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Neutron Production by Fusion in a Class of Anisotropic Plasmas

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ABSTRACT

The neutron production rate of a fusion reaction in a plasma of deuterium is calculated by a non-Maxwellian average, assuming the differential cross section is known. For the average it is assumed the ion velocity distribution of one plasma component is that of a beam and that of the other component is a Maxwell distribution. It is found that cross section anisotropy has no effect on neutron production rate or on the excitation function. An expression is obtained which contains the detailed effects of anisotropy on neutron speed and angle distributions. Prominent behavior of this expression is discussed. For purposes of numerical calculation, a Gamow form is assumed for the total cross section. Neutron production rate results are presented in a contour plot as a function of beam energy and plasma temperature over ranges of the parameters of most interest in thermonuclear research. Prominent behavior here is understood in the light of two well known subcases of this calculation: 1) neutron production by bombarding a cold target and 2) neutron production from a Maxwellian plasma. A qualitative analysis with order of magnitude estimates is made to consider the practical implications of the use of the assumed nonequilibrium distribution. It is shown that the assumed distribution may exist for small observation volumes and for short observation times in plasma regimes of thermonuclear interest. It is a general result of this analysis that an arbitrary plasma cannot be shown isotropic except by fine time resolution technique.

PROBLEM STATUS

This is a final report on one phase of the problem; work continues on the problem.

AUTHORIZATION

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NEUTRON PRODUCTION BY FUSION IN A CLASS OF ANISOTROPIC PLASMAS

1. INTRODUCTION

This report is concerned with the neutron production rate for a deuterium-deuterium fusion reaction. The calculation is based on a non-Maxwellian average of the reaction differential cross section. Specifically the average is performed over a prototype plasma ion distribution, the distribution due to a beam of deuterons undergoing nuclear reactions with a thermalized plasma of deuterons, called the beam-plasma configuration henceforth. This configuration might arise in a thermonuclear research plasma through turbulence or through some mechanism causing a local collective ion acceleration. It is called "prototype" because it is easily generalized to include all reactions between two components, one of which is thermalized (Maxwellian) and the other of which has an arbitrary ion velocity distribution. Thus general conclusions derived apply to a broad class of anisotropy problems in fusion plasmas. This calculation includes as a special case the more familiar one in which both components have the same Maxwell distribution. This case is developed below and agreement is obtained with previous results.[1] However, the applicability of these results is extended to include reactions with anistropic center of mass cross sections, the chief example of interest being the deuterium-deuterium cross section. It has been known at least since 1936[2] that

[1]W. R. Faust and E. G. Harris, The Energy Distribution of Neutrons Produced by a Thermonuclear Reaction, (U. S. Naval Research Laboratory Report 5131, 1958). Also a statement of results appears in Nuclear Fusion 1, 62 (1960).
[2]A. E. Kempton, B. C. Browne, and R. Maasdorp, Angular Distributions of the Protons and Neutrons Emitted in Some Transmutations of Deuterium, Proc. Roy. Soc. A157, 386 (1936).

the deuterium-deuterium reaction products do not have a random or flat angular distribution in the center of mass (c.m.) system. However, in calculating the production rates it has been customary[3] to ignore the cross section's angular dependence in order to facilitate the averaging process. Thus the present calculations are of interest because they avoid this restriction to isotropic cross sections and because they present the effects of cross section anisotropy in detail.

The deuterium-deuterium reaction has two branches for initial relative kinetic energies of thermonuclear range.

$$d + d \rightarrow \begin{cases} n + He^{3++} + Q_n \\ p + t + Q_p \end{cases}$$

The choice of the neutron branch for this investigation was arbitrary; both branches have been used diagnostically. The neutron branch ratio is often taken to be $\frac{1}{2}$; i.e., one neutron from every two reactions. However, recent measurements[4] show this ratio varies from about 0.5 at a deuteron relative kinetic energy of 20 keV to 0.6 at 350 keV. Since this variation is smooth, not too large and known, the neutron production rate serves as an indicator of the total thermonuclear reaction rate.

[3]Some examples of calculations assuming cross section isotropy either implicitly or explicitly are G. Gamow and E. Teller, Phys. Rev. 53, 608 (1938); W. B. Thompson, Proc. Phys. Soc. B70, 1 (1957); and Footnote 1.

[4]For a partial list of these results see R. B. Theus, W. I. McGarry, and L. A. Beach, Phys. Rev. Letters 14, 232 (1965). A more complete presentation has been accepted for publication in Nuclear Physics.

The neutron production rate is derived by an elementary statistical-mechanical development of the kinetics of a single deuterium-deuterium reaction. Laboratory (lab) ion velocity distributions are stated for the two interacting components, the beam and plasma, and the neutron production differential cross section in the center of mass system of a pair of ions is assumed known. The distributions and cross section lead, by certain rules of probability theory and by conservation of energy and momentum, to a statement of the neutron production rate in terms of a large set of independent variables. Averaging over uninteresting variables gives various neutron production rate distributions of interest and finally gives the total neutron rate as a function of beam speed and plasma temperature. As a result the effect of cross section anisotropy on neutron angular distribution is found to be very small. Furthermore these calculations show exactly that anisotropy has no effect on total neutron production rate.

From a practical standpoint, it is not neutron production rate (henceforth just "neutron rate") which is detected, but rather neutrons are counted over a finite observation time. Moreover, observed neutrons have a finite interaction volume as source. Theoretically one is free to impose arbitrary ion velocity distributions instantaneously at a point, but a calculation based on distributions greatly different from an equilibrium distribution would be sterile of practical import. To avoid this, some qualitative considerations of the effect of extending the assumption to finite observation times and volumes will be made. For example, magnetohydrodynamic development using the distributions of

the prototype reaction as initial conditions would, in many regimes of interest, result in shock fronts and turbulence after some time. Thus for practical application of these calculations one must either assume uniform and stationary conditions or else perform volume and time averages. If the former, the experiment is restricted to observation times much shorter than the relaxation time for the initial distributions. In the following, uniform and stationary conditions are assumed except in the analysis of the assumptions in section 3.

This investigation is motivated by a desire to elucidate parameters affecting thermonuclear reactions in plasmas. It is a parameter study and the effects described and parameters used need not be the ones most directly applicable to interpretation of experiments in order to be interesting. At first glance it may not seem that this study is at all interesting experimentally because many aspects of the ideal beam-plasma configuration are not subject to the control of the experimenter. However, the assumptions are carefully stated in section 3 and some of their practical implications are obtained qualitatively. It is shown that plasma anisotropy may exist over a range of many orders of magnitude of the observation time even when no anisotropy appears at longer observation times. In general the high temperatures and high rates of change in thermonuclear research plasmas imply that anisotropy must be assumed unless isotropy can be proved. In this light an assumption of isotropy without proof is seen as a possible aid to computation. As an approximation this assumption may lead to quick results but finding the degree of approximation of such a method probably requires completing more

detailed calculations under the more general assumption. On the other hand, under some circumstances this assumption yields exact results. It would be useful to be able to recognize such cases in advance, particularly when a formal criterion could be recognized to give the process a stamp of exactness. At one point a somewhat typical short cut is exhibited without finding a formal criterion.

In the following sections, after a brief statement of theoretical background and of conventions used, the main derivation proceeds as mentioned above. Construction of the neutron rate and the assumptions incident to the construction are the subjects of section 3. Section 4 treats the averaging process and digressions are made to develop the subcase of neutron production from a low temperature target and the subcase wherein both components have the same Maxwell distribution. All integrations performed are done exactly except in section 5 where numerical methods are used to obtain results on the total neutron rate. In the last section the more interesting neutron rate distributions are discussed and principal results are summarized.

2. BACKGROUND

Through probability considerations one knows that the joint probability of the occurrence of two or more independent events is the product of their separate probabilities. This product rule also applies when the probabilities are known through their probability densities provided the argument variables of these densities are independent. Furthermore one finds that the convolution of two or more probability densities gives

the combined probability density described as a function of a variable which is the sum of the arguments of the separate probability densities. This rule applies as well when the arguments are vectors.[5] For example, suppose one is given the velocity distributions of two populations of ions $f_1(\underset{\sim}{c}_1)$ [6] and $f_2(\underset{\sim}{c}_2)$ (specifically, probability density distributions in velocity, i.e., $f_1(\underset{\sim}{c})d^3c$ means the probability of finding a type 1 ion having a velocity in a neighborhood of $\underset{\sim}{c}$ bounded by an elemental volume of velocity space d^3c). Then if the populations are independent, the ion pair velocity distribution is $f_1(\underset{\sim}{c}_1)f_2(\underset{\sim}{c}_2)$. Also, the convolution rule gives the ion pair velocity distribution in relative velocity $\underset{\sim}{g} = \underset{\sim}{c}_1 - \underset{\sim}{c}_2$ as $\int f_1(\underset{\sim}{c})f_2(\underset{\sim}{c}-\underset{\sim}{g})d^3c = F(\underset{\sim}{g})$. The connection between these two joint distributions is easily seen. One merely restricts the joint distribution in the six dimensional $(\underset{\sim}{c}_1, \underset{\sim}{c}_2)$ space to the surfaces $\underset{\sim}{g} = \underset{\sim}{c}_1 - \underset{\sim}{c}_2$. This may be accomplished by the transformation of variables $\underset{\sim}{c}_2 \rightarrow \underset{\sim}{c}_2' = \underset{\sim}{c}_1 - \underset{\sim}{g}$ (with Jacobian $\frac{\partial \underset{\sim}{c}_2'}{\partial \underset{\sim}{c}_2} = 1$) and the integration over $\underset{\sim}{c}_1$. Or it may be accomplished by the integral transform

$$\iint \cdots \delta^3(\underset{\sim}{g} - \underset{\sim}{c}_1 + \underset{\sim}{c}_2)\, d^3c_1\, d^3c_2 , \qquad \text{thus}$$

$$\iint f_1(\underset{\sim}{c}_1)f_2(\underset{\sim}{c}_2)\,\delta^3(\underset{\sim}{g}-\underset{\sim}{c}_1+\underset{\sim}{c}_2)d^3c_1\,d^3c_2 = \int f_1(\underset{\sim}{c}_1)f_2(\underset{\sim}{c}_1-\underset{\sim}{g})d^3c_1 = F(\underset{\sim}{g}). \quad (1)$$

The rule for the combination of probability densities by convolution will be used as illustrated by (1) in the construction of the neutron rate to follow. The necessity of assuming the independence of the separate

[5] H. Cramer, Mathematical Methods of Statistics (Princeton 1946), pp. 285 and 140.
[6] Vector quantities are indicated by placing a tilde beneath the symbol.

probability densities entails physical consequences which will be obtained by qualitative considerations.

