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SUBJECT

REPORT ON

INFLUENCE OF DURATION OF IMPACT ON

DAMAGE TO SOME ELASTIC SYSTEMS

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NAVAL RESEARCH LABORATORY

BELLEVUE, D. C.

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NAVY DEPARTMENT

Report on

INFLUENCE OF DURATION OF IMPACT ON DAMAGE TO SOME ELASTIC SYSTEMS

RESTRICTED

NAVAL RESEARCH LABORATORY
ANACOSTIA STATION
WASHINGTON, D. C.

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ABSTRACT

In order to estimate the effects on mounted apparatus of shock or impact loading, simplified elastic systems may be taken to represent the actual structures, so as to make possible mathematical analysis. One such simplified system is a simple mass-spring system attached to the frame or table which is subjected to shock. An analysis of the probable damage done to this system by a drop test type of impact has been made previously. Another system is a cantilever beam, whose clamped end is subjected to a specified transverse impact. This system differs fundamentally from the simple mass-spring system in that its flexibility and mass are distributed, rather than lumped.

In this report a mathematical solution is obtained for the deflection and stresses in a cantilever beam, whose base is given a transverse impact of a type approximating that obtained in a drop test or other simplified shock loadings. It is found that the maximum stress, considered as a function of the duration of impact, differs considerably from those of the simple mass-spring system.

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Influence of Duration of Impact on

Damage to Some Elastic Systems

INTRODUCTION

(A) Authorization

1. This work was authorized by Bureau of Ships Project 1438/42 of 24 September 1941.

(B) Statement and Discussion of Problem

2. The proper protection and design of apparatus to enable it to withstand severe mechanical shock is a problem of foremost importance. Considerable has been learned of the nature of the mechanical shock that shipboard apparatus must be expected to withstand. Some of this knowledge has been applied in the construction of naval gear. Designers have, however, been handicapped by a lack of fundamental knowledge of the damaging effects of shock impulses of short time duration.

3. The problem of determining the damaging effects of shocks is of enormous scope. Both the varieties of shock and the types of apparatus are innumerable. Any discussion of this problem must therefore begin with many assumptions and simplications, which must always be kept in mind. An analysis of part of this problem was made by Dr. R. D. Mindlin (1)*, in which the damage effect was considered under conditions representing those of a "drop-test". An apparatus for the performance of a test of this nature is described by the American Standard Association.(2) Figure 1 shows a simplified picture of the arrangement considered by Mindlin. A heavy, rigid table of mass M has a light, rigid mass, m_1, flexibly mounted on it by spring k_1. A spring bumper, K, is attached to the lower side of M. The assembly falls a distance h and strikes an anvil plate. The direction of motion of M is reversed in a sinusoidal manner, and in a time determined principally by K and M. The whole assembly then moves away from the anvil, and is caught before rebounding.

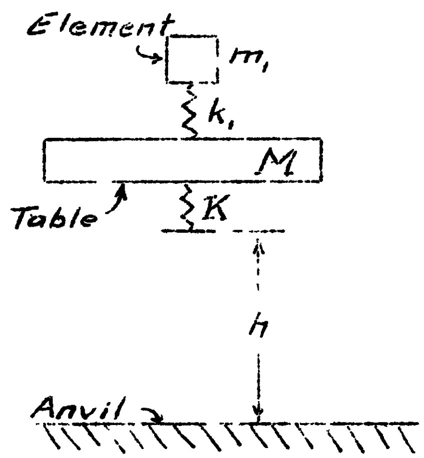

Figure 1

* Refer to Bibliography, Page 16

4. The motion of the mass m_1 relative to M is a function of their respective masses, the spring constants, and the striking velocity. The maximum relative displacement, which is the amount by which the spring k_1 is compressed, may be taken as a measure of the possible damaging effect on the system, of a given shock of this type. Thus for the system shown in Figure 1, an estimate of the probable damage to the mounted element, or to the spring k_1, may be obtained from the solution of the deflection of the spring.

5. Another case of fundamental interest is illustrated in Figure 2. A light cantilever beam is horizontally mounted on a rigid, heavy table, M. A spring bumper, K, is attached to the lower side of the table. The assembly is dropped, as in the previous case, and it is desired to determine the probability of damage to the beam.

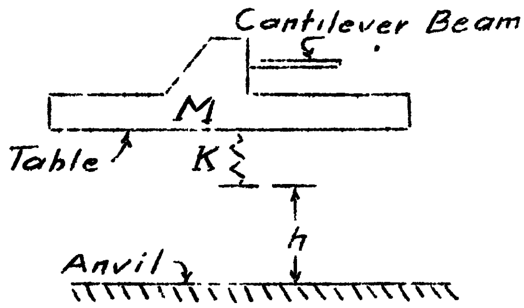

Figure 2

6. The probable damage to the beam by the impact is assumed to be determined by the maximum bending stress in the beam. Hence the criterion of the severity of the impact in this case is the curvature of the beam at its clamped end, rather than the displacement of the beam. Thus to evaluate the effect of the shock we have to obtain first the deflection curve of the beam, and from this derive the curvature at the clamped end, both during and following the impact. These are given as general functions of the table "stopping frequency" ω, the striking velocity V, and the elastic properties of the beam, by an analysis given later in this report.

