Naval Research Laboratory

Stennis Space Center, MS 39529-5004

NRL/FR/7440--07-10,153

Using Range and Range Rate for Relative Navigation

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September 5, 2007

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REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

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1. REPORT DATE *(DD-MM-YYYY)*
05-09-2007

2. REPORT TYPE
Formal Report

3. DATES COVERED *(From - To)*
September 2003 to July 2007

4. TITLE AND SUBTITLE

Using Range and Range Rate for Relative Navigation

5a. CONTRACT NUMBER

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER
0602435N

6. AUTHOR(S)

Brian S. Bourgeois

5d. PROJECT NUMBER

5e. TASK NUMBER
6636

5f. WORK UNIT NUMBER
6636

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

Naval Research Laboratory
Marine Geosciences Division
Stennis Space Center, MS 39529-5004

8. PERFORMING ORGANIZATION REPORT NUMBER

NRL/FR/7440--07-10,153

9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES)

Office of Naval Research
Suite 1425
875 North Randolph Street
Arlington, VA 22203-1995

10. SPONSOR / MONITOR'S ACRONYM(S)
ONR

11. SPONSOR / MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION / AVAILABILITY STATEMENT

Approved for public release; distribution is unlimited.

13. SUPPLEMENTARY NOTES

14. ABSTRACT

This report documents an investigation into the use of range and range rate as an approach for navigating an underwater vessel relative to another vessel for formation maneuvering tasks. Range and range rate between underwater vehicles can be estimated using elapsed time and Doppler shift measurements done by modern acoustic modems. This approach has the potential to allow coordinated team maneuvering using existing vehicle communications without the need for offboard positioning systems, surfacing for GPS fixes, and communicating individual vessel positions periodically throughout the team. The relationship between range rate and Doppler shift is derived and field results from a submarine tracking range are presented to support the hypothesis that this can be an effective measure of relative motion. Equations for range, range rate, and range acceleration are derived and these equations are examined numerically and graphically to determine characteristics that might be useful for relative navigation. The report ends with a summary of potential approaches revealed for using range and range rate measurements for relative navigation.

15. SUBJECT TERMS
Range rate Doppler Underwater navigation

16. SECURITY CLASSIFICATION OF:			**17. LIMITATION OF ABSTRACT**	**18. NUMBER OF PAGES**	**19a. NAME OF RESPONSIBLE PERSON**
					Brian Bourgeois
a. REPORT	**b. ABSTRACT**	**c. THIS PAGE**	UL	37	**19b. TELEPHONE NUMBER** *(include area code)*
Unclassified	Unclassified	Unclassified			(228) 688-5321

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std. Z39.18

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USING RANGE AND RANGE RATE FOR RELATIVE NAVIGATION

1. INTRODUCTION

We are interested in range and range rate between two vessels as readily measurable parameters that may be used to determine the relative position of one underwater vessel with respect to another. In situations where a vessel has poor positioning (using a magnetic compass for heading and propeller revolutions for speed, for example), the growth of position error is so large (greater than 1% of the distance traveled) that dead reckoning provides little useful information for vessel relative navigation. Likewise, errors in true heading and speed over ground measurements, if available, in each vessel may make it difficult for two vessels to assume the same velocity even if the velocity of the other is known. It is proposed in this report that range R and range rate $\dot{R} = dR/dt$ can be used by one vessel to determine the relative position and velocity of another vessel, despite errors in estimation of its own position and velocity. Examples include the following:

- When both $\dot{R}$ and $\ddot{R}$ (range acceleration) are zero then the vessel has matched its velocity to the other vessel and can use its own heading and speed measurements as the base course and speed in spite of deviations of its own measurements from true base course and speed.
- When $\dot{R}$ is a maximum or a minimum (as a vessel makes a slow turn) then the vessel is pointed directly away or towards the other vessel, respectively. Coupled with range, this allows the vessel to determine the relative position of the other vessel with respect to it.
- Range and range rate can also be used for collision avoidance. A constant negative range rate may result in vessel collision, so a small range and constant negative range rate can be used to detect imminent collision with another vessel.

Range is easily and accurately determined between two vessels by measuring the "time of flight" of a signal transmitted between vessels, either round-trip or one-way if clocks are synchronized [1]. Range rate can be measured using successive range measurements over time, or instantaneously by measuring the Doppler frequency shift of the signal transmitted between vessels [2]. Some issues and concerns regarding the ability to accurately measure range and range-rate include the following:

- The ability to have a direct line-of-sight measurement between vessels. This is impacted both by beam patterns of acoustic communication systems and water column properties.
- Vessel steadiness. The Doppler shift sensed by an acoustic modem is due to the relative motion between the modem itself and the other vessel's modem. This motion is composed of both the mean path of the vessel *and* of its motion about that path (yaw, pitch, and roll). If the vessel attitude is not steady then the sensed Doppler shift could be dominated by these motions and not accurately reflect the range rate between the two vessels.

Section 2 briefly reviews and illustrates some of the terminology that will be used throughout this report. Section 3 derives the equations for Doppler shift using the equations of vehicle motion, derives an equation for estimating range rate using the Doppler shift, and presents field test results that indicate the

Manuscript approved April 18, 2007.

Doppler shift can be a reasonable measure of vehicle range rate. In Section 4, equations that are used later in the text are derived from vehicle coordinates and motion for range, range rate, and range acceleration. In Section 5, these equations are analyzed to discern characteristics that can potentially be used for relative navigation. Finally, Section 6 summarizes potential approaches to using range and range rate revealed in this report.

2. TERMINOLOGY

This section delineates common terms that are used in this report and illustrates them in Figs. 1 and 2. Figure 1 depicts range between vessels, *R*, relative velocity ($\underline{v}_r$), the relative motion line (RML), and the closest point of approach (CPA). This figure shows the RML of vessel *v* with respect to vessel *h* that passes behind *h* for the indicated relative velocity between the vessels; similarly, the RML of *h* with respect to *v* would pass in front of *v* and point in the opposite direction. Figure 2 shows the special case where the two vessels are on a collision course and thus the RML of *v* with respect to *h* passes through *h*.

The following terms are used throughout the report:

- Range, *R* – the Euclidean distance between two vessels
- Range rate, $\dot{R}$ – the rate of change of Range, dR/dt
- Range rate rate, $\ddot{R}$, also Range Acceleration – the rate of change of range rate, $d\dot{R}/dt = d^2R/dt^2$
- Relative velocity, $\underline{v}_r = S_r\angle\theta_r$, where S_r is the relative speed and θ_r is the relative heading. Relative velocity describes how one vessel moves relative to a reference vessel. The relative velocity is defined as the vector difference between the vessel and the reference vessel velocities, $\underline{v}_v = S_v\angle\theta_v$ (vessel *v*) and $\underline{v}_h = S_h\angle\theta_h$ (vessel *h*), respectively. The relative velocity of *v* with respect to *h* is given by $\underline{v}_r = \underline{v}_v - \underline{v}_h$ and the relative velocity of *h* with respect to *v* is given by $\underline{v}_r = \underline{v}_h - \underline{v}_v$. The relative velocity vector is shown in the top right of Fig. 1 for the specified velocities of *v* and *h*.
- RML is the relative path upon which the other vessel appears to move and is what would be observed on a radar screen. RML is the dotted line in Fig. 1, and is parallel and in the same direction of the relative velocity vector.
- CPA is that point at which two moving vessels are closest to each other. Before CPA the vessels are closing (negative range rate), at the instant of CPA the range rate is zero, and after CPA the vessels are opening (positive range rate). When the two vessels are traveling at the same course and speed, we have a special case where the vessels are continuously at CPA (range rate and range acceleration are both zero). CPA is illustrated in Fig. 1 for the specified relative velocity and vehicle positions.
- CPA axis – for a fixed relative velocity between two vessels, this is a line through the reference vessel. At the instant a vessel is on this line, it is at CPA with the respect to the reference vessel.
- Collision course – this is the condition where vessels are before CPA and the CPA point has zero range between vessels. For this condition range rate is negative, constant and its magnitude is equal to the magnitude of the relative velocity between the vessels. CBDR (constant bearing, decreasing range) is a commonly used term to describe this condition. A collision course is a special case of the RML and is shown in Fig. 2.
- Anti-collision course – this is the condition where vessels are past CPA but the RML passes through the reference vehicle. For this condition, range rate is positive, constant, and equal to the magnitude of the relative velocity between the vessels.

- Collision axis – for a fixed relative velocity, this is the line through the reference vessel that corresponds to the collision and anti-collision courses. Any vessel that is traveling along this relative motion line with respect to another vessel is on a collision or anti-collision course with the reference vessel.

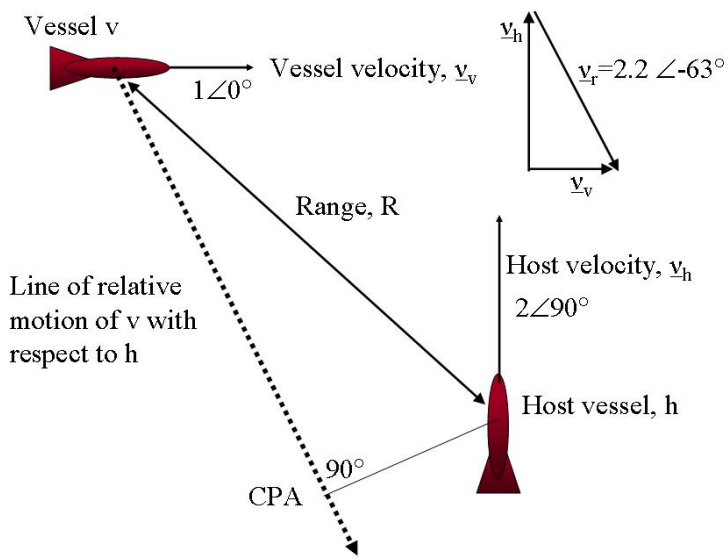

Fig. 1 — Relative motion between vessels

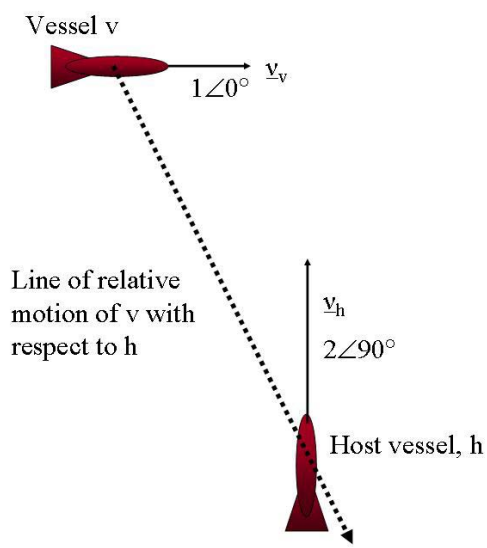

Fig. 2 — Depiction of a collision course

3. DOPPLER SHIFT AND RANGE RATE

In this section we present the equations for Doppler shift between two moving vessels, consider the impact of ocean currents on the shift, derive range rate as a function of Doppler shift, and finally present field measurements that indicate Doppler shift is a reasonable measure of range rate.

3.1 Derivation of Doppler Shift

From Ref. 3, Eq. 15-35a, the frequency received by a receiver *r* from a source *s* is given by

$$f_r = f_s \frac{c \pm u_r}{c \pm u_s}, \tag{1}$$

where f_s is the frequency of the source, c is the speed of sound in the medium, u_r is the speed of the receiver and u_s is the speed of the source. For the numerator the sign is positive if the receiver is moving towards the source and for the denominator the sign is negative when the source is moving towards the receiver (in both cases the result is an increase in f_r). We further note that the speed of the source and receiver used here refers to their relative speed along a line connecting the source and receiver and not to their individual speeds through the water.

