^2 - 1)^{1/2} \cos\theta \sum_{n=0}^{\infty} \cos n\varphi$$

$$\times \left\{ A_n \left[P^1_{n-1/2}(s) + 2(s - \cos\varphi)\, \dot{P}^1_{n-1/2}(s) \right] + (2 - \delta_{n0}) \left[Q^1_{n-1/2}(s) + 2(s - \cos\varphi)\, \dot{Q}^1_{n-1/2}(s) \right] \right\} \quad \text{(6a)}$$

$$\nabla U \cdot \hat{\boldsymbol{\varphi}} = \frac{\sqrt{2}}{\pi} H_0 (s - \cos\varphi)^{1/2} \cos\theta \sum_{n=0}^{\infty} \left[A_n P^1_{n-1/2}(s) + (2 - \delta_{n0}) Q^1_{n-1/2}(s) \right]$$

$$\times \left[\sin\varphi \cos n\varphi - 2n(s - \cos\varphi) \sin n\varphi \right] \quad \text{(6b)}$$

$$\nabla U \cdot \hat{\boldsymbol{\theta}} = - \frac{2\sqrt{2}}{\pi} H_0 \frac{(s - \cos\varphi)^{3/2}}{(s^2 - 1)^{1/2}} \sin\theta \sum_{n=0}^{\infty} \left[A_n P^1_{n-1/2}(s) + (2 - \delta_{n0})\, Q^1_{n-1/2}(s) \right] \cos n\varphi \,. \quad \text{(6c)}$$

The dotted functions signify differentiation; for example:

$$\dot{P}^1_{n-1/2}(s) = \frac{d}{ds} P^1_{n-1/2}(s) \,.$$

Applying the surface condition, $\partial U/\partial s|_{s=s_0} = 0$, leads to the following recurrence relations for the coefficients $\{A_n\}$:

$$A_0 \left[P^1_{-1/2}(s_0) + 2s_0\, \dot{P}^1_{-1/2}(s_0) \right] + Q^1_{-1/2}(s_0) + 2s_0\, \dot{Q}^1_{-1/2}(s_0) - A_1 \dot{P}^1_{1/2}(s_0) - 2\dot{Q}^1_{1/2}(s_0) = 0 \quad \text{(7a)}$$

and, for $n > 0$,

$$A_n \left[P^1_{n-1/2}(s_0) + 2s_0\, \dot{P}^1_{n-1/2}(s_0) \right] + 2 \left[Q^1_{n-1/2}(s_0) + 2s_0\, \dot{Q}^1_{n-1/2}(s_0) \right] - A_{n+1} \dot{P}^1_{n+1/2}(s_0)$$

$$- (1 + \delta_{n1}) A_{n-1} \dot{P}^1_{n-3/2}(s_0) - 2 \left[\dot{Q}^1_{n+1/2}(s_0) + \dot{Q}^1_{n-3/2}(s_0) \right] = 0 \,. \quad \text{(7b)}$$

Since these formulas are not sufficient to determine the $\{A_n\}$ uniquely, one additional condition (expressing the requirement $A_n \to 0$ as $n \to \infty$) is required. In Appendix A, the following relation is derived, using the fact that the induced field must be just strong enough to expel all flux from the torus:

$$\sum_{n=0}^{\infty} A_n \left[\left(n - \tfrac{1}{2}\right) P^1_{n-1/2}(s_0)\, Q^0_{n-3/2}(s_0) - \left(n + \tfrac{1}{2}\right) P^1_{n-3/2}(s_0)\, Q^0_{n-1/2}(s_0) \right] = - \frac{\pi^2}{8(s_0^2 - 1)} \,. \quad \text{(8)}$$

Although Eqs. (7) and (8) provide enough conditions to fix the $\{A_n\}$ uniquely, it does not follow that exact formulas for the coefficients can be found. In fact, Eq. (7b) is an inhomogeneous second-order difference equation with variable coefficients, which there is no general way of solving exactly. However, an algorithm suitable for digital computation which will find the coefficients to any desired degree of precision is described in Appendix B.