ANALYSIS OF DEFLECTION AND STRESS IN A CANTILEVER BEAM

7. The system considered is shown in Figure 2. The cantilever beam is assumed to be light in comparison with the table, so that the motion of the table during the impact is governed only by its own mass and the bumper spring constant. The motion of the table during the impact is taken as

$$u_L = \frac{V}{\omega} \sin\omega t \qquad (0 \leq t \leq \frac{\pi}{\omega})$$

where u_L = displacement of table (= displacement of clamped end of beam)

$V = \sqrt{2gh}$ = velocity of table just before impact

$\omega = \sqrt{\frac{K}{M}}$ (see Figure 2)

The quantity $\omega = \sqrt{\frac{K}{M}}$ will be referred to as the "stopping frequency" of the table. There is, of course, no continuous vibration of the table at this frequency. The duration time of the impact is $T = \frac{\pi}{\omega}$. It is assumed that after the impact is completed, i.e. for $t \geqslant \frac{\pi}{\omega}$, no forces act on the system.

8. The deflection curve of the beam is the solution of the partial differential equation

$$\frac{\partial^4 u}{\partial x^4} + \frac{w}{gEI} \frac{\partial^2 u}{\partial t^2} = 0$$

which satisfies the required boundary conditions, both during and following the impact. In this equation $u(x,t)$ is the displacement of the beam measured from its initial straight position at the start of the impact, w is the weight per unit length of the beam, E is Young's modulus, I is the cross-sectional moment of inertia of the beam, and g is the acceleration due to gravity. A complete solution of this problem is given in Appendix 1 of this report.

RESULTS AND CONCLUSION

9. A curve showing the maximum deflection of the spring, k_1, which is a measure of damage probability for the first considered case (see paragraph 4 and Fig. 1) is given in Plate 1. This curve was plotted from the equations derived by Mindlin.[1] For the interpretation of this graph it is best to consider a fixed arrangement of mounted apparatus, i.e. consider ω_1 a parameter of any fixed value. The variables considered are then x_m and ω. The time of impact is the half period corresponding to ω, which is equal to $\frac{\pi}{\omega}$. The graph indicates that the probability of damage (see paragraph 4) increases as ω increases until a maximum value is reached, with nothing unusual happening in the neighborhood of $\omega_1 = \omega$. For all practical considerations this maximum value is attained when $\omega \geqslant 3\omega_1$. This means that the probability of damage is independent of the impact time, if the impact time is short compared with the period of the flexibly mounted

apparatus. It means that, under those conditions, the change of velocity is the important factor, and that the magnitudes of accelerations involved, for a given velocity change, are of little importance.

10. In the above analysis the mass of the flexible mounting was neglected. Plate 2 illustrates the effect of impacts of various time durations on a cantilever beam, which is a system with distributed flexibility and mass. The bending stress at the clamped end of the cantilever beam is plotted as a function of the "stopping frequency" of the table. The impact time is, as before, equal to one half the period associated with this frequency, or $\frac{\pi}{\omega}$. The frequency and other properties of the beam should be considered as fixed parameters for any given set-up. The curve demonstrates that, for a given velocity change, the bending stresses continue to increase as the time of impact decreases. Thus high magnitude accelerations, of time duration short compared to the period of the beam, may be of considerable importance in this case in contrast to the previous case. It has been determined that the maximum bending stress at the clamped end occurs during the impact when $\omega/\alpha_1 < 1$, and after the impact when $\omega/\alpha_1 > 1$.* The curves of Plates 1 and 2 are quantitative in nature and provide means of estimating numerical values for practical problems that may be simplified to fit the assumed conditions. It should be noted that, as far as mathematical analysis is concerned, the present solution applies to many types of shock or impact loading of elastic structures, as well as to the "drop test" type of shock. For example, if a velocity -V were superimposed on all parts of the systems illustrated in Figs. 1 and 2, the motion of the table would approximate that of the anvil plate of a hammer-type shock machine. However, the questions of the suitability of using a sinusoidal displacement function, and of the interpretation of the analytical results, must be decided by experimental work. This is now in progress. Further discussion of the limitations of these results is given in Paragraphs 8 and 9 of the Appendix.

* α_1 is the fundamental frequency of free vibration of the beam.

APPENDIX I

ANALYSIS OF DEFLECTION AND STRESS IN A CANTILEVER BEAM

SUBJECTED TO A DROP-TEST TYPE OF TRANSVERSE IMPACT

1. It is required to determine the maximum bending stress in a cantilever beam, when the clamped end undergoes a specified change of motion. The physical conditions are illustrated in Figure 2. A beam of small mass is clamped in a horizontal position to a heavy table. The table is dropped on a flexible spring and the direction of motion of the table is reversed sinusoidally.

