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Direct Computation of Power Density and Energy Density Spectra

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ABSTRACT

Direct methods which utilize finite discrete Fourier transforms in the computation of both power and energy density spectra are examined. The methods are shown to be applicable to most physically realizable processes. The "intrinsic" windows involved in the methods and the variance of the resulting estimates are discussed. A generalized smoothing technique is developed which significantly reduces the variance of the estimates and greatly suppresses leakage from side lobes.

PROBLEM STATUS

This is an interim report on the problem; work is continuing.

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DIRECT COMPUTATION OF POWER DENSITY AND ENERGY DENSITY SPECTRA

INTRODUCTION

With the development of the fast Fourier transform (FFT) algorithm, it became feasible to compute the power density spectrum from a considerable number of data points in a direct manner without the intermediate explicit calculation of the autocovariance function, which heretofore had been standard.

Because of this basic change in the technique of computation of power density spectra, it becomes necessary to examine the assumptions, constraints, methods, and applicability of the new technique, as well as the relative merit of various subsequent smoothing operations. A portion of the investigation has been attacked piecemeal by various authors, often for specific cases. The purpose of this report is to present a consolidated treatment and to extend and generalize the treatment where possible.

The report is concerned with both the problem of computation of energy density spectra for aperiodic deterministic functions and the problem of computation and smoothing of power density spectra for stationary, ergodic, random functions. The Fourier sampling problem and the sampling variation inherent with random processes are both examined. The first section of this report presents the assumptions, sufficient conditions for applicability, and initial analysis of the technique. The next section describes the actual computation of power density and energy density spectra and the resulting frequency windows. The final section is concerned with several smoothing operations (one of which is particularly meritorious), the associated frequency windows, and the variance of the smoothed estimators. The details of the various computations and analyses, together with some background material, are presented in the Appendices.

INITIAL CONDITIONS AND ANALYSIS

Aperiodic (or Transient) Deterministic Functions

Consider a continuous time function $x(t)$ which is aperiodic (or transient), deterministic, and real. Denote the continuous Fourier transform* of $x(t)$ by $\psi(f)$. Assume that $\psi(f)$ is approximately band limited. A sufficient condition is that $\psi(f)$ approaches zero at least as rapidly as $\epsilon/(|f|^{\alpha})$ for $1/(2\Delta) \leq |f| < \infty$, where $\alpha > 1.0$, ϵ is some small constant, and Δ is a time interval. Assume that $x(t)$ is negligibly small outside an interval of T seconds. Here a sufficient condition is that $x(t)$ approaches zero at least as rapidly as $\nu/(|t|^{\beta})$ for t outside the interval of T seconds, where $\beta > 1.0$, and ν is some small constant.

It should be noted that a condition for the validity of the continuous Fourier transform requires that

*Defined in Appendix A.

$$\int_{-\infty}^{\infty} |x(t)|\, dt$$

be finite. Thus the assumption concerning $x(t)$ is nonrestrictive and is merely a convention regarding the size of the time interval during which $x(t)$ is nonnegligible. Similarly, the total energy which $x(t)$ represents is given by

$$\int_{-\infty}^{\infty} [x(t)]^2\, dt = \int_{-\infty}^{\infty} \psi(f)\, \psi^{\dagger}(f)\, df\,,$$

which must be finite*. Thus the assumption of approximate band limiting is realistic and simply involves a choice of the range of frequencies with appreciable energy.**

Let $x(t)$ be sampled at a rate of $1/\Delta$ samples per second during the T seconds with the convention that the first sample occurs at $t = 0$ and the Nth sample‡ at $t = T$. If we signify the continuous transform of the finite discrete time series $\{x_j = x(t_j = \Delta j)\}$ for $j = 0, 1, 2, \ldots, N-1$, by $\psi_b(f)$, then it is shown in Appendix B that, under the given sufficient conditions,

$$\psi_b(f) = \psi(f) + \gamma_1\,, \quad |f| \le 1/(2\Delta)\,, \tag{1}$$

where γ_1 can be made arbitrarily small. From Eq. B12 one has

$$\psi_b(f) = \begin{cases} \Delta \sum_{j=0}^{N-1} x_j \exp(-2\pi i f \Delta j), & |f| \le 1/(2\Delta)\,, \\ 0, & |f| > 1/(2\Delta)\,. \end{cases} \tag{2}$$

If we let

$$N\Delta = T' = T + \Delta \tag{3}$$

and consider the frequencies¶

$$f_k = k/T', \quad k = 0, 1, 2, \ldots, (N/2)-1 \tag{4}$$

then with these substitutions one has

$$\psi_b(f_k) = (T'/N) \sum_{j=0}^{N-1} x_j \exp(-2\pi i k j/N), \quad k = 0, 1, 2, \ldots, (N/2)-1\,, \tag{5}$$

*The $\dagger$ denotes conjugation.

**In Appendix B the sufficient conditions for approximate band limiting and negligibility of $x(t)$ are examined.

‡Note that the N samples occur in the closed time interval $[0, T]$ and that there are $N-1$ subintervals of duration Δ in the closed time interval $[0, T]$; thus $(N-1)\,\Delta = T$.

¶Note that the presentation by Swick (1) may lead to a slight misconception due to a failure to distinguish between the interval of time spanned by N samples, that is $(N-1)\,\Delta = T$, and the interval of time $T' = N\Delta$ used in all equations.

or

$$\psi_b(f_k) = T'F_k, \quad k = 0, 1, 2, \ldots, (N/2)-1, \tag{6}$$

where the coefficients F_k are the members of the sequence F which is defined* to be the discrete finite Fourier transform (DFT) of $\{x_j\}$. Thus the approximation to the continuous transform of $x(t)$, when evaluated for f equal to integer multiples of $1/T'$ and $|f| \leq 1/(2\Delta)$, is equal to T' times the corresponding member of DFT $\{x_j\}$.

Stationary, Ergodic, Random Functions

A continuous random process generates an infinite aggregate of continuous random functions. The observed continuous random time function $x(t)$ is to be regarded as one realization of the infinite ensemble of functions which might have been observed. An important characteristic of a stationary, ergodic, random process is its power density function $P(f)$ which is the continuous Fourier transform of the autocovariance function of the process

$$\rho(\tau) = E[(x^{\ell}(t) - \mu)(x^{\ell}(t+\tau) - \mu)], \tag{7}$$

where E represents an ensemble average, $x^{\ell}(t)$ is the ℓth member of the ensemble, and μ is the ensemble mean of the random process. Thus $P(f)$ is defined to be

$$P(f) = \int_{-\infty}^{\infty} \rho(\tau) \exp(-2\pi i f \tau)\, d\tau . \tag{8}$$

Since both the power density function and the autocovariance function are characteristic of the random process as a whole, they are deterministic functions. It is in this sense that both Appendix B and this portion of the report treat the Fourier sampling problem for the random process utilizing power density estimates which are based on an infinite number of realizations. The variation of the estimates arising in the random sampling problem (when only a finite number of realizations are available) is treated later on in the report.

Without loss in generality, let $\mu = 0$. If one uses as an estimate of $\rho(\tau = \Delta z)$ the function $C^{\ell}(\tau = \Delta z)$ given by

$$C^{\ell}(\tau = \Delta z) = \begin{cases} [1/(N-|z|)] \displaystyle\sum_{j=0}^{N-|z|-1} x_j^{\ell}\, x_j^{\ell} + |z|, & |z| = 0, 1, \ldots, N-1 \\ 0, & |z| \geq N \end{cases} \tag{9}$$

where $C^{\ell}(\tau = -\Delta z) = C^{\ell}(\tau = \Delta z)$, then it is obvious that

$$E[C^{\ell}(\tau = \Delta z)] = \begin{cases} \rho(\tau = \Delta z), & |z| = 0, 1, \ldots, N-1 \\ 0, & |z| \geq N. \end{cases} \tag{10}$$

Thus, $C^{\ell}(\tau = \Delta z)$ is an unbiased estimator of the autocovariance function at the lags $\tau = \Delta z$ for $|z| = 0, 1, 2, \ldots, N-1$.

*Similar to definitions in Appendix C.

Assume that $\rho(\tau)$ is negligibly small for $|\tau| > T$ seconds. A sufficient condition is that $\rho(\tau)$ approaches zero at least as rapidly as $\nu'/(|\tau|^{\beta'})$ for lags τ greater than T seconds, where $\beta' > 1.0$, and ν' is a small constant. The single realization $x(t)$ may be thought of as being composed of an infinite number of segments, each of length T. Because of the assumption about $\rho(\tau)$, the segments of $x(t)$ are approximately independent (for T large) and may be used in place of the member functions of the ensemble. This is merely another way of stating the stationary and ergodic constraints placed upon the random process. Assume that $P(f)$ is approximately band limited. Here a sufficient condition is that $P(f)$ approaches zero at least as rapidly as $\epsilon'/(|f|^{\alpha'})$ for $1/(2\Delta) \le |f| < \infty$, where $\alpha' > 1.0$, and ϵ' is some small constant. Let the sampling rate for $C^{\ell}(\tau)$ be $1/\Delta$. Note that exactly analogous arguments for the sensibility and practicality of the assumptions concerning $\rho(\tau)$ and $P(f)$ can be given as those presented in the previous subsection for the assumptions on $x(t)$ and $\psi(f)$.

If one computes $C^{\ell}(\tau = \Delta z)$ for $|z| = 0, 1, \ldots, N-1$ from each segment according to Eq. 9 and forms a finite discrete transform (this is the older "brute force" method)

$$P_e^{\ell}(f) = \begin{cases} \Delta \displaystyle\sum_{n=-N+1}^{N-1} C^{\ell}(\Delta n) \exp(-2\pi i f \Delta n), & |f| \le 1/(2\Delta) \\ 0, & |f| > 1/(2\Delta) \end{cases} \tag{11}$$

which is ensemble averaged

$$E\left[P_e^{\ell}(f)\right] = \begin{cases} \Delta \displaystyle\sum_{n=-N+1}^{N-1} E[C^{\ell}(\Delta n)] \exp(-2\pi i f \Delta n), & |f| \le 1/(2\Delta) \\ 0, & |f| > 1/(2\Delta), \end{cases}$$

and defines $P_b(f)$ to be equal to $E[P_e^{\ell}(f)]$, then

$$P_b(f) = \begin{cases} \Delta \displaystyle\sum_{n=-N+1}^{N-1} \rho(\Delta n) \exp(-2\pi i f \Delta n), & |f| \le 1/(2\Delta) \\ 0, & |f| > 1/(2\Delta). \end{cases} \tag{12}$$

It is shown in Appendix B that, under the sufficient conditions on $\rho(\tau)$ and $P(f)$,

$$P_b(f) = \begin{cases} P(f) + \gamma_2, & |f| \le 1/(2\Delta) \\ 0, & |f| > 1/(2\Delta) \end{cases} \tag{13}$$

where γ_2 can be made arbitrarily small*. It should be noted that an infinite number of segments (members of the ensemble) are required in the ensemble average, and thus an infinite amount of data are required to estimate $P(f)$ arbitrarily closely. A discussion

*Note that here γ_2 represents a greatest lower bound on the error in determining the power spectra from an infinite number of segments, whereas in Eq. 1 γ_1 represents an estimate of the error involved in approximating the Fourier transform of the deterministic function. The formula for γ_1 is given in Eq. B11; γ_2 is given by Eq. B13.

of the variance of the estimate when only a finite number of segments are available is deferred until later.