Considering the case where the source is moving away from the receiver and the receiver is moving towards the source we have

$$f_r = f_s \frac{c + u_r}{c + u_s} = f_s \frac{1 + u_r/c}{1 + u_s/c} = f_s \left(1 + u_r/c\right)\left(1 + u_s/c\right)^{-1}. \tag{2}$$

We can use the binomial expansion

$$\left(1 + x\right)^n = 1 + nx + \frac{n\left(n-1\right)}{2!} x^2 + \cdots \tag{3}$$

on the denominator, and assuming $u_r << c$, we obtain

$$\begin{aligned} f_r &= f_s \left(1 + u_r/c\right)\left(1 - u_s/c\right) \\ &= f_s \left[1 - u_s/c + u_r/c - u_s u_r/c^2\right] \\ &\approx f_s \left[1 + \frac{u_r - u_s}{c}\right] = f_s \left(1 + \frac{u_{rel}}{c}\right) \end{aligned} \tag{4}$$

where u_{rel} is the relative speed between the source and receiver, which will be positive when they are closing on each other and negative when they are opening. Identical results are obtained by starting with

the source moving toward the receiver and the receiver moving away. Solving for the relative speed we get

$$u_{rel} = c\left(\frac{f_r}{f_s} - 1\right). \tag{5}$$

Noting that the sense of u_{rel} is the opposite of $\dot{R}$ we get range rate in terms of the source frequency, received frequency and speed of sound in water:

$$\dot{R} = c\left(1 - \frac{f_r}{f_s}\right). \tag{6}$$

In the following three examples we will test three assumptions that have been made with this formulation: 1) there is negligible difference in received frequency whether the receiver or the source is moving, 2) the assumption that $u << c$ incurs minimal error, and 3) water current speed has negligible impact.

For the first example we will use a speed of sound in water of 1500 m/s, a source frequency of 15 kHz and a vessel speed of 10 m/s. We note that 1500 m/s is the nominal speed of sound in sea water, and for unmanned underwater vehicles (UUVs), 15 kHz is mid-range for underwater acoustic communications (which typically range from about 7 to 30 kHz) and this speed (20 knots) is very fast. If the source is moving towards a stationary receiver, Eq. (1) gives us

$$f_r = f_s \frac{c \pm u_r}{c \pm u_s} = 15000 \frac{1500 + 0}{1500 - 10} = 15100.67\,. \tag{7}$$

For the receiver moving towards a stationary source we get

$$f_r = f_s \frac{c \pm u_r}{c \pm u_s} = 15000 \frac{1500 + 10}{1500} = 15100\,, \tag{8}$$

showing that we have an error much less than 1%. For the second assumption, we can compare the result of Eq. (7) to the value computed by Eq. (4):

$$f_r = f_s \left[1 + \frac{u_r - u_s}{c}\right] = 15000\left[1 + \frac{10}{1500}\right] = 15100\,, \tag{9}$$

again showing negligible error. Finally, we test the assumption on current speed by including a current speed, u_c, of 2.5 m/s in Eq. (1) and using the previous values:

$$f_r = f_s \frac{\{c + u_c\} \pm u_r}{\{c + u_c\} \pm u_s} = 15000 \frac{1500 + 2.5}{1500 + 2.5 - 10} = 15100.5 \tag{10}$$

showing that the impact is negligible for nominal values.

From a practical application viewpoint we note that while we can expect to know the value of the source and received frequency quite well, the accuracy of the computed value of $\dot{R}$ depends upon the accuracy of the speed of sound. With typical maximum variations of 50 m/s of the speed of sound about the nominal value of 1500 m/s, maximum errors on the order of 3% will be seen for range rate if the nominal speed of sound is assumed. As will be shown in future sections, the absolute accuracy of the measurement may be less valuable than specific transition points (zero, maximum, minimum).

3.2 Field Data Results

In this section we present field data that supports the assertion that acoustic modem Doppler measurements may be used to make a reasonable estimate of a vehicle's range rate. Acoustic modems using broadband signaling schemes to maximize data throughput must be Doppler compensated to prevent incorrect decoding of bit streams in each frequency bin. Consequently, the Doppler shift in the received signal is measured within the modem and is available to be used to compute estimates of range rate. Figure 3 shows the track line of the USS *Pennsylvania* from June 2004 at the NUWC Keyport test range where trials were conducted with multiple platforms to confirm the proper Doppler compensation of communications between the vessels and a fixed modem. The blue line shows the vessel tracks as it made multiple passes by the fixed modem, whose position is shown by the red star.

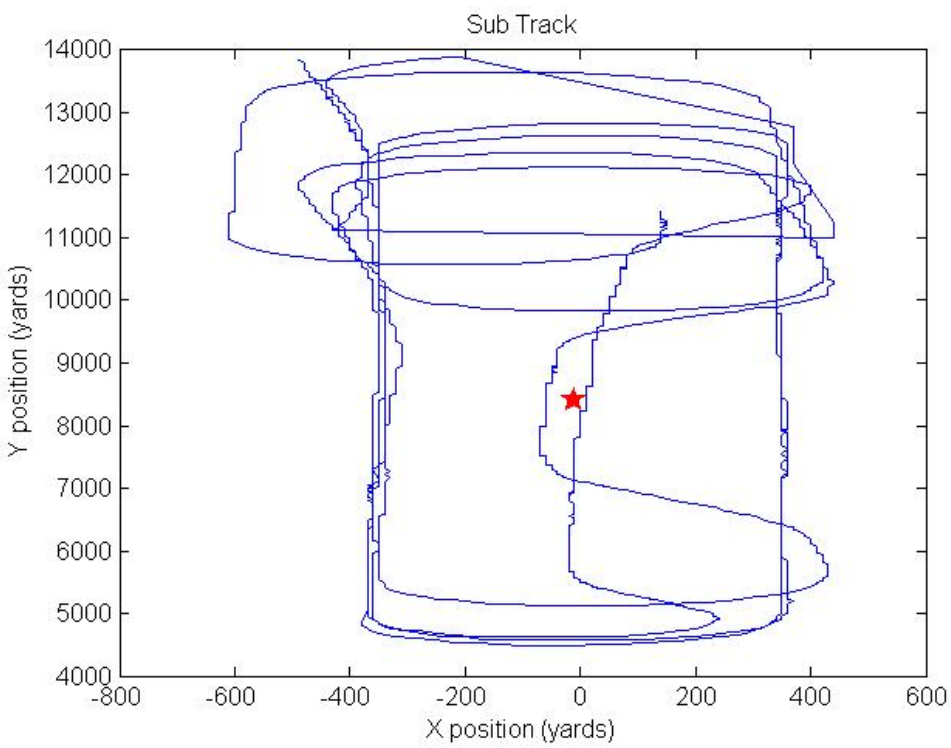

Fig. 3 — USS *Pennsylvania* test range track

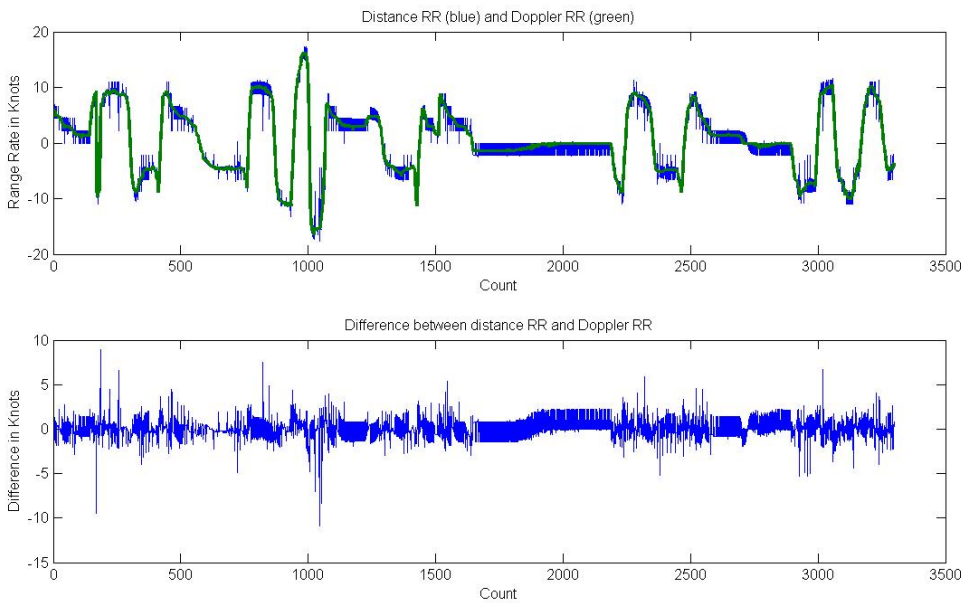

Fig. 4 — Range rate measurements

During these tests, the vessel position and speed was being measured and recorded by the test range, along with the modem derived range rate estimates. The upper plot in Fig. 4 shows the two range rate estimates: 1) computed from the vessel position vs. time and the fixed modem position (blue line) and 2) the acoustic modem estimate made using the Doppler measurement (green line). The lower plot shows the difference between the two estimates, showing fairly good agreement (typically within 1 knot). The Doppler estimated range rate may actually be much more accurate than indicated, as the test range had a range resolution of 10 yards and a sample rate of 8 s/sample. This yields a range rate granularity of 2.2 knots, and this is confirmed in Fig. 5 which shows a close up of the difference between the two measurements. We can see from the green line in Fig. 4 that the Doppler estimated range rate has a much finer resolution than that generated using the test range vessel positions and the jitter of the Doppler range rate data is 0.1 knots. This data supports the assertion that it is feasible to use Doppler shift as an estimate of range rate, at least for a very stable vessel.

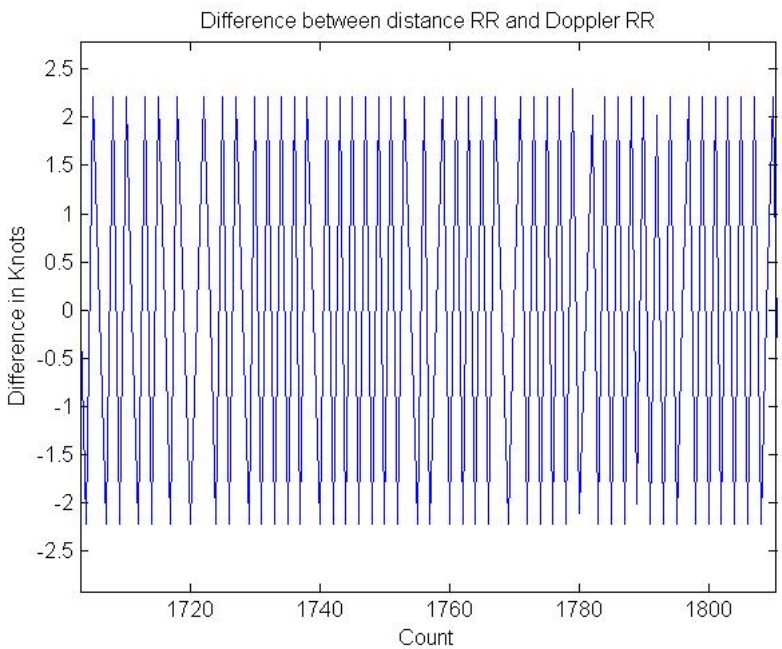

Fig. 5 — Range rate granularity

Figure 6 shows that transitions that occur in range rate as the vessel passes by the fixed modem. The top left plot shows the fixed modem (star) and the vessel track starting at the "X," which is initially heading almost directly away from the fixed modem and then turns and heads back towards it. The top right plot shows the change in range with time, first opening, peaking at the turn, closing until the vessel reaches CPA, and then opening again. The bottom left plot shows the vessel's heading (10° offset added for clarity) first to the south and then to the north. Range rate, estimated from the Doppler shift measured by the vessel's modem, is shown in the bottom right plot. We see that it is initially positive as the vessel is heading away from the fixed modem, passes through zero as the vessel turns and goes through a "transient" CPA point, becomes negative as the vessel heads towards the fixed modem, goes through zero at CPA and then becomes positive again as the distance to the fixed modem increases. This behavior will be explored analytically in the following sections.

