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GRAPHICAL DETERMINATION OF THE RESPONSE OF A SIMPLE LINE HYDROPHONE TO TWO SOURCES

C. W. Klee

Techniques Branch
Sound Division

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U. S. NAVAL RESEARCH LABORATORY
Washington, D.C.

ABSTRACT

A graphical determination of the line hydrophone's response to two sources was made by vector addition of elemental contributions. The results were then applied to the specific cases of a 3-wavelength line in the presence of (a) two equal-amplitude sources in phase and 180 degrees out of phase and (b) two sources in the amplitude ratio 1:2, in phase and 180 degrees out of phase. Plots were made for source separations from 0 degrees to 50 degrees. It was shown that at source separations smaller than a lobe width the hydrophone becomes susceptible to a pointing error such that the direction of maximum response may be outside the angle included by the sources. This error angle becomes larger as (a) the sources' amplitude difference becomes smaller, (b) their phase difference becomes larger, or (c) their separation becomes smaller. For a line hydrophone, the error angle has a maximum that is given by the value of θ that satisfies the equation

$$\left(\frac{1 + 2\cot^2\theta - (\pi^2 \ell^2/\lambda^2)\cos^2\theta}{(\pi\ell/\lambda)\sin\theta}\right)\sin\left[(\pi\ell/\lambda)\sin\theta\right]$$

$$-(1 + 2\cot^2\theta)\cos\left[(\pi\ell/\lambda)\sin\theta\right] = 0$$

where ℓ is the line length and λ is the wavelength. This angle amounts to about 12.5 degrees for a 3-wavelength line. Also, a derivation of the usual $(\sin\beta)/\beta$ pattern of the line hydrophone from a vector representation of the line's response was appended together with the second-order differentiation of $(\sin\beta)/\beta$.

PROBLEM STATUS

This report completes work on this phase of the problem; work is continuing on other phases of the problem.

AUTHORIZATION

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GRAPHICAL DETERMINATION OF THE RESPONSE OF A SIMPLE LINE HYDROPHONE TO TWO SOURCES

INTRODUCTION

The Techniques Branch has been interested recently in the response of directional hydrophones to multiple sources (especially two) because of an unusual and little appreciated phenomenon that occurs for particular phase relations of the sources. For these phase relations the "center of gravity" of the sources, or the direction the hydrophone's acoustic axis will point for maximum response, lies outside the angle included by the sources (1-3). This "error" in pointing is due to a disturbance of the phase front which we have called phase front tilt. For the case of two sources, the error will be maximum for equal source amplitudes received 180 degrees out of phase.

The present study was undertaken to examine the effect on a hydrophone of several sources from a sort of "beam pattern" point of view. That is, for each source configuration the hydrophone is rotated and its response noted as a function of the angle its acoustic axis makes with the source direction. The line hydrophone was chosen for this first effort because of its simplicity and because it is easy to generalize much of the information obtained. Also, it is a configuration we may use later in our experiments. The method of treatment will be a determination of hydrophone response by vector addition of all contributing elemental amplitudes after a manner set forth by Wood (4) and by Jenkins and White (5). The method gives a very clear physical picture of the origin of the line pattern.

GRAPHICAL TREATMENT OF THE LINE HYDROPHONE

If a line hydrophone is divided into a number of equal parts, say ten, then the amplitude r received from a point source P, a large distance away, by any one of these parts will be the same as that received by any other since all are of equal width. The phases will differ, though, unless P lies on the axis. The state of affairs is shown in Fig. 1. The difference in phase δ between contributions received by adjacent segments will be constant because each element is the same amount farther away (or nearer) as its neighbor. The resultant of all these contributions can be found by vector addition of the individual amplitudes, each vector making an angle with its neighbor equal to the phase difference. If P lies on the hydrophone axis, then the vector addition will appear as in Fig. 2a since all contributions are in phase. We will call the resultant R in general and the resultant along the axis R_o. If P lies off the axis, each of the ten equal amplitudes r is inclined at an angle δ with the preceding one, and the vector addition will appear as in Fig. 2b. Since the lengths of the elemental vectors remain unchanged, their accumulated "arc" length still has the magnitude R_o but their resultant is now R. If instead of dividing the line into ten elements we had divided it into an infinite number of equal elements, then the vectors r would become shorter but at the same time δ would decrease in the same proportion so that in the limit the vector diagram would approach the arc of a circle as shown in Fig. 2c. Here, the resultant amplitude R will be the same as before and equal to the chord of the arc R_o. The angle that would be made by tangents to the arc at its two extremities in Fig. 2c is the total phase difference of contributions received by opposite ends of the line. We shall call this phase angle 2β (i.e., $\beta = \Sigma\, \delta/2$). That the resultant R thus determined is in agreement with the usual $(\sin \beta)/\beta$ expression for the directional characteristic of a line hydrophone is shown in Appendix A.