Though distribution valued functions are used for convenience in the following, it is only necessary to deal with probability distributions which are continuous and which have continuous first derivatives. In this case, if $P(a)$ is the probability that some event occurs when the independent variable x, having a range of the whole real line, is less than a, and if $D(x)$ is the associated probability density, then they are related by the Riemann integral

$$P(a) = \int_{-\infty}^{a} D(x)\,dx. \tag{2}$$

Usually $P(a)$ is normalized by requiring $P(\infty)=1$. However other normalizations are often convenient. Consider the probability distribution over more than one linearly independent variable

$$P(a,b,c) = \int_{-\infty}^{a}\int_{-\infty}^{b}\int_{-\infty}^{c} D(x,y,z)\,dx\,dy\,dz \tag{3}$$

and let a be constant. Then the probability distribution over y and z, say

$$P_a(b,c) = \int_{-\infty}^{b}\int_{-\infty}^{c} D_a(y,z)\,dy\,dz = P(a,b,c), \tag{4}$$

is normalized by $P(\infty,\infty,\infty)=1$ and $P_a(\infty,\infty)$ need not be one. Having a normalized probability density, then for the average value of some physical quantity V, one obtains

$$\bar{V} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V(x,y,z)\,D(x,y,z)\,dx\,dy\,dz. \tag{5}$$

Furthermore if one is interested in the probability density distribution $\mathcal{V}(x)$ of V over x, which is such that

$$\bar{V} = \int_{-\infty}^{\infty} \mathcal{V}(x)\,dx, \tag{6}$$

it is simply the integrand of dx in (5) when x is integrated last.

$$\mathcal{V}(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V(x,y,z)\,D(x,y,z)\,dy\,dz. \tag{7}$$

This will be a means of discussing properties of the neutron rate below.

A physical quantity is so designated because it has known transformation properties, in particular, under translations and rotations. Since $\bar{V}$ remains unchanged upon transforming the independent variables, the transformation properties of any quantity $\mathcal{V}$, which is intermediate in the averaging process (5), may be deduced by transforming the integral (6). Assume a vector variable with transform $\underset{\sim}{x} = f(\underset{\sim}{x}')$. If the integrals are over the full range of $\underset{\sim}{x}$, then

$$\begin{aligned}\bar{V} = \int \mathcal{V}(\underset{\sim}{x})\,d^3x &= \int \mathcal{V}'(\underset{\sim}{x}')\,d^3x' \\ &= \int \mathcal{V}[f(\underset{\sim}{x}')]\,\frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{x}'}\,d^3x'\end{aligned} \tag{8}$$

or

$$\mathcal{V}'(\underset{\sim}{x}') = \mathcal{V}[f(\underset{\sim}{x}')]\,\frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{x}'}. \tag{9}$$

Thus $\mathcal{V}$ transforms by direct substitution and multiplication by the Jacobian, i.e., it transforms as a density. If the transformation is a translation or rotation, the Jacobian is one and the change of variables is especially simple. In the following, spherical polar coordinates are

convenient and the shift between these and Cartesian coordinates will be made without comment.

Use of the delta function will be made. It is useful to extend the usual rule to the following:

$$\int_a^c f(x)\,\delta(x-b)\,dx = f(b)\left[H(c-b)-H(a-b)\right], \qquad (10)$$

where H is the unit function, having the value 1 for positive argument and 0 for negative argument. This extension is particularly useful when a, b or c are functions of a parameter which is to be regarded as a variable in subsequent calculations. One also needs the decomposition of the unit function of a function in terms of the independent variable,

$$H[f(x)] = H[f(-\infty)] + \sum_i H(x-x_i)\,\mathrm{sgn}\,f'(x_i), \qquad (11)$$

$$f(x_i)=0,\ f'(x_i)\neq 0,$$

where

$$\mathrm{sgn}(x) = 2H(x)-1. \qquad (12)$$

Formally this is seen to be compatible with the delta function decomposition rule by functional differentiation.

$$\delta[f(x)] = \sum_i \delta(x-x_i)\,|f'(x_i)|^{-1},\ f(x_i)=0,\ f'(x_i)\neq 0. \qquad (13)$$

Neither (11) or (13) apply when $f(x)$ has zeroes of higher multiplicity than double but this is sufficient for the application below. The breakdown of the vector delta function for spherical polar coordinates is

$$\delta^3(\underset{\sim}{r} - \underset{\sim}{r}_0)\, dr\, d\theta\, d\varphi = \delta(r-r_0)\,\delta(\theta-\theta_0)\,\delta(\varphi-\varphi_0)\, dr\, d\theta\, d\varphi$$

$$= \delta(r-r_0)\,\delta(\cos\theta-\cos\theta_0)\,\delta(\varphi-\varphi_0)\, dr\, d(-\cos\theta)\, d\varphi, \quad (14)$$

where $\underset{\sim}{r}_0 = \underset{\sim}{r}_0(r_0, \theta_0, \varphi_0)$ and angles are referred to an arbitrary common base.

3. CONSTRUCTION OF THE NEUTRON RATE

If g is the relative velocity of a pair of deuterium ions, $\sigma(g)$ is the neutron production cross section, and $f(g)$ is the ion pair distribution, then the neutron production rate is the average over the ion pair distribution of the neutron production rate for a single pair $g\sigma(g)$.

$$R = \int f(g)\, g\, \sigma(g)\, dg. \quad (15)$$

This average can obviously be extended to include angular variables.

$$R = \int f(\underset{\sim}{g})\, g\, \frac{d\sigma}{d\Omega}(\underset{\sim}{g})\, d^3g. \quad (16)$$

Here it will be extended to include all variables of interest as arguments of neutron rate distributions and enough more to simplify transformations between reference frames. In this extension it is necessary to observe: 1) that the ion pair distribution is formulated in accordance with the above probability rules and 2) that in taking the average, the form (16) is maintained.

Consider two populations of deuterium ions, distinguished by the fact that they may be described by independent velocity distributions at a common space and time point. Assume the lab velocity distribution of the first population is that of a beam with velocity $\underset{\sim}{c}_0$, i.e.,

$$\nu_1 \, \delta^3(\underset{\sim}{c}_1 - \underset{\sim}{c}_0) \tag{17}$$

and that of the second is thermalized at temperature T, i.e.

$$\nu_2 \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} e^{-\frac{m}{2kT}\underset{\sim}{c}_2^2} , \tag{18}$$

where ν_1 and ν_2 are ion densities, k is Boltzmann's constant, and m is the deuteron mass. Both of these distributions are normalized.

$$\begin{gathered} \int \delta^3(\underset{\sim}{c}_1 - \underset{\sim}{c}_0)\, d^3c_1 = 1, \\ \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \int e^{-\frac{m}{2kT}c_2^2}\, d^3c_2 = 1. \end{gathered} \tag{19}$$

Based on the assumption of independence, the product of (17) and (18) gives the ion pair velocity distribution.[7]

The assumption of independence of the two coexisting ion populations can only be true as an approximation since there is a Coulomb interaction between ions (and electrons) of the two parts. The following qualitative discussion is offered to support the presumption that there is wide latitude for the application of such an approximation in plasma regimes of thermonuclear interest. Due to Coulomb scattering, there will be a

[7]Pairs consisting of two ions of the same population are excluded in this product. Their reactions may be calculated by simpler means as will be shown in a special case in the next section.

relaxation constant, the deuteron deflection time, definable as the time required to deflect half the deuterons in a zero-net-charge test stream through a right angle. For example, Montgomery and Tidman[8] calculate the deflection time for protons in a hydrogen plasma of density 10^{16} cm^{-3} and ion temperature 10^8 ^oK, as 4×10^{-5} sec. This time represents an upper limit on observation times for which the assumption is approximately valid. A possible lower limit is a nuclear reaction time, say (assuming compound nucleus) 10^{-15} sec. Thus there are many orders of magnitude of observation times for which the assumption applies. Similarly one may calculate limits on observation volume for generally the same result. Thus there appears to be a wide latitude for approximate application of the assumption of independence of two ion populations at a common space-time point. Furthermore, when coupled with the initial conditions for the present problem, which may be described as a severely anisotropic departure from the equilibrium velocity distribution, the assumption of independence may be rendered in practical effect as an assumption that the plasma is uniform and stationary over small regions of space and time.

Thus for practical application to interpretation of an experiment the present analysis would require considerations like those above in complete detail. Yet in contrast to the tenuous nature of the above remarks, a firm converse of practical significance follows. If one wished to demonstrate experimentally the nonexistence of anisotropic mechanisms as

[8]D. Montgomery and D. Tidman, Plasma Kinetic Theory (McGraw-Hill 1964), pp. 29-31.

neutron sources (as in the "real" thermonuclear neutron argument[9]), then it would be necessary to show neutron isotropy in a very fine time resolution.

The product of (17) and (18) gives the ion pair velocity distribution in lab velocities. This will now be transformed to the c.m. system of each pair, using

$$\left.\begin{aligned} \underset{\sim}{c}_1 &= \underset{\sim}{G} + \tfrac{1}{2}\underset{\sim}{g} \\ \underset{\sim}{c}_2 &= \underset{\sim}{G} - \tfrac{1}{2}\underset{\sim}{g} \\ \frac{\partial(\underset{\sim}{c}_1, \underset{\sim}{c}_2)}{\partial(\underset{\sim}{G}, \underset{\sim}{g})} &= 1.^{10} \end{aligned}\right\} \tag{20}$$

From the inverse transformation, $\underset{\sim}{G} = \underset{\sim}{c}_1 + \underset{\sim}{c}_2$, $\underset{\sim}{g} = \underset{\sim}{c}_1 - \underset{\sim}{c}_2$, one sees that $\underset{\sim}{G}$ is the pair c.m. velocity and $\underset{\sim}{g}$, the pair relative velocity. This gives the deuteron pair distribution

$$\nu_1 \nu_2 \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \delta^3\left(\underset{\sim}{G} + \tfrac{1}{2}\underset{\sim}{g} - \underset{\sim}{c}_0\right) e^{-\frac{m}{2kT}\left(\underset{\sim}{G} - \frac{1}{2}\underset{\sim}{g}\right)^2}. \tag{21}$$

In forming the single pair neutron rate, one requires the differential cross section for neutron production

$$\frac{d\sigma}{d\Omega}(g, \theta) \tag{22}$$

where θ is the neutron c.m. deflection angle relative to $\underset{\sim}{g}$ and $g = |\underset{\sim}{g}|$. This cross section is not a function of azimuth because it is assumed that neutron and deuteron spins have been averaged. The cross section

[9] L. A. Artsimovich, Controlled Thermonuclear Reactions, (Gordon and Breach 1964), pp. 153, 161, 203.
[10] S. Chapman and T. Cowling, The Mathematical Theory of Non-Uniform Gases (Cambridge 1952), 2nd ed., sections 5.2 and 1.411.

energy dependence is easily found from conservation of energy and momentum. The initial relative kinetic energy of the two deuterons is $\frac{1}{4}mg^2$, and if Q is the reaction energy, then the neutron emerges with a c.m. kinetic energy

$$E_0 = \alpha\left(Q + \tfrac{1}{4}mg^2\right) = \left(1 - \frac{\mu}{2m}\right)\left(Q + \tfrac{1}{4}mg^2\right), \tag{23}$$

where μ is the neutron mass and $\frac{1}{2}m$ is the reduced mass of a deuteron pair. The energy dependence then may be expressed by a delta function factor $\delta(E-E_0)$. Since it is a vector variable, it is convenient to use the neutron c.m. velocity $\underset{\sim}{u}(u,\theta,\varphi)$, rather than E and θ. Since $E = \frac{1}{2}\mu u^2$, one has $dE\,d\Omega = \frac{\mu}{u}d^3u$ and the cross section becomes[11]

$$\frac{d\sigma}{d\Omega}(g, \hat{g}\cdot\hat{u})\,\delta\left(\tfrac{1}{2}\mu u^2 - E_0(g)\right)\frac{\mu}{u}\,. \tag{24}$$

Multiplication by g gives the pair neutron rate. It should be noted that the direction of $\underset{\sim}{u}$ is defined by reference to $\underset{\sim}{g}$ rather than to an arbitrary common base.