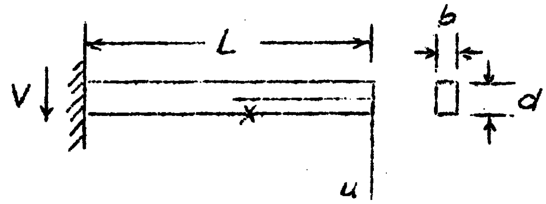

Figure 3

2. Some nomenclature to be used is given below:

t = time (t is taken as zero at the instant the table springs make contact with the anvil).

x = coordinate giving position along length of beam (x=0 at free end ; see (Fig. 3).

u(x,t) = displacement of beam, measured from its original straight position at the instant the table strikes the anvil (see Fig. 3)

$k^2 = \frac{gEI}{w}$

E = Young's modulus of material of beam

I = cross-sectional moment of inertia of beam $= bd^3/12$ (see Fig. 3)

b = width of beam

d = depth of beam

L = length of beam

w = weight of beam per unit length

g = acceleration due to gravity

V = striking velocity of table

$\omega = \sqrt{\frac{K}{M}}$ = "stopping frequency" of table

$T = \frac{\pi}{\omega}$ = duration time of impact

$\gamma^2 = \frac{L^2}{k}\omega$

$\beta_n (n=1,2\ldots\infty)$ = roots of frequency equation of cantilever, $1 + \cosh\beta_n \cos\beta_n = 0$

$\alpha_n = \frac{k}{L^2}\beta_n^2$ = natural frequencies of cantilever

Additional notations used in the analysis are defined as introduced.

3. The deflection curve of the beam is the solution of the equation

$$\frac{\partial^4 u}{\partial x^4} + \frac{1}{k^2}\frac{\partial^2 u}{\partial t^2} = 0 \qquad (1)$$

subject to the boundary conditions given below:

a. The initial and terminal conditions required during the impact are as follows:

$$t = 0 : \qquad u = 0, \qquad \frac{\partial u}{\partial t} = V \qquad (2a)$$

These express the conditions that the bar was initially straight and moving with a velocity V. For $t \leq \frac{\pi}{\omega}$, (the impact period), the conditions at the free end of the bar ($x = 0$), are:

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_o = \left(\frac{\partial^3 u}{\partial x^3}\right)_o = 0 \qquad (2b)$$

Also for $t \leq \frac{\pi}{\omega}$, at the clamped end of the bar ($x = L$), the conditions are assumed that

$$u_L = \frac{V}{\omega}\sin\omega t \;\; ; \;\; \left(\frac{\partial u}{\partial x}\right)_L = 0 \qquad (2c)$$

b. The conditions applying to the motion after the impact, $t \geq \frac{\pi}{\omega}$, may be simplified by taking a new time coordinate $t' = t - \frac{\pi}{\omega}$, so that $t' = 0$ corresponds to the instant the impact is completed; and by superimposing a velocity $+V$ on the entire system, so as to bring the table to rest. If the deflection and velocity functions of the beam at the completion of the impact are denoted as

$$u(x, \pi/\omega) = F(x)$$

$$\dot{u}(x, \pi/\omega) = G(x),$$

then the following conditions are required (using $u'(x,t')$ to denote the displacement after the impact):

$$t' = 0: \quad u'(x) = F(x) \qquad \dot{u}'(x) = G(x) + V \tag{4a}$$

$$x = 0 \text{ (free end)}: \left(\frac{\partial^2 u'}{\partial x^2}\right)_0 = \left(\frac{\partial^3 u'}{\partial x^2}\right)_0 = 0 \tag{4b}$$

$$x = L \text{ (clamped, fixed end)}: u'_L = \left(\frac{\partial u'}{\partial x}\right)_L = 0 \tag{4c}$$

4. The solution of equation (1) for the motion during the impact ($0 \leq t \leq \pi/\omega$) may be obtained by applying the Laplace transformation. The transform $\bar{u}(x,\lambda)$ of $u(x,t)$ is defined as[3]

$$\bar{u}(x,\lambda) = \int_0^\infty u(x,t)e^{-\lambda t}\, dt$$

Equation (1) transforms to

$$\frac{d^4\bar{u}}{dx^4} - \mu^4\bar{u} = \frac{V}{k^2}$$

$$\text{where } \mu^4 = -\frac{\lambda^2}{k^2}$$

The complete solution of this equation is

$$\bar{u} = \frac{V}{k^2} + A\cosh\mu x + B\sinh\mu x + C\cos\mu x + D\sin\mu x$$

where A, B, C, D are constants to be determined so as to satisfy the transformed conditions at the ends of the beam, derived from equations (3b) and (3c). Evaluating these constants, we obtain

$$\bar{u}(x,\lambda) = \frac{V}{\lambda^2} - \frac{V}{2\lambda^2(\lambda^2+\omega^2)} \frac{(\cosh\mu x+\cos\mu x)(\cosh\mu L+\cos\mu L)-(\sinh\mu x+\sin\mu x)(\sinh\mu L-\sin\mu L)}{1 + \cosh\mu L\cos\mu L}$$