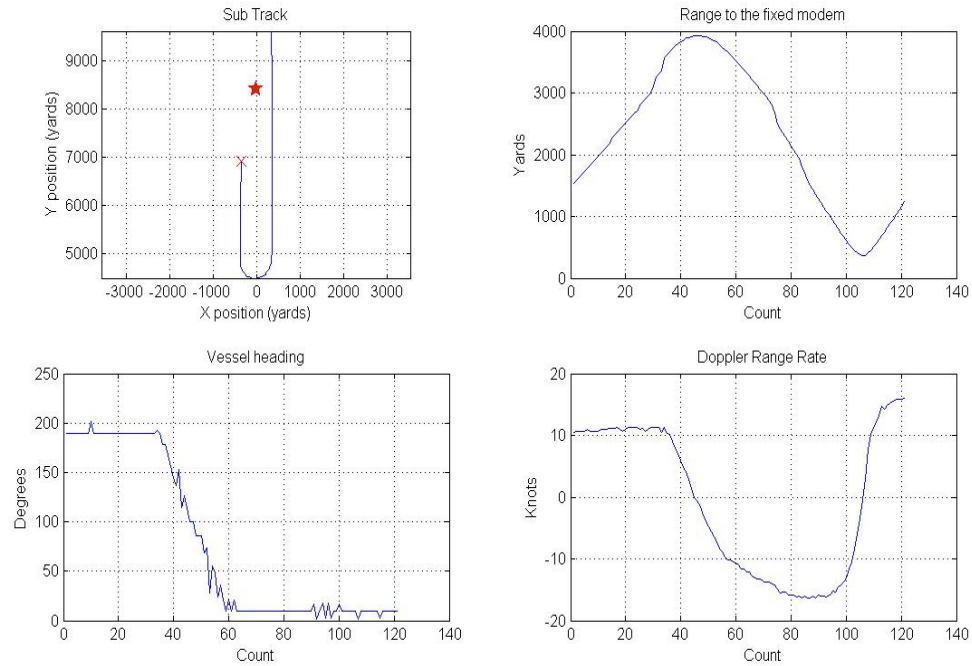

Fig. 6 — Range rate transition

4. DERIVATION OF RANGE RATE AND RANGE ACCELERATION

4.1 Range

In 2-D Cartesian coordinates, range (R) between a vessel (v) and the host vessel (h) is given by

$$R = \left[\left(x_v - x_h \right)^2 + \left(y_v - y_h \right)^2 \right]^{1/2}, \tag{11}$$

where (x_v, y_v) and (x_h, y_h) are changing with time, and R is always positive by definition. Figure 7 shows the nomenclature used for the derivation of range rate and range acceleration. Shown are the range between vessels, angle of v with respect to h, and positions of vessels v and h.

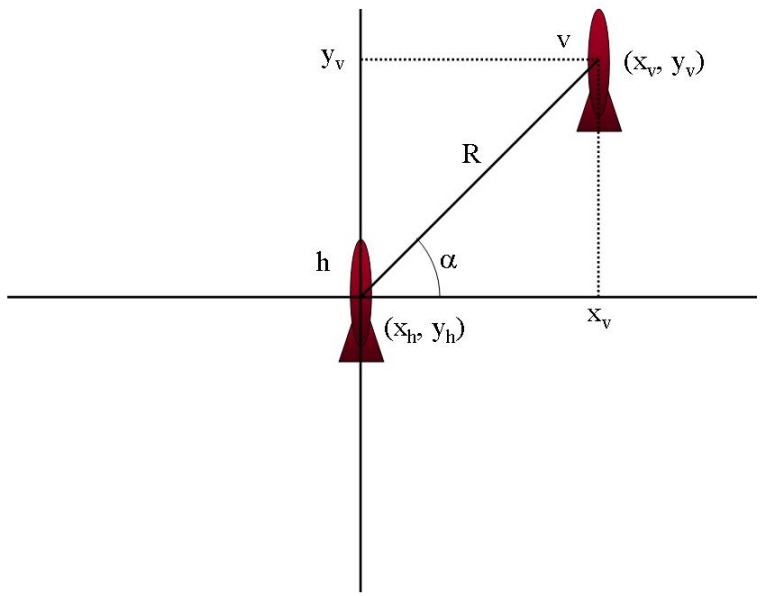

Fig. 7 — Range rate derivation nomenclature

4.2 Range Rate Derived from Range

Range rate $(\dot{R})$ is given by dR/dt

$$\dot{R} = dR/dt = \frac{2\left[(x_v - x_h)(v_v^x - v_h^x) + (y_v - y_h)(v_v^y - v_h^y)\right]}{2\left((x_v - x_h)^2 + (y_v - y_h)^2\right)^{1/2}}, \tag{12}$$

where:

S_v, θ_v are the speed and heading of vessel *v*

$v_v^x = S_v \cos\theta_v$ is the *x* component of the velocity of *v*

$v_v^y = S_v \sin\theta_v$ is the *y* component of the velocity of *v*

S_h, θ_h are the speed and heading of vessel *h*

$v_h^x = S_h \cos\theta_h$ is the *x* component of the velocity of *h*

$v_h^y = S_h \sin\theta_h$ is the *y* component of the velocity of *h*

By noting that the denominator is equal to *R*, Eq. (12) becomes

$$\dot{R} = \frac{\left[(x_v - x_h)(v_v^x - v_h^x) + (y_v - y_h)(v_v^y - v_h^y)\right]}{R}. \tag{13}$$

We can make our coordinate system relative to the host by setting $(x_h, y_h) = (0,0)$, which does not affect the velocity values, and Eq. (13) becomes

$$\dot{R} = \frac{\left[x_v (v_v^x - v_h^x) + y_v (v_v^y - v_h^y) \right]}{R}. \tag{14}$$

The relative velocity between vessels *v* and *h* with respect to *h*, $\underline{v}_r$, is given by

$$\underline{v}_r = \underline{v}_v - \underline{v}_h, \tag{15}$$

or in scalar form by $v_r^x = v_v^x - v_h^x$ and $v_r^y = v_v^y - v_h^y$.

Thus, Eq. (14) can be expressed as

$$\dot{R} = \frac{\left[x_v v_r^x + y_v v_r^y \right]}{R}. \tag{16}$$

Note that $\dot{R}$ is positive or negative depending upon both the relative position and relative velocity. Generalizing Eq. (16), we have $\dot{R} = f(x_v, y_v, \underline{v}_r)$, showing that $\dot{R}$ is a function of the relative position of *v* with respect to *h* and the relative velocity between the two vessels.

4.3 Range Rate Derived from Projection of Velocities

As discussed in Section 3, range rate can also be viewed at a particular instant in time as the sum of the projection of the two vessel velocities onto the line between the two vessels, as shown in Fig. 8. This illustration shows the projections of the velocity vector for vessel *v*, $\underline{v}_v$, and for vessel *h*, $\underline{v}_h$, onto the line between the two vessels.

In Fig. 8, S_h and θ_h are the speed and heading (polar coordinate system) of vessel *h*, which is at position (x_h, y_h), and S_v and θ_v are the speed and heading of vessel *v* which is at position (x_v, y_v). α is the angle of the line between vessel *h* and vessel *v* at the instant in time shown in the figure, which are distance *R* apart.

The projections of $\underline{v}_h$ and $\underline{v}_v$ onto the connecting line (at time *t*) are given respectively by

$$S_h \cos(\theta_h - \alpha)$$
$$S_v \cos(\theta_v - \alpha)$$

where $\cos\alpha = (x_v - x_h)/R$ and $\sin\alpha = (y_v - y_h)/R$.

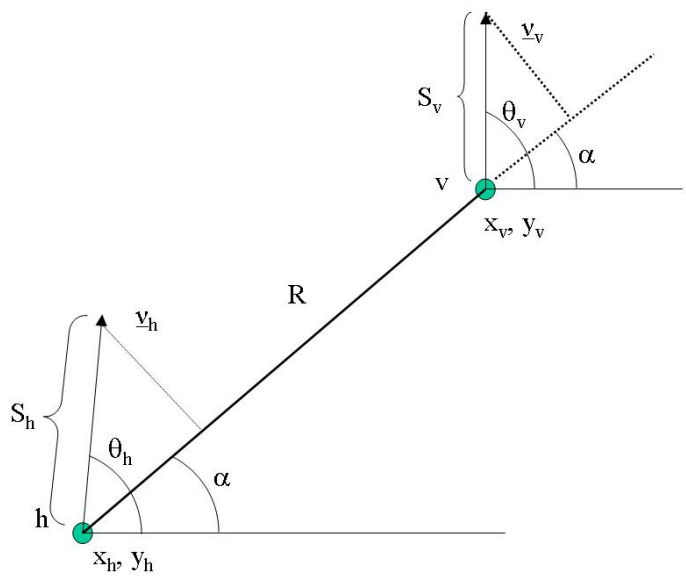

Fig. 8 — Velocity vector projections

Noting that velocity components pointing inward contribute to negative range rate and outward contribute to positive range rate, range rate is given by

$$\dot{R} = -S_h \cos(\theta_h - \alpha) + S_v \cos(\theta_v - \alpha)\,. \tag{17}$$

Using the equality $\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$ and substituting in the values for $\cos\alpha$ and $\sin\alpha$, we get

$$\dot{R} = -S_h\left(\cos\theta_h\left(\frac{x_v - x_h}{R}\right) + \sin\theta_h\left(\frac{y_v - y_h}{R}\right)\right) + S_v\left(\cos\theta_v\left(\frac{x_v - x_h}{R}\right) + \sin\theta_v\left(\frac{y_v - y_h}{R}\right)\right)$$

$$\dot{R} = \left(\frac{x_v - x_h}{R}\right)\left(S_v \cos\theta_v - S_h \cos\theta_h\right) + \left(\frac{y_v - y_h}{R}\right)\left(S_v \sin\theta_v - S_h \sin\theta_h\right).$$

Further substituting in the component values for the vessel velocity components yields

$$\dot{R} = \left(\frac{x_v - x_h}{R}\right)\left(v_v^x - v_h^x\right) + \left(\frac{y_v - y_h}{R}\right)\left(v_v^y - v_h^y\right),$$

or

$$\dot{R} = \left(\frac{x_v - x_h}{R}\right)\left(v_r^x\right) + \left(\frac{y_v - y_h}{R}\right)\left(v_r^y\right).$$

Setting the position of vessel *h* to zero yields

$$\dot{R} = \frac{\left[x_v v_r^x + y_v v_r^y \right]}{R}, \tag{18}$$

which is the same as Eq. (16).