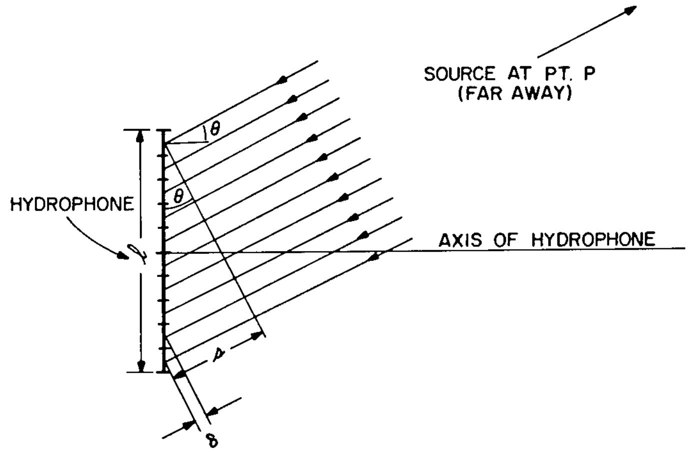

Fig. 1 - Line hydrophone considered as ten equal segments

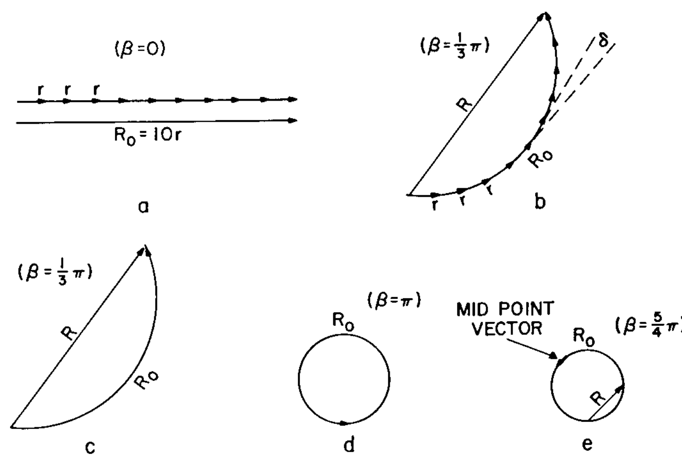

Fig. 2 - Response of a line hydrophone to a single source determined graphically by the vector addition of elemental contributions

The increment of signal received by the center element of the line is a convenient reference which we shall call the midpoint vector. The resultant always has the same phase angle as the midpoint vector (except as noted later) since there are as many elemental vectors advanced in phase from the midpoint as there are delayed in phase from the midpoint.

If now the hydrophone is rotated about its midpoint in a plane containing the source so that its acoustic axis moves away from the source, then R_o, though remaining the same length, will curve more and more until it forms a complete circle at $\beta = \pi$ as in Fig. 2d. The source is now lying in the first null of the line pattern, and R equals zero. As the hydrophone is rotated farther, R_o will wind up on itself and R will grow larger again corresponding to the second lobe. Now, however, R and the midpoint vector though parallel are oppositely directed, i.e., 180 degrees out of phase. This is shown in Fig. 2e and occurs whenever β has a value given by

$$(2n + 1)\,\pi < \beta < (2n + 2)\,\pi$$

where n is an integer. In other words, a signal is received in the same phase by every other lobe but 180 degrees out of phase by adjacent lobes.*

When two sources are present, the vector addition of all elemental amplitudes gives a result as shown in Fig. 3. Grouping together of all the contributions from each source separately results in two circular arcs with the same phase angle between midpoint vectors as exists between the two signals at the midpoint of the line hydrophone. The values of R_o will be determined by the respective signal strengths as received on axis for each source. It can be seen from Fig. 3 that (a) each source will lie at some angle off the hydrophone axis and have some on-axis signal strength that will give rise to its own β and R_o, (b) these will determine a resultant amplitude for each source, and (c) the total response of the hydrophone to the two sources will be the vector sum of these individual resultants with an angle between them equal either to (1) the phase angle α between signals that exists at the midpoint of the line if the sources both lie in even lobes or both in odd lobes or to (2) $\alpha + 180$ degrees if one source is in one of the even lobes and the other in one of the odd.

