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Abstract: A representation by a summable, analytic Eisenstein series is given for the standard Jacobian theta function. The result extends the known convergent Eisenstein series representations of the kth powers of the theta function for $k = 4, \ldots, 8$, due essentially to Hardy, to the case $k = 1$ by introducing a suitable summability method.

1. We consider the standard theta function[2]

$$\theta(z) \doteq \sum_{n=-\infty}^{+\infty} e^{-\pi z n^2} \qquad (z = x + iy,\ x > 0), \tag{1.1}$$

which is holomorphic in $x > 0$ and has the imaginary axis as its natural boundary: namely, it is known that on approaching a boundary point, say $z \to -i\rho$,

$$(z + i\rho)^{1/2}\, \theta(z) \to \omega(\rho) \qquad \left(|\arg(z + i\rho)^{1/2}| \leqslant \alpha < \frac{\pi}{4}\right), \tag{1.2}$$

where according to Hardy-Littlewood or Weyl

$$\omega(\rho) = 0 \quad \text{for irrational } \rho, \tag{1.3}$$

whereas if ρ is rational, $\omega(\rho)$ is a Gaussian sum

$$\omega(\rho) \doteq \frac{1}{2c} \sum_{r=1}^{2c} e^{i\pi\rho r^2} \quad \text{for } \rho = a/c \qquad (c > 0) \tag{1.4}$$

and $\omega(\rho)\sqrt{c}$ is an 8th root of unity if $(a, c) = 1$ and $a - c \equiv 1 \bmod 2$. This suggests trying to decompose the theta function into a partial fraction series with terms $\omega(\rho)\,(z + i\rho)^{-1/2}$ for the rational numbers ρ, where $|\arg\,(z + i\rho)^{1/2}\,| < \pi/4$. The analogous case of a kth power of $\theta(z)$ with natural k leads to a corresponding Eisenstein series which converges absolutely (only) when $k > 4$. There even exist the representations

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[2] We write $\doteq$ to define the left expression by the right hand one.

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$$\theta^k(z) = 1 + \sum_{\rho=a/c} \omega^k(\rho)\,(z + i\rho)^{-k/2} \qquad (k = 4, \ldots, 8), \tag{1.5}$$

which have been found by different motivations for the first time by Mordell and Hardy using the theory of modular functions. For $k = 4$ the Eisenstein partial fraction series in (1.5) is only conditionally convergent and has to be summed first over a (for fixed c) and then over c.

To overcome the convergence difficulties for $k < 4$, it is customary, following Hecke, to introduce the convergence-generating factor $|cz + ia|^{-w}$ and to use the modified Dirichlet-Eisenstein series (with $w = u + iv$)

$$H_k(z, w) = 1 + \sum_{\rho=a/c} \omega^k(\rho)\,(z + i\rho)^{-k/2}\,|cz + ia|^{-w} \qquad \left(u > 2 - \frac{k}{2}\right). \tag{1.6}$$

It has been shown by Maass that $H_k(z, w)$ for $k = 1, 2, 3$ has a single-valued analytic continuation as function of w to $w = 0$ with $H_k(z, 0) = \theta^k(z)$ for $k = 1, 3$ and with $H_2(z, 0) = 2\theta^2(z)$. Siegel [1] has shown that $H_1(z, w)$ is meromorphic in w and satisfies a simple functional equation relating it to $H_1(z, 1 - w)$. By (1.4) and the well-known reciprocity formula

$$\omega(\rho) = (i\rho)^{1/2}\,\omega(-1/\rho) \qquad (\rho \neq 0) \tag{1.7}$$

for the Gaussian sums, (1.6) implies $H_1(z + 2i, w) = H_1(z, w)$, in particular, and

$$h(1/z) = z^{1/2}h(z) \quad \text{for } h(z) \doteq x^{w/2}H_1(z, w), \quad (x = \operatorname{Re} z > 0) \tag{1.8}$$

so that $H_1(z, w)$ is essentially a (nonanalytic) modular form in z (at first for $u > 3/2$ and then, by analytic continuation, for all w), this being the motivation for the particular form of Hecke's factor.