By the Fourier-Mellin inversion theorem, (3) we have

$$u(x,t) = \int_{Br} \bar{u}\,(x,\lambda)e^{\lambda t}\,d\lambda$$

The evaluation of this line integral is obtained by transforming the path of integration to a closed contour, and applying the calculus of residues. Poles of the integrand are as follows:

(1) Double pole at $\lambda = 0$

(2) Simple poles at $\lambda = \pm i\omega$

(3) Simple poles at roots of $1 + \cosh\mu L\cos\mu L = 0$

Evaluating the residues at all these poles and adding, we obtain, finally

$$u = \frac{V\psi(x)}{2\omega}\sin\omega t + 2V\omega^2 \sum_{n=1}^{\infty} \frac{\phi_n(x)}{\beta_n\alpha_n(\omega^2-\alpha_n^2)} \sin\alpha_n t \qquad (6)$$

where

$$\phi_n(x) = \frac{(\cosh\beta_n\frac{x}{L}+\cos\beta_n\frac{x}{L})(\cosh\beta_n+\cos\beta_n)-(\sinh\beta_n\frac{x}{L}+\sin\beta_n\frac{x}{L})(\sinh\beta_n-\sin\beta_n)}{\cosh\beta_n\sin\beta_n-\sinh\beta_n\cos\beta_n}$$

$$\psi(x) = \frac{(\cosh\gamma\frac{x}{L}+\cos\gamma\frac{x}{L})(\cosh\gamma+\cos\gamma)-(\sinh\gamma\frac{x}{L}+\sin\gamma\frac{x}{L})(\sinh\gamma-\sin\gamma)}{1 + \cosh\gamma\cos\gamma}$$

β_n = the nth root of the frequency equation for a cantilever, $1 + \cosh\beta_n\cos\beta_n = 0$

$\alpha_n = \frac{k}{L^2}\beta_n^2$ = the nth natural frequency (radians/sec.) of the cantilever.

$\gamma = L\sqrt{\frac{\omega}{k}}$

It may be verified that this solution, eq. (6), satisfies the differential equation (1), and the initial and terminal conditions (3a) (3b) and (3c).

5. The solution of equation (1) for the motion following the impact ($t \geqslant \pi/\omega$, or $t' \geqslant o$) may be obtained directly in terms of the characteristic functions of the cantilever beam. The normalized characteristic functions for the cantilever are

$$U_n(x) = \frac{1}{\sqrt{L}}\left[\cosh\beta_n\frac{x}{L}+\cos\beta_n\frac{x}{L} - \frac{\sinh\beta_n-\sin\beta_n}{\cosh\beta_n+\cos\beta_n}\left(\sinh\beta_n\frac{x}{L}+\sin\beta_n\frac{x}{L}\right)\right] \tag{7}$$

where $1 + \cosh\beta_n\cos\beta_n = 0$

The solution is then taken as

$$u'(x,t') = \sum_{n=1}^{\infty} U_n(x)\left[P_n\cos\alpha_n t' + Q_n\sin\alpha_n t'\right] \tag{8}$$

where P_n, Q_n are constants to be determined so as to give the required displacement and velocity functions at $t' = 0$. The following relations exist:

$$u'(x,o) = \sum_{n=1}^{\infty} U_n(x)P_n = F(x) \tag{9a}$$

$$\dot{u}'(x,o) = \sum_{n=1}^{\infty} U_n(x)Q_n\alpha_n = G(x) + V \tag{9b}$$

where

$$F(x) = u(x,\pi/\omega) = 2V\omega^2 \sum_{n=1}^{\infty} \frac{U_n \sin \pi \frac{\alpha_n}{\omega}}{\beta_n \alpha_n (\omega^2 - \alpha_n^2)} \; \frac{\cosh\beta_n + \cos\beta_n}{\cosh\beta_n \sin\beta_n - \sinh\beta_n \cos\beta_n}$$

$$G(x) \equiv \dot{u}(x,\pi/\omega) = -\frac{V}{2}\left[\frac{(\cosh\gamma + \cos\gamma)(\cosh\gamma\frac{x}{L} + \cos\gamma\frac{x}{L})}{1 + \cosh\gamma\cos\gamma} - \frac{(\sinh\gamma - \sin\gamma)(\sinh\gamma\frac{x}{L} + \sin\gamma\frac{x}{L})}{1 + \cosh\gamma\cos\gamma}\right]$$

$$+ 2V\omega^2 \sum_{n=1}^{\infty} \frac{U_n \cos\frac{\pi\alpha_n}{\omega}}{\beta_n(\omega^2 - \alpha_n^2)} \; \frac{\cosh\beta_n + \cos\beta_n}{\cosh\beta_n \sin\beta_n - \sinh\beta_n \cos\beta_n}$$

Multiplying equations (9a) and (9b) by $U_n(x)$ and integrating from 0 to L, we have, since the U_n's are normalized,

$$P_n = \int_0^L F(x) U_n(x)\,dx$$

$$\alpha_n Q_n = \int_0^L \left[G(x) + V\right] U_n(x)\,dx$$

The integrals for P_n, Q_n may be evaluated by making use of the properties of orthogonality of the characteristic functions, and of the functions contained in F(x), G(x). The following result is finally obtained, after considerable simplification (taking t as defined originally):

$$u'(x,t) = 4V \sum_{n=1}^{\infty} \frac{\phi_n(x) \cos\frac{\pi}{2}\frac{\alpha_n}{\omega}}{\beta_n \alpha_n \left(1 - \frac{\alpha_n^2}{\omega^2}\right)} \sin\left(\alpha_n t - \frac{\pi\alpha_n}{2\omega}\right) \qquad (10)$$

where $\phi_n(x)$, β_n are the same as defined above for equation (6) .