4.4 Range Rate in Terms of Relative Bearing

To determine range rate as a function of the relative bearing of vessel *v* with respect to vessel *h,* Eq. (18) can be rewritten as

$$\dot{R} = \frac{x_v}{R}\left(v_r^x + v_r^y \frac{y_v}{x_v} \right) = \frac{x_v}{R}\left(v_r^x + v_r^y \tan\alpha \right) \tag{19}$$

since $y_v / x_v = \tan\alpha$. We can rewrite x_v / R as

$$\frac{x_v}{R} = \frac{x_v}{\sqrt{x_v^2 + y_v^2}} = \frac{x_v}{\sqrt{x_v^2 + x_v^2 \tan^2\alpha}} \tag{20}$$

since $y_v = x_v \tan\alpha$. Equation (20) can then be simplified to

$$\frac{x_v}{R} = \frac{x_v}{x_v\sqrt{1+\tan^2\alpha}} = \frac{1}{\sqrt{1+\tan^2\alpha}} = \cos\alpha \tag{21}$$

since $\sec^2\alpha = 1 + \tan^2\alpha$. Merging this with Eq. (19) yields:

$$\dot{R} = \cos\alpha\left(v_r^x + v_r^y \tan\alpha \right) = v_r^x \cos\alpha + v_r^y \sin\alpha, \tag{22}$$

showing that for a fixed instant in time, *t*, $\dot{R}$ is strictly a function of α and the relative velocity (v_r^x, v_r^y) and that it is independent of the distance between *v* and *h*. We note that Eq. (22) is the equation for a circle and substituting in the equations for v_r^x, v_r^y we have

$$\begin{aligned} \dot{R} &= \cos\alpha S_r \cos\theta_r + \sin\alpha S_r \sin\theta_r \\ &= S_r\left(\cos\alpha\cos\theta_r + \sin\alpha\sin\theta_r \right). \end{aligned} \tag{23}$$

But given the identity $\cos(\gamma - \beta) = \cos\gamma\cos\beta + \sin\gamma\sin\beta$, Eq. (22) becomes

$$\dot{R} = S_r \cos(\alpha - \theta_r). \tag{24}$$

From Eq. (24), we see that $\dot{R}$ is positive or negative depending upon the values of α and θ_r. In contrast to Eq. (16), we see that range rate is actually independent of the range between the vessels, and only a function of relative velocity and the bearing of *v* with respect to *h*.

4.5 Range Rate Rate (Acceleration)

From Eq. (13) we have

$$\dot{R} = \frac{\left[(x_v - x_h)(v_v^x - v_h^x) + (y_v - y_h)(v_v^y - v_h^y)\right]}{R}.$$

We are assuming that the vessel velocities are constant (derivative with respect to time is zero) so we can simplify this equation using Eq. (15)

$$\dot{R} = \left[(x_v - x_h)v_r^x + (y_v - y_h)v_r^y\right]R^{-1}.$$

Taking the derivative with respect to time yields

$$\begin{aligned} \frac{d\dot{R}}{dt} &= R^{-1}\frac{d\left[(x_v - x_h)v_r^x + (y_v - y_h)v_r^y\right]}{dt} + \left[(x_v - x_h)v_r^x + (y_v - y_h)v_r^y\right]\frac{dR^{-1}}{dt} \\ \frac{d\dot{R}}{dt} &= R^{-1}\frac{d\Phi}{dt} + \Phi\frac{dR^{-1}}{dt} \end{aligned}, \tag{25}$$

where $\Phi = \left[(x_v - x_h)v_r^x + (y_v - y_h)v_r^y\right]$.

Taking the derivative of Φ gives

$$\begin{aligned} \frac{d}{dt}\left[(x_v - x_h)v_r^x + (y_v - y_h)v_r^y\right] &= \left[\left(\frac{dx_v}{dt} - \frac{dx_h}{dt}\right)v_r^x + \left(\frac{dy_v}{dt} - \frac{dy_h}{dt}\right)v_r^y\right] \\ \frac{d\Phi}{dt} &= \left[\left(v_r^x\right)^2 + \left(v_r^y\right)^2\right] = S_r^2 \end{aligned} \tag{26}$$

where S_r is the relative speed between the vessels. The derivative of R^{-1} yields

$$\begin{aligned} \frac{dR^{-1}}{dt} &= -R^{-2}\left(\tfrac{1}{2}R^{-1}\right)\left[2(x_v - x_h)(v_v^x - v_h^x) + 2(y_v - y_h)(v_v^y - v_h^y)\right] \\ &= -R^{-3}\left[(x_v - x_h)v_r^x + (y_v - y_h)v_r^y\right] \\ &= -R^{-3}\Phi. \end{aligned} \tag{27}$$

Combining Eqs. (25) through (27) yields

$$\ddot{R} = \frac{\left(v_r^x\right)^2 + \left(v_r^y\right)^2}{R} - \frac{\left[(x_v - x_h)v_r^x + (y_v - y_h)v_r^y\right]^2}{R^3}. \tag{28}$$

We can make our coordinate system relative to the host by setting $(x_h, y_h) = (0,0)$, which gives range acceleration in terms of the range between *v* and *h*, the relative position of *v* with respect to *h*, and the relative velocity:

$$\ddot{R} = \frac{\left(v_r^x\right)^2 + \left(v_r^y\right)^2}{R} - \frac{\left[x_v v_r^x + y_v v_r^y\right]^2}{R^3}. \tag{29}$$

Both fractions in Eq. (29) are necessarily always positive, so to determine the sign of $\ddot{R}$ we need to know the relative magnitude of each fraction. Analyzing the sign of Eq. (29) further and setting $A = v_r^x$ and $B = v_r^y$ we can rewrite it as follows:

$$\ddot{R} = \frac{R^2A^2 + R^2B^2 - \left[x_v^2A^2 + y_v^2B^2 + x_v y_v AB\right]}{R^3} \tag{30}$$

Rearranging terms gives

$$\ddot{R} = \frac{\left(R^2 - x_v^2\right)A^2 + \left(R^2 - y_v^2\right)B^2 - x_v y_v AB}{R^3} \tag{31}$$

but $R^2 = x_v^2 + y_v^2$, so Eq. (31) becomes

$$\ddot{R} = \frac{y_v^2A^2 + x_v^2B^2 - x_v y_v AB}{R^3}. \tag{32}$$

However,

$$\left(y_v A - x_v B\right)^2 = y_v^2A^2 + x_v^2B^2 - 2x_v y_v AB \tag{33}$$

is always positive and must be less than the numerator in Eq. (32), hence Eq. (29) must always be positive. The meaning of this can be illustrated as follows: as *v* is approaching CPA with *h*, its range rate must be negative, but will pass through zero at CPA and go positive after CPA; consequently, range acceleration must be positive. Note that this uses the assumption made above for the derivation of Eq. (29) that the relative velocity is constant (vessel velocities are constant), so we would necessarily see negative range acceleration occur only with changing vessel velocities.

5. RANGE RATE AND RANGE ACCELERATION CHARACTERISTICS

5.1 Zero Range Rate

From Eqs. (18) and (24), it is evident that range rate is zero when the velocity of the two vessels is matched. This is a stable condition since $\ddot{R} = 0$. For $\ddot{R} \neq 0$, we have a transient condition that occurs only when the vessels pass through CPA. Setting Eq. (18) to zero yields

$$\begin{aligned} 0 &= x_v v_r^x + y_v v_r^y \\ x_v v_r^x &= -y_v v_r^y \\ \frac{x_v}{y_v} &= -\frac{v_r^y}{v_r^x} \end{aligned}$$

Note that $\frac{x_v}{y_v}$ is the cotangent of the angle α of vessel *v* with respect to vessel *h*. Note also that $\frac{v_r^y}{v_r^x}$ is the tangent of the angle of the relative velocity vector, θ_r. Since $-\tan(\alpha) = \cot(\alpha - 90)$ we have

$$\cot\alpha = -\tan\theta_r = \cot(\theta_r - 90)\,, \tag{34}$$

and since $\cot\gamma = \cot\left(\gamma \pm 180\right)$, we have

$$\cot\alpha = \cot\left(\theta_r \pm 90\right). \tag{35}$$

Equation (35) describes the condition where the line between the vessels is perpendicular to the relative velocity vector, which is by definition the CPA point. It is worthwhile to note that when vessel velocities are matched that the vessels are continuously at CPA. The same result is obtained using Eq. (24) since the cosine is only equal to zero for $\alpha - \theta_r = \pm 90^\circ$.

5.2 Zero Range Acceleration

From Eq. (28), it is evident that $\ddot{R} = 0$ occurs when vessel velocities are matched (relative velocity is zero). To examine other conditions for which $\ddot{R} = 0$, we set $\ddot{R}$ to zero in Eq. (28) and multiply through by R^3 yielding

$$R^2\left[\left(v_r^x\right)^2 + \left(v_r^y\right)^2\right] = \left[(x_v - x_h)v_r^x + (y_v - y_h)v_r^y\right]^2. \tag{36}$$

Expanding the left side gives

$$\begin{aligned} R^2\left[\left(v_r^x\right)^2 + \left(v_r^y\right)^2\right] &= \left(\left(x_v - x_h\right)^2 + \left(y_v - y_h\right)^2\right)\left(\left(v_r^x\right)^2 + \left(v_r^y\right)^2\right) \\ &= \left(x_v - x_h\right)^2\left(v_r^x\right)^2 + \left(y_v - y_h\right)^2\left(v_r^x\right)^2 + \left(x_v - x_h\right)^2\left(v_r^y\right)^2 + \left(y_v - y_h\right)^2\left(v_r^y\right)^2.^2 \end{aligned} \tag{37}$$

Expanding the right side yields:

$$\left(x_v - x_h\right)^2 \left(v_r^x\right)^2 + \left(y_v - y_h\right)^2 \left(v_r^y\right)^2 + 2\left(x_v - x_h\right)\left(y_v - y_h\right) v_r^x v_r^y . \tag{38}$$

Combining Eqs. (37) and (38) and eliminating common terms gives

$$\left(y_v - y_h\right)^2 \left(v_r^x\right)^2 + \left(x_v - x_h\right)^2 \left(v_r^y\right)^2 = 2\left(x_v - x_h\right)\left(y_v - y_h\right) v_r^x v_r^y . \tag{39}$$

By making the substitutions

$$A = x_v - x_h, B = y_v - y_h, C = v_r^x, \text{and } D = v_r^y ,$$

Eq. (39) becomes

$$B^2C^2 + A^2D^2 = 2ABCD . \tag{40}$$

But we note that

$$(AD - BC)^2 = B^2C^2 + A^2D^2 - 2ABCD .$$

And substituting this result into Eq. (40) yields

$$(AD - BC)^2 + 2ABCD = 2ABCD . \tag{41}$$

Thus $(AD - BC)^2$ must be equal to zero and $AD = BC$ for range acceleration equal to zero with the relative velocity constant (the initial assumption). This yields the equation

$$(x_v - x_h)v_r^y = (y_v - y_h)v_r^x , \tag{42}$$

which can be rewritten as

$$\frac{(y_v - y_h)}{(x_v - x_h)} = \frac{v_r^y}{v_r^x} . \tag{43}$$

For a fixed relative velocity the right side of Eq. (43) is a constant and if we assume vessel *h*'s position is the origin, we have the equation of a line:

$$y_v = \frac{v_r^y}{v_r^x} x_v = \theta_r x_v . \tag{44}$$

Equation (44) defines the collision axis, i.e., the condition when vessel *v* is traveling along a path parallel to the relative velocity vector and through vessel *h*'s position. For this condition, range rate is also constant, non-zero and equal to the magnitude of the relative velocity. We will see in the following

section that for large R, $\ddot{R}$ may be measurably zero, which could make it difficult to distinguish a collision course at long ranges.