The relation $\theta(z) = H_1(z, 0)$ shows that in its entire domain of existence, the right half plane $x > 0$, $\theta(z)$ can be characterized by and constructed from its singular main parts $\omega(\rho)\,(z + i\rho)^{-1/2}$ belonging to rational points $-i\rho$ on the imaginary axis with $\rho = a/c$, $(a, c) = 1$, $a - c \equiv 1 \bmod 2$, and $c > 0$. However, $H_1(z, 0)$ cannot be obtained simply by letting $w \to 0$ in the right-hand expression of (1.6) for $k = 1$, but only by analytic continuation of $H_1(z, w)$ to $w = 0$, thereby exhibiting poles in $u \geqslant 0$ (the first at $w = 1$) and in the process dissolving the terms $\omega(\rho)\,(z + i\rho)^{-1/2}$; compare with Siegel [1]. When $k = 3$, for instance, the situation is more favorable, since the series for $H_3(z, w)$ in (1.6) converges conditionally for $u > 0$ (when summed first over a and then over c), and also $H_3(z, w) \to \theta^3(z)$ for $w \to 0$ $(u > 0)$, as was shown by Bateman [2]. But $\theta^3(z)$ still appears as the limit function of a family of nonanalytic functions of z, which is a disadvantage for certain desirable applications of a partial fraction series of $\theta^3(z)$.

The main goal of this report is to derive the following result on the existence of a summable, analytic, Eisenstein series for the theta function.

THEOREM. *There exists an entire function $p(\tau)$, with $p(0) = 1$ and $p(\tau) = 0(\tau^{-n})$ for every natural n as $\tau \to +\infty$ through the reals, such that*

$$\theta(z) = 1 + \lim_{\tau \to +0} \left[\sum_{\rho = a/c} \omega(\rho)\, (z + i\rho)^{-1/2}\, p(\tau \|\rho\|) \right] \qquad (x > 0) \tag{1.9}$$

$$= \lim_{\tau \to +0} \left[\sum_{\rho = a/c}^{*} \omega(\rho) \left[(z + i\rho)^{-1/2} + (i\rho z + 1)^{-1/2} \right] p(\tau \|\rho\|) \right] \text{ (summed over odd } c \text{ only)},$$

where $\|\rho\| \doteq (a^2 + c^2)^{1/4}$. *The convergence as* $\tau \to +0$ *is uniform in* z *on every compact set contained in* $x = \operatorname{Re} z > 0$.

The interest in the representation (1.9) lies in the fact that it directly exhibits one of the fundamental properties of $\theta(z)$, which is not immediately visible in (1.1), namely that $\theta(1/z) = z^{1/2}\theta(z)$. However, the periodicity $\theta(z + 2i) = \theta(z)$ is not directly visible in (1.9), though it is obvious in (1.1). The "summability function" $p(z)$ involves essentially only the Riemann zeta function, as expected, and it will be constructed explicitly in the following proof (clearly, it is not unique).

For the proof of the theorem we introduce a series $T(z, w)$ similar to (1.6) but with the convergence-generating factor $|c + ia|^{-w}$, which is different from Hecke's, thus giving up the periodicity in z. Analytic continuation via Fourier expansion again yields poles in $u \geqslant 0$ and the value $\theta(z)$ at $w = 0$. We then introduce a suitable contour integral (over w) involving $T(z, w)$ and a parameter τ, such that $T(z, 0)$ appears as its only residue in $u \geqslant 0$, and the remainder can be readily estimated. Calculating this integral, on the other hand, termwise from the series $T(z, w)$ leads to (1.9).

2. Again we let $z = x + iy$ and $w = u + iv$ $(x > 0)$ and consider the Dirichlet series

$$T(z, w) \doteq \sum_{\rho = a/c} \omega(\rho)\, (z + i\rho)^{-1/2}\, |c + ia|^{-w} \qquad (u > 3/2), \tag{2.1}$$

which for fixed z is to be analytically continued with respect to w. The summation is extended over all rational numbers $\rho = a/c$ with $c > 0$, i.e., $(a, c) = 1$. Then $|\omega(\rho)| \leqslant c^{-1/2}$ by (1.4), so that the series in (2.1) converges absolutely for $u > 3/2$. Hence, rearranging and using the periodicity of $\omega(\rho)$,

$$T(z, w) = \sum_{\rho \bmod 2} \omega(\rho) c^{-w} \sum_{k=-\infty}^{+\infty} (z + i\rho + 2ik)^{-1/2}\, |1 + i\rho + 2ik|^{-w}.$$