CALCULATION OF THE MAGNETIC MOMENT

The magnetic moment resulting from a current density distribution $\boldsymbol{J}$ in a volume V is given (6) by*

$$\boldsymbol{m} = \frac{1}{2} \int_V \boldsymbol{r} \times \boldsymbol{J} \, dV$$

where the vector $\boldsymbol{r}$ gives the location of the source point with respect to an arbitrary origin. Since currents flow only on the surface of the superconducting torus, its magnetic moment can be written as an area integral:

$$\boldsymbol{m} = \frac{1}{2} \int_A \boldsymbol{r} \times \boldsymbol{K} \, dA$$

where $\boldsymbol{K}$ is the surface current density. In terms of the field at the surface of the torus, $\boldsymbol{K}$ is given (6) by

$$\begin{aligned} \boldsymbol{K} &= -\hat{\boldsymbol{s}} \times \boldsymbol{H} \big|_{s=s_0} \\ &= (H_\theta \hat{\boldsymbol{\varphi}} - H_\varphi \hat{\boldsymbol{\theta}}) \big|_{s=s_0} . \end{aligned}$$

The position vector $\boldsymbol{r}$ to a point on the surface, with respect to the origin of the toroidal coordinate system, is given by

$$\boldsymbol{r} = -\frac{a}{s_0 - \cos\varphi} \left[(s_0^2 - 1)^{1/2} \cos\varphi \, \hat{\boldsymbol{s}} + s_0 \sin\varphi \, \hat{\boldsymbol{\varphi}} \right].$$

The y and z components of $\boldsymbol{m}$ vanish; the x component of the integrand is:

$$\boldsymbol{r} \times \boldsymbol{K} \cdot \hat{\boldsymbol{x}} = \frac{a}{s_0 - \cos\varphi} \left[H_\varphi \sin\varphi \cos\theta + H_\theta (s_0^2 - 1)^{1/2} \cos\varphi \sin\theta \right] .$$

An element of area is:

$$dA = \left(\frac{a}{s_0 - \cos\varphi} \right)^2 (s_0^2 - 1)^{1/2} \, d\varphi \, d\theta .$$

The moment is given by the integral:

$$m = |\boldsymbol{m}| = \frac{a^3 (s_0^2 - 1)^{1/2}}{2} \int_0^{2\pi} d\theta \int_0^{2\pi} \frac{d\varphi}{(s_0 - \cos\varphi)^3} \left[H_\varphi \sin\varphi \cos\theta + (s_0^2 - 1)^{1/2} H_\theta \cos\varphi \sin\theta \right].$$

In this expression, H_θ and H_φ are evaluated at $s = s_0$.

Now if $\boldsymbol{H}$ is divided into the two parts $\boldsymbol{H} = \boldsymbol{H}_0 + \boldsymbol{H}_1$, where $\boldsymbol{H}_1 = -\nabla U_1$, the integral above will divide into two integrals: the first (m_0) involving the components of $\boldsymbol{H}_0$, and the second (m_1) involving the components of $\boldsymbol{H}_1$. The first integral is $-H_0 V$, where V is the volume of the torus. (See Appendix C.) The second part of m can be written,† after integrating over θ:

*In this report all equations are in rationalized MKS units.

†The following mathematical steps, along with some others in this paper, involve such manipulations as the interchange of order of summation and integration, or differentiation under an integral. It will be implicitly assumed that such procedures are valid when applied to the functions appearing here.

$$m_1 = 2\sqrt{2}\, a^3 H_0 (s_0^2 - 1)^{1/2} \int_0^{\pi} \frac{d\varphi}{(s_0 - \cos\varphi)^3} \left\{ (s_0 - \cos\varphi)^{3/2} \sum_{n=0}^{\infty} A_n P^1_{n-1/2}(s_0) \cos n\varphi \cos\varphi \right.$$

$$\left. - \frac{1}{2} (s_0 - \cos\varphi)^{1/2} \sum_{n=0}^{\infty} A_n P^1_{n-1/2}(s_0) \left[\sin\varphi \cos n\varphi - 2n(s_0 - \cos\varphi) \sin n\varphi \right] \sin\varphi \right\}$$

$$= 2\sqrt{2}\, a^3 H_0 (s_0^2 - 1)^{1/2} \sum_{n=0}^{\infty} A_n P^1_{n-1/2}(s_0) \left\{ \int_0^{\pi} \frac{\cos n\varphi \cos\varphi + n \sin n\varphi \sin\varphi}{(s_0 - \cos\varphi)^{3/2}} d\varphi - \frac{1}{2} \int_0^{\pi} \frac{\sin^2\varphi \cos n\varphi}{(s_0 - \cos\varphi)^{5/2}} d\varphi \right\}.$$

When the integrands are reduced by means of trigonometric identities, these integrals can be evaluated in terms of ring functions and their derivatives. The result, after using some identities for Legendre functions (7) is:

$$m_1 = \frac{16}{3} a^3 H_0 \sum_{n=0}^{\infty} A_n P^1_{n-1/2}(s_0) \left\{ \left(n - \frac{1}{2}\right) Q^1_{n+1/2}(s_0) - \left(n + \frac{1}{2}\right) Q^1_{n-3/2}(s_0) \right\}. \tag{9}$$

(Note that $Q^1_{-3/2}(s) = Q^1_{1/2}(s)$.)