It should be noted that the above result gives the motion of the beam (after the impact) which would be seen by an observer moving with the beam, at constant velocity -V. It may be verified that this result, equation (10), at $t = \pi/\omega$, gives the same displacement and velocity as does the solution for the motion during impact (eq.6).

NUMERICAL EVALUATION

6. The curvature at the clamped end of the cantilever is first obtained from equation (6) and (10) as follows:

(a) During impact:

$$\frac{k}{V}\left(\frac{\partial^2 u}{\partial x^2}\right)_L = \left[\frac{\sinh\gamma\sin\gamma}{1+\cosh\gamma\cos\gamma}\right]\sin\omega t + 4\sum_{n=1}^{\infty}\frac{\sin\alpha_n t}{\beta_n\left(1-\frac{\alpha_n^2}{\omega^2}\right)}\left[\frac{\sinh\beta_n\sin\beta_n}{\cosh\beta_n\sin\beta_n-\sinh\beta_n\cos\beta_n}\right] \quad (11a)$$

(b) After impact:

$$\frac{k}{V}\left(\frac{\partial^2 u}{\partial x^2}\right)_L = 8\sum_{n=1}^{\infty}\frac{\cos\frac{\pi\alpha_n}{2\omega}}{\beta_n\left(1-\frac{\alpha_n^2}{\omega^2}\right)}\left[\frac{\sinh\beta_n\sin\beta_n}{\cos\beta_n\sin\beta_n-\sinh\beta_n\cos\beta_n}\right]\sin\left(\alpha_n t-\frac{\pi\alpha_n}{2\pi}\right) \quad (11b)$$

These equations cannot be applied when $\omega = \alpha_p$, i.e. when the "stopping frequency" of the table is equal to any one of the natural frequencies, since in the above form they lead to indeterminacies in these cases. The limiting values in these "resonance" cases may be shown to be finite, and to be given by the following:

(a) During impact, for $\omega = \alpha_p$, $\gamma = \beta_p = \sqrt{\frac{\omega L^2}{k}}$

$$\frac{k}{V}\left(\frac{\partial^2 u'}{\partial x^2}\right)_L = \left\{2\sqrt{\frac{\omega k}{L^2}}\, t \cos\omega t + \left[\operatorname{cotanh}\gamma + \cot\gamma - \frac{4}{\gamma}\right]\sin\omega t\right\}\frac{\sinh\gamma \sin\gamma}{\sinh\gamma \cos\gamma - \cosh\gamma \sin\gamma}$$

$$+ 4\sum_{n\neq p}^{\infty} \frac{\sin\alpha_n t}{\beta_n\left(1-\frac{\alpha_n^2}{\omega^2}\right)}\left[\frac{\sinh\beta_n \sin\beta_n}{\cosh\beta_n \sin\beta_n - \sinh\beta_n \cos\beta_n}\right] \qquad (12a)$$

(b) After impact, for $\omega = \alpha_p$, $\gamma = \beta_p = \sqrt{\frac{\omega L^2}{k}}$,

$$\frac{k}{V}\left(\frac{\partial^2 u'}{\partial x^2}\right)_L = \frac{2\pi}{\gamma}\left[\frac{\sinh\gamma \sin\gamma}{\cosh\gamma \sin\gamma - \sinh\gamma \cos\gamma}\right]\cos\omega t$$

$$+ 8\sum_{n\neq p}^{\infty} \frac{\cos\frac{\pi\alpha_n}{2\omega}}{\beta_n\left(1-\frac{\alpha_n^2}{\omega^2}\right)}\left[\frac{\sinh\beta_n \sin\beta_n}{\cosh\beta_n \sin\beta_n - \sinh\beta_n \cos\beta_n}\right]\sin\left(\alpha_n t - \frac{\pi\alpha_n}{2\omega}\right) \qquad (12b)$$

It may be checked that the above equations for the curvatures before and after impact agree at $t = \pi/\omega$.