The inner series here is uniformly convergent on $0 \leqslant \rho \leqslant 2$, is periodic in ρ with period 2, and is holomorphic on $0 \leqslant \rho \leqslant 2$, if we replace the absolute value according to

$$|1 + i\rho| = (1 + \rho^2)^{1/2}; \left[\text{with } |\arg (1 + \rho^2)^{1/2}| < \frac{\pi}{4} \text{ for } |\operatorname{Im} \rho| < 1 \right].$$

Thus it has a Fourier expansion, absolutely convergent for real ρ, and we obtain

$$T(z, w) = \sum_{\rho \bmod 2} \omega(\rho) c^{-w} \sum_{n=-\infty}^{+\infty} b(n; z, w) e^{-i\pi\rho n} \qquad \left(u > \frac{3}{2}\right), \tag{2.2}$$

where

$$b(n; z, w) \doteq \frac{1}{2} \int_{-\infty}^{+\infty} e^{i\pi n t} (z + it)^{-1/2} (1 + t^2)^{-w/2} \, dt. \tag{2.3}$$

These integrals exist for $u > 1/2$. If $n \neq 0$, it is seen, after partial integration and shifting the line of integration into the upper or lower half plane (through $t = (i/2) \min(x, 1) \operatorname{sgn} n$; $z = x + iy$ is fixed with $x > 0$), that $b(n; z, w)$ is holomorphic in $u > -1/2$ and that amply

$$b(n; z, w) = O\left\{(1 + |w|) \exp\left[-(\pi/2)|n| \min(x, 1)\right]\right\} \text{ uniformly for } |n| \geqslant 1, u \geqslant -\frac{1}{4}. \tag{2.4}$$

Therefore the repeated series in (2.2) is absolutely convergent (only for $u > 3/2$), and by allowed exchange of summations

$$T(z, w) = \sum_{n=-\infty}^{+\infty} b(n; z, w) \, q(n, w) \qquad (u > 3/2), \tag{2.5}$$

where

$$q(n, w) \doteq \sum_{\rho \bmod 2} \omega(\rho) c^{-w} e^{-i\pi\rho n} \qquad (u > 3/2). \tag{2.6}$$

From (2.3) for $n = 0$ (with Γ denoting the gamma function)

$$b(0; z, w) = \frac{1}{2} \int_{-\infty}^{+\infty} (1 + t^2)^{-w/2} [(z + it)^{-1/2} - (it)^{-1/2}] \, dt + \frac{\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{w}{2} - \frac{1}{4}\right)}{\Gamma\left(\frac{w}{2}\right) \sqrt{8}}, \; (u > -\tfrac{1}{2}) \tag{2.7}$$

so that $b(0; z, w)$ is holomorphic in $u > -1/2$, except for a simple pole at $w = 1/2$. For $n \neq 0$ we have already seen that $b(n; z, w)$ is holomorphic in $u > -1/2$. Now we calculate for negative n

$$b(n; z, 0) = \frac{1}{4\pi n} \int_{-\infty - ic}^{+\infty - ic} e^{i\pi n t} (z + it)^{-3/2} \, dt = O(e^{\pi n c}) \text{ for every } c \geqslant 0;$$

thus

$$b(n; z, 0) = 0 \text{ for } n \leqslant 0 \quad (x > 0), \tag{2.8}$$

the case $n = 0$ following from (2.7). For positive n we get

$$b(n; z, 0) = \frac{e^{-\pi n z}}{4\pi i n} \int_{z - i\infty}^{z + i\infty} e^{\pi n w} w^{-3/2} \, dw = n^{-1/2} e^{-\pi n z}. \quad (n > 0, x > 0). \tag{2.9}$$