The resulting total neutron rate is in the form (16).

$$R = \mu v_1 v_2 \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \int e^{-\frac{m}{2kT}\left(\underset{\sim}{G} - \frac{1}{2}\underset{\sim}{g}\right)^2} \delta^3\left(\underset{\sim}{G} + \tfrac{1}{2}\underset{\sim}{g} - \underset{\sim}{c}_0\right) \times \frac{d\sigma}{d\Omega}\,\delta\left(\tfrac{1}{2}\mu u^2 - E_0\right) d^3G\,d^3g\,d^3u. \tag{25}$$

This is equivalent to

$$R = \mu v_1 v_2 \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \int e^{-\frac{m}{2kT}\left(\underset{\sim}{G} - \frac{1}{2}\underset{\sim}{g}\right)^2} \delta^3\left(\underset{\sim}{G} + \tfrac{1}{2}\underset{\sim}{g} - \underset{\sim}{c}_0\right) \times \frac{d\sigma}{d\Omega}\,\delta\left(\tfrac{1}{2}\mu u^2 - E_0\right)\delta^3\left(\underset{\sim}{v} - \underset{\sim}{G} - \underset{\sim}{u}\right) d^3G\,d^3g\,d^3u\,d^3v. \tag{26}$$

where the integral is over the 12 dimensional velocity space. The latter

[11]The caret denotes unit vectors.

has the advantage of containing the transform to neutron lab velocity $\underset{\sim}{v} = \underset{\sim}{u} + \underset{\sim}{G}$. It may be observed that this transform and (20) as well, are expressions of conservation of momentum.

4. THE AVERAGING PROCESS

Mathematically the order of performing the integrals forming the average, or total, neutron rate is arbitrary. The order used here is influenced by the desire to identify distributions over variables of interest, namely, deuteron relative speed g, neutron lab speed v, and neutron lab deflection angle θ_v. Thus the integrals over these variables are held to the last. Convenience also plays a role in determining this order. Thus, the integration over neutron velocity is most naturally performed in the system in which the beam particles are at rest (beam system - see Fig. 1).

To proceed with the averaging, one first integrates (26) over $\underset{\sim}{G}$, obtaining

$$R = \mu v_1 v_2 \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \int e^{-\frac{m}{2kT}(\underset{\sim}{c}_0 - \underset{\sim}{g})^2} \frac{d\sigma}{d\Omega}\, \delta\left(\tfrac{1}{2}\mu u^2 - E_0\right)$$
$$\times\, \delta^3\left(\underset{\sim}{v} - \underset{\sim}{u} - \underset{\sim}{c}_0 + \tfrac{1}{2}\underset{\sim}{g}\right) \frac{g}{u}\, d^3g\, d^3u\, d^3v. \tag{27}$$

Next $\underset{\sim}{u}$ is integrated. Since $\frac{d\sigma}{d\Omega}$ is a function of the polar angle of $\underset{\sim}{u}$, that is $\theta_u = \theta = \cos^{-1}\hat{g}\cdot\hat{u}$, the delta function breakdown (14) is used. Define the neutron velocity in the beam system (beam neutron velocity) by

$$\underset{\sim}{w} = \underset{\sim}{v} - \underset{\sim}{c}_0, \tag{28}$$

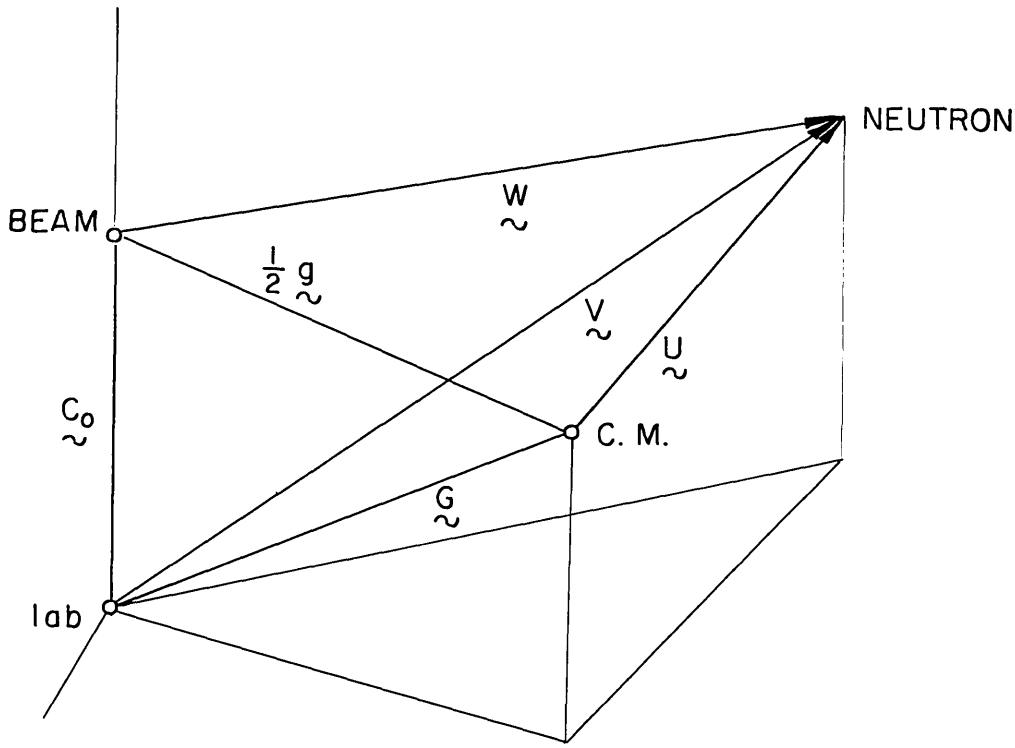

Fig. 1 - Neutron Velocity Reference Systems

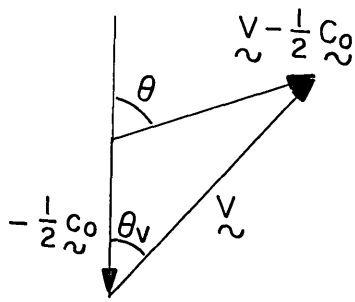

Fig. 2 - Neutron Deflection, Cold Target Case

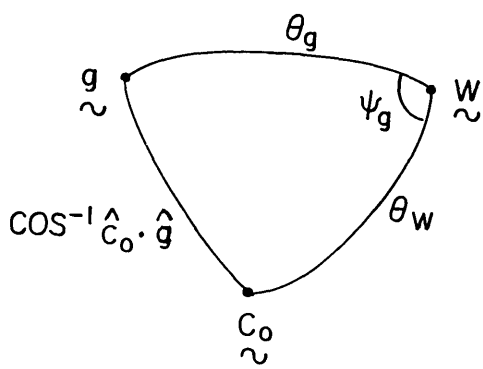

Fig. 3 - Polar Axis Shift

and let $$\underset{\sim}{u}_0 = \underset{\sim}{v} - \underset{\sim}{c}_0 + \tfrac{1}{2}\underset{\sim}{g} = \underset{\sim}{w} + \tfrac{1}{2}\underset{\sim}{g}. \tag{29}$$

Then $$\cos\theta_0 = \hat{g}\cdot\hat{u}_0 = \frac{1}{g u_0}\left(u_0^2 + \tfrac{1}{4}g^2 - w^2\right), \tag{30}$$

and one obtains

$$R = \mu v_1 v_2 \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \int e^{-\frac{m}{2kT}(\underset{\sim}{c}_0 - \underset{\sim}{g})^2} \frac{d\sigma}{d\Omega}(g, \hat{g}\cdot\hat{u}_0) \times \delta\left(\tfrac{1}{2}\mu u_0^2 - E_0\right) \frac{g}{u_0}\, d^3g\, d^3v. \tag{31}$$

It is interesting now to digress to consider the zero temperature limit. As the thermal motion of plasma ions becomes negligible, it is evident that the beam-plasma production of neutrons degenerates to neutron production from a solid target. It should be noted that this limit can be taken at any point in the averaging process.

Using the familiar representation of the delta function [12]

$$\delta(x) = \frac{1}{\sqrt{\pi}} \lim_{\epsilon\to 0} \frac{1}{\sqrt{\epsilon}}\, e^{-\frac{x^2}{\epsilon}}, \tag{32}$$

one sees that

$$\lim_{T\to 0} \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} e^{-\frac{m}{2kT}(\underset{\sim}{c}_0 - \underset{\sim}{g})^2} = \delta^3(\underset{\sim}{c}_0 - \underset{\sim}{g}) \tag{33}$$

Thus from (31)

$$\lim_{T\to 0} R = \mu v_1 v_2 \int \delta^3(\underset{\sim}{c}_0 - \underset{\sim}{g}) \frac{d\sigma}{d\Omega}(g, \hat{g}\cdot\hat{u}_0)\, |\underset{\sim}{v} - \underset{\sim}{c}_0 + \tfrac{1}{2}\underset{\sim}{g}|^{-1} \times \delta\left[\tfrac{1}{2}\mu\left(\underset{\sim}{v} - \underset{\sim}{c}_0 + \tfrac{1}{2}\underset{\sim}{g}\right)^2 - \alpha\left(Q + \tfrac{1}{4}mg^2\right)\right] g\, d^3g\, d^3v. \tag{34}$$

[12] E. Merzbacher, Quantum Mechanics (Wiley 1961), p. 82.

Preparatory to the integration over $\underset{\sim}{g}$, it is useful to fix the reference for polar angles since this simplifies the integration. It is evident that this may be done with no loss in generality since it is equivalent to a rotation of the coordinate system. Taking the direction of $\underset{\sim}{c}_0$ as the Z axis, one notes (see Fig. 2)

$$\begin{aligned} \hat{g}\cdot\hat{u}_0\Big]_{g=c_0} &= \frac{1}{c_0 u_0}\left(u_0^2+\tfrac{1}{4}c_0^2-w^2\right) \\ &= \frac{1}{c_0|\underset{\sim}{v}-\frac{1}{2}c_0|}\left[(\underset{\sim}{v}-\tfrac{1}{2}\underset{\sim}{c}_0)^2+\tfrac{1}{4}c_0^2-(\underset{\sim}{v}-\underset{\sim}{c}_0)^2\right] \\ &= |\underset{\sim}{v}-\tfrac{1}{2}\underset{\sim}{c}_0|^{-1}\left(v\cos\theta_v-\tfrac{1}{2}c_0\right) \\ &= \cos\theta \end{aligned} \tag{35}$$

The notation here is somewhat loose. This θ and that of (22) are both neutron deflection angles, but they have different reference systems. Integrating $\underset{\sim}{g}$, one obtains

$$\begin{aligned} \lim_{T\to 0} R = \mu v_1 v_2 c_0 \int \frac{d\sigma}{d\Omega}(c_0,\theta)\, \delta\left[\tfrac{1}{2}\mu(\underset{\sim}{v}-\tfrac{1}{2}\underset{\sim}{c}_0)^2-\alpha(Q+\tfrac{1}{4}mc_0^2)\right] \\ \times\, |\underset{\sim}{v}-\tfrac{1}{2}\underset{\sim}{c}_0|^{-1} d^3v. \end{aligned} \tag{36}$$

Since there is no dependence on the $\underset{\sim}{v}$ azimuth φ_v , integrating this gives a factor of 2π .