DIRECT COMPUTATION OF ENERGY DENSITY AND POWER DENSITY SPECTRA

Energy Density Spectra for Aperiodic Deterministic Functions

The energy density spectrum of the aperiodic function $x(t)$ is given by Lee (2) as

$$\Gamma(f) = \psi(f)\psi^{\dagger}(f) \tag{14}$$

where $\psi(f)$ is the continuous Fourier transform of $x(t)$. From the previous section and Appendix B we know that, to within an arbitrarily small quantity γ_1,

$$\psi(f_k) = T'F_k, \; k = 0, 1, \ldots, (N/2)-1, \text{ and } f_k = k/T'. \tag{15}$$

Thus, upon neglecting terms of order one or higher in γ_1, we have

$$\Gamma(f_k) = (T')^2 F_k F_k^{\dagger}, \; k = 0, 1, \ldots, (N/2)-1. \tag{16}$$

The right-hand side of Eq. 16 can be thought of as an approximation to the average energy density in a band of width $1/T'$ centered at the frequency $f_k = k/T'$. It can be shown that

$$(T')^2 F_k F_k^{\dagger} \approx \int_{-1/(2\Delta)}^{1/(2\Delta)} \Gamma(f_k - f)\, WE(f)\, df, \; k = 0, 1, \ldots, (N/2)-1, \tag{17}$$

where the window* is

$$WE(f) = \begin{cases} \Delta \sin[2\pi f \Delta(N-0.5)]/\sin(\pi f \Delta), & |f| \le 1/(2\Delta) \\ 0, & |f| > 1/(2\Delta) \end{cases} \tag{18}$$

and

$$\int_{-\infty}^{\infty} WE(f)\, df = 1.$$

Figure 1 is a scaled plot of the energy density window versus $N\Delta f$ for the case where $N = 1024$ and $\Delta = 0.2$.

Power Density Spectra for Stationary, Ergodic, Random Functions

Rather than applying the "brute force" method of the first section, let us utilize the discrete finite Fourier transform of the ℓth segment with length T (N samples) of the

* $WE(f)$ is called a "window" because it effectively acts as a weighting function (or variable transmission window) throughout the frequency domain.

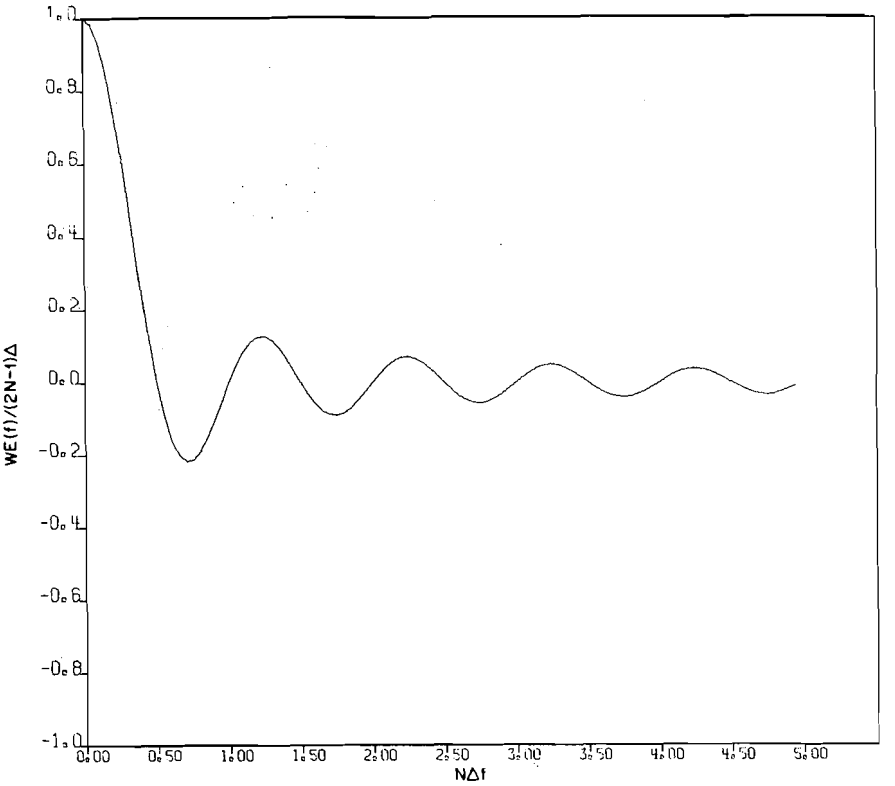

Fig. 1 - Plot of $WE(f)/(2N-1)\Delta$ vs $N\Delta f$ for the case $N = 1024$ and $\Delta = 0.2$ sec.

realization $x(t)$ to compute the power density spectrum. The direct approach is considerably faster (3) than the old method. Thus we have

$$F_k^{\ell} = (1/N) \sum_{j=0}^{N-1} x_j^{\ell} \exp(-2\pi i f_k \Delta j), \; k = 0, 1, \ldots, (N/2)-1. \tag{19}$$

The constraint that $k = 0, 1, \ldots, (N/2)-1$ will be understood to apply to the equations that follow. Consider the expression , which will be shown to be an estimator of power density,

$$S_k^{\ell} = (1/T') \left(T' F_k^{\ell} \, T' \left(F_k^{\ell}\right)^{\dagger}\right). \tag{20}$$

In Appendix D it is shown that

$$S_k^{\ell} = \Delta \sum_{z=-N+1}^{N-1} \left(1 - (|z|/N)\right) C^{\ell}(\Delta z) \exp(-2\pi i (k/N) z) \tag{21}$$

where $C^{\ell}(\Delta z)$ is defined in Eq. 9. Equation 21 can be rewritten as

$$S_k^{\ell} = \Delta \sum_{z=-N+1}^{N-1} [1-(\Delta|z|/T')]\ C^{\ell}(\Delta z)\ \exp(-2\pi i f_k \Delta z) \tag{22}$$

where $f_k = k/(N\Delta) = k/T'$. An ensemble average of Eq. 22, utilizing Eq. 10, gives

$$E\left(S_k^{\ell}\right) = \Delta \sum_{z=-N+1}^{N-1} [1-(\Delta|z|/T')]\,\rho(\Delta z)\ \exp(-2\pi i f_k \Delta z) \tag{23}$$

$$= \int_{-\infty}^{\infty} \left[\Delta \sum_{z=-N+1}^{N-1} \delta(\tau-\Delta z)\right] [1-(|\tau|/T')]\ \rho(\tau)\ \exp(-2\pi i f_k \tau) d\tau. \tag{24}$$

Application of the convolution theorem gives rise to

$$E\left(S_k^{\ell}\right) = \int_{-1/(2\Delta)}^{1/(2\Delta)} P(f_k - f)\ \mathrm{WBD}(f)\ df,\ k = 0, 1, \ldots, (N/2)-1, \tag{25}$$

where

$$\mathrm{WBD}(f) = \int_{-\infty}^{\infty} \left[\Delta \sum_{z=-N+1}^{N-1} \delta(\tau-\Delta z)\right] [1-(|\tau|/T')]\ \exp(-2\pi i f \tau)\ d\tau \tag{26}$$

$$= \Delta \sum_{z=-\infty}^{\infty} \mathrm{QBD}(\Delta z)\ \exp(-2\pi i f \Delta z)$$

$$= \begin{cases} (\Delta^2/T')\ [\sin(T'\pi f)/\sin(\Delta\pi f)]^2, & |f| \le 1/(2\Delta) \\ 0, & |f| > 1/(2\Delta) \end{cases} \tag{27}$$

and

$$\int_{-\infty}^{\infty} \mathrm{WBD}(f)\, df = 1,$$

and where $\mathrm{QBD}(\Delta z)$ is defined in Table A2, Appendix A. A plot of the frequency window makes obvious the interpretation of the expected value of S_k^{ℓ} as an estimate* of the average power density in a band of width $1/T'$ centered at the frequency $f_k = k/T'$.

Figure 2 is a scaled plot of the window $\mathrm{WBD}(f)$ versus $N\Delta f$ for the case $N = 1024$ and $\Delta = 0.2$ sec.

*In the direct method, the power density estimates are calculated directly from the finite discrete Fourier transform according to Eq. 20. However, it can be seen from Eq. 21 that this involves an implicit calculation of the estimator of the autocovariance function.

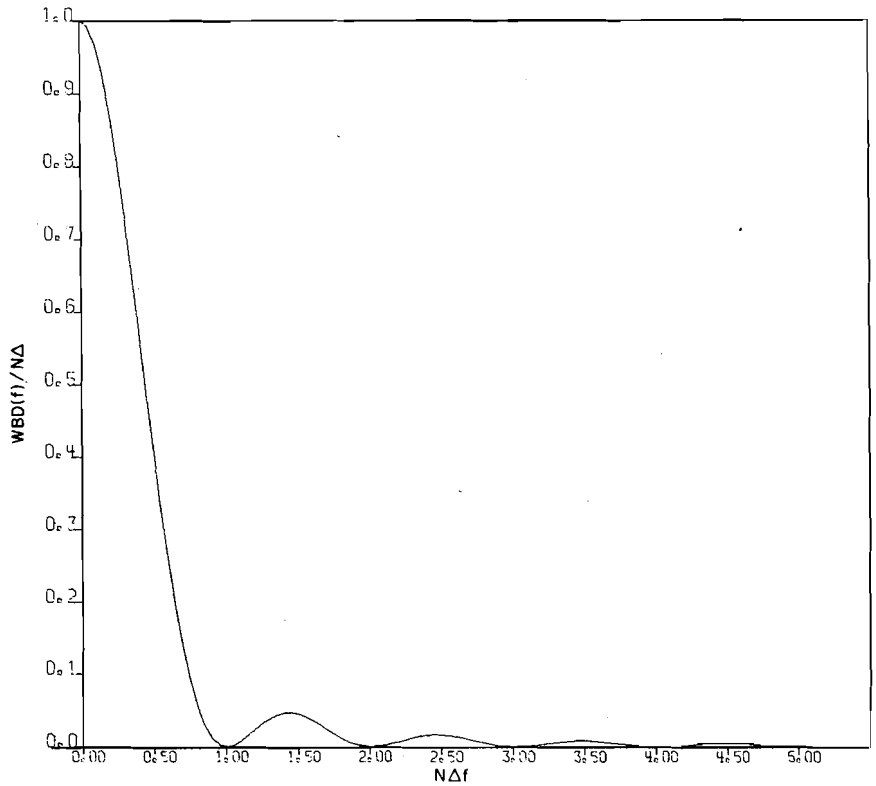

Fig. 2 - Plot of WBD(f)/NΔ vs NΔf for the case N = 1024 and Δ = 0.2

The spectral window WBD(f) is similar to a slit of width 1/T', so for very large T' (i.e., large N) it is reasonable to assume that P(f) is approximately constant over the slit. Hence one has the approximation

$$E(S_k^\ell) \approx P(f_k) \int_{-1/(2\Delta)}^{1/(2\Delta)} WBD(f)\, df = P(f_k) \tag{28}$$

for large T . This type of argument relating the window to a slit will be used again and referred to as a slit-window argument.