5.3 Limit of Range Rate as *R* Goes to Infinity

We note that $R \to \infty$ is equivalent to $t \to \pm\infty$, so we substitute Eq. (11) into Eq. (18) to give

$$\dot{R} = \frac{\left[x_v v_r^x + y_v v_r^y\right]}{\left[x_v^{\ 2} + y_v^{\ 2}\right]^{1/2}}. \tag{45}$$

However, $x_v = v_r^x t + x_0$ and $y_v = v_r^y t + y_0$, where position of v at $t = 0$ is (x_0, y_0). Substituting into Eq. (45) gives

$$\dot{R} = \frac{\left[(v_r^x t + x_0)v_r^x + (v_r^y t + y_0)v_r^y\right]}{\left[(v_r^x t + x_0)^2 + (v_r^y t + y_0)^2\right]^{1/2}}. \tag{46}$$

As $t \to \pm\infty$ x_0, y_0 become insignificant, so Eq. (46) becomes

$$\dot{R} = \frac{\left[(v_r^x t)v_r^x + (v_r^y t)v_r^y\right]}{\left[(v_r^x t)^2 + (v_r^y t)^2\right]^{1/2}}. \tag{47}$$

Factoring out t and taking the limit, we have

$$\lim_{t\to\infty} \dot{R} = \frac{t\left[(v_r^x)^2 + (v_r^y)^2\right]}{t\left[(v_r^x)^2 + (v_r^y)^2\right]^{1/2}} = \pm\left[(v_r^x)^2 + (v_r^y)^2\right]^{1/2}, \tag{48}$$

or

$$\lim_{R\to\infty} \dot{R} = \pm S_r. \tag{49}$$

Equation (49) tells us that, for a fixed relative velocity, as R becomes large, the magnitude of range rate asymptotically approaches the relative speed between the vessels. For this condition, $\ddot{R}$ does not actually reach zero, but becomes so small as range increases that it is measurably zero.

5.4 Range Rate Terminal Angle

From Eq. (24) we have $\dot{R}$ in terms of α and the relative velocity between vessels:

$$\dot{R} = S_r \cos(\alpha - \theta_r).$$

Since this equation is not a function of R, $\dot{R}$ goes to its maximum and minimum value not as $R \to \infty$, but as α reaches its terminal angle. Computing the derivative of $\dot{R}$ with respect to α yields

$$\frac{\partial \dot{R}}{\partial \alpha} = -S_r \sin\left(\alpha - \theta_r\right). \tag{50}$$

To determine the terminal value of α, we start with the following equation:

$$\tan\alpha = \frac{y_v}{x_v}.$$

Substituting for (x_v, y_v) we have

$$\tan\alpha = \frac{v_r^y t + y_v(0)}{v_r^x t + x_v(0)}.$$

Taking the limit yields

$$\lim_{t\to\pm\infty} \tan\alpha = \lim_{t\to\pm\infty} \frac{v_r^y t + y_v(0)}{v_r^x t + x_v(0)} = \lim_{t\to\pm\infty} \frac{v_r^y t}{v_r^x t} = \frac{v_r^y}{v_r^x} = \tan\theta_r, \tag{51}$$

indicating that the terminal angle for α is the same as the relative velocity angle. Since $\tan\gamma = \tan(\gamma - 180)$, $\alpha = \theta_r, \theta_r - 180$ is the limit as $t \to \pm\infty$.

5.5 Range at Terminal Angle

For a given relative velocity, we can make a piecewise linear approximation of range rate using the slope of the range rate curve at CPA given by Eq. (57) in the next section. Assuming t = 0 at CPA we have

$$\begin{aligned} \dot{R}(t) &= -S_r; t < -t_T \\ \dot{R}(t) &= S_r; t > t_T \\ \dot{R}(t) &= \frac{S_r^2}{R_{cpa}} t; -t_T \le t \le t_T \end{aligned} \tag{52}$$

where t_T is the time at which range rate reaches its terminal angle. Beyond the terminal angle we know that $\left|\dot{R}\right| = S_r$, so we can solve Eq. (52) for t_T:

$$\begin{aligned} S_r &= \frac{S_r^2}{R_{cpa}} t_T \\ t_T &= \frac{R_{cpa}}{S_r}. \end{aligned} \tag{53}$$

Using the approximation of range rate in Eq. (52), we can also make a piecewise linear approximation for range:

$$
\begin{aligned}
R(t) &= R_T - S_r t; t < -t_T \\
R(t) &= R_T + S_r t; t > t_T \\
R(t) &= R_{cpa} + \frac{S_r^2}{R_{cpa}} t^2; -t_T \le t \le t_T
\end{aligned}
\tag{54}
$$

where R_T is the range at the terminal angle. Substituting Eqs. (53) into (54) gives

$$
\begin{aligned}
R_T &= R(t_T) = R_{cpa} + \frac{S_r^2}{R_{cpa}} \frac{R_{cpa}^2}{S_r^2} \\
R_T &= 2R_{cpa},
\end{aligned}
\tag{55}
$$

showing that the vessel reaches its terminal angle at a range approximately twice the range at CPA. Due to the curvature of the range rate function, the actual range at the terminal angle is slightly larger, but Eq. (55) gives a fairly decent approximation since range acceleration is very small near the terminal angle.

The utility of this relationship is that (assuming constant vessel velocities, as we might see with one vessel joining a formation) range rate will be nearly constant and equal to the relative speed at long ranges, and when we start to see an appreciable change in range rate we can estimate the range at CPA using Eq. (55). Given that and our estimate of relative speed, we can then use Eq. (53) to estimate when CPA will be reached.

5.6 Behavior about CPA

5.6.1 CPA Examples

The following two examples serve to illustrate some of the relationships developed above. Figure 9 shows the behavior of $R, \dot{R}, \ddot{R}, \alpha$ as vessel *v* passes *h* with a relative velocity of 7.1 m/s $\angle$ 45° and a CPA of 71 m. This figure illustrates that for a small CPA the range rate is nearly constant until the point of CPA and passes through zero very quickly. For this example, *v* starts in the third quadrant relative to *h* and passes just below it at CPA. Initially, *v* is far from *h* and $\dot{R}$ is negative, constant, and equal in magnitude to the relative speed. Also, $\ddot{R}$ is measurably zero. As *v* approaches CPA with *h,* we see both range rate and range acceleration increase. At CPA, $\dot{R}$ passes through zero and $\ddot{R}$ reaches its maximum. This behavior is consistent with the conclusion earlier that for a fixed relative velocity the range acceleration must be always positive.

Since the CPA range is small in this example the slope of $\dot{R}$ is large at CPA. The green curve in the top right plot shows the piecewise linear approximation of range rate, using the estimate that the terminal angle is reached at a range twice that of the CPA range. The approximation is seen to be fairly accurate for a small CPA range. The horizontal green line seen in the bottom right plot shows that the angle from *h* to *v* at CPA is –45°, which is perpendicular to the relative velocity direction as expected. As *v* passes beyond CPA, $\dot{R}$ becomes positive, and as *R* becomes large, $\dot{R}$ asymptotically approaches the magnitude of the relative speed. We also see in the bottom right plot that as *R* becomes large, the angle of *v* from *h* asymptotically approaches the angle of the relative velocity vector and its reciprocal (45°, –135°).

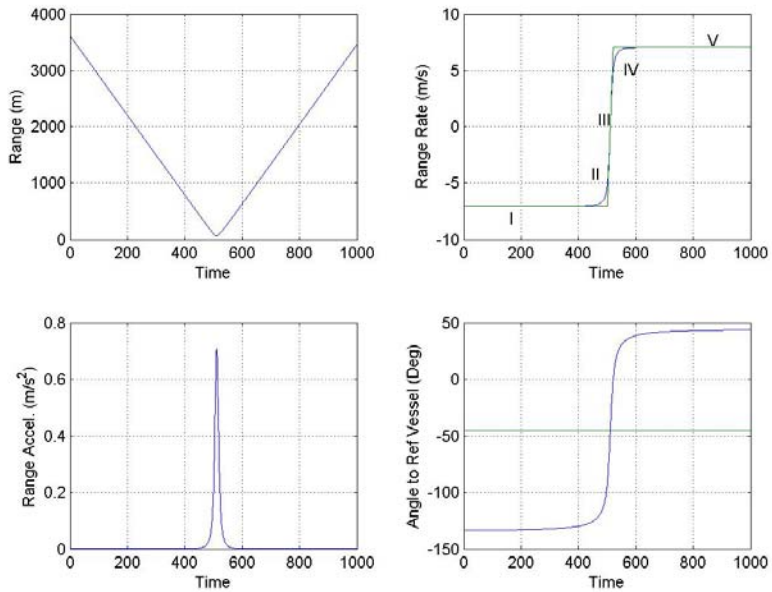

Fig. 9 — Range rate for a 71-m CPA

Figure 10 shows an example where the CPA range is much larger, resulting in a much shallower slope of $\dot{R}$ at CPA. This figure illustrates that for a large CPA, the range rate changes relatively gradually but still exponentially approaches a terminal value at each range extreme. As in Fig. 9, the top right graph in Fig. 10 shows the piecewise linear approximation to range rate. While the approximation is not as accurate as it was for the example with a small CPA, it still provides a useful metric.

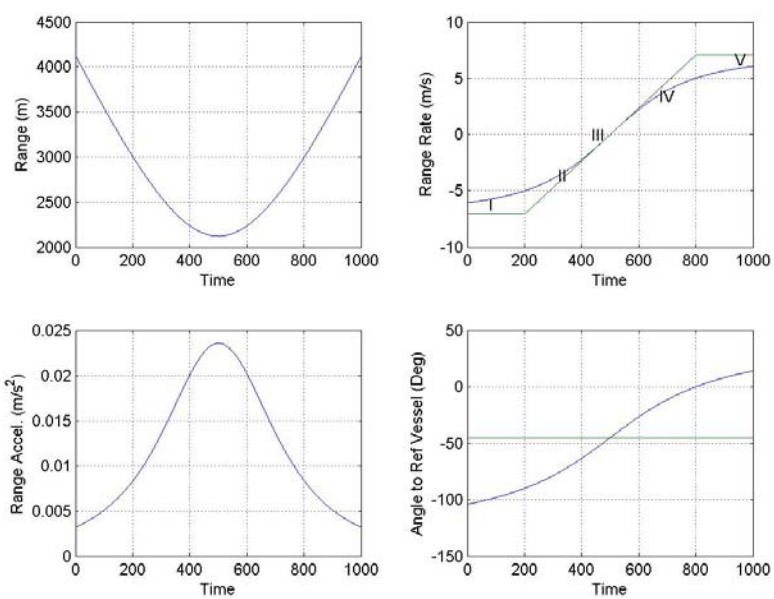

Fig. 10 — Range rate for a 2121-m CPA

5.6.2 Slope of Range Rate at CPA

Equation (45) reveals that range rate is a sigmoid function, and as will be shown in the following, its slope at the inflection point ($\dot{R} = 0$) is dependent upon the range between the two vessels at CPA. For a collision course (CPA range = 0) the curve is vertical at the inflection point. Rewriting Eq. (29) in terms of $\dot{R}$ gives us

$$\ddot{R} = \frac{\left(v_r^x\right)^2 + \left(v_r^y\right)^2}{R} - \frac{1}{R}\dot{R}^2 . \tag{56}$$

The slope of $\dot{R}$ at the inflection point is given by Eq. (56) for $\dot{R} = 0$, which yields

$$\ddot{R}_{CPA} = \frac{\left(v_r^x\right)^2 + \left(v_r^y\right)^2}{R_{CPA}} = \frac{S_r^2}{R_{CPA}} . \tag{57}$$

Equation (57) shows that the slope of range rate at its inflection point is a function of both the range at CPA and the relative speed between vessels. The slope increases with the square of the relative speed, and decreases with an increase of the range at CPA. It also confirms that the curve is vertical for $R_{cpa} = 0$. The angle of the $\dot{R}$ curve at CPA is given by

$$\tan^{-1} \frac{S_r^2}{R_{CPA}} , \tag{58}$$

where $R_{cpa} = \left(x_{cpa}^2 + y_{cpa}^2\right)^{1/2}$.