The remaining two integrations will be done in both orders to show the flexibility allowed by the use of the delta function. In order to integrate v first, the delta function must be written in the form $\delta(v-v_i)$ using (13). For $f(v_i)=0$ one has

$$v^2 - v c_0 \cos\theta_v + \tfrac{1}{4}c_0^2 - u_1^2(c_0) = 0, \tag{37}$$

where $$u_1^2(c_0) = \frac{2\alpha}{\mu}\left(Q + \frac{1}{4} m c_0^2\right). \tag{38}$$

Roots of (37) are

$$\begin{aligned} V_\pm &= \frac{1}{2} c_0 \cos\theta_V \pm \frac{1}{2}\left(c_0^2 \cos^2\theta_V - c_0^2 + 4 u_1^2(c_0)\right)^{\frac{1}{2}} \\ &= \frac{1}{2} c_0 \cos\theta_V \pm \left(u_1^2(c_0) - \frac{1}{4} c_0^2 \sin^2\theta_V\right)^{\frac{1}{2}}. \end{aligned} \tag{39}$$

The second term does not become imaginary since, using (38) and (23),

$$\begin{aligned} u_1^2(c_0) - \frac{1}{4} c_0^2 \sin^2\theta_V &= \frac{2\alpha}{\mu} Q + \frac{\alpha}{2}\frac{m}{\mu} c_0^2 - \frac{1}{4} c_0^2 \sin^2\theta_V \\ &= \frac{2\alpha}{\mu} Q + \frac{1}{4} c_0^2 \left(2\frac{m}{\mu} - 1 - \sin^2\theta_V\right), \end{aligned} \tag{40}$$

and since $m > \mu$. Furthermore even with $Q = 0$, the second term in (39) is always greater than the first term.

$$\begin{aligned} \left(u_1^2(c_0) - \frac{1}{4} c_0^2 \sin^2\theta_V\right)^{\frac{1}{2}}\Big]_{Q=0} &= \frac{1}{2} c_0 \left(2\frac{m}{\mu} - 1 - \sin^2\theta_V\right)^{\frac{1}{2}}, \quad m > \mu \\ &> \frac{1}{2} c_0 \cos\theta_V. \end{aligned} \tag{41}$$

Since V is a positive quantity, the root V_- is rejected. Then, the $|f'(V_i)|$ factor is

$$\left|\frac{1}{2}\mu\left(2V - c_0 \cos\theta_V\right)\right|_{V=V_+} = \mu\left(u_1^2(c_0) - \frac{1}{4} c_0^2 \sin^2\theta_V\right)^{\frac{1}{2}} \tag{42}$$

and the delta function becomes

$$\delta\left[\frac{1}{2}\mu\left(\underset{\sim}{V} - \frac{1}{2}\underset{\sim}{c_0}\right)^2 - \frac{1}{2}\mu u_1^2(c_0)\right] = \frac{1}{\mu}\left(u_1^2(c_0) - \frac{1}{4} c_0^2 \sin^2\theta_V\right)^{-\frac{1}{2}} \delta(V - V_+). \tag{43}$$

Integration of V gives

$$\begin{aligned} \lim_{T\to 0} R &= 2\pi v_1 v_2 c_0 \int \frac{d\sigma}{d\Omega}(c_0, \theta(\theta_V)) \left(u_1^2(c_0) - \frac{1}{4} c_0^2 \sin^2\theta_V\right)^{-\frac{1}{2}} \frac{V_+^2}{u_1(c_0)} \sin\theta_V \, d\theta_V \\ &= 2\pi v_1 v_2 c_0 \int \frac{d\sigma}{d\Omega}(c_0, \theta(\theta_V)) \frac{\left[\frac{c_0}{2u_1}\cos\theta_V + \left(1 - \left(\frac{c_0}{2u_1}\right)^2 \sin^2\theta_V\right)^{\frac{1}{2}}\right]^2}{\left(1 - \left(\frac{c_0}{2u_1}\right)^2 \sin^2\theta_V\right)^{\frac{1}{2}}} \sin\theta_V \, d\theta_V \\ &= 2\pi v_1 v_2 c_0 \int \frac{d\sigma}{d\Omega}(c_0, \theta) \sin\theta \, d\theta. \end{aligned} \tag{44}$$

The last step occurs because the fraction in the second line is recognized as the reciprocal of the factor which transforms a cross section from the lab to the c.m. system[13]. The θ integration gives the total cross section and

$$\lim_{T\to 0} R = \nu_1 \nu_2 c_0 \sigma(c_0) \tag{45}$$

This is the required result when it is not necessary to perform an average. It should be remarked that this particular case does not exhibit the full utility of the delta function. If one were dealing with the case in which a reaction product has a velocity smaller than the c.m. velocity (either $Q \leq 0$ or $\alpha \leq \frac{1}{2}$ or both), then the lab deflection angle is restricted to a forward cone. This maximum angle would arise naturally here as the point which separated the two branches of V in (39). Two terms would be obtained for the delta function in V and the restrictions in the limits of succeeding integrals would be carried by unit functions using (10).

Integration of θ_V first is simpler. After the integration of φ_V following (36), the delta function may be written in the form $\delta(\cos\theta_V - \cos\theta_{V0})$, thus

$$\delta\left[\tfrac{1}{2}\mu\left(\underset{\sim}{v} - \tfrac{1}{2}\underset{\sim}{c}_0\right)^2 - \tfrac{1}{2}\mu u_1^2(c_0)\right] = \frac{1}{\mu V c_0}\,\delta(\cos\theta_V - \cos\theta_{V0}), \tag{46}$$

[13]See for example, E. Amaldi, Handbuch der Physik (Springer-Verlag 1959), Vol. XXXVIII/2, p. 95. Note the misprint in Eq. 33.20. The third term in the denominator should not be under the radical.

where $$\cos\theta_{V0} = \frac{1}{Vc_0}\left(V^2 + \frac{1}{4}c_0^2 - u_1^2(c_0)\right). \tag{47}$$

Substitution in the differential cross section may be accomplished by defining

$$\begin{aligned}\cos\theta_0 &= \cos\theta\Big]_{\theta_V=\theta_{V0}} = \left|\underset{\sim}{V} - \tfrac{1}{2}\underset{\sim}{c_0}\right|^{-1}\left(V\cos\theta_V - \tfrac{1}{2}c_0\right)\Big]_{\theta_V=\theta_{V0}} \\ &= \frac{1}{u_1(c_0)}\left[\frac{1}{c_0}\left(V^2 + \frac{1}{4}c_0^2 - u_1^2(c_0)\right) - \frac{1}{2}c_0\right] \\ &= \frac{1}{c_0 u_1(c_0)}\left(V^2 - \frac{1}{4}c_0^2 - u_1^2(c_0)\right),\end{aligned} \tag{48}$$

using (35), (47), and (38). Now (10) must be used for the integration of $\cos\theta_V$ from -1 to 1.

$$\lim_{T\to 0} R = 2\pi v_1 v_2 \int \frac{d\sigma}{d\Omega}(c_0,\theta_0)\left[H(1-\cos\theta_{V0}) - H(-1-\cos\theta_{V0})\right]\frac{V}{u_1}\,dV. \tag{49}$$

In performing the V integration, it is convenient to change variables since from (48)

$$\frac{V}{u_1(c_0)}\,dV = c_0\,d(\cos\theta_0). \tag{50}$$

Now the unit functions in (49) are functions of V and provide limits for the V integration and hence for the $\cos\theta_0$ integration. These limits are easily obtained from (48) or Fig. 2. When $\cos\theta_{V0}$ has the values ± 1, $\cos\theta_0$ has the same value. Hence (49) becomes

$$\lim_{T\to 0} R = 2\pi v_1 v_2 c_0 \int_{-1}^{1} \frac{d\sigma}{d\Omega}(c_0,\cos\theta_0)\,d(\cos\theta_0) \tag{51}$$

and (45) is obtained.

This digression also gives an example of a type of anisotropy which will be mostly ignored in the following. This is that introduced by the use of an anistropic probe, a beam. The results of such anisotropy are conceptually simple, being attributable to the c.m. to lab transformation. Formally such results are contained in the transformation factor in (44), which is rather complex even for this simple example. Further formal development of this effect is therefore avoided.

To return to the averaging process, after (31) one may integrate over the angles of $\underset{\sim}{q}$, that is over θ_q and φ_q. Generally as in the digression, it is convenient to take $\underset{\sim}{c_0}$ as the Z axis. However, for these two integrations let $\underset{\sim}{w}$ be the Z axis. Aside from $\hat{q}\cdot\hat{u}_0$, which it will be convenient to ignore at first[14], the only θ_q and φ_q dependence in (31) is in the exponential and delta functions. This choice of Z axis restricts the θ_q dependence to the delta function and the φ_q dependence to the exponential. Using the law of cosines of spherical trigonometry

$$\hat{c}_0\cdot\hat{q} = \cos\theta_w \cos\theta_q + \sin\theta_w \sin\theta_q \cos\varphi_q \tag{52}$$

Figure 3 is a polar diagram showing the relationship of these angles. The points labeled $\underset{\sim}{q}$, $\underset{\sim}{w}$ and $\underset{\sim}{c_0}$ represent the penetrations of these vectors of a unit sphere centered at their common origin. The arcs represent great circles subtending the indicated angles. In this labeling of the angle between $\underset{\sim}{c_0}$ and $\underset{\sim}{w}$, one anticipates the shift back to $\underset{\sim}{c_0}$

[14]See the remarks following Eq. (60).

as Z axis after the two integrations. For the φ_g integral[15]

$$\int_0^{2\pi} \exp\left(\frac{m}{kT} c_0 g \sin\theta_w \sin\theta_g \cos\varphi_g\right) d\varphi_g = 2\pi I_0\left(\frac{m}{kT} c_0 g \sin\theta_w \sin\theta_g\right), \quad (53)$$

where I_0 is a Bessel function of purely imaginary argument. This gives from (31)

$$R = 2\pi \mu v_1 v_2 \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \int e^{-\frac{m}{2kT}(c_0^2+g^2)} \frac{d\sigma}{d\Omega}(g, \hat{g}\cdot\hat{u}_0)$$
$$\times \exp\left(\frac{m}{kT} c_0 g \cos\theta_w \cos\theta_g\right) I_0\left(\frac{m}{kT} c_0 g \sin\theta_w \sin\theta_g\right)$$
$$\times \delta\left[\tfrac{1}{2}\mu\left(w^2 + wg\cos\theta_g + \tfrac{1}{4}g^2\right) - E_0\right] \frac{g^3}{u_0} dg\, d(\cos\theta_g)\, d^3v. \quad (54)$$