Up to this point, it has been assumed that an infinite number of segments were available for the ensemble average. Let us now consider the case where only one segment of length T_1 ($M = rN$ samples where r is an integer) is obtained and where the S_k^1 for $k = 0, 1, \ldots, (M/2)-1$ are computed from the segment. Here the single values S_k^1 are used in place of the ensemble averages; the equations to this point still apply with T_1' replacing T' and M replacing N. Jenkins and Watts (4) provide a derivation, under the relatively weak restriction that the random process is linear but not necessarily normal, of the approximate variance and covariance of S_k^1, that is (for $i \neq j$)

$$\begin{aligned} \text{Covariance } (S_i^1, S_j^1) \approx \Delta^2 P(f_i)P(f_j) \Big(&\{\sin[\pi M\Delta(f_i + f_j)]/M \sin[\pi\Delta(f_i + f_j)]\}^2 \\ &+ \{\sin[\pi M\Delta(f_i - f_j)]/M \sin[\pi\Delta(f_i - f_j)]\}^2 \Big) \end{aligned} \tag{29}$$

and

$$\text{Variance}\,(S_k^1) \approx P^2(f_k),\ k = 0,\ 1,\ \ldots,\ (M/2)-1. \tag{30}$$

They also show that, for M large, $2S_k^1/P(f_k)$ has approximately a χ^2 distribution with two degrees of freedom for $k \geq 1$, and that $S_0^1/P(0)$ has approximately a χ^2 distribution with one degree of freedom.

Several features of Eq. 29 and 30 should be noted: (a) the covariance between two spectral estimators is of order $(1/T_1')^2$ for $i \neq j$ and thus, for T_1' large, the S_k^i are approximately independent random variables; (b) the estimates are not consistent estimates of the power density spectrum, since the variance of the estimates does not approach zero as N approaches infinity; and (c) the standard deviation of the estimates is 100% of its mean value.

SMOOTHING THE POWER DENSITY SPECTRUM ESTIMATES

In order to reduce the variance of the S_k^1 estimates, some smoothing technique must be applied. A straightforward method by Bartlett (5) of exchanging resolution for decreased variance is to subdivide the single record of length T_1 into r nonoverlapping segments each of length T (N samples). Compute the S_k^ℓ for each segment ($\ell = 1, 2, \ldots, r$) and then average over the segments

$$S_k = (1/r) \sum_{\ell=1}^{r} S_k^\ell,\ k = 0,\ 1,\ \ldots,\ (N/2)-1. \tag{31}$$

It is obvious from Eqs. 25 and 31 that

$$E(S_k) = \int_{-1/(2\Delta)}^{1/(2\Delta)} P(f_k - f)\ \mathrm{WBD}(f)\,df,\ k = 0,\ 1,\ \ldots,\ (N/2)-1, \tag{32}$$

where WBD(f) is given by Eq. 27. A similar slit-window argument gives

$$E(S_k) \approx P(f_k),\ T \text{ large and } k = 0,\ 1,\ \ldots,\ (N/2)-1. \tag{33}$$

The constraint that $k = 0, 1, \ldots, (N/2)-1$ will be understood to apply to all equations that follow unless specifically excepted.

Since $N\Delta$ is larger than the time during which the autocovariance remains nonnegligible, the S_k^i and S_k^j are approximately independent for $i \neq j$. So

$$\mathrm{Var}\,(S_k) = (1/r^2) \sum_{\ell=1}^{r} \mathrm{Var}\left(S_k^\ell\right) \approx (1/r)\ P^2(f_k). \tag{34}$$

The single record of length T_1 provides estimates of the average power density in a band width $1/(rN\Delta)$ centered at the frequencies $f_k = k/(rN\Delta)$ for $k = 0, 1, \ldots, (rN/2)-1$ with variance $P^2(f_k)$, whereas the r segments give rise to the S_k which are estimates of the

average power density in a band of width $1/(N\Delta)$ centered at frequencies $f_k = k/(N\Delta)$ for $k = 0, 1, \ldots, (N/2)-1$ with variance $(1/r)\, P^2(f_k)$.

Now if a reduction of side lobe leakage is deemed necessary, one has the choice of either smoothing the estimates S_k or smoothing the $T'F_k^{\ell}$ at the intermediate stage before computation of the power density estimates.

As an example of the former choice, consider the following typical smoothing operation applied to the S_k:

$$V_k = d_{-1} S_{k-1} + d_0 S_k + d_1 S_{k+1}, \quad 1 \le k \le (N/2)-2, \tag{35}$$

where $d_{-1} = d_1 = 1/4$ and $d_0 = 1/2$. It is shown in Appendix E that

$$E(V_k) = \int_{[-1/(2\Delta)]-(1/T')}^{[1/(2\Delta)]+(1/T')} P(f_k - f)\, K(f)\, df, \quad 1 \le k \le (N/2)-2, \tag{36}$$

where

$$K(f) = \sum_{p=-1}^{1} d_p\, \mathrm{WBD}[f + (p/T')], \quad |f + (p/T')| \le 1/(2\Delta), \tag{37}$$

and $\mathrm{WBD}(f)$ is given in Eq. 27.

The window $K(f)$ is shown scaled in Fig. 3. It is graphed versus $N\Delta f$ for the case $N = 1024$ and $\Delta = 0.2$ sec. Using a slit window argument and the fact that $d_{-1} + d_0 + d_1 = 1$ one has

$$E(V_k) \approx P(f_k), \quad N \text{ large and } 1 \le k \le (N/2)-2. \tag{38}$$

Because the S_k are approximately independent for large N, the variance is given by

$$\mathrm{Var}(V_k) \approx \left(\sum_{p=-1}^{1} d_p^2\right) (1/r)\, P^2(f_k) = [3/(8r)]\, P^2(f_k), \quad 1 \le k \le (N/2)-2. \tag{39}$$

On the other hand, if the estimates $T'F_k^{\ell}$ are smoothed for each segment and power density subsequently computed, one has*

$$E\left\{U_k^{\ell} = (C_0/T')\, B^{\ell}(f_k) \left[B^{\ell}(f_k)\right]^{\dagger}\right\} = C_0 \int_{-\infty}^{\infty} P(f_k - f)\, Q(f)\, df \tag{40}$$

where

$$B^{\ell}(f) = \int_{-\infty}^{\infty} T'F^{\ell}(f-r)\, H(r)\, dr \tag{41}$$

*The details of the analysis leading to Eqs. 40 through 43 are given in Appendix F.

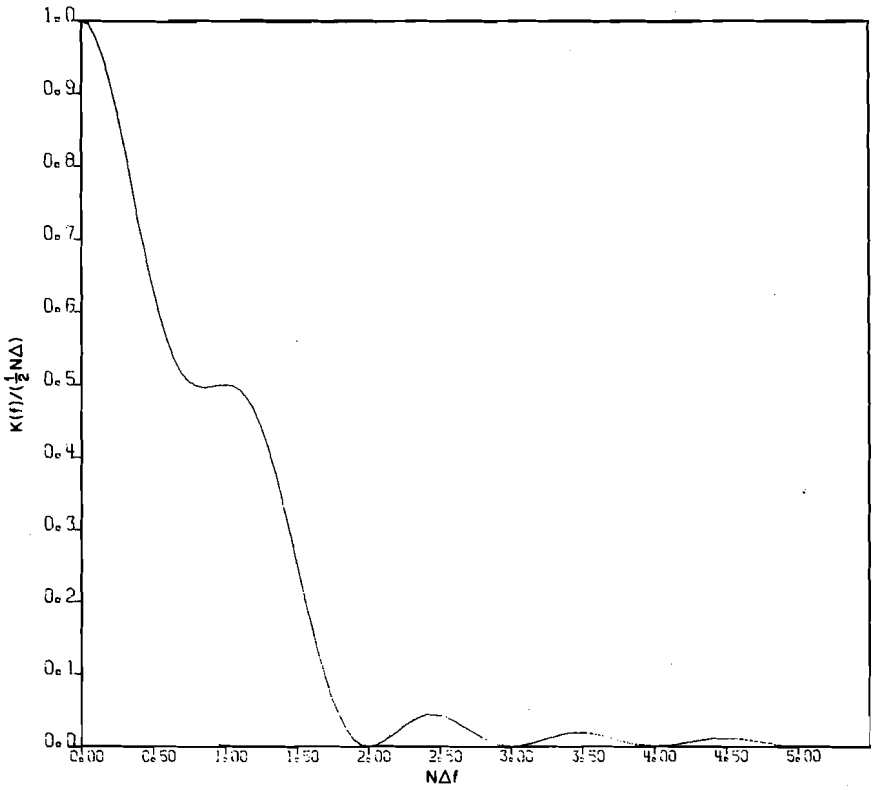

Fig. 3 - Plot of $K(f)/((1/2)N\Delta)$ vs $N\Delta f$ for the case $N = 1024$ and $\Delta = 0.2$ sec.

and C_0 is a normalizing constant given by

$$C_0 = \left(\int_{-\infty}^{\infty} Q(f)\, df \right)^{-1} \tag{42}$$

where the frequency window $Q(f)$ is given by

$$Q(f) = \begin{cases} (1/T')\, H_D(f)\, H_D^{\dagger}(f), & |f| \le 1/(2\Delta) \\ 0, & |f| > 1/(2\Delta). \end{cases} \tag{43}$$

Here $H_D(f)$ and $H(f)$ are the discrete finite Fourier transform and the continuous Fourier transform, respectively, of the time window $h(t)$ of the smoothing operation.