7. A convenient dimensionless form for these results may be obtained by rewriting them in terms of the ratios

$$f \equiv \frac{\omega}{\alpha_1} = \frac{\text{"stopping frequency" of table}}{\text{fundamental frequency of cantilever}}$$

$$\tau \equiv \frac{t}{\pi/\omega} = \frac{\text{time after start of impact}}{\text{duration time of impact}}$$

The bending stress at the clamped end is given by

$$\sigma_o = \frac{Ed}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_L$$

where d is the depth of the beam. We now write

$$\frac{k}{V}\left(\frac{\partial^2 u}{\partial x^2}\right)_L = \frac{1}{V}\,\frac{\alpha_1 L^2}{\beta_1^2}\cdot\frac{2}{Ed}\,\sigma_o = \frac{2}{\beta_1^2}\,\frac{L^2}{Ed}\,\frac{\alpha_1}{V}\,\sigma_o = .569\,\frac{L^2\alpha_1}{EdV}\,\sigma_o$$

The equations expressing the severity of the impact are, finally:

(a) During impact ($o \leq \tau \leq 1$)

$$\frac{2}{\beta_1^2}\,\frac{L^2}{Ed}\,\frac{\alpha_1}{V}\,\sigma_o = A_o \sin\pi t + \sum_{n=1}^{\infty} \frac{A_n \sin \frac{\pi}{f}\frac{\beta_n^2}{\beta_1^2}\tau}{1 - \frac{1}{f^2}\frac{\beta_n^4}{\beta_1^4}} \tag{13a}$$

(b) After impact ($\tau \geqslant 1$)

$$\frac{2}{\beta_1^2}\,\frac{L^2}{Ed}\,\frac{\alpha_1}{V}\,\sigma_o = \sum_{n=1}^{\infty} \frac{2\,A_n \cos\frac{\pi}{2f}\frac{\beta_n^2}{\beta_1^2}}{1 - \frac{1}{f^2}\frac{\beta_n^4}{\beta_1^4}}\,\sin\left[\frac{\pi}{f}\frac{\beta_n^2}{\beta_1^2}\left(\tau - \frac{1}{2}\right)\right] \tag{13b}$$

where $A_o = \dfrac{\sinh\beta_1\sqrt{f}\,\sin\beta_1\sqrt{f}}{1 + \cosh\beta_1\sqrt{f}\,\cos\beta_1\sqrt{f}}$

$$A_n = \frac{4}{\beta_n}\,\frac{1}{\text{cotanh}\beta_n - \cot\beta_n}$$

In the"resonance" cases, $\omega = \alpha_p$, these are replaced by the following equations:

(a) During impact, for $\omega = \alpha_p$ $(0 \leq \tau \leq 1)$

$$\frac{2L^2\alpha_1}{\beta_1{}^2 EdV}\sigma_o = -\frac{\beta_p A_p}{4}\left\{\frac{2\pi\tau}{\beta_p}\cos\pi\tau + \left[\text{cotanh}\beta_p + \cot\beta_p - \frac{4}{\beta_p}\right]\sin\pi\tau\right\}$$

$$+\sum_{n\neq p}^{\infty}\frac{A_n}{1-\frac{1}{f^2}\frac{\beta_n{}^4}{\beta_1{}^4}}\sin\frac{\pi}{f}\frac{\beta_n{}^2}{\beta_1{}^2} \qquad (14a)$$

(b) After impact, for $\omega = \alpha_p$ $(\tau \geq 1)$

$$\frac{2L^2\alpha_1}{\beta_1{}^2 EdV}\sigma_o = -\frac{\pi A_p}{2}\cos\pi\tau$$

$$+\sum_{n\neq p}^{\infty}\frac{2A_n\cos\frac{\pi}{2f}\frac{\beta_n{}^2}{\beta_1{}^2}}{1-\frac{1}{f^2}\frac{\beta_n{}^4}{\beta_1{}^4}}\sin\left[\frac{\pi}{f}\frac{\beta_n{}^2}{\beta_1{}^2}\left(\tau-\frac{1}{2}\right)\right] \qquad (14b)$$

In all of the above, the values of β_n are the roots of the cantilever frequency equation: $1 + \cosh\beta_n\cos\beta_n = 0$. Their values are [4]:

$\beta_1 = 1.87510$

$\beta_2 = 4.69410$

$\beta_3 = 7.85476$

$n \geq 4, \; \beta_n = \frac{2n-1}{2}\pi$

8. It is found, by plotting σ_o vs. τ for a series of values of f, that the maximum stress occurs during the impact when $f < 1$ while the maximum stress occurs after the impact, for $f > 1$ Figure 2 gives a plot of the maximum bending stress at the clamped end of the beam for a range

of f from 0.1 to 100. These maximum values of stress, after the impact, have been obtained by adding the amplitudes of the component oscillations, without regard for their phase relationships. The solution (equations (13a) and (14a)) shows that, as f is increased, the amplitudes of the higher harmonics are increased, and it is this greater prominence of the high frequency beam vibrations at larger values of f, that causes the maximum stress to continue to increase with f, as shown in Figure 2. It should be noted that although it is justifiable to take the maximum stress as simply the sum of the amplitudes of the component harmonics when f is not very large, it is not permissible to continue this up to arbitrarily large values of f. If f is large, say 1000, the above solution indicates that the natural frequencies of the beam which will be prominent in the stress response, will be those up to about the 20th harmonic, (since $\alpha_{20}/\alpha_1 = \beta_{20}^2/\beta_1^2 \cong 1000$). However, these high harmonics will not contribute to the bending stresses in the beam in the manner predicted by the above analysis, due to the existence of internal damping, shear, and rotatory inertia effects which are important in the high harmonics, but negligible in the low harmonics. In fact, the differential equation (1) for the transverse vibrations of a beam, which was used as the basis of this analysis, is an approximate equation, valid only at sufficiently low frequencies. However, for the range of f from 0 to 100, which is of practical interest, the method used to calculate the maximum bending stresses probably gives a good approximation to the actual values which would be obtained in a beam under the assumed impact conditions.