RESULTS

The results of the magnetic moment calculation, for several values of the radius ratio s_0, are presented in Table 1 and Fig. 3 in terms of the ratio

$$\bar{\chi}_\perp = \frac{m_0 + m_1}{H_0 V} = -1 + \frac{m_1}{H_0 V}$$

Table 1
Mean Equivalent Transverse and Axial Susceptibilities*

$s_0 = R/r$	$\bar{\chi}_\perp$	$\bar{\chi}_\parallel$
1.1	-1.3583	—
1.2	-1.3595	-2.074
1.4	-1.3673	-2.0797
1.6	-1.3780	-2.0761
2.0	-1.3997	-2.0639
3.0	-1.4372	-2.0400
5.0	-1.4692	-2.0195
7.0	-1.4816	-2.0116
10.0	-1.4896	-2.0066
.	.	.
.	.	.
.	.	.
∞	-1.5000	-2.0000

*To convert susceptibility in rationalized MKS units to Gaussian units, divide by 4π. To normalize the moment values to the volume of a sphere of radius R, rather than to the volume of the torus, multiply by

$$2\pi^2 R r^2 / (4/3)\pi R^3 = 3\pi / 2 s_0^2 .$$

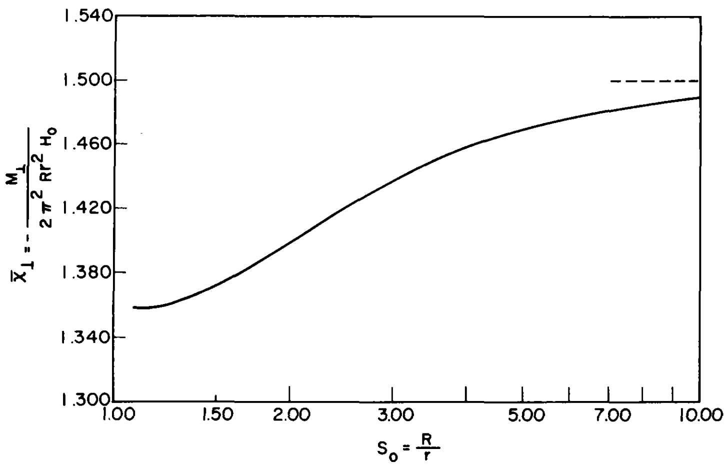


Fig. 3 - Mean transverse susceptibility of torus vs radius ratio

where $\bar{\chi}_\perp$ represents a "mean equivalent transverse susceptibility" for the superconducting torus. The analogous quantity $\bar{\chi}_\parallel$, for an axisymmetric applied field, is also tabulated for comparison.*

Table 2 and Fig. 4 give maximum values of the components of the field $\mathbf{H}(s, \varphi, \theta)$ on the surface of the torus for several values of s_0. The column labeled H_{rim} lists the maximum magnitude of the field on the outer rim of the torus, $|H_\theta(s_0, 0, \pi/2)|$. The column labeled H_{max} lists the maximum field value on the entire surface, $|H_\varphi(s_0, \varphi_{max}, 0)|$,

Table 2
Maximum Outer Rim Fields and Maximum Surface Fields with Locations
(Transverse Applied Field)

s_0	H_{rim}	H_{max}	φ_{max}	ψ_{max}
1.1	1.3552 H_0	1.5815 H_0	0.4297	90° 0'
1.2	1.3457	1.5987	0.5857	90° 0'
1.4	1.3279	1.6347	0.7880	91° 2'
1.6	1.3114	1.6705	0.9286	92°23'
2.0	1.2819	1.7356	1.1147	94°23'
3.0	1.2253	1.8428	1.3271	95°45'
5.0	1.1581	1.9272	1.4502	94°41'
7.0	1.1210	1.9576	1.4909	93°39'
10.0	1.0892	1.9765	1.5173	92°41'
⋮	⋮	⋮	⋮	⋮
∞	1.0000	2.0000	1.5708	90° 0'

*Values in the $\bar{\chi}_\parallel$ column were derived from the calculations of Ref. 1 and represent only that part of the magnetic moment arising from the expulsion of flux from the material of the ring. Additional axisymmetric contributions to the moment arise from persistent currents and depend on the value of the magnetic flux through the central opening of the torus when it becomes superconducting (8).