The variance of the smoothed estimator U_k^{ℓ} has been shown by Sloane (6) to be $P^2(f_k)$. The effective bandwidth of the smoothing window $C_0 Q(f)$ is given by a technique described by Jenkins and Watts (4) for non-rectangular windows. For an estimator based on a spectral window which is not rectangular, the bandwidth b of a window W is defined as the width of the rectangular window having unit area which gives the same variance as W. This bandwidth is

$$b[W(f)] = \left(\int_{-\infty}^{\infty} [W(f)]^2 df\right)^{-1} \tag{44}$$

where $W(f)$ is any nonrectangular spectral window. Thus the technique gives

$$b[C_0 Q(f)] = \left(C_0^2 \int_{-\infty}^{\infty} Q^2(f)\, df\right)^{-1}, \tag{45}$$

and we know

$$\mathrm{Var}\left(U_k^{\ell}\right) = P^2(f_k). \tag{46}$$

Now averaging across the segments provides

$$U_k = (1/r) \sum_{\ell=1}^{r} U_k^{\ell}. \tag{47}$$

From Eqs. 40 and 47, we have

$$E(U_k) = E\left(U_k^{\ell}\right) = C_0 \int_{-\infty}^{\infty} P(f_k - f)\, Q(f)\, df, \tag{48}$$

and again a slit-window argument gives

$$E(U_k) \simeq P(f_k), \text{ N large.} \tag{49}$$

Since the U_k^{ℓ} are independent for different segments, from Eq. 47 we have

$$\mathrm{Var}(U_k) = (1/r)\, \mathrm{Var}\left(U_k^{\ell}\right) = (1/r) P^2(f_k) \tag{50}$$

Construction of the quantity R, which is the ratio of the effective bandwidth of the smoothing window to the effective bandwidth of the "intrinsic" window, $WBD(f)$, gives one an estimation of the change in resolution due to the smoothing. Thus we have

$$R = b[C_0 Q(f)]/b[WBD(f)] \tag{51}$$

Now Eqs. 40-51 and Appendix F have dealt with a generalized smoothing operation applied to the estimates $T'F_k^{\ell}$. If we take the specific case of the Hanning smoothing* operation and determine the above relations for the case $N = 1024$ and $\Delta = 0.2$ sec, we have

$$B^{\ell}(f_k) = d_{-1}T'F^{\ell}_{k-1} + d_0T'F^{\ell}_k + d_1T'F^{\ell}_{k+1} \tag{52}$$

and

$$C_0 = 8/3,\ d_{-1} = d_1 = -1/4,\ d_0 = 1/2. \tag{53}$$

(Note that the minus sign for the d_{-1} and d_1 coefficients corresponds to the case where both the data and the smoothing time window extend from $t = 0$ to $t=T$). Also,

$$C_0Q(f) = \begin{cases} [8/(3T')]\,HAD_1(f)\,HAD_1^{\dagger}(f), & |f| \le 1/(2\Delta) \\ 0, & |f| > 1/(2\Delta) \end{cases} \tag{54}$$

($HAD_1(f)$ is given in Table A2)

and

$$b[WBD(f)] \approx 3/(2T'), \tag{55}$$

$$b[C_0Q(f)] \approx 1/(0.481T'), \tag{56}$$

$$E(U_k) \approx P(f_k), \tag{57}$$

$$\mathrm{Var}\,(U_k) \approx (1/r)P^2(f_k) \tag{58}$$

and

$$R \approx 1.39 \tag{59}$$

Figure 4 provides a scaled graph of the window $C_0Q(f)$ in Eq. 54 versus $N\Delta f$ for the case $N = 1024$ and $\Delta = 0.2$ seconds.

Now let us discuss the various smoothing techniques. The method due to Bartlett is a component of both of the other techniques and leads to a reduction in variance by a factor of r, the number of segments. With the provisions that the decrease in resolution by a factor of r can be tolerated for the specific problem and that the time length of each segment still satisfies the requirement of the sufficiency condition, this method can be useful.

The second method which involves smoothing the power spectrum estimates S_k is the smoothing method normally used when the power estimates have been obtained via the Blackman and Tukey approach. In this sense the second method is the "standard" method. When this method, utilizing a typical smoothing operation, was applied to the power spectrum estimates obtained via the direct approach detailed in this report, it was observed that a reduction in variance occured. However, in Fig. 3 it can be seen that this method suffers in two respects: first, the failure to suppress* the side lobes with the resultant "leakage" of power from adjacent bands produces a possibly severe bias for the smoothed estimates; and second, the introduction of a "shoulder" on the main lobe at $N\Delta f \approx 1.0$ in the example considered leads to similar consequences. These disadvantages arise from the convolution of the "intrinsic" window (the $WBD(f)$ window) produced in the direct approach with the smoothing window. On the other hand the Blackman and Tukey approach has an effective "intrinsic" window ($Q_0(f)$ in their notation)

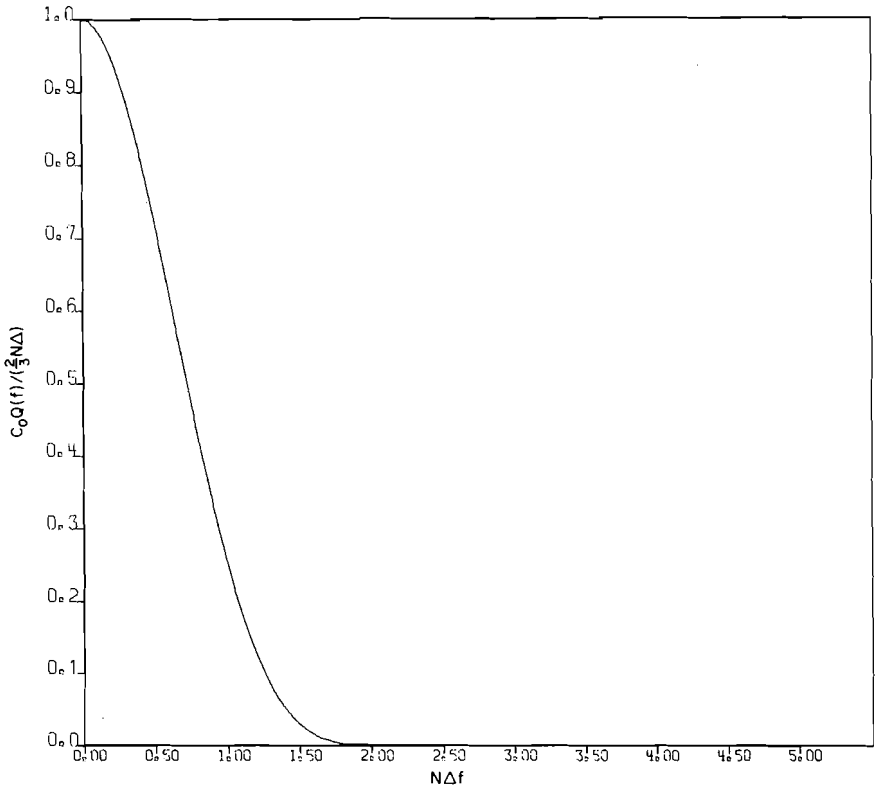

Fig. 4 - Plot of $C_oQ(f)/(2/3\ N\Delta)$ vs $N\Delta f$ for the case $N = 1024$ and $\Delta = 0.2$ sec.

which, when convolved with the smoothing window, simply gives the smoothing window again. Because of these two disadvantages, use of this method may be hazardous.

Next, let us consider the third method which involves smoothing the estimates $T'F_k^{\ell}$ for each segment and subsequent computation of power density spectra. It was observed that the choice of Hanning smoothing, power spectrum computation, and averaging across segments resulted in a significant reduction of variance by a factor of r. It can be seen from Fig. 4 that this method provides a relatively sharp window with the additional advantage of almost total suppression of side lobes.* This general method gives very good overall results. (The choice of Hanning was arbitrary, and other smoothing windows such as Parzens could be used with very good results.) As in any spectral analysis scheme, the smoothing window and technique to be used are those which are appropriate for the particular signal characteristics.

Figure 5 contains the graphs of the windows for each of the methods, plus the standard Hanning window for reference versus $N\Delta f$. All of the windows have been normalized so that their value is 1 for $N\Delta f = 0$. This particular normalization applies to the figures only. This allows one to see the relative side lobe height and obtain some feeling for the relative widths of the windows. The exact widths and window configurations can, of course, be obtained from the formulas given above.

*In numerical calculations, the first side lobe had a maximum value which was more than 31 dB down in comparison with the value at $N\Delta f = 0$. The 10 log ratio definition of dB was used.

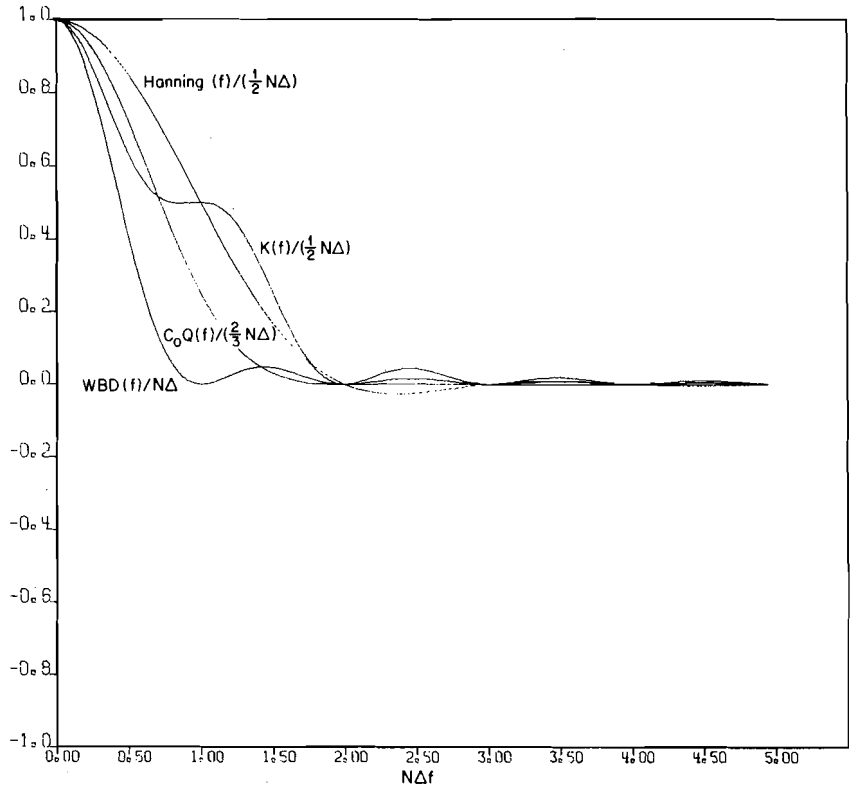

Fig. 5 - Plot of WBD(f)/NΔ, $C_0Q(f)/((2/3)N\Delta)$, K(f)/((1/2)NΔ), and Hanning (f)/((1/2)NΔ) vs NΔf for case N = 1024 and Δ = 0.2 sec.