To simplify the θ_g dependence, one applies the identity [16]

$$\exp\left(\frac{m}{kT} c_0 g \cos\theta_w \cos\theta_g\right) I_0\left(\frac{m}{kT} c_0 g \sin\theta_w \sin\theta_g\right)$$
$$= \sum_{j=0}^{\infty} (2j+1)\, \mathcal{J}_j\left(\frac{m}{kT} c_0 g\right) P_j(\cos\theta_w)\, P_j(\cos\theta_g), \quad (55)$$

where $\mathcal{J}_j$ is a spherical Bessel function of purely imaginary argument. The delta function may be written

$$\delta\left[\tfrac{1}{2}\mu\left(w^2 + wg\cos\theta_g + \tfrac{1}{4}g^2\right) - E_0\right] = \frac{2}{\mu w g}\, \delta(\cos\theta_g - \cos\theta_{g0}), \quad (56)$$

where
$$\cos\theta_{g0} = \frac{1}{wg}\left(u_1^2(g) - w^2 - \tfrac{1}{4}g^2\right), \quad (57)$$

using (38). Now using (10) for the integration of $\cos\theta_g$, one obtains from (54)

[15] M. Abramowitz and I. Stegun (Eds.), Handbook of Mathematical Functions (National Bureau of Standards 1964), Eq. 9.6.16.
[16] A. Erdelyi (Ed.), Higher Transcendental Functions (McGraw-Hill 1953), Vol. 2, Eq. 7.15(43).

$$R = 4\pi v_1 v_2 \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \int e^{-\frac{m}{2kT}(c_0^2+g^2)} \frac{d\sigma}{d\Omega}(g, \hat{g}\cdot\hat{u}_1(g))$$
$$\times \sum_{j=0}^{\infty} (2j+1) \mathcal{I}_j\left(\frac{m}{kT} c_0 g\right) P_j(\cos\theta_w) P_j(\cos\theta_{g0})$$
$$\times \left[H(1-\cos\theta_{g0}) - H(-1-\cos\theta_{g0})\right] \frac{g^2}{w\, u_1(g)}\, dg\, d^3w. \tag{58}$$

Here d^3v has been replaced by d^3w to complete the transformation to the beam system. This transform is given by (28) and has a Jacobian of unity. Also the use of $\underset{\sim}{c}_0$ as the Z axis may now be resumed with no essential relabeling.

The integration over θ_w and φ_w may now be done. Since there is no function of φ_w, this integration yields a factor of 2π. The only functions of θ_w are the Legendre polynomials which have the useful orthogonality property

$$(2j+1)\int_{-1}^{1} P_j(\mu)\, d\mu = 2\,\delta_{0j} \tag{59}$$

Then since $\mathcal{I}_0(z) = z^{-1}\sinh z$ [17], one has from (58)

$$R = 8\pi v_1 v_2 \left(\frac{m}{2\pi kT}\right)^{\frac{1}{2}} \int e^{-\frac{m}{2kT}(c_0^2+g^2)} \frac{d\sigma}{d\Omega}(g, \hat{g}\cdot\hat{u}_1)$$
$$\times \sinh\left(\frac{m}{kT} c_0 g\right) \left[H(1-\cos\theta_{g0}) - H(-1-\cos\theta_{g0})\right] \frac{g\,w}{c_0 u_1}\, dg\, dw. \tag{60}$$

Though the neglect of the angular dependence of the differential cross section does not make it entirely credible, this result is exact. This point was arrived at by a short cut for which it was necessary to forego neutron angular information. This process is acceptable from a phenomenological position. One feels that when precise angular information

[17] M. Abramowitz and I. Stegun, op.cit., Eq. 10.2.13.

is obtained, allowing the construction of a differential cross section, it must be compatible with total cross sections already known. Thus one is not surprised at a process which simply by passes the more precise portion of the available information. It seems a formal justification of this process should be possible but for the present the justification to follow must suffice.

The result (60) will now be derived in more detail, obtaining the neutron angular distribution in the process. It is convenient to develop the differential cross section in a series of Legendre polynomials. This may be done with complete generality for functions of interest here using the completeness property of the Legendre polynomials. Moreover, it is also practical to do so inasmuch as experimental results for restricted energy ranges are often presented using only the first few terms of this development.

$$\frac{d\sigma}{d\Omega}(q,\theta) = \frac{1}{4\pi}\sum_{n=0}(2n+1)\,B_n(q)\,P_n(\cos\theta). \tag{61}$$

One notes the coefficient $B_0(q)$ is the total cross section

$$\sigma(q) = \int_{4\pi}\frac{d\sigma}{d\Omega}(q,\theta)\,d\Omega = B_0(q). \tag{62}$$

In order to shift the Legendre polynomials of (61) to a more convenient origin, consider the identity[18]

$$P_n(\cos\psi_1) = \left(\frac{\sin\psi_1}{\sin\psi_2}\right)^n \sum_{k=0}^{n}\binom{n}{k}\left(\frac{\sin(\psi_2-\psi_1)}{\sin\psi_1}\right)^{n-k} P_k(\cos\psi_2). \tag{63}$$

[18]E. Rainville, Special Functions (MacMillan 1960), Eq. 95(5).

This will be applied to the triangle $\underset{\sim}{u}_0 = \underset{\sim}{W} + \frac{1}{2}\underset{\sim}{g}$ in Fig. 1. Let $\psi_1 = \theta = \cos^{-1}\hat{g}\cdot\hat{u}_0$ and $\psi_2 = \theta_g = \cos^{-1}\hat{g}\cdot\hat{W}$ (using $\underset{\sim}{W}$ as the Z axis). Then $\psi_2 - \psi_1 = \cos^{-1}\hat{W}\cdot\hat{u}_0$ and the law of sines is

$$\frac{\sin\psi_1}{W} = \frac{\sin(\pi-\psi_2)}{u_0} = \frac{\sin(\psi_2-\psi_1)}{\frac{1}{2}g}$$

or

$$\frac{\sin\psi_1}{\sin\psi_2} = \frac{W}{u_0}, \quad \frac{\sin(\psi_2-\psi_1)}{\sin\psi_1} = \frac{g}{2W}. \tag{64}$$

This gives for (63)

$$P_n(\cos\theta) = \left(\frac{W}{u_0}\right)^n \sum_{k=0}^{n}\binom{n}{k}\left(\frac{g}{2W}\right)^{n-k} P_k(\cos\theta_g), \tag{65}$$

which gives for (61)

$$\frac{d\sigma}{d\Omega}(g,\theta) = \frac{1}{4\pi}\sum_{n=0}(2n+1)B_n(g)\left(\frac{W}{u_0}\right)^n \sum_{k=0}^{n}\binom{n}{k}\left(\frac{g}{2W}\right)^{n-k} P_k(\cos\theta_g). \tag{66}$$

The φ_g integration proceeds exactly as before and this expression may be substituted directly in (54). For the $\cos\theta_g$ integration however the delta function (56) is developed in Legendre polynomials

$$\frac{2}{\mu W g}\,\delta(\cos\theta_g - \cos\theta_{go}) = \frac{2}{\mu W g}\sum_{\ell=0}^{\infty}(\ell+\tfrac{1}{2})P_\ell(\cos\theta_g)P_\ell(\cos\theta_{go}). \tag{67}$$

Thus using (55), (66) and (67) one has instead of (54)

$$\begin{aligned} R = v_1 v_2 \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \int e^{-\frac{m}{2kT}(c_0^2+g^2)} & \sum_{\ell=0}^{\infty}(\ell+\tfrac{1}{2})P_\ell(\cos\theta_g)P_\ell(\cos\theta_{go}) \\ & \times \sum_{n=0}(2n+1)B_n(g)\left(\frac{W}{u_0}\right)^n \sum_{k=0}^{n}\binom{n}{k}\left(\frac{g}{2W}\right)^{n-k} P_k(\cos\theta_g) \\ & \times \sum_{j=0}^{\infty}(2j+1)\mathcal{J}_j\left(\frac{m}{kT}c_0 g\right)P_j(\cos\theta_w)P_j(\cos\theta_g)\frac{g^2}{W u_0}\,dg\,d(\cos\theta_g)\,d^3W. \end{aligned} \tag{68}$$

Each term in this multiple sum is a product of three Legendre polynomials in $\cos\theta_g$ and this is the only θ_g dependence. For this integration, one may apply the identity

$$\int_{-1}^{1} P_j P_k P_l \, d(\cos\theta_g) = \frac{2}{2l+1} (jk00|l0)^2 \tag{69}$$

which is a special case of a formula derived algebraically by Gaunt[19]. The quantity $(jk00|l0)$ is a Clebsch-Gordan coefficient which vanishes unless the indices are integers satisfying the triangle relations

$$j \leq k+l, \quad k \leq l+j, \quad l \leq j+k \tag{70}$$

and having an even sum. As to non-zero values[20], those of use here are

$$(0k00|l0) = \delta_{kl} \tag{71}$$

Applying (69) to (68), one obtains

$$R = v_1 v_2 \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \int e^{-\frac{m}{2kT}(c_0^2+g^2)} \sum_{n=0} (2n+1) B_n(g) \left(\frac{w}{u_1}\right)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{g}{2w}\right)^{n-k}$$

$$\times \sum_{j=0}^{\infty} (2j+1) \mathcal{J}_j\left(\frac{m}{kT} cg\right) P_j(\cos\theta_w) \sum_{l=0}^{\infty} P_l(\cos\theta_{g0}) (jk00|l0)^2$$

$$\times \left[H(1-\cos\theta_{g0}) - H(-1-\cos\theta_{g0})\right] \frac{g^2}{wu_1} \, dg \, d^3w, \tag{72}$$

where $\quad u_1^2 = u_1^2(g) = \frac{2\alpha}{\mu}\left(Q + \frac{1}{4} m g^2\right),$

and $\quad \cos\theta_{g0} = \frac{1}{wg}\left(u_1^2 - w^2 - \frac{1}{4} g^2\right).$

[19] J. A. Gaunt, Phil. Trans. Roy. Soc. London, A 228, 195 (1929), Eq. (14). A more recent, brief derivation is given by M. A. Preston, Physics of the Nucleus (Addison-Wesley 1962), p. 614, Eq. (A-69).

[20] For an explicit formula see L. C. Biedenharn, J. M. Blatt and M. E. Rose, Rev. Mod. Phys. 24, 250 (1952), Eqs. (2)-(5).

This expression contains the details of the neutron angular distribution. It is cylindrically symmetric about the beam direction since there is no φ_w dependence. However, no analytical way was found of obtaining this angular information in the absence of other variables. Instead some qualitative remarks will be made in the last section. Of course, given the $B_n(g)$, numerical integration of w and g will give the neutron angular distribution in the beam system. Also, there is no difficulty numerically in shifting to the lab system.

The φ_w integration yields a factor of 2π and the $\cos\theta_w$ integration is done using (59). Then using (71), the sum over ℓ may be performed. The resulting series in $P_k(\cos\theta_{go})$ is then reduced to one in $P_n(\hat{g}\cdot\hat{u}_1)$ using (65). This again gives (60), thus confirming the briefer development.