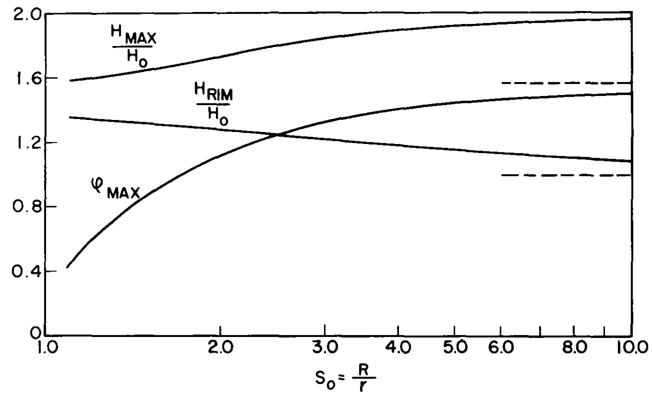


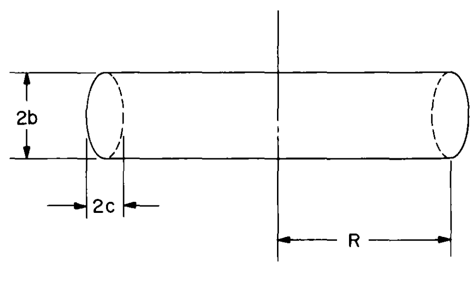
Fig. 4 - Maximum surface field vs radius ratio, maximum rim field vs radius ratio, and φ location (in radians) of maximum surface field vs radius ratio

where the values of φ_{max} are given in radians in the next column. The last column gives the corresponding value of the angle ψ (see Fig. 2), where

$$\cos \psi_{max} = \frac{s_0 \cos \varphi_{max} - 1}{s_0 - \cos \varphi_{max}}.$$

Entries are included in Tables 1 and 2 for the extreme case $s_0 \to \infty$, where the torus becomes a wire loop of infinitesimal thickness. Exact limiting values of the fields and moments for such a loop are easily calculated since the problem reduces to that of a uniformly magnetized infinite cylinder (9). In fact, the calculation can even be generalized to the case of a loop of elliptical cross section, with semiaxes b and c, as shown in Fig. 5.

Fig. 5 - Wire Loop of Elliptical Cross Section. As $R/b \to \infty$ and $R/c \to \infty$, the mean equivalent susceptibilities of this superconducting loop approach the limits $\bar{\chi}_{\parallel} \to -\left(1 + \frac{c}{b}\right)$ $\bar{\chi}_{\perp} \to -\left(1 + \frac{b}{2c}\right)$. For the case of a circular cross section ($b = c = r$), Table 1 shows that these limits are very nearly reached when $R = 10r$.



The susceptibility and fields can be represented by the following approximate formulas, where the maximum relative error for $s_0 \geq 1.1$ is given in each case:

$$\bar{\chi}_{\perp} = -1.5 + \frac{3 \ln s_0 + 1}{7.6\, s_0^2} \qquad \epsilon < 0.002 \quad (< 0.001 \text{ for } s_0 \geq 2)$$

$$\frac{H_{rim}}{H_0} = 1 + \frac{s_0^{1/2}}{(s_0 + 1)^{3/2}} \left[1 + \frac{e^{-4.4/s_0} + 0.06}{2.4 s_0} \right] \qquad \epsilon < 0.0005$$

$$\frac{H_{max}}{H_0} = 2 - \frac{8}{9}\,\frac{s_0 + 1}{s_0^3}\left[\ln s_0 + 0.096\right] - 40\,e^{-5.12 s_0} \qquad \epsilon < 0.002$$
$$(< 0.001 \text{ for } s_0 \geq 2).$$

ACKNOWLEDGMENTS

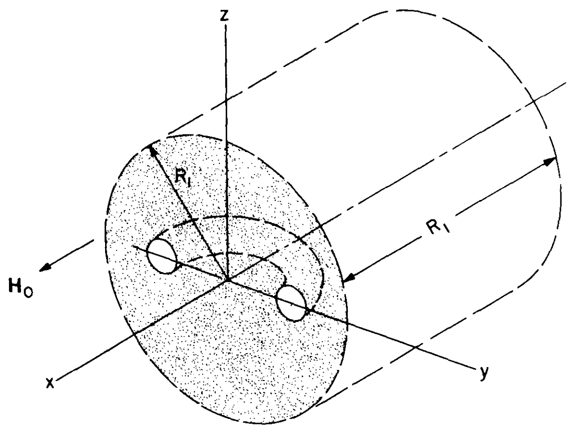
We thank Dr. Jules de Launay for a helpful discussion of this problem. We also thank the Atmosphere and Astrophysics Division of this laboratory for the use of its IBM 1620 computer.