SUMMARY

The methods of "direct" calculation of power density spectra and energy density spectra were found to be applicable to most physically realizable processes. In order to make the above statement, the following steps were taken: (a) the implicit basic assumptions of approximate band limiting and negligibility of the basic data outside the given time interval are rewritten as two sufficient conditions and are shown to be non-restrictive for practical physical processes; (b) since the "direct" methods involve discrete finite Fourier transforms, the Fourier sampling problem is minutely examined in the light of the sufficient conditions, and formulas are derived for the error factors γ_1 and γ_2. (For the case of the random process, γ_2 is an estimate of the greatest lower bound on the error in determining the power density spectra from an infinite number of realizations; whereas in the case of the deterministic function, γ_1 represents an estimate of the error in approximating the energy density spectra.); and (c) it is shown that both γ_1 and γ_2 can be made arbitrarily small through the proper choice of the sampling parameters.

The methods of "direct" calculation of both power density and energy density spectra are described, and the resulting "intrinsic" frequency windows WE(f) and WBD(f), for the energy density case and the power density case, respectively, are presented. The practical situation of a finite number of realizations with the sampling variation inherent in random processes is examined. In order to reduce variance, several smoothing operations are considered.

A generalized smoothing technique (which involves smoothing at an intermediate stage prior to the computation of power density) is developed. With the choice of Hanning smoothing and averaging across segments, this technique results in a significant reduction of variance, a relatively sharp window, and almost total suppression of side lobes.

REFERENCES

1. Swick, D.A., "Discrete Finite Fourier Transforms: a Tutorial Approach," NRL Report 6557, 1967.

2. Lee, Y.W., "Statistical Theory of Communication," New York: Wiley & Sons, 1960.

3. Palmer, L.B., "The Fourier Transform and Some of its Applications," NRL Report 7170, 1970.

4. Jenkins, G.M., and Watts, D.G., "Spectral Analysis and Its Applications," San Francisco: Holden-Day, 1968.

5. Bartlett, M., "An Introduction to Stochastic Processes," Cambridge, England; Cambridge University Press, 1955.

6. Sloane, E.A., "Comparison of Linearly and Quadratically Modified Spectral Estimates of Gaussian Signals," IEEE, AU-17, 133 (1969).

7. Hinich, M.J., and Clay, C.S., "The Application of the Discrete Fourier Transform in the Estimation of Power Spectra, Coherence, and Bispectra of Geophysical Data," Rev. Geophys., 6:347-363 (1968).

APPENDIX A

CONTINUOUS AND DISCRETE FOURIER TRANSFORMS

The continuous Fourier transform of a function $y(t)$ is defined to be

$$B(f) = \int_{-\infty}^{\infty} y(t) \exp(-2\pi i f t)\, dt \tag{A1}$$

$$= F(y), \tag{A2}$$

and similarly the inverse Fourier transform of $B(f)$ is defined as

$$y(t) = \int_{-\infty}^{\infty} B(f) \exp(+2\pi i f t)\, dt \tag{A3}$$

$$= F^{-1}(B). \tag{A4}$$

It is easy to show that F and F^{-1} are inverse operations, that is,

$$FF^{-1} = F^{-1}F = I \tag{A5}$$

where I is the identify operator. Table A1 lists some useful functions and their transforms.

Table A1
Continuous Transform Pairs. In this table $\delta(t)$ is the Dirac delta function, $QB(t)$ and $WB(f)$ are the Bartlett time and frequency windows, and $QH(t)$ and $HA(f)$ are the Hanning time and frequency windows.

$y(t)$	$B(f)$
$\Delta \sum_{n=-\infty}^{\infty} \delta(t-n\Delta)$	$\sum_{\ell=-\infty}^{\infty} \delta[f-(\ell/\Delta)]$
$y(t-\alpha)$	$B(f) \exp(-2\pi i \alpha f)$
$QB(t) = \begin{cases} [1-(\lvert t\rvert/T)], & \lvert t\rvert \le T \\ 0, & \lvert t\rvert > T \end{cases}$	$WB(f) = T\{\sin(\pi f T)/(\pi f T)\}^2$
$QH(t) = \begin{cases} (1/2)\,[1+\cos(2\pi t/T)], & \lvert t\rvert \le T/2 \\ 0, & \lvert t\rvert > T/2 \end{cases}$	$HA(f) = (T/2)\,\{\sin(\pi f T)/(\pi f T) + \sin(\pi T[f+(1/T)])/(2\pi T[f+(1/T)]) + \sin(\pi T[f-(1/T)])/(2\pi T[f-(1/T)])\}$

The convolution of $G_1(f)$ and $G_2(f)$ is defined to be

$$G_1(f) * G_2(f) = \int_{-\infty}^{\infty} G_1(f-r)\, G_2(r)\, dr. \tag{A6}$$

It is a well-known theorem that if $G(f)$, $G_1(f)$, and $G_2(f)$ are the transforms of $z(t)$, $z_1(t)$, and $z_2(t)$, respectively, and if $z(t) = z_1(t) \cdot z_2(t)$, then

$$G(f) = G_1(f) * G_2(f). \tag{A7}$$

Similarly, if $G(f) = G_1(f) \cdot G_2(f)$, then

$$z(t) = z_1(t) * z_2(t) = \int_{-\infty}^{\infty} z_1(t-s)\, z_2(s)\, ds. \tag{A8}$$

The discrete Fourier transform analogous to the continuous Fourier transform given in Eq. A1 is defined to be

$$BD(f) = \begin{cases} \Delta \sum_{t_j=-\infty}^{\infty} YD(t_j) \exp(-2\pi i f t_j), & |f| \leq 1/(2\Delta) \\ 0, & |f| > 1/(2\Delta) \end{cases} \tag{A9}$$

where t_j is a discrete time variable (that is, $t_j = \Delta j$) and the interval between successive values of t_j is Δ. Table A2 lists some functions and their discrete transforms.

Table A2

Discrete Transform Pairs. In this table $QBD(t_j)$ and $WBD(f)$ are the discrete Barlett time and frequency windows, $QHD(t_j)$ and $HAD(f)$ are the discrete Hanning time and frequency windows, and $QHD_1(t_j)$ and $HAD_1(f)$ are the discrete Hanning time and frequency windows with a shift along the time axis.

$YD(t_j)$	$BD(f)$
$QBD(t_j = \Delta j) = \begin{cases} [1-(\lvert\Delta j\rvert/T')], & \lvert\Delta j\rvert \le T' \\ 0, & \lvert\Delta j\rvert > T' \end{cases}$	$WBD(f) = \begin{cases} [(\Delta^2)/T'][\sin(T'\pi f)/\sin(\Delta\pi f)]^2, & \lvert f\rvert \le 1/(2\Delta) \\ 0, & \lvert f\rvert > 1/(2\Delta) \end{cases}$
$QHD\left(t_j = \Delta j + [\Delta/2]\right) = \begin{cases} (1/2)\left(1 + \cos\dfrac{2\pi[\Delta j + (\Delta/2)]}{T'}\right), \\ \quad \text{for } \lvert\Delta j\rvert \le T'/2 \\ 0, \text{ for } \lvert\Delta j\rvert > T'/2 \end{cases}$	$HAD(f) = \begin{cases} \Delta \displaystyle\sum_{\ell=-1}^{1} c_\ell \left(\dfrac{\sin T'\pi[f+(\ell/T')]}{\sin\Delta\pi[f+(\ell/T')]}\right), & \lvert f\rvert \le 1/(2\Delta) \\ 0, & \lvert f\rvert > 1/2\Delta) \end{cases}$
$QHD_1\left(t_j = \Delta j + [\Delta/2]\right) = \begin{cases} (1/2)\left(1 + \cos\dfrac{2\pi[\Delta j + (\Delta/2) - (T'/2)]}{T'}\right), \\ \quad \text{for } 0 \le \Delta j \le T' \\ 0, \text{ for } \Delta j < 0 \text{ or } \Delta j > T' \end{cases}$	$HAD_1(f) = \begin{cases} \exp(-\pi i f N\Delta)\Delta \displaystyle\sum_{\ell=-1}^{1} d_\ell \left(\dfrac{\sin T'\pi[f+(\ell/T')]}{\sin\Delta\pi[f+(\ell/T')]}\right), & \lvert f\rvert \le 1/(2\Delta) \\ 0, & \lvert f\rvert > 1/(2\Delta) \end{cases}$
where $c_{-1} = c_1 = 1/4$, $d_{-1} = d_1 = +1/4$, $c_0 = d_0 = 1/2$, and $j = -\infty, \ldots, -2, -1, 0, 1, 2, \ldots, \infty$	

APPENDIX B

FOURIER SAMPLING PROBLEM

Since in practice only a finite number of samples of the continuous function may be available, this appendix derives a relationship, subject to two sufficient conditions, between the continuous Fourier transform of the function and the finite discrete Fourier transform of the set of samples.