3. We now turn to the analytic continuation and estimation of the expressions $q(n, w)$, defined in (2.6), to the left of $u = 3/2$. By (2.6), (2.4), and absolute convergence,

$$q(n, w) = \sum_{c=1}^{\infty} c^{-w} \sum_{\substack{a=1 \\ (a,c)=1}}^{2c} \omega(\rho) e^{-i\pi\rho n} = \zeta^{-1}(w) \sum_{c=1}^{\infty} c^{-w} \sum_{a=1}^{2c} \omega(\rho) e^{-i\pi\rho n}$$

$$= \zeta^{-1}(w) \sum_{c=1}^{\infty} N(n, 2c) c^{-w} \quad \text{with } N(n, m) \doteq \sum_{\substack{r=1 \\ r^2 \equiv n \bmod m}}^{m} 1, \tag{3.1}$$

where ζ denotes the Riemann zeta function. Since $N(n, km) = N(n, k)N(n, m)$ whenever $(k, m) = 1$, we have (certainly for $u > 2$)

$$2^{-w} \sum_{c=1}^{\infty} N(n, 2c) c^{-w} = \left[\sum_{k=1}^{\infty} N(n, 2^k) 2^{-kw} \right] \left[\prod_{p>2} \sum_{k=0}^{\infty} N(n, p^k) p^{-kw} \right], \tag{3.2}$$

the product extended over primes p. The enumeration of the solutions of quadratic congruences modulo a prime power is known (it follows from the structure of the prime residue class group modulo p^k); namely, if $n \neq 0$, let n^* be the discriminant of the algebraic number field generated by $\sqrt{n}$ (over the rationals), let $m = m(p, n)$ be the largest integer such that p^{2m} divides $4n/n^*$, and let

$$\chi(r, n) \doteq \left(\frac{n^*}{r}\right)$$

be the Kronecker symbol ($\chi = 0$ if $(r, n^*) > 1$). Then from the factorization

$$n = n_0 4^{\tilde{m}} \prod_{p>2} p^{2m(p,\, n)} \text{ with squarefree } n_0 \text{ and } \tilde{m} \doteq m(2, n) - \chi^2(2, n)$$

we have

$$\sum_{k=1}^{\infty} N(n, 2^k) 2^{-kw} = \sum_{k=1}^{2\tilde{m}} 2^{[k/2]-kw} + 2^{\tilde{m}-(2\tilde{m}+1)w} \left\{ 1 + 2\chi^2(2,n) \left[2^{-w} + (1 + \chi(2,n)) \sum_{k=2}^{\infty} 2^{-kw} \right] \right\}$$

$$= 2^{-w}(1 - \chi(2, n) 2^{-w})^{-1} h(2, n; w),$$

where

$$h(2, n; w) \doteq 1 + \chi(2, n) + [2^{1-w} - \chi(2, n)] \sum_{k=0}^{2m-1} 2^{[k/2]-kw}, \; m = m(2, n), \tag{3.3}$$

and for odd p

$$\sum_{k=0}^{\infty} N(n, p^k) p^{-kw} = \sum_{k=0}^{2m} p^{[k/2]-kw} + p^{m-(2m+1)w}(1 + \chi(p, n)) \left\{ 1 + \chi^2(p, n) \sum_{k=1}^{\infty} p^{-kw} \right\}$$

$$= (1 + p^{-w})(1 - \chi(p, n) p^{-w})^{-1} h(n, p; w) \quad (m = m(p, n)),$$

where

$$h(p, n; w) \doteq p^{m(1-2w)} + (1 - \chi(p, n)p^{-w}) \sum_{k=0}^{m-1} p^{k(1-2w)} \quad (p > 2). \tag{3.4}$$

Finally, if $n = 0$, we clearly have $N(0, p^k) = p^{[k/2]}$ for all $k \geqslant 0$, $p \geqslant 2$. Thus

$$\sum_{k=1}^{\infty} N(0, 2^k)2^{-kw} = 2^{-w} \frac{1 + 2^{1-w}}{1 - 2^{1-2w}},$$

and

$$\sum_{k=0}^{\infty} N(0, p^k)p^{-kw} = \frac{1 + p^{-w}}{1 - p^{1-2w}} \qquad (p > 2).$$

Using (3.1) and (3.2), we obtain for $n = 0$

$$q(0, w) = \frac{(1 + 2^{1-w})\, \zeta(2w - 1)}{(1 + 2^{-w})\, \zeta(2w)}, \tag{3.5}$$

and for $n \neq 0$, since by (3.4) $h(p, n; w) = 1$ when $m(p, n) = 0$,

$$q(n, w) = \frac{L(w; n)}{(1 + 2^{-w})\, \zeta(2w)} \prod_{p \mid 2n} h(p, n; w) \quad (n \neq 0), \tag{3.6}$$

where (for $u > 1$)

$$L(w; n) \doteq \prod_{p} (1 - \chi(p, n)p^{-w})^{-1} = \sum_{k=1}^{\infty} \chi(k, n)k^{-w} = L(w, n^*). \tag{3.7}$$