As preparation for the w integration, the unit functions must be written as functions of w. Since for a positive $H(ax) = H(x)$, one has

$$H(1-\cos\theta_{go}) - H(-1-\cos\theta_{go})$$
$$= H(w^2+wg+\tfrac{1}{4}g^2-u_1^2) - H(w^2-wg+\tfrac{1}{4}g^2-u_1^2). \tag{73}$$

Applying (11) to the first unit function, since $w \geq 0$, one has

$$H[f(w)] = H[f(0)] + \sum_i H(w-w_i)\,\mathrm{sgn}\, f'(w_i),$$
$$f(w_i) = 0, \quad f'(w_i) \neq 0 \tag{74}$$

where
$$f(w) = w^2 + wg + \tfrac{1}{4}g^2 - u_1^2. \tag{75}$$

Hence
$$w_i = -\tfrac{1}{2}g \pm u_1 \quad \text{or} \quad w_1 = u_1 - \tfrac{1}{2}g \tag{76}$$

since $w \geq 0$. Now

$$f(0) = \frac{1}{4}g^2 - u_1^2 = \frac{1}{4}g^2\left(1 - 2\alpha\frac{m}{\mu}\right) - \frac{2\alpha}{\mu}Q = \frac{1}{4}g^2\left[1 - 2\frac{m}{\mu}\left(1 - \frac{\mu}{2m}\right)\right] - \frac{2\alpha}{\mu}Q$$
$$= \frac{1}{2}g^2\left(1 - \frac{m}{\mu}\right) - \frac{2\alpha}{\mu}Q < 0, \tag{77}$$

using (38), (23) and $m > \mu$. Thus $H[f(0)] = 0$. Next

$$f'(w_1) = 2w + g\Big]_{w_1} = 2u_1 > 0 \tag{78}$$

since $Q > 0$. Thus $\operatorname{sgn} f'(w_1) = +1$ and

$$H\left(w^2 + wg + \frac{1}{4}g^2 - u_1^2\right) = H\left(w - u_1 + \frac{1}{2}g\right). \tag{79}$$

Similarly

$$H\left(w^2 - wg + \frac{1}{4}g^2 - u_1^2\right) = H\left(w - u_1 - \frac{1}{2}g\right). \tag{80}$$

This gives for (60)

$$R = 2v_1 v_2 \left(\frac{m}{2\pi kT}\right)^{\frac{1}{2}} \int e^{-\frac{m}{2kT}(c_0^2 + g^2)} \sum_{n=0} (2n+1) B_n(g) P_n(\hat{g}\cdot\hat{u}_1)$$
$$\times \sinh\left(\frac{m}{kT}c_0 g\right)\left[H\left(w - u_1 + \frac{1}{2}g\right) - H\left(w - u_1 - \frac{1}{2}g\right)\right]\frac{gw}{c_0 u_1}\, dg\, dw, \tag{81}$$

where

$$\hat{g}\cdot\hat{u}_1 = \frac{1}{gu_1}\left(u_1^2 + \frac{1}{4}g^2 - w^2\right). \tag{82}$$

The unit functions give the limits $u_1 - \frac{1}{2}g$ to $u_1 + \frac{1}{2}g$ for the w integration. Now note from (82)

$$\frac{w}{u_1}\, dw = -\frac{1}{2}g\, d(\hat{g}\cdot\hat{u}_1), \tag{83}$$

and when W has the above limits, $\hat{g}\cdot\hat{u}_1$ has the limits 1 and -1, respectively. Instead of W then, one integrates $\hat{g}\cdot\hat{u}_1$ and uses (62) to obtain

$$R = 2\nu_1\nu_2 \left(\frac{m}{2\pi kT}\right)^{\frac{1}{2}} \int e^{-\frac{m}{2kT}(c_0^2+g^2)} \sinh\left(\frac{m}{kT}c_0 g\right)\sigma(g)\,\frac{g^2}{c_0}\,dg. \qquad (84)$$

The integrand is the distribution of neutrons with respect to pair relative speed. If it were written in terms of pair relative energy, it would be the excitation function. Evaluation of the location of the maximum of this function shows what deuteron relative energy contributes the most thermonuclear reactions (provided one disregards the departure of the branch ratio from one-half). It is noteworthy that the cross section coefficients which describe the anisotropy, $B_n, n \neq 0$, have dropped out due to the orthogonality property of the Legendre polynomials. The excitation function is therefore unaffected by cross section anisotropy.

Before taking up the last integration, it is interesting to develop the special case of neutron production from a Maxwellian plasma. The process to be used is illustrative of the use of any distribution for the first ion population, say f_1 . If one were applying such a process at a sufficiently early stage of the beam-plasma development that the beam distribution still appeared as $\delta^3(\underset{\sim}{c}_1-\underset{\sim}{c}_0)$, then the process obviously is

$$\int f_1(\underset{\sim}{c}_0)\,\delta^3(\underset{\sim}{c}_1-\underset{\sim}{c}_0)\,d^3c_0 = f_1(\underset{\sim}{c}_1). \qquad (85)$$

This assumes f_1 is properly normalized and that the delta function is the only function of $\underset{\sim}{c}_1$ present. Now since $\underset{\sim}{c}_0$ has been held constant,

this same process may be applied at any point in the development. It amounts to a convolution of the beam-plasma neutron rate with the desired distribution written in terms of $\underset{\sim}{c}_0$.

The point of this digression is to take f_1 to be the same Maxwell distribution as for the second ion population. The process will be applied at the stage of (84). It is multiplied by

$$\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} e^{-\frac{m}{2kT}c_0^2} \tag{86}$$

and integrated over $\underset{\sim}{c}_0$. The result is the neutron production rate of a Maxwellian plasma R_M. In (84) the only functions of c_0 are the exponential, the hyperbolic sine, and a factor c_0^{-1}, so the relevant integral is

$$\begin{aligned}
&\left(\frac{m}{2\pi kT}\right)^2 \int e^{-\frac{m}{kT}c_0^2} \sinh\left(\frac{m}{kT}c_0 g\right) \frac{d^3c_0}{c_0} \\
&= 4\pi \left(\frac{m}{2\pi kT}\right)^2 \int_0^\infty e^{-\frac{m}{kT}c_0^2} \sinh\left(\frac{m}{kT}c_0 g\right) c_0\, dc_0 \\
&= 2\pi \left(\frac{m}{2\pi kT}\right)^2 \int_{-\infty}^\infty e^{-\frac{m}{kT}c_0^2} \sinh\left(\frac{m}{kT}c_0 g\right) c_0\, dc_0,
\end{aligned}$$

and integrating by parts

$$\begin{aligned}
&= \pi g \left(\frac{m}{2\pi kT}\right)^2 \int_{-\infty}^\infty e^{-\frac{m}{kT}c_0^2} \cosh\left(\frac{m}{kT}c_0 g\right) dc_0 \\
&= \tfrac{1}{2}\pi g \left(\frac{m}{2\pi kT}\right)^2 \int_{-\infty}^\infty \left(e^{-\frac{m}{kT}(c_0^2 - c_0 g)} + e^{-\frac{m}{kT}(c_0^2 + c_0 g)}\right) dc_0 \\
&= \tfrac{1}{2}\pi g \left(\frac{m}{2\pi kT}\right)^2 e^{\frac{m}{4kT}g^2} \int_{-\infty}^\infty \left(e^{-\frac{m}{kT}(c_0 - \frac{1}{2}g)^2} + e^{-\frac{m}{kT}(c_0 + \frac{1}{2}g)^2}\right) dc_0 \\
&= \pi g \left(\frac{m}{2\pi kT}\right)^2 e^{\frac{m}{4kT}g^2} \int_{-\infty}^\infty e^{-\frac{m}{kT}x^2} dx \\
&= \frac{1}{4\sqrt{\pi}}\, g \left(\frac{m}{kT}\right)^{\frac{3}{2}} e^{\frac{m}{4kT}g^2}.
\end{aligned} \tag{87}$$

Also, in (84) the pair density $\nu_1 \nu_2$ must be replaced by $\frac{1}{2}\nu^2$ where ν is the total ion density. This gives an increased neutron rate because now that the two populations are not distinguished, neutron production <u>within</u> each component must be counted. The factor $\frac{1}{2}\nu^2$ is the pair density counted without regard to order, except for a slight inaccuracy in that this also includes "self-pairs". This inaccuracy is insignificant for densities and volumes of interest here. One then obtains

$$R_M = \frac{\nu^2}{4\sqrt{\pi}} \left(\frac{m}{kT}\right)^{\frac{3}{2}} \int e^{-\frac{m}{4kT} g^2} \sigma(g)\, g^3\, dg \tag{88}$$

in agreement with Eq. (13) of Faust and Harris[21]. However, the present approach shows their assumption of cross section isotropy is not required.

5. THE TOTAL NEUTRON RATE

Upon integrating the relative ion velocity, (84) yields the total neutron rate. At this point, the total cross section is still in general form. To proceed further one must at least specify the functional form of the cross section. The most accurate procedure at this point would simply be to use experimental data for the cross section. However, the nature of the calculation to follow allows considerable latitude here. Hence, to simplify the numerical work and continue in the spirit of a parameter study, the Gamow form for the total cross section has been adopted[22].

[21]See Footnote 1, the first reference.
[22]G. Gamow and C. L. Critchfield, Theory of Atomic Nucleus and Nuclear Energy-Sources (Oxford 1949), p. 266.

$$\sigma(g) = \frac{a}{g^2} \exp\left(-\frac{b}{g}\right). \tag{89}$$

Here a and b are constants obtained by fitting experimental data. This gives for the total neutron rate.

$$R = 2a v_1 v_2 \left(\frac{m}{2\pi kT}\right)^{\frac{1}{2}} \frac{1}{c_0} \int_0^\infty e^{-\frac{m}{2kT}(c_0^2+g^2)-\frac{b}{g}} \sinh\left(\frac{m}{kT} c_0 g\right) dg. \tag{90}$$

To obtain the integral in dimensionless form, substitute

$$x = \frac{g}{b}, \quad c = \frac{c_0}{b}, \quad t = \frac{kT}{mb^2}. \tag{91}$$

Then

$$R = \frac{2a}{bc} v_1 v_2 (2\pi t)^{-\frac{1}{2}} \int_0^\infty e^{-\frac{1}{2t}(x^2+c^2)-\frac{1}{x}} \sinh\left(\frac{c}{t} x\right) dx. \tag{92}$$

In the last section, it was noted that anisotropy of cross section contributes nothing to the total neutron rate. Furthermore the total neutron rate of a Maxwellian plasma is well known[23]. Thus, the major effect in (92) which remains to be elucidated is that peculiar to the class of reactions typified by the prototype reaction, the effect of a beam-like anisotropy. To exhibit this a comparison will now be made between the total neutron rate from a beam-plasma configuration and the total neutron rate from a Maxwellian plasma of the same total density and temperature.