Following Blackman and Tukey*, let us represent the operation of sampling data throughout all time, at equally spaced times with interval Δ, by the infinite Dirac comb

$$\Delta \sum_{n=-\infty}^{\infty} \delta(t-n\Delta),$$

and the operation of obtaining a finite number N (even) of equally spaced samples by the finite Dirac comb

$$\Delta \sum_{n=-N/2}^{(N/2)-1} \delta(t-\Delta n).$$

The transform of the continuous, aperiodic, deterministic function $x(t)$, which has been sampled throughout all time, is given by

$$\psi_a(f) = \begin{cases} \int_{-\infty}^{\infty} \left[\Delta \cdot \sum_{n=-\infty}^{\infty} \delta(t-\Delta n)\right] x(t) \exp(-2\pi i f t)\, dt, & |f| \leq 1/(2\Delta) \\ 0, & |f| > 1/(2\Delta). \end{cases} \tag{B1}$$

From Table A1 and Eq. A7 one has [for $|f| \leq 1/(2\Delta)$]

$$\psi_a(f) = \left\{ \sum_{\ell=-\infty}^{\infty} \delta[f-(\ell/\Delta)] \right\} * \psi(f)$$

$$= \int_{-\infty}^{\infty} \left\{ \sum_{\ell=-\infty}^{\infty} \delta[f-(\ell/\Delta)-r] \right\} \psi(r)\, dr$$

*R. Blackman and J. W. Tukey, "The Measurement of Power Spectra," New York: Dover, 1959.

which upon integrating gives

$$\psi_a(f) = \psi(f) + \sum_{\ell=1}^{\infty} \left\{\psi[f-(\ell/\Delta)] + \psi[f + (\ell/\Delta)]\right\}, \quad |f| \leq 1/(2\Delta) \tag{B2}$$

The transform of the N samples of $x(t)$ is given by

$$\psi_b(f) = \begin{cases} \int_{-\infty}^{\infty} \left[\Delta \cdot \sum_{n=-N/2}^{(N/2)-1} \delta(t-\Delta n)\right] x(t) \exp(-2\pi i f t)\, dt, & |f| \leq 1/(2\Delta) \\ 0, \quad |f| > 1/(2\Delta). \end{cases} \tag{B3}$$

We will consider frequencies f such that $0 \leq |f| \leq 1/(2\Delta)$ only. Upon comparing with Eq. B1 we have

$$\psi_b(f) = \psi_a(f) - \int_{-\infty}^{\infty} \left[\Delta \cdot \sum_{n=-\infty}^{-(N/2)-1} \delta(t-\Delta n) + \Delta \cdot \sum_{n=N/2}^{\infty} \delta(t-\Delta n)\right] x(t) \exp(-2\pi i f t)\, dt$$

$$= \psi_a(f) - \Delta \sum_{n=N/2}^{\infty} x(\Delta n) \exp(-2\pi i f \Delta n) - \Delta \sum_{n=-\infty}^{-(N/2)-1} x(\Delta n) \exp(-2\pi i f \Delta n). \tag{B4}$$

Substitution from Eq. B2 gives

$$\psi_b(f) = \psi(f) + \sum_{\ell=1}^{\infty} \left\{\psi[f-(\ell/\Delta)] + \psi[f+(\ell/\Delta)]\right\}$$

$$-\Delta \sum_{n=N/2}^{\infty} x(\Delta n) \exp(-2\pi i f \Delta n) - \Delta \sum_{n=-\infty}^{-(N/2)-1} x(\Delta n) \exp(-2\pi i f \Delta n). \tag{B5}$$

The summation over ℓ in Eq. B5 gives the contribution to ψ_b due to aliasing, whereas the summations over n give the contributions from the outer skirts of the transient function.

Assume that $\psi(f)$ approaches zero at least as rapidly as $\epsilon/(|f|^{\alpha})$ for $1/(2\Delta) \leq |f| \leq \infty$, where $\alpha > 1.0$, and ϵ is some small constant. This is the condition for approximate band limiting. Assume that $x(t)$ approaches zero at least as rapidly as $\nu/(|t|^{\beta})$ for t outside the interval contained in the N samples, where $\beta > 1.0$, and ν is some small constant. This is the condition for the negligibility of $x(t)$.

Application of these conditions in Eq. B5 gives

$$\psi_b(f)-\psi(f) \le \epsilon \sum_{\ell=1}^{\infty}\left\{(-1)^{-\alpha}\,[(\ell/\Delta)-f]^{-\alpha} + [(\ell/\Delta)+f]^{-\alpha}\right\}$$
$$-[\nu/(\Delta^{\beta-1})]\left(\sum_{n=N/2}^{\infty} n^{-\beta}\exp(-2\pi i f\Delta n) + (-1)^{-\beta}\sum_{n=(N/2)+1}^{\infty} n^{-\beta}\exp(+2\pi i f\Delta n)\right). \quad \text{(B6)}$$

Let

$$f' = 1/\Delta \quad \text{(B7)}$$

be the sampling frequency. After a slight rearrangement in Eq. B6 we have

$$|\psi(f)-\psi_b(f)| < [|\epsilon|\,(f')^{-\alpha}]\sum_{\ell=1}^{\infty}[(\ell-\Delta f)^{-\alpha}+(\ell+\Delta f)^{-\alpha}] + [2|\nu|(f')^{\beta-1}]\sum_{n=N/2}^{\infty} n^{-\beta} \quad \text{(B8)}$$

Now $(\ell\pm\Delta f)\ge[\ell-(1/2)]$ and $[\ell-(1/2)]^{-\alpha} \le (2/\ell)^{\alpha}$ and finally

$$\sum_{n=N/2}^{\infty} n^{-\beta} \le \int_{x=[(N/2)-1]}^{\infty} x^{-\beta}\,dx.$$

Substitution of these relations into Eq. B8 gives

$$|\psi(f)-\psi_b(f)| < \left\{|+\epsilon|/[(f')^{\alpha}]\right\}(2^{\alpha+1})\,\zeta(\alpha) + [2|\nu|(f')^{\beta-1}]\int_{x=[(N/2)-1]}^{\infty} x^{-\beta}\,dx, \quad \text{(B9)}$$

where

$$\zeta(\alpha) = \sum_{\ell=1}^{\infty} \ell^{-\alpha}$$

is the Riemann-Zeta function when $\alpha > 1.0$. Upon completion of the integration in the last term one has

$$|\psi(f)-\psi_b(f)| < |+\epsilon|\,(f')^{-\alpha}\,(2^{\alpha+1})\,\zeta(\alpha) + \left(2|\nu|(\beta-1)^{-1}(f')^{\beta-1}\right)[(N/2)-1]^{-\beta+1} \quad \text{(B10)}$$

Define $T' = N\Delta = N/f'$, that is, T' equals the time interval T for the N samples plus Δ. Substitution for N in Eq. B10 gives (after modification)

$$|\psi(f)-\psi_b(f)| < \frac{|+\epsilon|\,(2^{\alpha+1})\,\zeta(\alpha)}{(f')^{\alpha}} + \frac{2^{\beta}\,|\nu|}{(\beta-1)\,(T'-2\Delta)^{\beta-1}} = \gamma_1. \quad \text{(B11)}$$

The Riemann-Zeta function is bounded for $\alpha > 1.0$ *. Now by choosing the sampling frequency high enough, the first term on the right of Eq. B11 can be made as small as desired. Similarly, by choosing the time interval T large enough, the second term on the right of Eq. B11 can be made as small as desired. Thus the conditions on $\psi(f)$ and $x(t)$ are sufficient conditions for the representation of $\psi(f)$ by $\psi_b(f)$ to within an arbitrarily small quantity for $0 \le |f| \le 1/(2\Delta)$.

Next let us establish a convention as regards zero phase and call the time that the first nonnegligible sample was obtained $t = 0$. Then the finite Dirac comb becomes

$$\Delta \cdot \sum_{j=0}^{N-1} \delta(t-\Delta j)$$

and $\psi_b(f)$ is given by

$$\psi_b(f) = \begin{cases} \Delta \cdot \sum_{j=0}^{N-1} x(\Delta j) \exp(-2\pi i f \Delta j), & |f| \le 1/(2\Delta) \\ 0, & |f > 1/(2\Delta). \end{cases} \tag{B12}$$

Although the entire discussion in the preceeding portion of Appendix B is germane when a stationary, ergodic, random process is involved, some "translation" of ideas and symbols is needed.

If one "translates" according to

$$x(t) \rightarrow \rho(\tau) \quad \text{[defined in Eq. 7]}$$

$$\psi(f) \rightarrow P(f) \quad \text{[defined in Eq. 8]}$$

$$\psi_b(f) \rightarrow P_b(f) \quad \text{[defined in Eq. 12]}$$

$$\psi_a(f) \rightarrow P_a(f) = \begin{cases} \Delta \sum_{n=-\infty}^{\infty} \rho(\Delta n) \exp(-2\pi i f \Delta n), & |f| \le 1/(2\Delta) \\ 0, & |f| > 1/(2\Delta) \end{cases}$$

$$N \rightarrow M = 2N-1^{\dagger}$$

then in the foregoing discussion one has results exactly applicable to the random process case. As examples, one has

$$T' = M\Delta \text{ and } T = (M-1)\Delta$$

*See E. Jahnke and F. Emde, "Tables of Functions," New York:Dover, 1945.

†With the convention that the first nonnegligible sample occurs at $t = (-N+1)\Delta$ and the last nonnegligible sample occurs at $t = (N-1)\Delta$

and Eq. B11 would become

$$|P(f) - P_b(f)| < \frac{|+\epsilon'|(2^{\alpha'+1})\,\zeta(\alpha')}{(f')^{\alpha'}} + \frac{2^{\beta'}\,|\nu'|}{(\beta'-1)\,(T'-\Delta)^{\beta'-1}} = \gamma_2 ; \tag{B13}$$

finally Eq. B12 would become

$$P_b(f) = \begin{cases} \Delta \cdot \sum_{j=-N+1}^{N-1} \rho(\Delta j)\, \exp(-2\pi i f \Delta j), & |f| \le 1/(2\Delta) \\ 0, & |f| > 1/(2\Delta). \end{cases} \tag{B14}$$

APPENDIX C

DISCRETE FOURIER TRANSFORM AND FAST FOURIER TRANSFORM

Here we present a definition of the discrete finite Fourier transform and several related formulae. A more detailed discussion may be found elsewhere (Ref. 3).

Consider the sequence $\{x_j\}$ $(j = 0, 1, \ldots, N-1)$ of N complex numbers as a point in N-space $[C^N]$ and denote the sequence by X, i.e.,

$$X = \{x_0, x_1, \ldots, x_{N-1}\}. \tag{C1}$$

Let

$$W = \exp(2\pi i/N). \tag{C2}$$

Now the N vectors b_k $(k = 0, 1, \ldots, N-1)$, defined to be

$$b_k = (1, W^k, W^{2k}, W^{3k}, \ldots, W^{(N-1)k}) \tag{C3}$$

form a basis for C^N. So X is also given by

$$X = \sum_{k=0}^{N-1} A_k b_k \tag{C4}$$

where the A_k are complex numbers (note that the coefficients A_k are unique for a given basis) obtained by

$$A_j = (1/N) \sum_{k=0}^{N-1} x_k W^{-jk}, \quad j = 0, 1, \ldots, N-1. \tag{C5}$$

The sequence $A = \{A_0, A_1, \ldots, A_{N-1}\}$ in Eq. C5 is defined to be the discrete finite Fourier transform (DFT) of X. The x_k are related to the coefficients A_j by

$$x_k = \sum_{j=0}^{N-1} A_j W^{jk}, \quad k = 0, 1, \ldots, N-1. \tag{C6}$$

The operation generating the sequence X as in Eq. C6 is defined to be the inverse discrete finite Fourier transform (IDFT) of A. It should be noted that the original sequence X (which can be thought of as a time series) is represented exactly by the sequence A at the N points of interest. If there were sampling errors or experimental errors in the x_i, then these will be retained in the A_i.