5.6.3 Position and Bearing at CPA

To compute (x_{cpa}, y_{cpa}), which are illustrated in Fig. 11, we note the following:

$$\tan\theta_r = \frac{y_{cpa} - y_o}{x_{cpa} - x_o}, \tan\beta = \frac{y_{cpa}}{x_{cpa}} ,$$

and at CPA $\theta_r = \beta \pm 90^\circ$. Given that x_o, y_o, θ_r are known, we can then compute y_{cpa} using

$$y_{cpa} = x_{cpa} \tan\beta . \tag{59}$$

From the equations above we have

$$\tan\theta_r \left(x_{cpa} - x_o\right) = \left(y_{cpa} - y_o\right) = x_{cpa} \tan\beta - y_o ,$$

and solving this for x_{cpa} yields

$$x_{cpa} = \frac{\left(x_o \tan\theta_r - y_o\right)}{\left(\tan\theta_r - \tan\beta\right)}, \tag{60}$$

which allows determination of the coordinates knowing the relative velocity and an initial position relative to the host vessel.

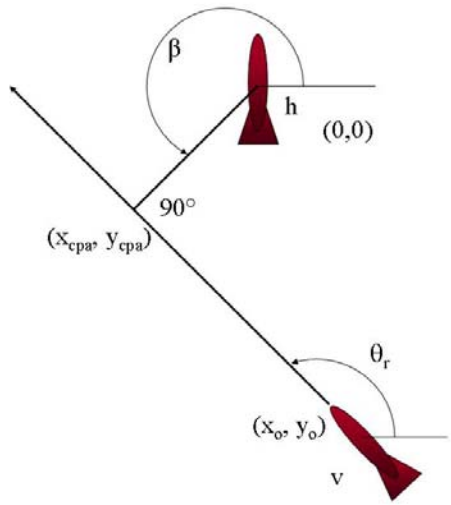

Fig. 11 — CPA computation nomenclature

5.6.4 Range Acceleration at CPA

Since $\dot{R}$ has an inflection point at CPA, we expect that range acceleration will be maximum at this point. To prove this, we solve for the derivative of range acceleration with respect to time and evaluate it at the CPA point. From before we have

$$\ddot{R} = \frac{1}{R}\left(\left(v_r^x\right)^2 + \left(v_r^y\right)^2 - \dot{R}^2\right).$$

Solving for the derivative of range acceleration,

$$\dddot{R} = \frac{d\ddot{R}}{dt} = \frac{dR^{-1}}{dt}\left(\left(v_r^x\right)^2 + \left(v_r^y\right)^2 - \dot{R}^2\right) + \frac{1}{R}\frac{d}{dt}\left(\left(v_r^x\right)^2 + \left(v_r^y\right)^2 - \dot{R}^2\right). \tag{61}$$

From before we have

$$\frac{dR^{-1}}{dt} = -R^{-3}\left(x_v v_r^x + y_v v_r^y\right) = \frac{-\dot{R}}{R^2}, \tag{62}$$

so the first part of Eq. (61) is

$$\frac{-\dot{R}}{R^2}\left(\left(v_r^x\right)^2+\left(v_r^y\right)^2-\dot{R}^2\right). \tag{63}$$

Taking the derivative in the second part of Eq. (61) gives

$$\begin{aligned}\frac{d}{dt}\left(\left(v_r^x\right)^2+\left(v_r^y\right)^2-\dot{R}^2\right)&=-2\dot{R}\frac{d\dot{R}}{dt}=-2\dot{R}\ddot{R}\\&=\frac{-2\dot{R}}{R}\left(\left(v_r^x\right)^2+\left(v_r^y\right)^2-\dot{R}^2\right)\end{aligned}. \tag{64}$$

Combining Eqs. (61), (63), and (64) yields

$$\begin{aligned}\dddot{R}=\frac{d\ddot{R}}{dt}&=\frac{-\dot{R}}{R^2}\left(\left(v_r^x\right)^2+\left(v_r^y\right)^2-\dot{R}^2\right)-\frac{2\dot{R}}{R^2}\left(\left(v_r^x\right)^2+\left(v_r^y\right)^2-\dot{R}^2\right)\\&=\frac{-3\dot{R}}{R^2}\left(\left(v_r^x\right)^2+\left(v_r^y\right)^2-\dot{R}^2\right)\end{aligned}, \tag{65}$$

which is zero at CPA since $\dot{R}=0$. q.e.d.

5.7 Range Rate as a Function of *v* Bearing (α)

Figure 12 uses Eq. (24), $\dot{R}=S_r\cos(\alpha-\theta_r)$, to plot $\dot{R}$ as a function of α for a fixed relative velocity and illustrates how range rate behaves as a function of bearing. Equation (44) is used to define the collision axis and, given the fact that Eq. (24) is zero only when $\alpha=\theta_r\pm 90^\circ$, gives us the CPA axis. In this figure, vessel *h* has a speed of 12 m/s and a heading of 45°, while vessel *v* has a speed of 11 m/s, and a heading of 30°, yielding a relative velocity of 3.2 m/s ∠−71°. The direction of the relative velocity vector for this example is indicated by the red circle shown in the top two plots. The graph at the top left shows the behavior of range rate with respect to the bearing of vessel *v* from vessel *h,* i.e., we are holding vessel *h* still and moving vessel *v*'s position around *h,* and we note as before that this behavior is independent of the range between the two vessels. This curve forms a cardioid since (as seen in the bottom left plot) $\dot{R}$ takes on negative values for a range of bearings. The largest range rate is seen to be of 3.2 m/s ∠−71°, corresponding to the position of *v* that puts it along a line of motion through *h,* i.e., the anti-collision course. This is shown in the graph on the bottom left, which indicates bearings of vessel *v* corresponding to the collision and anti-collision directions. The graph at the top right shows the absolute value of the range rate and shows more clearly the collision (109°) and anti-collision course (−71°). It also shows both the collision axis and the CPA axis, the latter showing the two relative bearings (19° and 199°) corresponding to the line of instantaneous CPA points upon which *v* would be at CPA with respect to *h* (for the specified relative velocity). The partial derivative of range rate with respect to bearing is shown on the bottom right, and we see that the two CPA bearings correspond to the maximum and minimum (range rate passes through zero) as would be expected and the collision and anti-collision courses correspond to the zero crossings (range rate is maximum and minimum).

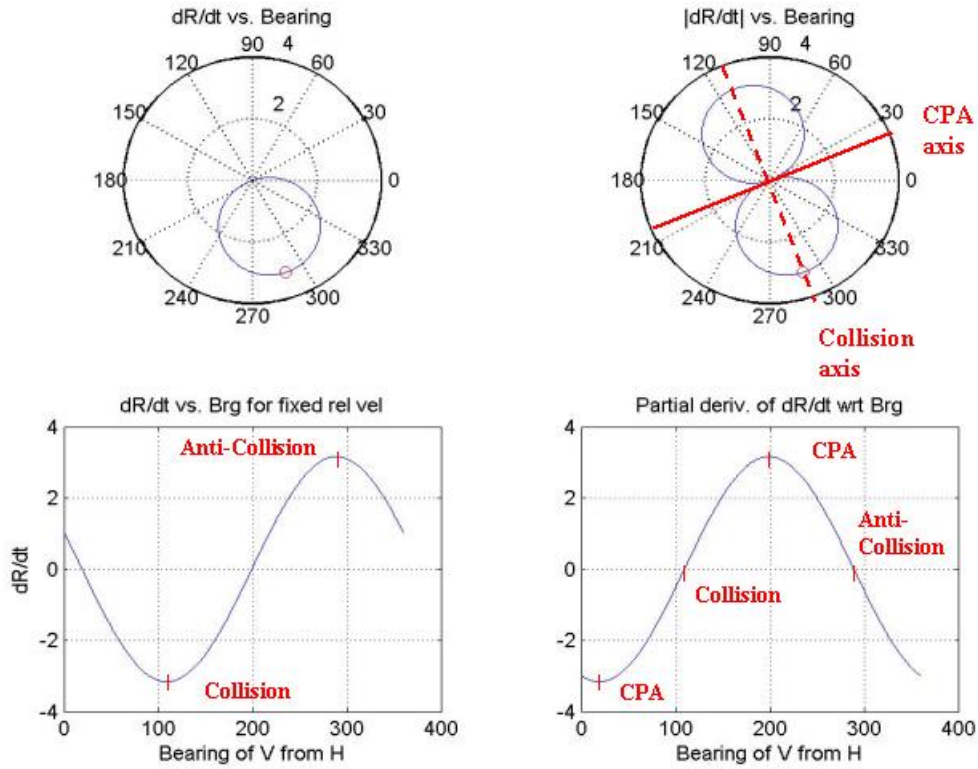

Fig. 12 — Range rate as a function of bearing

Additional observations that can be made from Fig. 12 include the following:

- $\dot{R}$ is symmetrical about both the collision axis and the CPA axis.
- Changes in relative velocity between the two vessels rotates the cardioid about the origin
- Due to the shape of the curve near the collision axis, a change in α results in only a small change in $\dot{R}$. As an example, let us assume a variation of ± 10° in α about the collision axis. For the stated relative velocity of 3.2 m/s $\angle -71°$, the range rate at these angles will be 3.151 m/s, only a 0.05 m/s variation. Recall that α approaches the relative velocity bearing θ_r in the limit, so this corresponds to conditions where $\dot{R} \to S_r$ for large *R*.
- With the exception of the maximum and minimum values of $\dot{R}$, Fig. 12 shows that there are two possible bearings for every value of $\dot{R}$, symmetrical about the collision axis. This illustrates the significance of Eq. (24), in that if we know the relative velocity between *v* and *h* and can measure the value of $\dot{R}$, we can compute the two possible values of α. Potentially, a maneuver by vessel *v* could be used to determine which of the two angles is correct.

5.8 Range Rate as a Function of *v* Position

Figure 13 was generated using Eq. (45) and shows $\dot{R}$ plotted as a function of x_v, y_v for a fixed relative velocity between *v* and *h*. In this figure we have a moving coordinate system whose origin is

vessel *h* and three "test" positions for vessel *v* are shown. In this example, we have the simple case where the two vessel headings are the same and vessel *v* is moving faster. As can be seen from the figure, the minimum $\dot{R}$ (maximum closing rate) occurs when *v* is directly behind *h* and on a collision course, and the maximum $\dot{R}$ (maximum opening rate) occurs when *v* is directly ahead of *h* and on an anti-collision course. It is further seen that $\dot{R}$ becomes zero along the line y = 0, which corresponds to the CPA axis (note that the area in the center of the plot where *R* is close to zero has been clipped due to the discontinuity in $\dot{R}$).