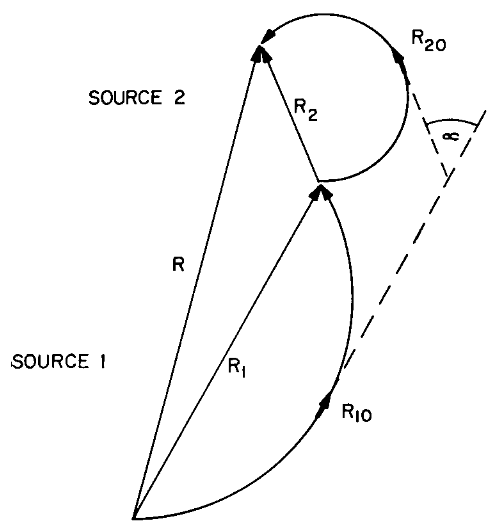

Fig. 3 - Response of a line hydrophone to two sources determined graphically by the vector addition of elemental contributions

RESPONSE OF A 3λ LINE TO TWO SOURCES OF VARIOUS PHASE, AMPLITUDE, AND SEPARATION

The voltage pattern resulting from the rotation of a 3-wavelength line in the plane of two equal-amplitude sources was determined graphically and appears in Fig. 4. Results are shown for a phase angle between sources of 0 and 180 degrees. Similar plots were made for sources in the amplitude ratio 1:2 and appear in Fig. 5. Neither figure shows plots for source separations much greater than the 40-degree main-lobe width of the

*This 180-degree phase change from lobe to lobe is taken care of when working a problem analytically by the fact that $(\sin\beta)/\beta$ changes sign every time β increases π radians.

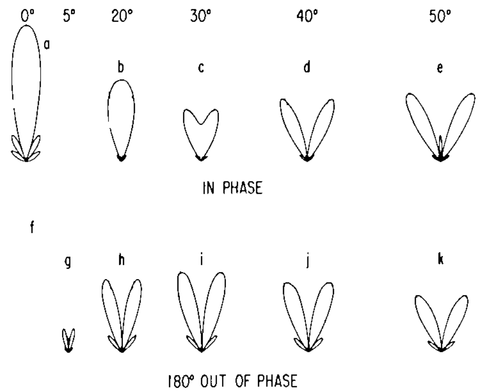

Fig. 4 - Pattern of the voltage, in the plane of the hydrophone and sources, resulting from the rotation of a 3-wavelength line hydrophone in the presence of two sources of equal amplitude. The straight lines indicate the source positions.

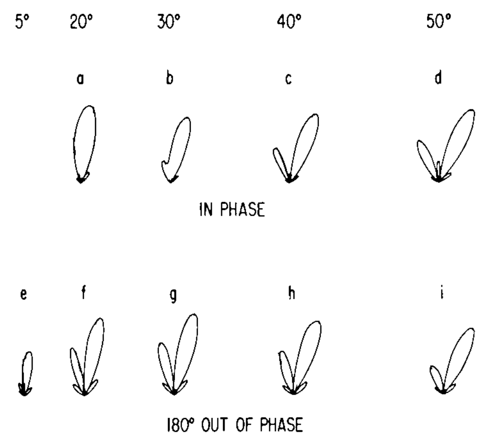

Fig. 5 - Pattern of the voltage, in the plane of the hydrophone and sources, resulting from the rotation of a 3-wavelength line hydrophone in the presence of two sources of amplitude ratio 1:2. The larger amplitude source is on the right side. The straight lines indicate the source positions.

3λ line. Sources separated by more than a lobe width are not interesting, because (if side lobes are ignored) their effects never combine at any point; they are simply received as two single sources. Since the 0-degree separation of sources of Fig. 4a yields, in effect, a single source, this figure shows the ordinary beam pattern of the 3λ line.