Now the Kronecker symbol is the principal character only when $n^* = 1$, i.e., when $n = k^2$ (k an integer), and then $L(w; k^2) = \zeta(w)$. Therefore, by (3.5), (3.6), and the foregoing, $q(n, w)$ is holomorphic in $u \geqslant 1/2$, except for a simple pole at $w = 1$ when n is a perfect square, including 0.

The further analytic continuation of $q(n, w)$ to the left of $u = 1/2$ follows from the functional equations of the L-series

$$g(w + e_n)\, L(w; n) = |n^*|^{(1/2)-w}\, g(1 - w + e_n)\, L(1 - w; n), \tag{3.8}$$

where

$$g(2w) \doteq \pi^{-w}\, \Gamma(w), \; e_n \doteq \frac{1}{2}(1 - \operatorname{sgn} n) \quad (n \neq 0)$$

(since $\chi(k, n)$ is a primitive character modulo $|n^*|$), and from the relations

$$h(p, n; w) = p^{m(1-2w)}\, h(p, n; 1 - w) \text{ with } m = m(p, n) \quad (p \geqslant 2),$$

which follow from (3.3) and (3.4) by elementary calculation. Thus using (3.6) and the definition of $m(p, n)$, we obtain the functional relations

$$q^*(n, w) = 2|n|^{(1/2)-w}(2\pi)^{-w}\Gamma(w)\, q^*(n, 1-w) \sin\frac{\pi}{2}(w + e_n) \quad (n \neq 0), \tag{3.9}$$

where

$$q^*(n, w) \doteq q(n, w)\,(1 + 2^{-w})\,\varphi(2w),\ \varphi(w) \doteq g(w)\,\zeta(w) = \varphi(1-w). \tag{3.10}$$

Collecting the information from Eqs. (3.3) to (3.10), we get the following lemma.

LEMMA 1. *$q(n, w)$ and $q^*(n, w)$ are meromorphic functions of w for each n. In $u > -1$, $q^*(n, w)$ has poles (and then of order 1) at most at $w = 1$ (if $n = k^2 \geqslant 0$) and at $w = 0$ (if $n = k^2$ or $n \leqslant 0$), whereas $q(n, w)$ has infinitely many other poles but is holomorphic at $w = 0$ for every n, assuming (in particular) the values*

$$q(k^2, 0) = 2k > 0,\ q(n, 0) = 0 \text{ for } n > 0,\ n \neq k^2. \tag{3.11}$$

Equation (3.11) follows for $n = k^2$, i.e., $n^* = 1$, from (3.6), (3.3), and (3.4), since $h(2, k^2; 0) = 2^{m+1}$ especially, and for $n \neq k^2$ from (3.8).

Next we establish some needed estimates.

LEMMA 2. *For $n \neq 0$ we have, uniformly in n and $w = u + iv$,*

$$\left.\begin{aligned} q(n, w) &= O(|n|(1+v)) \text{ for } u \geqslant \tfrac{1}{2},\ v \geqslant 0,\ n \neq k^2, \\ (w-1)\,q(n, w) &= O(|n|(1+v)^2) \text{ for } u \geqslant \tfrac{1}{2},\ v \geqslant 0,\ n = k^2, \end{aligned}\right\} \tag{3.12}$$

and

$$\left.\begin{aligned} q^*(n, w) &= O(|n|^b\,(1+v)^b\, e^{-\pi v/2}) \text{ for } -\tfrac{1}{2} \leqslant u \leqslant 2,\ v \geqslant 0, \\ |w-1| &\geqslant \tfrac{1}{4},\ |w| \geqslant \tfrac{1}{4} \text{ with a suitable constant } b > 1, \end{aligned}\right\} \tag{3.13}$$

and

$$\left.\begin{aligned} (w-1)\,q^*(n, w) &= O(|n|^b) \text{ in } |w-1| \leqslant \tfrac{1}{4}, \\ wq^*(n, w) &= O(|n|^b) \text{ in } |w| \leqslant \tfrac{1}{4}. \end{aligned}\right\} \tag{3.14}$$