9. With this restriction on the analytical results, the variation of maximum stress with duration of impact is found to be basically different from that of the simple one-degree of freedom system, in which the deflection becomes independent of the shortness of the impact duration at a value of f equal to about 3. This conclusion, and the extension of these results to impacts of arbitrarily short duration, must be checked by experimental work, which is now in progress.

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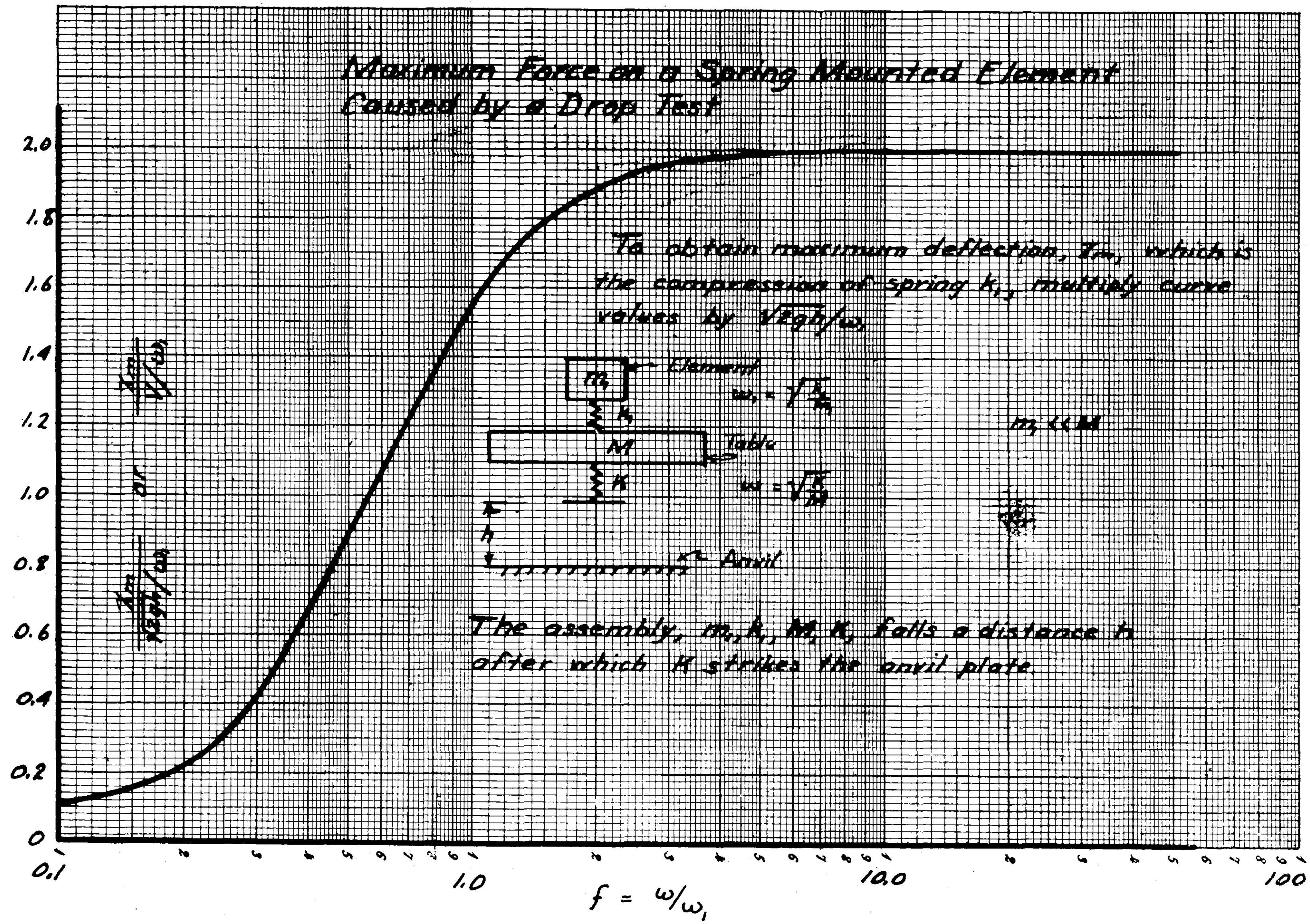
Maximum Force on a Spring Mounted Element
Caused by a Drop Test
To obtain maximum deflection, x_m, which is the compression of spring k_1, multiply curve values by $\sqrt{2gh}/\omega_1$
Element
m_1
k_1
$\omega_1 = \sqrt{k_1/m_1}$
Table
M
K
$\omega = \sqrt{K/M}$
$m_1 << M$
h
Anvil
The assembly, m_1, k_1, M, K, falls a distance h after which K strikes the anvil plate.
$\frac{x_m}{V/\omega_1}$ or $\frac{x_m}{\sqrt{2gh}/\omega_1}$
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
0.1
1.0
10.0
100
$f = \omega/\omega_1$