Several of the plots exhibit a pointing error of the hydrophone that was described previously. If we consider the angle that the direction of maximum response makes with a direction midway between sources, then the pointing error may be defined as the amount by which this angle exceeds half the source separation. Such an error does not usually occur unless the sources are received nearly 180 degrees out of phase and then only if they are close enough together so that the hydrophone cannot resolve them as separate sources in the usual sense,* at least they must be within a lobe width of one another. Side lobes in the beam pattern of the line can give rise to slight departures from this rule. For example, in Figs. 4d and 5c small pointing errors can be seen in the pattern of sources that are 40 degrees apart and in phase (an error of about 6% of the source separation in each case).

*Usually one is dealing with a response pattern that is an average resulting from many different phase relations of the sources. For the special case of a 180-degree phase difference, the sources will appear resolved down to the smallest of separations.

In Figs. 4c and 5b it can be seen that the hydrophone is just beginning to resolve the two sources at 30 degrees. In Figs. 4h and 5f the pointing error (3 degrees and 2 degrees respectively) is just beginning to make an appearance and increases as the source separation gets smaller as in Figs. 4g and 5e. Also, the error becomes greater as the sources approach equality of amplitude as illustrated by comparison of Figs. 4g and 5e; the error is about 10 degrees in Fig. 4g and about 5 degrees in Fig. 5e. The error always appears on the side of the higher amplitude source as in Figs. 5e and 5f accompanied by a lower level maximum on the low amplitude side.* If the sources are of equal amplitude, there will be two errors symmetrically disposed to either side of the sources as in Figs. 4g and 4h.

MAXIMUM POSSIBLE ERROR ANGLE CAUSED BY TWO SOURCES

We have evidence from previous work that, in continuation of the trends described above, the maximum possible error in pointing for a hydrophone in the presence of two sources occurs when the sources are 180 degrees out of phase, are of equal amplitude, and when the distance between them becomes vanishingly small. Of course, the combined amplitude of the sources also falls off with separation, becoming zero when they coincide. Such a pair of sources comprise the classical doublet which has received much attention down through the years in acoustics literature (see, for instance, Ref. 6 or Ref. 7).

It will be of interest to determine the error angle caused by the doublet. Since the sources are 180 degrees out of phase, their contributions, if received in the same lobe, will subtract, so that their resultant will be maximum where their difference is greatest. Thus, the maximum response from the doublet will occur at the point in the hydrophone's pattern where the radius vector is changing most rapidly with angle. For the line pattern, this occurs at an inflection point in the main lobe at some angle short of the null (less than one-half lobe width off axis). This angle can easily be found as a real root of the equation gotten by setting the second derivative of $(\sin\beta)/\beta$ equal to zero. In Appendix B, $(\sin\beta)/\beta$ is twice differentiated with respect to θ, yielding

$$\frac{d^2}{d\theta^2}\left(\frac{R}{R_o}\right) = \left(\frac{1 + 2\cot^2\theta - (\pi^2 \ell^2/\lambda^2)\cos^2\theta}{(\pi\ell/\lambda)\sin\theta}\right)\sin\left[(\pi\ell/\lambda)\sin\theta\right] - (1 + 2\cot^2\theta)\cos\left[(\pi\ell/\lambda)\sin\theta\right]$$

where

θ = angle that received rays make with the hydrophone's acoustic axis

ℓ = length of line

λ = wavelength.

Although this expression cannot be set equal to zero and solved explicitly for θ, it can be graphed in specific cases as in Fig. 6. In this figure $d^2/d\theta^2[(\sin\beta)/\beta]$ has been plotted for the case of a 3λ line for values of θ ranging from 0 to 20 degrees ($\beta = 0$ to π radians).† It can be seen in Fig. 6 that $d^2/d\theta^2[(\sin\beta)/\beta]$ becomes zero at $\theta \approx 12.5$ degrees. This angle, therefore, represents the maximum error for a 3λ line in the presence of two sources.

*Although the lower level maximum also fails to coincide with the true source position and could be considered in error, it is excluded from consideration in the present study, attention being given only to the highest maximum in the response pattern.

†A value for $\theta = 0$ was obtained analytically by R. E. Morden which helped pin down the shape of the curve at the low θ end.