Appendix A

DERIVATION OF SUPPLEMENTARY CONDITION FOR COEFFICIENTS

Equation (8) can be derived by using Gauss's law for the magnetic field. Let the torus lie as shown in Fig. A1, with the applied field $\mathbf{H}_0$ in the positive x direction. Construct a Gaussian cylinder along the negative x axis, the base at $x = 0$, with radius and length R_1, where R_1 tends to infinity. The net flux of the magnetic field through this surface must be zero.

Fig. A1 - A cylindrical Gaussian surface, centered on the x axis, is used to derive Eq. (9). The integral $\int \mathbf{H}_1 \cdot d\mathbf{A}$ over the shaded area is calculated as $R_1 \to \infty$.



This flux can be written

$$\pi R_1^2 H_0 - (\pi R_1^2 - 2\pi r^2) H_0 + \int \mathbf{H}_1 \cdot d\mathbf{A} = 0$$

the first term representing the flux of $\mathbf{H}_0$ crossing the end of the cylinder at $x = -R_1$ and the second representing the flux of $\mathbf{H}_0$ passing through the end at $x = 0$. The integral covers the whole surface of the cylinder except where $x = 0$ and $s > s_0$. Thus the fact that $\mathbf{H} = 0$ within the torus is taken into account. Now as $R_1 \to \infty$, $\mathbf{H}_1$ will vanish on the surface of the cylinder except on the $x = 0$ end. If the integral of $\mathbf{H}_1$ over the y-z plane is denoted by Φ_1, then

$$\Phi_1 = \iint_{\substack{x=0 \\ s \le s_0}} H_1 \, dy \, dz$$

$$= 2\pi r^2 H_0 .$$

This clearly expresses the fact that the flux of $\mathbf{H}_0$ which is expelled from the torus when it becomes superconducting must reappear outside.

Substituting the expression derived previously for H_θ, and integrating over the half plane $\theta = 0$ gives:

$$\frac{1}{2}\,\Phi_1 = \int_0^{2\pi} d\varphi \int_1^{s_0} \frac{ds}{(s^2-1)^{1/2}} \left(\frac{a}{s-\cos\varphi}\right)^2 \left[-\frac{2\sqrt{2}}{\pi} H_0 \frac{(s-\cos\varphi)^{3/2}}{(s^2-1)^{1/2}} \sum_{n=0}^{\infty} A_n P^1_{n-1/2}(s) \cos n\varphi\right]$$

$$= -\frac{4\sqrt{2}}{\pi} H_0 a^2 \sum_{n=0}^{\infty} A_n \int_1^{s_0} \frac{P^1_{n-1/2}(s)\,ds}{s^2-1} \int_0^{\pi} \frac{\cos n\varphi\, d\varphi}{(s-\cos\varphi)^{1/2}}$$

$$= -\frac{8}{\pi} H_0 a^2 \sum_{n=0}^{\infty} A_n \int_1^{s_0} P^1_{n-1/2}(s)\, Q^0_{n-1/2}(s)\, \frac{ds}{s^2-1}\,.$$

If these integrals are evaluated,* the result, when Φ_1 is equated to

$$2\pi r^2 H_0 = \frac{2\pi a^2 H_0}{s_0^2-1}\,,$$

is

$$\sum_{n=0}^{\infty} A_n \left[\left(n-\frac{1}{2}\right) P^1_{n-1/2}(s_0)\, Q^0_{n-3/2}(s_0) - \left(n+\frac{1}{2}\right) P^1_{n-3/2}(s_0)\, Q^0_{n-1/2}(s_0)\right] = \frac{-\pi^2}{8(s_0^2-1)}\,. \qquad (8)$$

*See Erdélyi et al. (4), p. 169.

Appendix B

NUMERICAL CALCULATION OF COEFFICIENTS

To calculate the coefficients $\{A_n\}$, the ring functions and their derivatives must be evaluated for values of n greater than those in existing tables.* Methods of doing this, using the recurrence relations for Legendre functions (Eqs. 5), have been described by other authors (11). Successive applications of the recurrence formula must be made in the direction of stability:† increasing n for the $\{P^m_{n-1/2}\}$ and decreasing n for the $\{Q^m_{n-1/2}\}$.