Plate 1

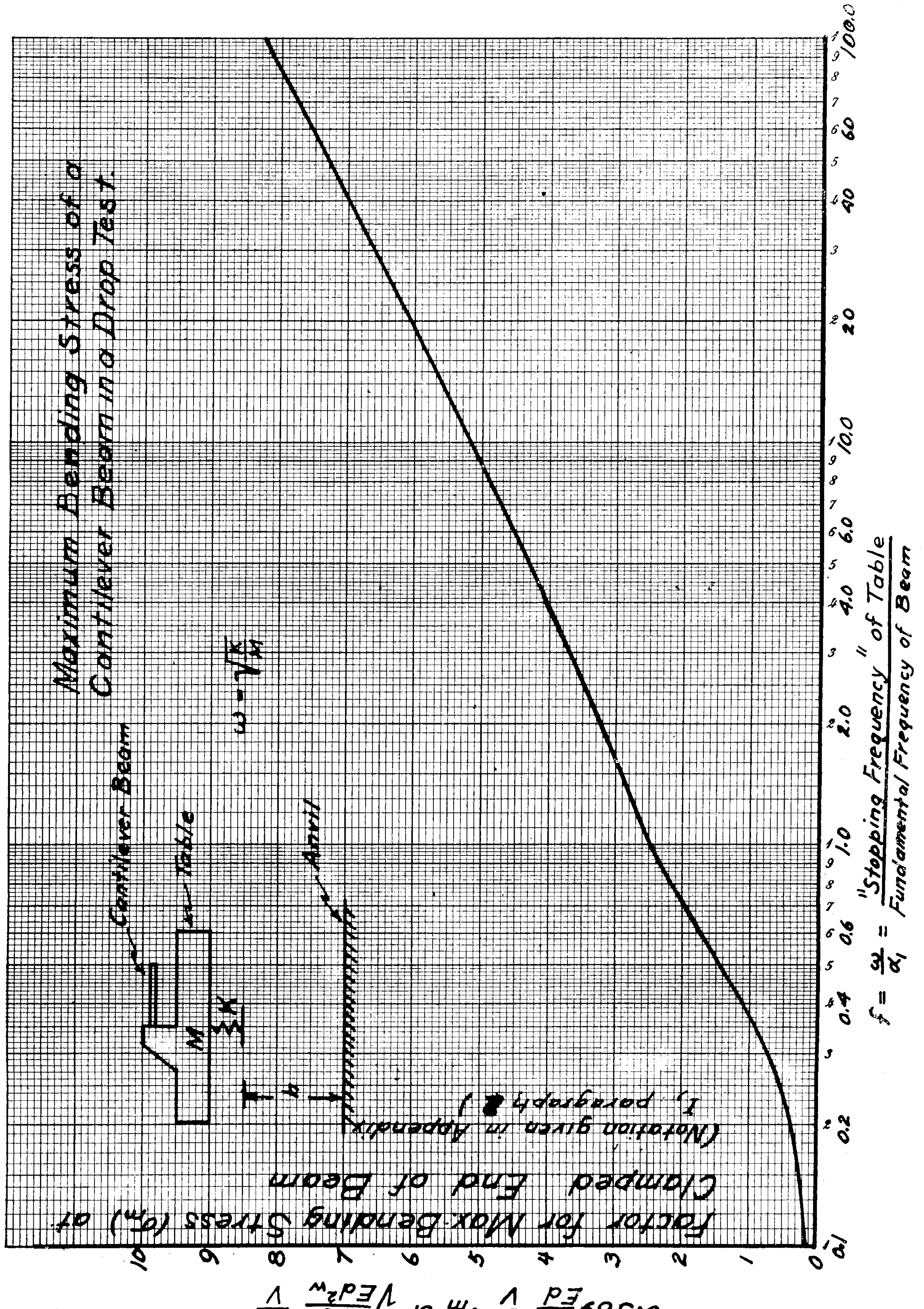
Maximum Bending Stress of a
Cantilever Beam in a Drop Test.
Cantilever Beam
Table
Anvil
M
K
h
$\omega = \sqrt{\frac{K}{M}}$
(Notation given in Appendix I, paragraph 2)
Factor for Max. Bending Stress (σ_m) at Clamped End of Beam
$0.569 \frac{\ell^2}{Ed} \frac{\alpha_1}{V} \sigma_m$ or $\sqrt{\frac{4Ig}{Ed^2 w}} \frac{\sigma_m}{V}$
$f = \frac{\omega}{\alpha_1} = \frac{\text{"Stopping Frequency" of Table}}{\text{Fundamental Frequency of Beam}}$
0
1
2
3
4
5
6
7
8
9
10
0.1
0.2
0.4
0.6
1.0
2.0
4.0
6.0
10.0
20
40
60
100.0

Plate 2