From (88) one finds for a Maxwellian plasma of ion density v

[23] It is calculated in each of the references of Footnote 3.

and temperature T the total neutron rate

$$R_M = \frac{\nu^2}{4\sqrt{\pi}} \frac{a}{b} t^{-\frac{3}{2}} \int_0^\infty e^{-\frac{x^2}{4t} - \frac{1}{x}} x\,dx. \tag{93}$$

Now in the beam-plasma configuration, neutrons arise not only from the interaction of the beam with the plasma, but also from the plasma's interaction with itself. The latter may be calculated from (93) by replacing ν by ν_2; this neutron rate will be designated R_P. Since the total ion densities are to be the same for this comparison, we have

$$\nu = \nu_1 + \nu_2 \tag{94}$$

One may define the beam-plasma density ratio $\beta = \nu_1/\nu_2$. Thus the desired comparison is

$$\rho(c, t, \beta) = (R_P + R)/R_M$$
$$= (1+\beta)^{-2} \left\{ 1 + \beta \frac{\frac{2}{c}(2\pi t)^{-\frac{1}{2}} \int_0^\infty e^{-\frac{1}{2t}(x^2+c^2) - \frac{1}{x}} \sinh\left(\frac{c}{t}x\right) dx}{\frac{1}{\sqrt{2\pi}}(2t)^{-\frac{3}{2}} \int_0^\infty e^{-\frac{x^2}{4t} - \frac{1}{x}} x\,dx} \right\}. \tag{95}$$

The comparison is independent of the total ion density and the Gamow cross section constants. It is of particular interest to ask for what beam velocities can the total neutron rate of the beam-plasma configuration exceed that of the Maxwellian plasma of equal density and temperature. To best answer this the beam plasma ratio β may be adjusted to achieve maximum ρ. The form of ρ in β is

$$\rho = (1+\beta)^{-2}(1+\beta A) \tag{96}$$

For an extremum $\left.\frac{\partial \rho}{\partial \beta}\right]_{\beta_0} = 0$ or $\beta_0 = 1 - \frac{2}{A}$. (97)

Since $0 \leq \beta$, an extremum exists only for $A \geq 2$ and since $\left.\frac{\partial^2 \rho}{\partial \beta^2}\right]_{\beta_0} < 0$, it is a maximum when it exists. Also $\beta_0 \leq 1$. The maximum is

$$\rho_M = \rho(c, t, \beta_0) = \frac{A^2}{4(A-1)} \quad , \; A \geq 2. \tag{98}$$

The answer to the question then is presented in Fig. 4, a contour plot of $A(c,t)$. When for some (c,t) , $A > 2$, then the beam-plasma configuration will produce more neutrons than a Maxwellian plasma of equal density and temperature, provided the optimum beam-plasma ratio is used.

This presentation, Fig. 4, is based on numerical evaluation of A on a mesh of 143 points of (c,t) in the ranges plotted. In the numerical integration, Simpson's rule was used with binary partition until the new sum differed from the previous one by less than a fixed relative tolerance. The first partition was chosen near the maximum point of the integrand to avoid premature tolerance satisfaction. End points were chosen where the integrand fell below a finer relative tolerance than the previous one. This tolerance was so chosen that varying it changed the integral by less than the previous tolerance for the case of slowest convergence, $(c,t) = (1,1)$. End points and first partition point were recomputed for each point (c,t) . These procedures were arrived at through a knowledge of the form of the integrand in the numerator of A gained by use of the Laplace integral estimate technique. It has a single maximum somewhat to the right of $x = c$, determined by solution of the transcendental equation,

$$x_0^3 - c\,x_0^2 \coth\left(\frac{c}{t} x_0\right) - t = 0. \tag{99}$$

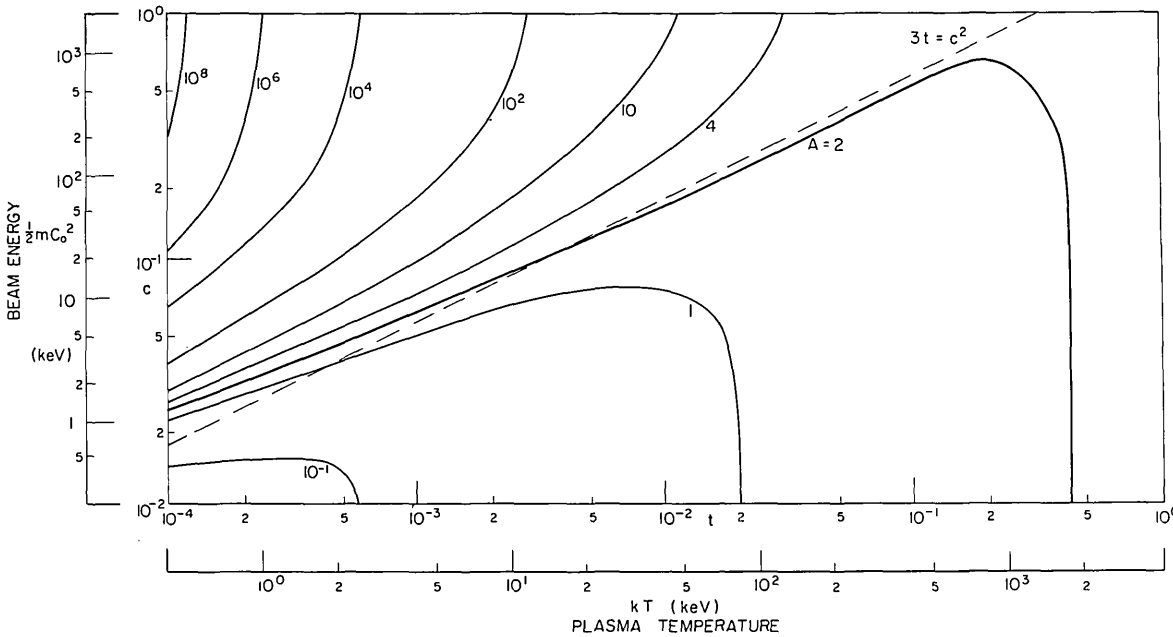

Fig. 4 - Comparison of Total Neutron Rate for Beam-Plasma and Maxwellian Plasma Distribution

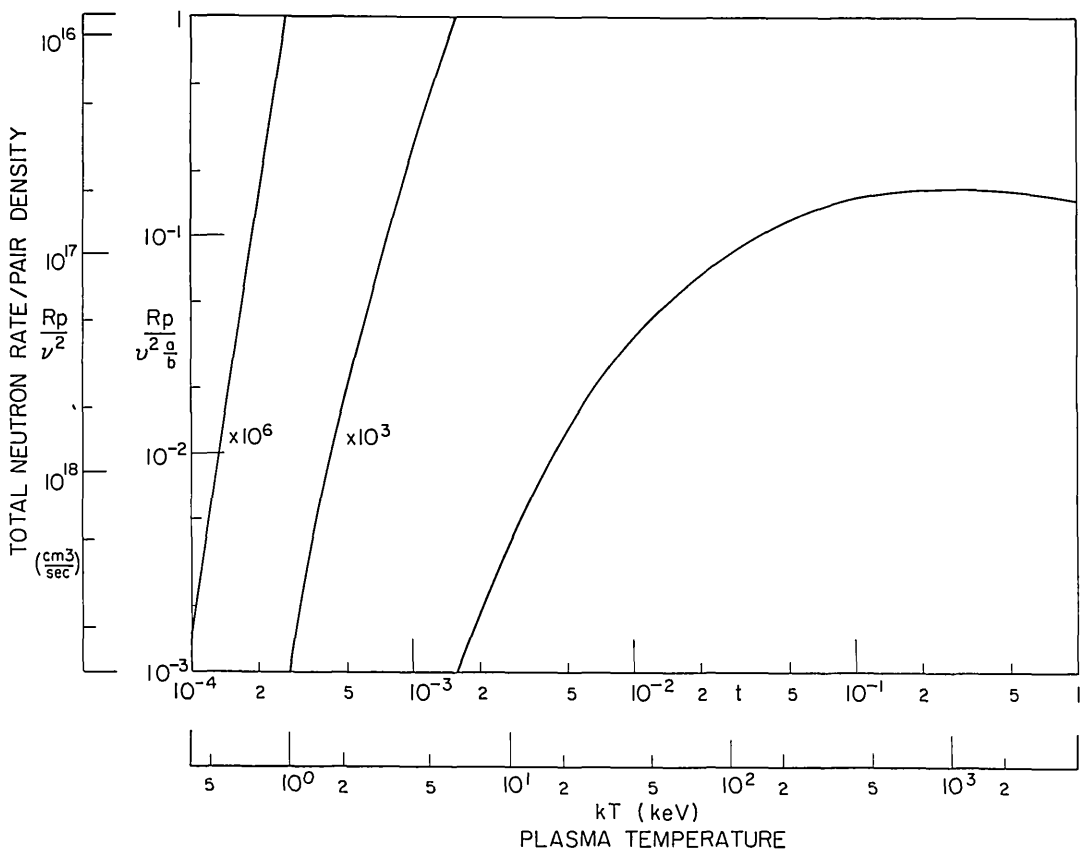

Fig. 5 - Total Neutron Rate of a Maxwellian Plasma

The width of this peak is controlled largely by t and the maximum is quite broad for $t \gtrsim 1$. These calculations were performed on the NAREC computer using NELIAC compiler.

Along side the dimensionless scales are scales in practical units. The latter are dependent on the values of the Gamow cross section constants. The constants used here were obtained from the Theus, McGarry and Beach[4] data applied to the neutron branch of the deuterium-deuterium reaction by curve fitting. This data was given in deuteron lab energy in 11 points from 20 to 350 keV. The root-mean-square relative error for this fit is

$$\left[\Sigma_i \left(\frac{\sigma_i'}{\sigma_i} - 1 \right)^2 / \Sigma_j j \right]^{\frac{1}{2}} = .1486. \tag{100}$$

This rather large ratio is indicative of the fact that a fit of the Gamow form to neutron data is a force fit which neglects or at least smooths over the energy variations of the neutron branch ratio. This number would indicate rather poor justification for the severe extrapolation of the fit which is required in the evaluation of the integrals in A , were it not for two circumstances. The first is the lack of dependence of A on a and b . Moreover, since A is a ratio in which both numerator and denominator depend on the form of the cross section, there is an implication that A would be nearly independent of small departures from the assumed form. The second is that the region of greatest contribution to these integrals is in the region of greatest confidence except when $c, t \gtrsim 1$. The values obtained by this fit are

$$a' = 182.1 \text{ barns-keV}, \; b' = 45.85 \, (\text{keV})^{\frac{1}{2}}. \tag{101}$$

Using relations between c.m. and lab systems and (91), the relationships of the scales are given by

$$\tfrac{1}{2} m c_0^2 = \tfrac{1}{2} m b^2 c^2 = b'^2 c^2 = 2102 c^2 \text{ keV}, \tag{102}$$

$$kT = m b^2 t = 2 b'^2 t = 4202 \, t \text{ keV}, \tag{103}$$

and

$$\frac{a}{b} = \frac{a'}{b'} \sqrt{\frac{2}{m}} = 1.230 \times 10^{-16} \, \frac{\text{cm}^3}{\text{sec}}. \tag{104}$$

From the practical scales one sees, for example, that a beam-plasma configuration with a 10 keV temperature requires a beam energy of more than 15.4 keV and a favorable beam-plasma density ratio to produce more neutrons than a Maxwellian plasma of equal density and temperature. Since the average energy of a plasma ion is $\frac{3}{2} kT$ or 15 keV, this beam-plasma configuration makes a slightly less efficient use of ion energy in the production of neutrons than a pure Maxwellian plasma. To extend the observation of efficiency to other temperatures, one may draw the line,

$$c^2 = 3t \quad \text{or} \quad \tfrac{1}{2} m c_0^2 = \tfrac{3}{2} kT. \tag{105}$$

Below this line, beam ions have less energy than plasma ions. Thus, at temperatures where $A = 2$ falls below $c^2 = 3t$, above 15 keV, it is possible to increase neutron production efficiency with a beam-plasma configuration. The increase allowed, however, is rather small due to the small variation in A in this region. In fact the major feature of

Fig. 4 is that at low temperatures raising the beam energy can increase neutron production very significantly.