When these functions are known up to a sufficiently large value of n, and substituted in Eqs. (7a) and (7b), then all the equations can be reduced to two-term recurrence formulas.

First, Eq. (7a) is of the form $b_0 A_0 + c_0 A_1 = d_0$. If this equation is solved for A_0 and substituted into the second relation, involving A_0, A_1, and A_2, that relation can be reduced to the form $b_1 A_1 + c_1 A_2 = d_1$. By continuing in this manner, a set of two-term relations is derived:

$$b_n A_n + c_n A_{n+1} = d_n \qquad n = 0, 1, 2, \ldots, N.$$

Now if a suitable approximate value for A_{n+1} is chosen ($A_{N+1} = 0$ or, better, $A_{N+1} = A_N / e^{2u_0}$, where $e^{u_0} = s_0 + \sqrt{s_0^2 - 1}$), these two-term relations can be used to compute successively each A_N, $n = N, N-1, \ldots, 0$. The backward direction of recurrence is, as might be expected, the direction of stability for this computation. Thus, the computed values of the $\{A_n\}$ for small values of n depend very little on the initial choice for A_{N+1}, provided N is chosen large enough. A sufficiently large choice for the least favorable case computed here ($s_0 = 1.1$) was $N = 30$, which gives at least six significant figures for each of the first twelve coefficients.

The auxiliary relation

$$\sum_{n=0}^{N} A_n \left[\left(n - \frac{1}{2}\right) P^1_{n-1/2}(s_0)\, Q^0_{n-3/2}(s_0) - \left(n + \frac{1}{2}\right) P^1_{n-3/2}(s_0)\, Q^0_{n-1/2}(s_0)\right] \approx \frac{-\pi^2}{8(s_0^2 - 1)}$$

(a truncation of Eq. (8)) then serves as a check on the process.

*Reference 10 tabulates the functions for the first six values of n.
†See Abramowitz and Stegun (7), p. xiii.

Appendix C

CALCULATION OF THE MAGNETIC MOMENT

It was asserted in the text that if

$$m_0 = \frac{a^3}{2}(s_0^2 - 1)^{1/2} \int_0^{2\pi} d\theta \int_0^{2\pi} \frac{d\varphi}{(s_0 - \cos\varphi)^3} \left[H_{0\varphi} \sin\varphi \cos\theta + (s_0^2 - 1)^{1/2} H_{0\theta} \cos\varphi \sin\theta\right]$$

where $\mathbf{H}_0 = H_0 \hat{\mathbf{x}}$, then

$$m_0 = -2\pi^2 a^3 H_0 \frac{s_0}{(s_0^2 - 1)^{3/2}} = -H_0 V$$

where V is the volume of the torus.

The simplest way to calculate the integral is to start with the θ and φ components of $\mathbf{H}_0$ in their closed form:

$$H_{0\varphi} = -H_0 \cos\theta \frac{(s_0^2 - 1)^{1/2}}{s_0 - \cos\varphi} \sin\varphi$$

$$H_{0\theta} = -H_0 \sin\theta$$

giving

$$\begin{aligned}
\mathbf{m}_0 &= -2H_0 a^3 (s_0^2 - 1) \int_0^{\pi} d\theta \int_0^{\pi} \frac{d\varphi}{(s_0 - \cos\varphi)^3} \left[\frac{\sin^2\varphi \cos^2\theta}{s_0 - \cos\varphi} + \cos\varphi \sin^2\theta\right] \\
&= -H_0 \pi a^3 (s_0^2 - 1) \int_0^{\pi} \left[\frac{\sin^2\varphi}{(s_0 - \cos\varphi)^4} + \frac{\cos\varphi}{(s_0 - \cos\varphi)^3}\right] d\varphi \\
&= -2\pi^2 a^3 H_0 \frac{s_0}{(s_0^2 - 1)^{3/2}} = -H_0 V
\end{aligned}$$

where $V = 2\pi^2 R r^2$ is the volume of the torus.

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The magnetic constants of an ideal superconducting torus, located in a uniform magnetic field perpendicular to its axis of symmetry, are calculated from a scalar potential expansion in toroidal Legendre functions. An algorithm for finding the expansion coefficients, suitable for digital computation, is included. Graphs and approximate formulas are given for the induced magnetic moment of the torus and the maximum field values (with locations) on its surface, as functions of its radius ratio.

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