The constant b has not necessarily the same value in its four appearances here, and the estimates given are ample for convenience. To prove (3.12), we look at (3.6) and use the known estimates for the L-series of (3.7):

$$L(w; n) = O(|n^*|^{1/2}\,(1+v)^{1/2}) \text{ for } u \geqslant \tfrac{1}{2},\ v \geqslant 0,\ n^* \neq 1,$$

$$(w-1)\,L(w; n) = O((1+v)^{3/2}) \text{ for } u \geqslant \tfrac{1}{2},\ v \geqslant 0,\ n^* = 1,$$

(see for instance [3]), $\zeta^{-1}(2w) = O[\log(2 + |v|)]$ in $u \geqslant 1/2$, and finally, from (3.3) and (3.4) with $m = m(p, n)$ for the various prime divisors of $4n/n^*$,

$$\Big|\prod_{p|2n} h(p, n; w)\Big| \leqslant 3\prod_p \sum_{r=0}^{2m} p^{r[(1/2)-u]} \leqslant 3 \sum_{d|(4n/n^*)} d^{\{(1/2)-u\}} = O\left[\left(\frac{n}{n^*}\right)^{1/2}\right]$$

for $u \geqslant 1/2$. Equation (3.13) follows for $u \geqslant 1/2$ from (3.10) and (3.6), for $u < 1/2$ from (3.9), using (3.12), or the estimates leading to it, and from Stirling's asymptotic formula for $\Gamma(w)$ on vertical lines. Equation (3.14) follows from (3.12) using (3.10), respectively (3.9). Thus the lemma is proved.

4. With regard to (2.5), (3.5), (3.6), and (3.10), we introduce the function

$$T^*(z, w) \doteq (1 + 2^{-w})\, \varphi(2w) T(z, w) = \sum_{n=-\infty}^{+\infty} b(n; z, w) q^*(n, w). \tag{4.1}$$

LEMMA 3. *For every z in $x > 0$ $T^*(z, w)$ is a meromorphic function of w in $u \geqslant -1/4$ (at least) where it has poles exactly at $w = 0$, $w = 1/2$, and $w = 1$, each of order 1. Its residue at $w = 0$ is $1 - \theta(z) = -T(z, 0)$, and away from the poles*

$$T^*(z, w) = O\left[(1 + |v|)^b e^{-\pi|v|/2}\right] \text{ for } -\frac{1}{4} \leqslant u \leqslant 2, \tag{4.2}$$

with a suitable constant $b > 1$, uniformly in $x \geqslant x_0$ for any $x_0 > 0$.

This lemma follows from (2.7), (3.10) for $n = 0$, (3.5), and Stirling's formula, and further from (2.4), Lemmas 1 and 2, and, since $2w\varphi(2w) \to -1$ as $w \to 0$, from (3.10), (3.11) and (2.8), (2.9).

Next we consider the integral with a parameter $\tau \neq 0$

$$I(z, \tau) \doteq \frac{-1}{2\pi i} \int_{2-i\infty}^{2+i\infty} T^*(z, w) \sin \pi w\, \Gamma(2w)\, (1 - 2w)\, \tau^{-2w}\, dw, \tag{4.3}$$

which exists by Lemma 3 for τ in the angle $|\arg \tau| < \pi/4$. Again by Lemma 3 (and Stirling's formula), we move the contour to $u = -1/4$, pick up the only residue at $w = 0$, and estimate the resulting integral to obtain

$$I(z, \tau) = \frac{\pi}{2}\, [\theta(z) - 1] + O(|\tau|^{1/2}), |\arg \tau| \leqslant \frac{\pi}{8} \tag{4.4}$$

uniformly in every half plane $x \geqslant x_0 > 0$. On the other hand, by absolute convergence in (2.1)

$$T(z, w) = \sum_{n=1}^{\infty} T_n(z) n^{-w/2}, \quad T_n(z) \doteq \sum_{\substack{a^2+c^2=n \\ (a,c)=1}} \omega(\rho)\, (z + i\rho)^{-1/2}, \tag{4.5}$$