The denominator of A, Fig. 5, is presented to allow the calculation of other quantities from Fig. 4. Because of the lack of confidence previously expressed in the value in (104), it is not intended for comparison with experiment. However, it does remain within a factor of 2 in ranges for which experimental values are available[24]. Using Fig. 5, it is possible to compare the neutron rate of an optimum beam-plasma configuration with that of a Maxwellian plasma of equal density and average ion energy. Such a comparison is not applicable for $A \leq 2$. From this comparison, one finds the most efficient beam-plasma configuration in the ranges plotted, excluding those near $A=2$ where $\beta_0 \simeq 0$, is near $(c,t) = (.5, 10^{-4})$. Because of the low temperature, the cold target limit of the beam-plasma configuration is a good approximation. The Maxwellian plasma having equal density and average energy has $t = 4 \times 10^{-2}$ and produces 1.5 times as many neutrons per second. Thus, even the optimum beam-plasma configuration makes a less efficient use of particles and their kinetic energy than a Maxwellian plasma.

6. DISCUSSION AND SUMMARY

Knowing the behavior of the total neutron rate as a function of beam speed and plasma temperature, one may regress and discuss the qualitative behavior of the neutron rate distributions as functions of g,

[24] J. L. Tuck, Nuclear Fusion 1, 201 (1961).

W and θ_W , using equations (84), (81), and (72). In principle, the desired behavior for each of these variables individually may be obtained by appropriate averages of the equations listed. However, because these averages do not yield to general analytical methods, except those used, the approach of this discussion will be to abstract known behavior from these equations in order to isolate prominent behavior in the desired variables. In retrospect, the main effects of relevant averages can be casually approximated using the sizes of parameters describing the reaction.

The behavior of the integrand of (84), the neutron rate distribution in g , was mentioned in the last section. It has a maximum determined by (99) to be near c_0 but larger, and the width of this maximum is controlled by kT . Now consider (81) and let g be such as to give this maximum. The effect of the unit functions is to restrict the range of W to $u_1 - \frac{1}{2}g \leq W \leq u_1 + \frac{1}{2}g$ or, using (38), to

$$\sqrt{\frac{2\alpha}{\mu}\left(Q+\frac{1}{4}mg^2\right)} - \frac{1}{2}g \leq W \leq \sqrt{\frac{2\alpha}{\mu}\left(Q+\frac{1}{4}mg^2\right)} + \frac{1}{2}g. \qquad (106)$$

Now $Q \cong$ 3.26 MeV, $\alpha \cong$ 3/4 so that one may take $kT << \alpha Q$ without disregarding any temperatures of interest. Then, as g varies within a half width (controlled by kT) of the maximum, the allowed variation of W is relatively small. Within this variation one can infer a two peaked structure for the neutron rate as a function of W , knowing the behavior of the B_n . Because the reacting particles are identical, the angular distribution of the products must be symmetrical about the $\frac{1}{2}\pi$ deflection

cone. This requires all odd ordered coefficients to vanish. The behavior of B_0 has been accounted for. Experimentally, it has been found sufficient to obtain only B_2 and B_4 in the lab energy range 20 to 350 keV[4]. In general B_4 is small throughout this range. B_2 however is significant at the low end and very substantial at the high end, giving a nearly $\cos^2\theta$ distribution. Thus, the Legendre polynomial expansion in (81) has maxima when $\cos^{-1}\hat{g}\cdot\hat{u}_1$ is near 0 or π. Hence, in general the neutron rate distribution in W has two peaks very near $\sqrt{\frac{2\alpha}{\mu}Q}$, the smaller being associated with small deflection angles. Actually, this applies strictly only when $\frac{1}{2}mc_0^2 > kT$. Suppose $\frac{1}{2}mc_0^2 < kT$, then the maximum of the neutron rate in g will be near $g=0$ but the width of the maximum will be relatively large. In this case, the two peaks of the neutron rate in W may fuse and become indistinguishable.

Knowing the conspicuous behavior of neutron rate in g and W, one may now turn to (72) in the hope of extracting angular behavior. Consider the $n=0$ term

$$B_0(g)\sum_{j=0}^{\infty}(2j+1)\mathcal{J}_j\left(\frac{m}{kT}c_0 g\right)P_j(\cos\theta_w)P_j(\cos\theta_{g0}). \tag{107}$$

The $\mathcal{J}_j$ have rising exponential behavior (eventually overcome by the Gaussian factor) with $\mathcal{J}_0(0)=1$ but $\mathcal{J}_j(0)=0,\ j\neq 0$. When $\frac{m}{kT}c_0 g$ is small $\left(\frac{1}{2}mc_0^2 < kT\right)$, the $\mathcal{J}_0$ term is the most important and the neutron distribution is spherical. As $\frac{m}{kT}c_0 g$ becomes larger $\left(\frac{1}{2}mc_0^2 > kT\right)$, the higher order terms contribute. Then the two Legendre polynomials contribute most when their arguments behave similarly.

Thus, the distribution in θ_W becomes more and more like that of θ_{go}. The latter is related to $\cos^{-1}\hat{g}\cdot\hat{u}_1$ by (65) with $u_0 \to u_1$ and $\cos\theta \to \hat{g}\cdot\hat{u}_1$. Because high beam energies are of little interest, one may require $\frac{1}{2}mc_0^2 < \alpha Q$. Then in (65), the ratio $\frac{W}{u_1}$ is near unity but $\frac{g}{2W}$ is small and the $k=n$ term dominates. Thus for small shifts of origin, (65) has the effect of passing on nearly the same distribution. Hence the neutron distribution in θ_W closely resembles that in $\cos^{-1}\hat{g}\cdot\hat{u}_1$, i.e., it is a $\cos^2\theta$ distribution with maxima at 0 and π. This result is not changed in character by consideration of terms of higher order in n. It may be seen that of these two cases, the spherical behavior is that expected in the pure Maxwellian plasma subcase, whereas $\cos^2\theta$ behavior is expected in the cold target subcase.

This report has considered neutron production by fusion in a plasma having an extremely anisotropic ion distribution, called the beam-plasma configuration. Any plasma may be analyzed in terms of this one whenever it is possible to separate the plasma into an isotropic and an anisotropic component. The beam-plasma configuration is thus a prototype for a large class of anisotropic plasmas. This configuration was described by statistical-mechanical distributions, and analysis of necessary assumptions showed that the configuration could be applied for small volumes and short times in many plasma regimes of thermonuclear interest. Then using the fusion reaction neutron cross section, the total neutron rate was obtained as an average. A numerical evaluation was made particularly to bring out neutron rate behavior as a function of beam energy and the

results were exhibited in a contour plot. Certain neutron rate distributions of thermonuclear interest were interpreted in terms of two subcases, neutron production from a cold target and from a Maxwellian plasma. Cross section anisotropy was found to have no effect on total neutron rate or on the excitation function. Expressions were obtained giving the detailed effects of anisotropy on neutron rate, speed and angular distributions. The expectation from symmetry argument, that the neutron distribution is cylindrical, was confirmed in detail. Introduction of experimental data showed anisotropic effects are small since the large reaction energy dominates in cases of interest. In practice then the use of neutron measurements to determine plasma anisotropy requires both energy and time resolution.

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INDEX OF SYMBOLS

Symbol	Name, Page of first appearance or definition
A	Comparison ratio, β independent part (see ρ), 34.
a, a'	Gamow cross section parameter, c.m. system (prime denotes lab system), 33, 38.
α	Neutron's share of the reaction energy, 14.
B_n	Coefficients in the Legendre polynomial expansion of $\frac{d\sigma}{d\Omega}$, 25.
b, b'	Gamow cross section parameter, c.m. system (prime denotes lab system), 33, 38.
β, β_0	Ratio of beam and plasma ion densities[25], 34.
c	Beam speed, unitless, 33.
$\underset{\sim}{c}_0, \underset{\sim}{c}_1, \underset{\sim}{c}_2$	Ion velocity, lab system (subscript 0 denotes beam,[26]), 11.
$\frac{d\sigma}{d\Omega}$	Differential cross section for $D(d,n)He^3$, c.m. system, 13.
δ, δ^3	Delta function, vector delta function, 9, 10.
E, E_0	Neutron energy, c.m. system[27], 14.
$\underset{\sim}{G}$	Velocity of the c.m. of an ion pair, 13.
$\underset{\sim}{g}(g, \theta_g, \varphi_g)$	Relative velocity of an ion pair[27], 13.
H	Unit function, 9.
I_0	A cylindrical Bessel function of purely imaginary argument, 23.

[25]Subscript denotes optimum value.
[26]Subscripts 1 and 2 refer to plasma components.
[27]Subscript 0 denotes use as a zero of a delta function argument.

$\mathcal{L}_j$	A spherical Bessel function of purely imaginary argument, 23.
k	Boltzmann's constant, 11.
m	Deuteron mass, 11.
μ	Neutron mass, 14.
ν, ν_1, ν_2	Ion density (no subscript denotes total ion density,[26]) 34, 11.
P_n	A Legendre polynomial, 23.
Q	Reaction energy for $D(d,n)He^3$, 14.
R, R_M, R_P	Neutron production rate (no subscript denotes non-Maxwellian beam-plasma contribution, M denotes pure Maxwell distribution, P denotes the Maxwellian contribution within the total beam-plasma production), 10, 14, 31, 34.
ρ, ρ_M	Ratio of the total beam-plasma neutron rate to that of a Maxwellian plasma of the same total density and temperature, 34.
sgn	Sign function, 9.
σ	Total cross section for $D(d,n)He^3$, 10, 25.
T	Temperature of Maxwellian component, 11.
t	Temperature, dimensionless, 33.
θ, θ_0	Neutron c.m. deflection angle[27], 13, 17, 18.
$\underset{\sim}{u}, \underset{\sim}{u}_0, \underset{\sim}{u}_1$	Neutron c.m. velocity (subscripts number uses as a zero of the argument of various delta functions), 14, 17, 19.
$\underset{\sim}{v}(v, \theta_v, \varphi_v)$	Neutron lab velocity, 15.
$\underset{\sim}{w}(w, \theta_w, \varphi_w)$	Neutron beam system velocity, 15.
X, X_0	Dimensionless form of g [25], 33, 35.

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13. ABSTRACT

The neutron production rate of a fusion reaction in a plasma of deuterium is calculated by a non-Maxwellian average, assuming the differential cross section is known. For the average it is assumed the ion velocity distribution of one plasma component is that of a beam and that of the other component is a Maxwell distribution. It is found that cross section anisotropy has no effect on neutron production rate or on the excitation function. An expression is obtained which contains the detailed effects of anisotropy on neutron speed and angle distributions. Prominent behavior of this expression is discussed. For purposes of numerical calculation, a Gamow form is assumed for the total cross section. Neutron production rate results are presented in a contour plot as a function of beam energy and plasma temperature over ranges of the parameters of most interest in thermonuclear research. Prominent behavior here is understood in the light of two well known subcases of this calculation: (1) neutron production by bombarding a cold target and (2) neutron production from a Maxwellian plasma. A qualitative analysis with order of magnitude estimates is made to consider the practical implications of the use of the assumed nonequilibrium distribution. It is shown that the assumed distribution may exist for small observation volumes and for short observation times in plasma regimes of thermonuclear interest. It is a general result of this analysis that an arbitrary plasma cannot be shown isotropic except by fine time resolution technique.

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