The sequence $Y = \{Y_0, Y_1, \ldots, Y_{N-1}\}$ obtained from the sequence X by

$$Y_j = \sum_{k=0}^{N-1} x_k W^{-jk}, \quad j = 0, 1, \ldots, N-1 \tag{C7}$$

is often defined symbolically to be the fast Fourier transform (FFT) of X, although in actuality the term fast Fourier transform refers to a particularly efficient algorithm for calculating the sequence Y according to Eq. C7.

APPENDIX D

A SIMPLE COUNTING PROBLEM

This appendix concerns itself with the simple counting problem incurred in the direct computation of power density spectra.

Consider the equation

$$S_k^\ell = (1/T')\ T'F_k^\ell T' F_k^{\ell\,\dagger}$$

$$= (T'/N^2)\left(\sum_{n=0}^{N-1} \left(x_n^\ell\right)^\dagger \quad \exp\,(2\pi i k n/N)\right)\left(\sum_{m=0}^{N-1} x_m^\ell \exp\,(-2\pi i k m/N)\right). \tag{D1}$$

Let

$$m - n = z\,. \tag{D2}$$

After the multiplication indicated in Eq. D1 the resulting terms will have an exponential portion whose argument is $-2\pi ikz/N$. Table D1 gives the values of z which arise from this multiplication. Note that the diagonals of the table are lines of constant z. If one regroups the terms according to values of z and sums first over the diagonals of constant z, one has

$$\exp\,(-2\pi i k z/N) \sum_{\alpha=0}^{N-1-z} \left(x_\alpha^\ell\right)^\dagger x_{\alpha+z}^\ell\,, \quad z \geq 0 \tag{D3}$$

and

$$\exp\,(-2\pi i k z/N) \sum_{\beta=0}^{N-1-|z|} \left(x_{\beta+|z|}^\ell\right)^\dagger x_\beta^\ell\,, \quad z < 0\,. \tag{D4}$$

Define

$$C^\ell(\Delta z) = [1/(N-|z|)] \sum_{j=0}^{N-1-|z|} \left(x_j^\ell\right)^\dagger x_{j+|z|} \tag{D5}$$

where $C^\ell(\Delta z) = C^{\ell\dagger}(-\Delta z)$ for $z \geq 0$. The use of Eq. D5 allows both Eqs. D3 and D4 to be represented by the following type of term:

$$(N-|z|)\ C^\ell(\Delta z)\ \exp\,(-2\pi i k z/N)\,. \tag{D6}$$

Finally if one sums Eq. D6 over the range of z and multiplies by $T'/(N^2)$ one has

$$S_k^{\ell} = (T'/N^2) \sum_{z=-N+1}^{N-1} (N-|z|)\, C^{\ell}(\Delta z)\, \exp(-2\pi i k z/N) \tag{D7}$$

or

$$S_k^{\ell} = \Delta \sum_{z=-N+1}^{N-1} [1-(|z|/N)]\, C^{\ell}(\Delta z)\, \exp(-2\pi i k z/N) \tag{D8}$$

where $T' = N\Delta$.

Table D1
z Values

	$(x_0^{\ell})^{\dagger} W^{(0)}$	$(x_1^{\ell})^{\dagger} W^{k}$	$(x_2^{\ell})^{\dagger} W^{2k}$	$(x_3^{\ell})^{\dagger} W^{3k}$	...	$(x_n^{\ell})^{\dagger} W^{nk}$	...	$(x_{N-1}^{\ell})^{\dagger} W^{(N-1)k}$
$x_0^{\ell} W^{-k(0)}$	$Z = 0$	-1	-2	-3	...	$-n$	...	$-(N-1)$
$x_1^{\ell} W^{-k}$	1	0	-1	-2	...	$1-n$	...	$1-(N-1)$
$x_2^{\ell} W^{-2k}$	2	1	0	-1	...	$2-n$	...	$2-(N-1)$
$x_3^{\ell} W^{-3k}$	3	2	1	0	...	$3-n$	...	$3-(N-1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$x_m^{\ell} W^{-mk}$	m	$(m-1)$	$(m-2)$	$(m-3)$	...	$(m-n)$	...	$m-(N-1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$x_{N-1}^{\ell} W^{-(N-1)k}$	$N-1$	$(N-1)-1$	$(N-1)-2$	$(N-1)-3$	...	$(N-1)-n$	...	0

APPENDIX E

WINDOW RESULTING FROM A TYPICAL SMOOTHING APPLIED AFTER THE COMPUTATION OF POWER SPECTRA

Consider the equation

$$V_k = (1/4)S_{k-1} + (1/2)S_k + (1/4)S_{k+1} \tag{E1}$$

which, when ensemble averaged, gives

$$E(V_k) = (1/4)E(S_{k-1}) + (1/2)E(S_k) + (1/4)E(S_{k+1}). \tag{E2}$$

Substitution from Eq. 32 provides

$$E(V_k) = \int_{-1/(2\Delta)}^{1/(2\Delta)} (1/4)P(f_{k-1}-f)\ \mathrm{WBD}(f)\ df + \int_{-1/(2\Delta)}^{1/(2\Delta)} (1/2)P(f_k-f)\ \mathrm{WBD}(f)\ df$$

$$+ \int_{-1/(2\Delta)}^{1/(2\Delta)} (1/4)P(f_{k+1}-f)\ \mathrm{WBD}(f)\ df. \tag{E3}$$

Recall that $f_k = k/T'$, and let $y_1 = f + (1/T')$ and $y_2 = f - (1/T')$. With these changes of variables, Eq. E3 becomes

$$E(V_k) = \int_{[-1/(2\Delta)]+(1/T')}^{[1/(2\Delta)]+(1/T')} (1/4)P(f_k-y_1)\ \mathrm{WBD}[y_1-(1/T')]\ dy_1 + \int_{-1/(2\Delta)}^{1/(2\Delta)} (1/2)P(f_k-f)\ \mathrm{WBD}(f)\ df$$

$$+ \int_{-[1/(2\Delta)]-(1/T')}^{[1/(2\Delta)]-(1/T')} (1/4)P(f_k-y_2)\ \mathrm{WBD}[y_2+(1/T')]\ dy_2. \tag{E4}$$

Now y_1, y_2 and f are all dummy variables of integration and could be replaced with a single dummy variable of integration, say g. So Eq. E4 becomes

$$E(V_k) = \int_{-[1/(2\Delta)]-(1/T')}^{[1/2\Delta)]+(1/T')} P(f_k-g)\,K(g)\ dg \tag{E5}$$

where

$$K(g) = \sum_{p=-1}^{1} d_p \, WBD[g + (p/T')], \quad |g + (p/T')| \leq 1/(2\Delta) \tag{E6}$$

and $d_{-1} = d_1 = 1/4$, $d_0 = 1/2$. Although it is a very small correction, it should be remembered that the three different portions of $K(g)$ have slightly different regions in the frequency domain for which they are nonzero.

APPENDIX F

WINDOW RESULTING FROM SMOOTHING BEFORE THE COMPUTATION OF POWER SPECTRA

For simplicity of calculation, Appendix F begins with the convention that the N samples occur during the time interval $-T/2$ to $T/2$. The main body of the text uses the convention that the same N samples occur during the time interval 0 to T. This shift of the time axis merely introduces a phase factor (recall Table A1). However, this phase factor does not influence the results of Appendix F because the true power density, the power density estimates, and the resulting smoothing spectral window (see Eqs. F16 and F17) are all independent of the initial phase factor.

Consider the expression

$$A^{\ell}(f) = T'F^{\ell}(f) = \begin{cases} \displaystyle\int_{-\infty}^{\infty} \left\{ \Delta \sum_{j=-N/2}^{(N/2)-1} \delta[t-\Delta j-(\Delta/2)] \right\} x^{\ell}(t) \exp(-2\pi i ft)\, dt, & |f| \le 1/(2\Delta) \\ 0, \quad |f| > |f| \; 1/(2\Delta) \end{cases}$$

which is the discrete finite Fourier transform of the ℓth segment with length T (N samples) multiplied by $T' = N\Delta$. With the change of convention as to the time of occurance of the first sample, $A^{\ell}(f)$ becomes

$$A^{\ell}(f) = \exp\left(2\pi i\left(\frac{N-1}{2}\right)\Delta f\right)\Delta \sum_{j=0}^{N-1} x^{\ell}(\Delta j) \exp(-2\pi i f \Delta j). \tag{F1}$$

Let $B^{\ell}(f)$ be the convolution of $A^{\ell}(f)$, with the frequency window $H(f)$, of the smoothing function, i.e.,

$$B^{\ell}(f) = \int_{-\infty}^{\infty} A^{\ell}(f-r)\, H(r)\, dr \tag{F2}$$

and

$$B^{\ell\dagger}(f) = \int_{-\infty}^{\infty} A^{\ell\dagger}(f-s)\, H^{\dagger}(s)\, ds. \tag{F3}$$

It should be noted that the limits of the integrals in Eq. F2 and F3 are finite due to $A^{\ell}(f)$. Having smoothed the transforms, let us consider the equation which gives rise to power spectra:

$$E\left[(1/T')B^{\ell}(f)\, B^{\ell\dagger}(f)\right] = (1/T') \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(r)H^{\dagger}(s) E\left[A^{\ell}(f-r)\, A^{\ell\dagger}(f-s)\right] dr\, ds. \tag{F4}$$

Now from Eq. F1 one has

$$A^{\ell}(f-r)\ A^{\ell^{\dagger}}(f-s) = \Delta^2 \exp(-2\pi i(r-s)\Delta(N-1)/2)\left[\left(\sum_{j=0}^{N-1} x^{\ell}(\Delta j)\ \exp(-2\pi i a j \Delta)\right)\right.$$
$$\left.\cdot\left(\sum_{k=0}^{N-1} x^{\ell^{\dagger}}(\Delta k)\ \exp(2\pi i b k \Delta)\right)\right] \tag{F5}$$

where $a = f-r$, and $b = f-s$. After the multiplication indicated in Eq. F5 the resulting terms will contain a product $x^{\ell}(x^{\ell})^{\dagger}$ for which the argument of x^{ℓ} minus the argument of $(x^{\ell})^{\dagger}$ is set equal to $z\Delta$. Table F1 gives the value of z in parentheses and lists the exponential portion of each term as the exponential of W.* Summing first over diagonals of constant z we have

$$W^{-az\Delta} \sum_{\alpha=0}^{N-1-|z|} x^{\ell}(\alpha\Delta)\ x^{\ell^{\dagger}}(\alpha\Delta + |z|\Delta)\, W^{(b-a)(\alpha+|z|)\Delta},\quad z \le 0$$

and

$$W^{-bz\Delta} \sum_{\alpha=0}^{N-1-|z|} x^{\ell^{\dagger}}(\alpha\Delta)\ x^{\ell}(\alpha\Delta + |z|\Delta)\, W^{(b-a)(\alpha+|z|)\Delta},\quad z \ge 0.$$