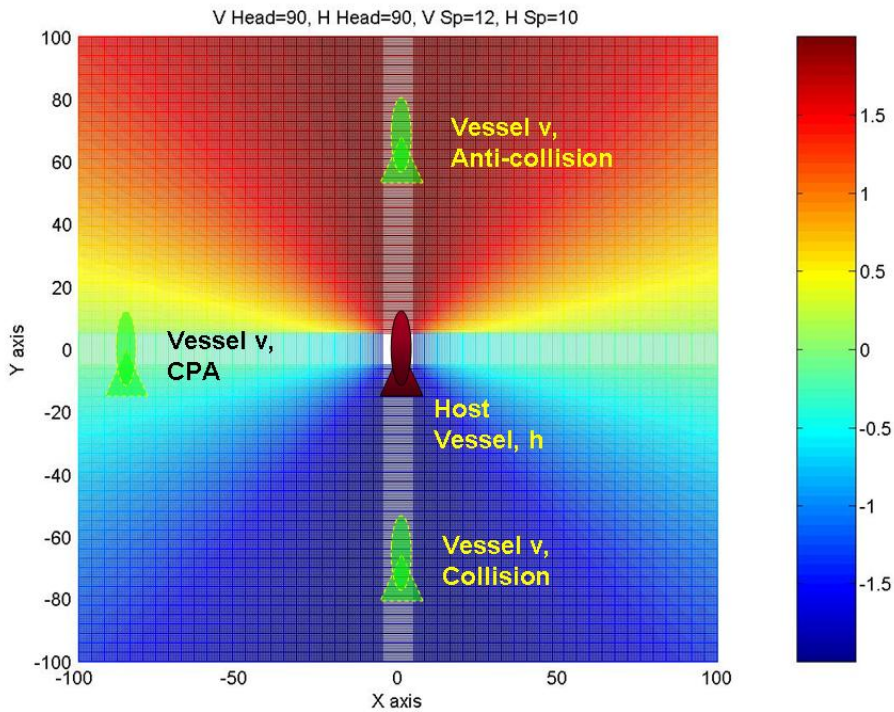

Fig. 13 — Range rate as a function of *v* position with respect to *h*, same headings

Figure 14 shows an example where *h* and *v* are at the same speed, but the heading of *h* is 0° and the heading of *v* is 40°. Computing the relative velocity between *h* and *v* yields the collision axis whose slope is –70°, shown in yellow. For a fixed relative velocity between *h* and *v*, the max/min of $\dot{R}$ as a function of the position of *v* with respect to *h* corresponds to the collision axis as shown in the figure. Two "test" positions for vessel *v* are shown, with the vessel *v* symbol pointing in the direction of its true heading, to depict the collision and anti-collision conditions. From the perspective of vessel *h*, the vessel shown in the collision condition would move along the yellow line towards vessel *h*. Note that for *v* not on a collision or anti-collision course, the RML will not pass through (0,0). As predicted, the CPA axis ($\dot{R}$ = 0, shown in black) is perpendicular to the collision axis and is at an angle of 20° with respect to the X axis. Two "test" positions for vessel *v* at CPA are shown, one ahead and one behind vessel *h*. In both cases the line of relative motion would pass through vessel *v* and be parallel to the collision axis, and each would appear to move up along their RMLs from the perspective of vessel *h*.

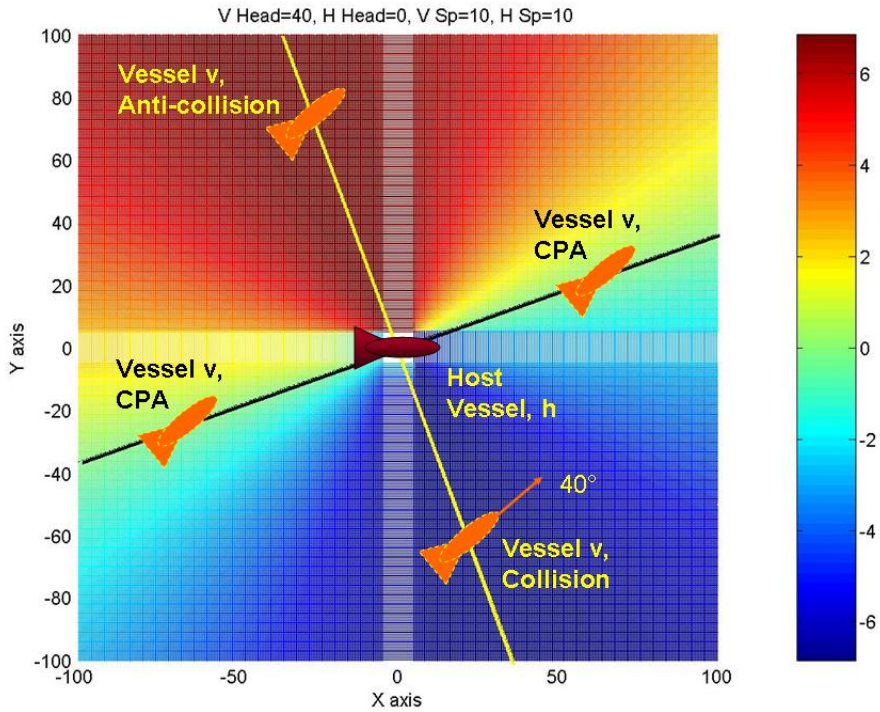

Fig. 14 — Range rate as a function of *v* position with respect to *h*, same speeds

5.9 Range Rate as a Function of *v* Heading and Speed

In the previous two sections we examine the impact on range rate of vessel *v*'s position and bearing with respect to vessel *h* for a constant relative velocity, primarily to help illustrate the relationships involved. While vessel *v* cannot make a rapid change in either range or bearing with respect to *h*, it can control its course and speed and thus cause measurable changes in range rate, assuming the host is maintaining a constant course and speed. Thus it is of interest to examine if the resulting changes in range rate can be used to make some direct inferences about the values of $(x_v, y_v, \underline{v}_h)$. Expressed mathematically, we would like to determine the following function, if it exists:

$$f\left(\frac{d\dot{R}}{dS_v}, \frac{d\dot{R}}{d\theta_v}\right) = (x_v, y_v, \underline{v}_h),$$

where

S_v = vessel *v* speed, and

θ_v = vessel *v* heading.

To accomplish this, let us rewrite Eq. (14) in terms of the variables we can control, (S_v, θ_v):

$$\begin{aligned}\dot{R} &= \left[x_v(S_v\cos\theta_v - S_h\cos\theta_h) + y_v(S_v\sin\theta_v - S_h\sin\theta_h)\right]/R \\ &= \left[S_v(x_v\cos\theta_v + y_v\sin\theta_v) - S_h(x_v\cos\theta_h + y_v\sin\theta_h)\right]/R.\end{aligned} \tag{66}$$

Validity check: If $\theta_v = \theta_h$ and $S_v = S_h$ then $\dot{R} = 0$.

Unfortunately, it is not readily apparent that Eq. (66) can be simplified to reveal any fundamental characteristics that may be monopolized for relative navigation, so we will examine this relation numerically in the following figure.

Plotting Eq. (66), Fig. 15 shows $\dot{R}$ as a function of *v* course and speed, with respect to a fixed position relative to *h,* shown in the inset. Note that this plot applies only to the instant of time that vessel *v* is at this relative position with respect to *h,* which will change unless it is at the same course and speed as *h,* and that the line of relative motion of *v* with respect to *h* will change accordingly with changes in the speed and heading of v. We see from this plot that the maximum $\dot{R}$ occurs when *v*'s course is 45° (black line) and the minimum $\dot{R}$ occurs on the reciprocal course of –135° (yellow line); the magnitude of $\dot{R}$ increases or decreases along these lines with the velocity of *v*. Note also that with respect to *v* heading and speed that $\dot{R}$ is not symmetrical — the maximum is 5 and the minimum is –20. This figure also demonstrates that for the position specified, *v* may be ahead or behind CPA ($\dot{R}$ negative or positive) depending upon its course and speed. Of particular interest is the fact that the maximum range rate occurs at the instant in time that vessel *v* is pointing directly away from *h* and that the minimum range rate occurs at the instant in time that vessel *v* is pointing directly towards *h.* The navigation implication of this is that vessel *v* could determine the direction of *h* by executing a turn and looking for the maximum or minimum range rate.

The blue line in Fig. 15 represents the CPA curve and the point marked on it corresponds to vessels *v* and *h* having the same course and speed (stable CPA point). Every other point on this line corresponds to transient CPA conditions where, for the particular vessel *v* course and speed indicated, it is at CPA at the instant in time that it is at the shown relative position with respect to vessel *h.* The red area bordered by the CPA curve corresponds to positive values of $\dot{R}$ and hence, for the headings and speeds indicated, vessel *v* is past CPA and opening. For the headings and speeds outside of the CPA curve, $\dot{R}$ is negative, and thus vessel *v* is before CPA and closing on vessel *h.* The CPA curve shows us that, for sufficiently high *v* speed, if vessel *v* executes a turn that it will see range rate pass through zero twice, in addition to a maximum and minimum value. However, we can see from the figure that the heading of *v* at which this occurs also depends upon the speed of *v*.

Finally, it is also apparent from this plot that along the vessel *v* heading lines corresponding to –45° and 135° (90° out from the lines pointing directly towards and away from vessel *h*) $\dot{R}$ is constant with changes in vessel *v* speed. This corresponds to the two conditions where vessel *v*'s course is perpendicular to line between *v* and *h* at the instant in time that vessel *v* is at the indicated relative position. It is not readily apparent that this particular relationship will provide any useful navigational insight.

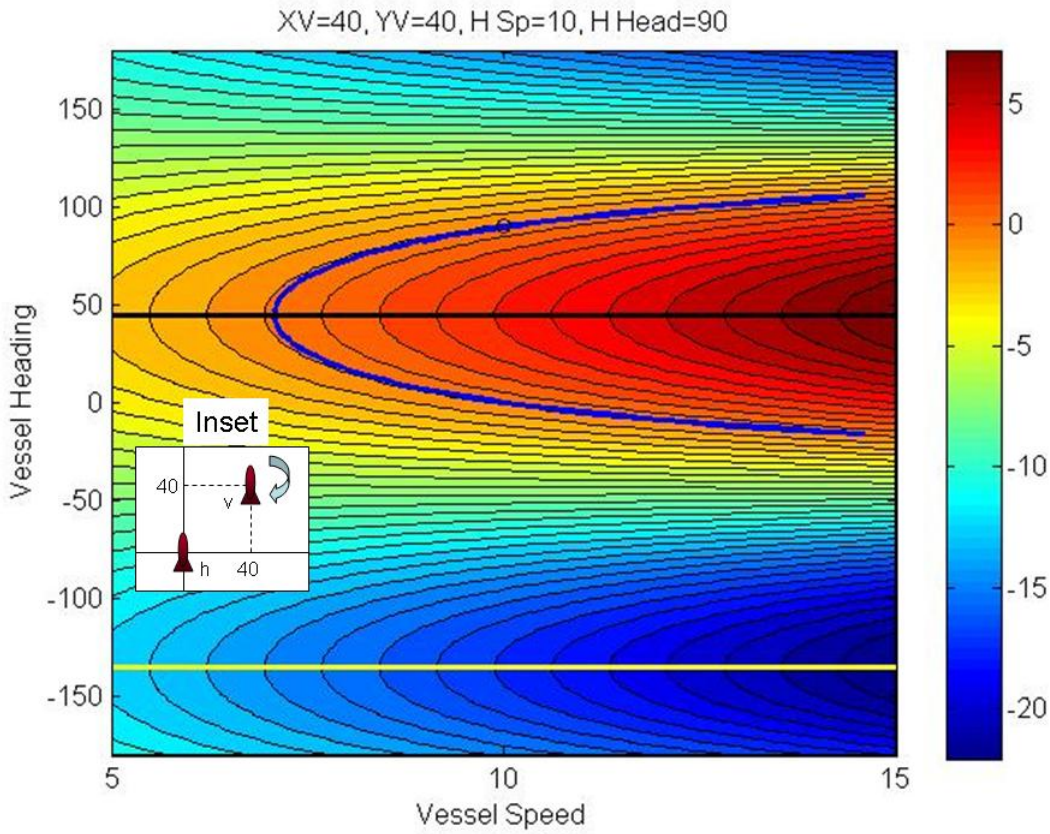

Fig. 15 — dR/dt as a function of *v*'s heading (deg)/speed (m/s) at a fixed relative position