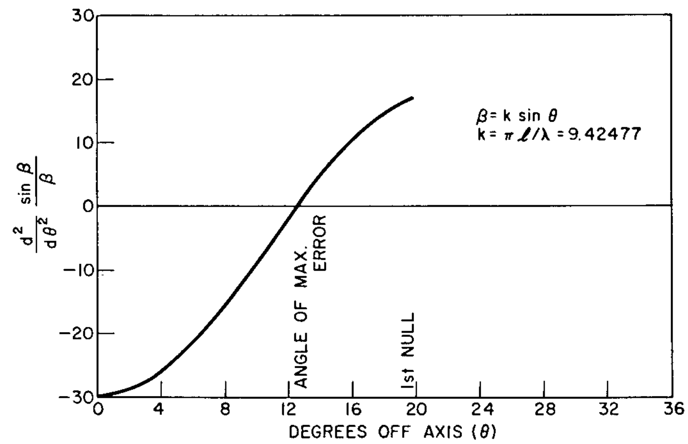

Fig. 6 - The second derivative of the $(\sin\ \beta)/\beta$ pattern for a 3-wavelength line hydrophone

SUMMARY

Although the number of plots made here is sufficient to provide only a sketchy picture, the patterns illustrate the following conclusions that we had reached previously from other evidence.

The hydrophone acted upon by two sources simultaneously, receives them simply as two single sources (ignoring side lobes) if they are beyond a lobe width apart. Their effects do not combine. As they are moved closer than a lobe width, the hydrophone soon loses the ability to resolve them and the contributions from both combine to form a single pattern which will now be determined by the relative phase and separation of the sources as well as their amplitudes. From this point on, as the sources are moved closer, the hydrophone becomes susceptible to a pointing error such that the direction of maximum response may lie outside the angle included by the sources. This pointing error is negligible, however, unless the sources are more than, say, 120 degrees out of phase. The error will appear on the side of the higher amplitude source, or if the sources are equal, two errors will exist symmetrically disposed to either side of the sources. The error will become larger as (a) the sources' amplitude difference becomes smaller, (b) their phase difference becomes larger, or (c) their separation becomes smaller. The error will be maximum when the sources are equal in amplitude, 180 degrees out of phase, and separated by a vanishingly small distance (i.e., when they take the form of the classical doublet).

The value of the maximum error angle for a line hydrophone is a value of θ that satisfies the equation

$$\left(\frac{1 + 2\cot^2\theta - (\pi^2 \ell^2/\lambda^2)\cos^2\theta}{(\pi\ell/\lambda)\sin\theta}\right)\sin\left[(\pi\ell/\lambda)\sin\theta\right] - (1 + 2\cot^2\theta)\cos\left[(\pi\ell/\lambda)\sin\theta\right] = 0$$

which is the angle off axis of an inflection point in the $(\sin\ \beta)/\beta$ pattern. This amounts to about 12.5 degrees for a 3λ line. Also, because of side lobes, small pointing errors are possible even when the sources are in phase.

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APPENDIX A

DERIVATION OF $(\sin\beta)/\beta$ FROM THE RATIO OF CHORD TO ARC IN FIG. 2c

The angle 2β, previously defined, is also the angle subtended by the arc R_o, as shown in Fig. A1. The ratio of the resultant amplitude R at any point to that on axis R_o is the ratio of the chord to the arc. If q is taken as the radius of arc in Fig. A1, then the arc is given by

$$R_o = 2\beta q$$

where β is in radians. Also, from the geometry of Fig. A1

$$\sin\beta = \frac{R/2}{q}$$

or the chord is given by

$$R = 2q\sin\beta.$$

The ratio of the chord to the arc is then

$$\frac{\text{chord}}{\text{arc}} = \frac{R}{R_o} = \frac{2q\sin\beta}{2\beta q} = \frac{\sin\beta}{\beta}$$

which is the well-known pattern for a line hydrophone. To find β, we observe in Fig. 1 that the bottom of the line is a distance s farther away from P than the top of the line. This represents a phase difference of 2β according to the definition of β. Now s may be expressed as

$$s = \ell\sin\theta,$$

where ℓ is the length of the line and θ is the angle that rays from P make with the axis. This distance can be converted to phase difference by multiplying by $2\pi/\lambda$ thusly:

$$2\beta = \frac{s\,2\pi}{\lambda} = \frac{(\ell\sin\theta)\,2\pi}{\lambda}$$

or

$$\beta = \frac{\pi\ell\sin\theta}{\lambda}.$$

Rewriting the previously expressed ratio,

$$\frac{R}{R_o} = \frac{\sin\beta}{\beta} = \frac{\sin\left[(\pi\ell/\lambda)\sin\theta\right]}{(\pi\ell/\lambda)\sin\theta}.$$