for $u > 3/2$, and

$$|T_n(z)| \leqslant \sum_{a^2+c^2=n} |cz + ia|^{-1/2} = O(n^{-(1/4)+\epsilon}) \quad (\epsilon > 0, x > 0),$$

so that the Dirichlet series in (4.5) converges uniformly on $u = 2$. Moreover, for $|\arg \tau| < \pi/4$, and observing the definition of φ in (3.10) and (3.8),

$$\int_{-\infty}^{+\infty} |(1 + 2^{-w})\, \varphi(2w) \sin \pi w\, \Gamma(2w)\, (1 - 2w)\, \tau^{-2w}|\, dv < \infty \quad (w = 2 + iv).$$

Therefore, on substituting from (4.5) into (4.1) and (4.3), exchange of integration and summation is justified and yields

$$I(z, \tau) = \frac{\pi}{2} \sum_{n=1}^{\infty} T_n(z) p(\tau n^{1/4}), \quad |\arg \tau| < \frac{\pi}{4}, \tag{4.6}$$

where

$$\frac{\pi}{2} p(\tau) \doteq \frac{-1}{2\pi i} \int_{2-i\infty}^{2+i\infty} (1 + 2^{-w})\, \varphi(2w) \sin \pi w\, \Gamma(2w)\, (1 - 2w)\, \tau^{-2w}\, dw. \tag{4.7}$$

In this integral we may move the contour to the right and obtain

$$|p(\tau)| < K_n |\tau|^{-n} \text{ in } |\arg \tau| \leqslant \frac{\pi}{8} \text{ for every } n \geqslant 4, \tag{4.8}$$

where K_n is independent of τ. If we move the contour to $u = -N < 0$, we encounter simple poles at $w = 0$ and $w = -m/2$, $0 < m < 2N$, m odd. Hence

$$\pi p(\tau) = \pi - \sum_{m=1}^{2N-1} (1 + 2^{m/2})\, \varphi(-m) \left(\sin \frac{\pi m}{2}\right) \frac{1 + m}{m!}\, \tau^m + I_N.$$

Since for $w = -N + iv$

$$|\sin \pi w\, \Gamma(2w)| = |\Gamma(2iv) \sinh v| \prod_{k=1}^{2N} |k - 2iv|^{-1} = O\left[\frac{1}{(2N)!}\right]$$

and

$$\varphi(2w) = \varphi(1 - 2w) = O\left[(1 + v)^N e^{-\pi v/2}\right] \quad (v \geqslant 0)$$

uniformly in N and v, we obtain $I_N = O\,[|2\tau|^{2N}\, \Gamma(N + 2)\, \Gamma^{-1}(2N)]$ for fixed τ in $|\arg \tau| \leqslant \pi/8$. Letting $N \to \infty$, we get

$$p(\tau) = 1 + \frac{2}{\pi\tau} \sum_{n=1}^{\infty} (1 + 2^{n-(1/2)}) \frac{(-1)^n\, n!}{(2n - 1)!}\, \zeta(2n) \left(\frac{\tau^2}{\pi}\right)^n, \tag{4.9}$$

an entire function of τ, whose coefficients can be expressed by Bernoulli numbers. From (4.4), (4.5), and (4.6) we obtain

$$\theta(z) = 1 + \sum_{n=1}^{\infty} p(\tau n^{1/4}) \sum_{a^2+c^2=n} \omega(\rho)\, (z + i\rho)^{-1/2} + O(\tau^{1/2}) \quad (\tau > 0) \tag{4.10}$$

uniformly in every half plane $x \geqslant x_0 > 0$. This implies (1.9), since the double series over (a, c) is absolutely convergent by (4.8). Observing (4.8) and (4.9), the proof of the theorem is thus complete.

It is clear that the idea to employ a suitable integral, like (4.3), to obtain a summable Eisenstein series is applicable to other modular forms besides $\theta(z)$ of (absolutely small) negative dimension as well.

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13. ABSTRACT

A representation by a summable, analytic Eisenstein series is given for the standard Jacobian theta function. The result extends the known convergent Eisenstein series representations of the kth powers of the theta function for $k = 4, \ldots, 8$, due essentially to Hardy, to the case $k = 1$ by introducing a suitable summability method.

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