Next, summing over values of z we have

$$A^{\ell}(f-r)\ A^{\ell^{\dagger}}(f-s) = \Delta^2 \exp(-2\pi i(r-s)(N-1)\Delta/2)\left[\left(\sum_{z=0}^{N-1} W^{-bz\Delta} \sum_{\alpha=0}^{N-1-|z|} x^{\ell^{\dagger}}(\alpha\Delta)\right.\right.$$
$$\left.\cdot\ x^{\ell}(\alpha\Delta+|z|\Delta)\, W^{(b-a)(\alpha+|z|)\Delta}\right) + \left(\sum_{z=-N+1}^{-1} W^{-az\Delta}\right.$$
$$\left.\left.\cdot \sum_{\alpha=0}^{N-1-|z|} x^{\ell}(\alpha\Delta)\ x^{\ell^{\dagger}}(\alpha\Delta+|z|\Delta)\, W^{(b-a)(\alpha+|z|)\Delta}\right)\right] \tag{F6}$$

Ensemble averaging gives (recall Eq. 7)

$$E[A^{\ell}(f-r)\ A^{\ell^{\dagger}}(f-s)] = \Delta^2 \exp(-\pi i(r-s)(N-1)\Delta)\left[\left(\sum_{z=0}^{N-1} W^{-bz\Delta}\rho(\Delta z) \sum_{\alpha=0}^{N-1-|z|} W^{(b-a)(\alpha+|z|)\Delta}\right)\right.$$
$$\left. + \left(\sum_{z=-N+1}^{-1} W^{-az\Delta}\ \rho(\Delta z) \sum_{\alpha=0}^{N-1-|z|} W^{(b-a)(\alpha+|z|)\Delta}\right)\right]$$

*For the table $W = e^{2\pi i}$

Table F1
Z Values and Exponential Portion of Terms

	$\left(x_0^{\ell}\right)^{\dagger} W^{b\Delta 0}$	$\left(x_1^{\ell}\right)^{\dagger} W^{b\Delta}$	$\left(x_2^{\ell}\right)^{\dagger} W^{2b\Delta}$	...	$\left(x_{N-1}^{\ell}\right)^{\dagger} W^{(N-1)b\Delta}$
$x_0^{\ell} W^{-a\Delta 0}$	$(z=0) W^{(b-a)\Delta 0}$	$(-1) W^{(b-a)\Delta + a\Delta}$	$(-2) W^{2(b-a)\Delta + 2a\Delta}$	...	$(-N+1) W^{(N-1)(b-a)\Delta + (N-1)a\Delta}$
$x_1^{\ell} W^{-a\Delta}$	$(1) W^{(b-a)\Delta - b\Delta}$	$(0) W^{(b-a)\Delta}$	$(-1) W^{2(b-a)\Delta + a\Delta}$	...	$(-N+2) W^{(N-1)(b-a)\Delta + (N-2)a\Delta}$
$x_2^{\ell} W^{-2a\Delta}$	$(2) W^{2(b-a)\Delta - 2b\Delta}$	$(1) W^{2(b-a)\Delta - b\Delta}$	$(0) W^{2(b-a)\Delta}$	...	$(-N+3) W^{(N-1)(b-a)\Delta + (N-3)a\Delta}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$x_{N-1}^{\ell} W^{-(N-1)a\Delta}$	$(N-1) W^{(N-1)(b-a)\Delta - (N-1)b\Delta}$	$(N-2) W^{(N-1)(b-a)\Delta - (N-2)b\Delta}$	$(N-3) W^{(N-1)(b-a)\Delta - (N-3)b\Delta}$	...	$(0) W^{(N-1)(b-a)\Delta}$

which can be rearranged to give

$$E[A^{\ell}(f-r)\ A^{\ell\dagger}(f-s)] = \Delta^2 \exp(-\pi i(r-s)(N-1)\Delta)\left[\sum_{z=0}^{N-1} W^{-az\Delta}\ \rho(\Delta z) \sum_{\alpha=0}^{N-1-|z|} W^{(b-a)\alpha\Delta}\right.$$

$$\left. + \sum_{z=-N+1}^{-1} W^{-bz\Delta}\rho(\Delta z) \sum_{\alpha=0}^{N-1-|z|} W^{(b-a)\alpha\Delta}\right] \tag{F7}$$

It is easy to show that

$$\sum_{\alpha=0}^{N-1-|z|} W^{(b-a)\alpha\Delta} = \exp(-\pi i(a-b)(N-|z|-1)\Delta)\left(\frac{\sin[\pi\Delta(N-|z|)(a-b)]}{\sin[\pi\Delta(a-b)]}\right) \tag{F8}$$

Utilizing Eq. F8 and recalling the definition of a and b, Eq. F7 becomes

$$E\left[A^{\ell}(f-r)\ A^{\ell\dagger}(f-s)\right] = \Delta^2\left(\sum_{z=-N+1}^{N-1} \rho(\Delta z)\left(\frac{\sin[\pi\Delta(N-|z|)(a-b)]}{\sin[\pi\Delta(a-b)]}\right)\right.$$

$$\left.\cdot \exp\left(-2\pi i\left(f-\frac{(r+s)}{2}\right)\Delta z\right)\right). \tag{F9}$$

Equation F9 can be rewritten as

$$E\left[A^{\ell}(f-r)\ A^{\ell\dagger}(f-s)\right] = \int_{-\infty}^{\infty} \Delta\left[\Delta \sum_{z=-N+1}^{N-1} \delta(t-\Delta z)\right]\rho(t)\ \frac{\sin[\pi(T'-|t|)(s-r)]}{\sin[\pi\Delta(s-r)]}\ \exp\left(-2\pi i\left(f-\frac{r+s}{2}\right)t\right)dt$$

which, because of the convolution theorem, is

$$E\left[A^{\ell}(f-r)\ A^{\ell\dagger}(f-s)\right] = \int_{-\infty}^{\infty} P(x)\Delta I\left(f-\frac{r+s}{2}-x\right)dx \tag{F10}$$

where

$$I(f) = \int_{-\infty}^{\infty}\left[\Delta \sum_{z=-N+1}^{N-1} \delta(t-\Delta z)\right]\frac{\sin[\pi(T'-|t|)(s-r)]}{\sin[\pi\Delta(s-r)]}\ \exp(-2\pi i f t)\ dt,$$

$$= \Delta\left(\frac{\sin[\pi(T'/2)(s-r+2f)]\ \sin[\pi(T'/2)(s-r-2f)]}{\sin[\pi(\Delta/2)(s-r+2f)]\ \sin[\pi(\Delta/2)(s-r-2f)]}\right), \tag{F11}$$

Substitution from Eqs. F10 and F11 into Eq. F4 gives

$$E\left[(1/T')B^{\ell}(f)\,B^{\ell\dagger}(f)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1/T')H(r)H^{\dagger}(s)\left[\int_{-\infty}^{\infty}P(x)\Delta^2\right.$$

$$\left.\cdot\left(\frac{\sin[\pi T'(f-s-x)]\,\sin[\pi T'(f-r-x)]}{\sin[\pi\Delta(f-s-x)]\,\sin[\pi\Delta(f-r-x)]}\right)dx\right]dr\,ds$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}P(x)\,(1/T')\left[H(r)\,\Delta\left(\frac{\sin[\pi T'(f-x-r)]}{\sin[\pi\Delta(f-x-r)]}\right)\right]$$

$$\cdot\left[H^{\dagger}(s)\,\Delta\left(\frac{\sin[\pi T'(f-x-s)]}{\sin[\pi\Delta(f-x-s)]}\right)\right]dx\,dr\,ds \tag{F12}$$

The interchange of the order of integration is justifiable due to the nature of the functions involved and the effective finite limits of integration. So consider first the integral over r,

$$\int_{-\infty}^{-\infty}H(r)\,\Delta\left(\frac{\sin[\pi T'(f-x-r)]}{\sin[\pi\Delta(f-x-r)]}\right)dr = \int_{-\infty}^{\infty}\Delta\sum_{j=-N/2}^{(N/2)-1}\delta[t-\Delta j-(\Delta/2)]\,h(t)\exp(-2\pi i(f-x)t)\,dt,$$

$$= \text{Discrete finite Fourier transform of } h(t), \tag{F13}$$

$$= \begin{cases} H_D(f-x), & |f-x| \le 1/(2\Delta) \\ 0, & |f-x| > 1/(2\Delta). \end{cases} \tag{F14}$$

Exactly analogous results apply for the integral over s; applying these results in Eq. F12 we have

$$E[(1/T')B^{\ell}(f)\,B^{\ell\dagger}(f)] = \int_{f-[1/(2\Delta)]}^{f+[1/(2\Delta)]}P(x)\,(1/T')H_D(f-x)\,H_D^{\dagger}(f-x)\,dx$$

$$= \int_{-1/(2\Delta)}^{1/(2\Delta)}P(f-y)\,(1/T')\,H_D(y)H_D^{\dagger}(y)\,dy, \tag{F15}$$

It should be noted that T' is necessary from both the standpoint of definition of power density spectrum and from the standpoint of a dimensionalizing and normalizing constant. So finally we have

$$E[(1/T')B^{\ell}(f)\,B^{\ell\dagger}(f)] = \int_{-\infty}^{\infty}P(f-y)\,Q(y)\,dy \tag{F16}$$

where the frequency window is

$$Q(y) = \begin{cases} (1/T')\,H_D(y)\,H_D^{\dagger}(y), & |y| \le 1/(2\Delta) \\ 0, & |y| > 1/(2\Delta). \end{cases} \tag{F17}$$

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13. ABSTRACT

Direct methods which utilize finite discrete Fourier transforms in the computation of both power and energy density spectra are examined. The methods are shown to be applicable to most physically realizable processes. The "intrinsic" windows involved in the methods and the variance of the resulting estimates are discussed. A generalized smoothing technique is developed which significantly reduces the variance of the estimates and greatly suppresses leakage from side lobes.

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