5.9.1 Partial Derivative of Range Rate with Respect to Speed

Exploring Eq. (66) further, we take the derivative of $\dot{R}$ with respect to *v* speed, S_v, to obtain

$$\frac{d\dot{R}}{dS_v} = \left(x_v \cos\theta_v + y_v \sin\theta_v\right) / R. \tag{67}$$

Equation (67) tells us that if S_v increases, $\dot{R}$ may increase or decrease depending on the values of x_v, y_v, θ_v. It is interesting to note that $\frac{d\dot{R}}{dS_v}$ is independent of $\underline{v}_h$. Table 1 shows whether $\dot{R}$ increases or decreases with an increase in S_v.

Table 1 — Change In Range Rate with a Change in Vessel Speed

	Q1: $x_v > 0, y_v > 0$	Q2: $x_v < 0, y_v > 0$	Q3: $x_v < 0, y_v < 0$	Q4: $x_v > 0, y_v < 0$
$0 < \theta_v < 90$	+	+/–	–	+/–
$90 < \theta_v < 180$	+/–	+	+/–	–
$180 < \theta_v < 270$	–	+/–	+	+/–
$270 < \theta_v < 360$	+/–	–	+/–	+

For example, if *v* is in Q1 with respect to *h* and its heading is between 0° and 90°, then an increase in S_v will result in an increase in $\dot{R}$.

Looking at the converse, assume θ_v is 45°. If we increase S_v and $\dot{R}$ increases, then we can only conclude that *v* is not in Q3 with respect to *h* since in Q2 and Q4 $\dot{R}$ may increase or decrease depending upon the values of x_v, y_v. An inefficient approach to determining what quadrant *v* is in would be to steer a course in three different quadrants, execute a speed increase, and thus determine which three quadrants *v* is not in.

Ignoring the discontinuity at $R = 0$, we can set Eq. (67) equal to zero to determine max/min:

$$\begin{aligned} &x_v \cos\theta_v + y_v \sin\theta_v = 0 \\ &\frac{\sin\theta_v}{\cos\theta_v} = \tan\theta_v = -\frac{x_v}{y_v} = -\cot\alpha \end{aligned} \tag{68}$$

where α is the true bearing of *v* with respect to *h*. But $\tan\alpha = -\cot(\alpha - 90)$ so Eq. (68) becomes

$$-\cot(\theta_v - 90) = -\cot\alpha\,, \tag{69}$$

and thus

$$\theta_v = \alpha \pm 90\,. \tag{70}$$

Note that the second derivative of $\dot{R}$ with respect to S_v is zero, so Eq. (70) could correspond to a maximum, minimum, or inflection point. However, we note from the analysis before that this condition only occurs when *v* is at CPA with respect to *h*, at which point $\dot{R}$ passes through zero. Since $\dot{R} < 0$ before reaching CPA and $\dot{R} > 0$ after CPA, then this is an inflection point. Unfortunately, however, this provides no new insight.

5.9.2 Partial Derivative of Range Rate with Respect to Heading

The derivative of $\dot{R}$ with respect to vessel heading θ_v is given by

$$\frac{d\dot{R}}{d\theta_v} = S_v \left(y_v \cos\theta_v - x_v \sin\theta_v \right) / R \,. \tag{71}$$

Table 2 gives the corresponding change in range rate for an increase in heading angle (CCW rotation) and as before can only provide information on which coordinate vessel *v* is not in with respect *h* for a minor change in heading.

Table 2 — Change in Range Rate with a Change in Vessel Heading

	Q1: $x_v > 0, y_v > 0$	Q2: $x_v < 0, y_v > 0$	Q3: $x_v < 0, y_v < 0$	Q4: $x_v > 0, y_v < 0$
$0 < \theta_v < 90$	–	+/–	+	+/–
$90 < \theta_v < 180$	+/–	–	+/–	+
$180 < \theta_v < 270$	+	+/–	–	+/–
$270 < \theta_v < 360$	+/–	+	+/–	–

Setting Eq. (71) equal to zero to examine the max/min yields

$$\frac{\sin\theta_v}{\cos\theta_v} = \tan\theta_v = \frac{y_v}{x_v} = \tan\alpha \,; \tag{72}$$

thus,

$$\theta_v = \pm\alpha \tag{73}$$

since $\tan\alpha = \tan\left(\alpha - 180\right)$. The second derivative yields

$$\frac{d^2(\dot{R})}{d\theta_v^{\,2}} = -S_v \left(x_v \cos\theta_v + y_v \sin\theta_v \right) / R \,, \tag{74}$$

which, depending upon the values of x_v, y_v, θ_v could be positive, negative, or zero. Intuitively, Eq. (73) indicates that if the course of *v* is equal to the true bearing from *h* to *v*, then $\dot{R}$ is a maximum, and on a reciprocal course, $\dot{R}$ is a minimum. This too provides no new insight but does provide analytic proof of the observation made from the numerical analysis that the max/min occurs when *v* is pointing directly away from or towards vessel *h*.

For example, if *v* is in Q1 (*x*,*y* >0) and on a heading of 45°, then Eq. (74) is negative. If the true bearing from *h* to *v* is also 45°, then $\dot{R}$ is at its maximum. If we take the reciprocal course, then both $\cos\theta_v$ and $\sin\theta_v$ are negative and Eq. (74) is positive, so $\dot{R}$ is a maximum. As previously mentioned,

this result is particularly important since it indicates that if we steer *v* until it finds the maximum or minimum $\dot{R}$, we now know the range and bearing to *h*.

SUMMARY

Table 3 enumerates six vessel conditions, dependent upon the possible values of range rate and range acceleration, which are derived and/or demonstrated in the previous sections. Condition I corresponds to the benign situation where vessel headings and speeds are matched, which is evident from Eq. (12). Conditions II and III correspond to the collision and anti-collision courses from Eq. (44) and also to the situation where the vessels are far apart with respect to CPA as revealed by Eq. (49). Condition IV is defined by Eqs. (24) and (35), and Figs. 9 and 10 illustrate that range acceleration is maximum for this condition. Conditions V and VI are also illustrated in Figs. 9 and 10.

Table 3 — Range Rate and Range Acceleration Conditions

	$\ddot{R} = 0$	$\ddot{R} > 0$
$\dot{R} = 0$	Condition I Course and speed matched, at CPA	Condition IV Passing through CPA $\ddot{R}$ = maximum
$\dot{R} < 0$ (closing)	Condition II Collision course, $\dot{R} = -S_r$ or Vessels are far apart $(R \to \infty)$	Condition V Before CPA $\ddot{R}$ increasing
$\dot{R} > 0$ (opening)	Condition III Anti-collision course, $\dot{R} = S_r$ or Vessels are far apart $(R \to \infty)$	Condition VI After CPA $\ddot{R}$ decreasing

Reviewing those observations that may be helpful to relative navigation, we see from Eq. (24) and Fig. 12 that for any measured value of $\dot{R}$ there are two possible values of α, the angle of vessel *v* from vessel *h,* except when vessel *v* is on the collision axis. Thus, given a measurement of range rate and the estimated velocities of vessels *v* and *h,* vessel *v* can estimate its bearing with respect to vessel *h*; the ambiguity can be solved using a priori information about the vessel relative positions or through a course change by *v*. From Eq. (35) we learned that range rate passes through zero when vessel *v* reaches CPA with respect to vessel *h.* But by observing Fig. 15 we see that the angle from *h* to *v* at which this occurs depends upon vessel *v*'s speed and so this condition does not readily provide any information regarding the relative position of *h* from *v*.

Looking at the limit of range rate as range becomes large, we saw in Eq. (49) that range rate asymptotically approaches the relative speed between vessels *v* and *h,* assuming they are holding constant velocities. While this condition is difficult to distinguish between the collision and anti-collision course conditions, it may be useful for navigation. In both cases (collision course or long range), we have from Eqs. (44) and (51) the result that the relative bearing from vessel *h* to vessel *v* is the same as their relative velocity angle; thus, given estimates of both vessel *h* and *v* velocities and the detection of a constant range rate by *v,* vessel *v* can estimate the bearing to vessel *h*. Also, consider the situation where vessel *v* is attempting to join a vessel formation and has been assigned a station relative to vessel *h.* Since it is maneuvering towards a point relatively close to *h,* at a far distance its range rate with respect to *h* will be the relative speed; if the range rate is constant but not equal in value to the relative speed (based on the estimated velocities of *v* and *h*) then an error must exist in the estimated vessel velocities. As was shown

earlier, once *v* gets to within twice the CPA range the range rate begins to increase. This change in range rate could be used as a cue by vessel *v* to reassess its position relative to vessel *h,* and to estimate the range at CPA and take corrective action as necessary to avoid a collision.

Perhaps the most promising observation of this study, made from Fig. 15, was that at a fixed position of vessel *v* relative to vessel *h,* if vessel *v*'s heading is rotated through 360° we would see $\dot{R}$ pass through a maximum, a minimum, and zero twice. More importantly, $\dot{R}$ maximum corresponds to *v* pointing directly away from *h* and $\dot{R}$ minimum corresponds to *v* pointing directly toward *h.* Assuming that estimates of $\dot{R}$ can be made relatively quickly from Doppler shift, it may be practical for vessel *v* to estimate the relative bearing of vessel *h* by executing a turn. Moreover, this approach does not depend upon the accuracy of the range rate measurement itself. Finally, we examined the partial derivatives of range rate with respect to changes in vessel *v* heading and speed to look for additional relationships that might provide navigational cues. Unfortunately, the only insight provided was the determination that a change in either would only isolate the position to three out of four possible quadrants with respect to vessel *h.*

In conclusion, this study has revealed the possibility of using the Doppler shift in underwater acoustic communications between two vessels to estimate their relative position. Future work on this topic includes computer simulation of the approaches proposed above, determination of achievable range rate resolution using Doppler shift estimates in conventional underwater acoustic modems, and the impact of vessel stability and measurement time interval on the ability to use the Doppler measurement as an estimate of range rate. Extending these results to three dimensions is straightforward and left to the interested reader.

ACKNOWLEDGMENTS

The author thanks Doug Ray with the Naval Undersea Warfare Center in Keyport, WA, who provided submarine tracking data and Dale Green with Teledyne-Benthos who computed the corresponding range rate estimates from acoustic modem Doppler measurements on the submarines. This work was sponsored by the Office of Naval Research under Program Element 0602435N.

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