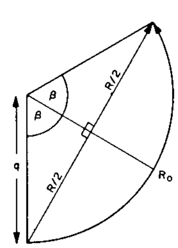

Fig. A1 - Redrawing of Fig. 2c showing 2β as the angle subtended by the arc R_o

APPENDIX B

DETERMINATION OF THE INFLECTION POINTS FOR THE $(\sin \beta)/\beta$ PATTERN OF A LINE

The beam pattern of a line hydrophone is given by

$$\frac{R}{R_o} = \frac{\sin \beta}{\beta} \tag{B1}$$

where

$\beta = k \sin \theta$

θ = the angle that received rays make with the hydrophone axis

$k = \pi \ell / \lambda$

ℓ = length of line

λ = wavelength

R = amplitude at angle θ

R_o = amplitude on axis.

The inflection points of Eq. (B1) are gotten by setting its second derivative with respect to θ equal to zero and solving for θ. Using the standard form for $d/d\theta(u/v)$, where $u = \sin \beta$ and $v = \beta = k \sin \theta$, we get

$$\frac{d}{d\theta}\left(\frac{R}{R_o}\right) = \frac{1}{\beta}\frac{d}{d\theta} \sin \beta - \frac{\cot \theta \sin \beta}{\beta}. \tag{B2}$$

The first term of (B2) is differentiated using the standard form

$$\frac{du}{d\theta} = \frac{du}{dv}\frac{dv}{d\theta}$$

where $v = \beta = k \sin \theta$ and $u = \sin v$:

$$\begin{aligned} \frac{d \sin \beta}{d\theta} &= \frac{d}{d\beta} \sin \beta \frac{d}{d\theta} (k \sin \theta) \\ &= \cos \beta \, (k \cos \theta) \end{aligned} \tag{B3}$$

so that (B2) becomes

$$\frac{d}{d\theta}\left(\frac{R}{R_o}\right) = \cot \theta \cos \beta - \frac{\cot \theta \sin \beta}{\beta}. \tag{B4}$$

Taking the derivative of (B4) we have

$$\frac{d^2}{d\theta^2}\left(\frac{R}{R_o}\right) = \frac{d}{d\theta} (\cot \theta \cos \beta) - \frac{d}{d\theta}\left(\frac{\cot \theta \sin \beta}{\beta}\right). \tag{B5}$$

Using the standard form for $d/d\theta(uv)$, where $u = \cot \theta$ and $v = \cos \beta$, the first term in (B5) is differentiated, yielding

$$\frac{d}{d\theta} (\cot \theta \cos \beta) = \cot \theta \frac{d}{d\theta} \cos \beta - \csc^2 \theta \cos \beta. \tag{B6}$$

The $\cos\beta$ factor is differentiated similarly to $\sin\beta$ in (B3):

$$\frac{d}{d\theta}\cos\beta = \frac{d}{d\beta}\cos\beta\,\frac{d}{d\theta}(k\sin\theta)$$

$$= -\sin\beta\,(k\cos\theta). \tag{B7}$$

Therefore (B6) becomes

$$\frac{d}{d\theta}(\cot\theta\cos\beta) = -k^2\cos^2\theta\,\frac{\sin\beta}{\beta} - \csc^2\theta\cos\beta. \tag{B8}$$

Differentiating the second term of (B5) similarly to the first yields

$$\frac{d}{d\theta}\left(\cot\theta\,\frac{\sin\beta}{\beta}\right) = \cot\theta\,\frac{d}{d\theta}\left(\frac{\sin\beta}{\beta}\right) - \csc^2\theta\,\frac{\sin\beta}{\beta}. \tag{B9}$$

Now, $(\sin\beta)/\beta$ is iaentical with (B1), and its differentiation already appears in (B4). Substituting this in (B9), we have

$$\frac{d}{d\theta}\left(\cot\theta\,\frac{\sin\beta}{\beta}\right) = \cot^2\theta\cos\beta - \cot^2\theta\,\frac{\sin\beta}{\beta} - \frac{\csc^2\theta\sin\beta}{\beta}. \tag{B10}$$

Substituting (B8) and (B10) in (B5), we have

$$\frac{d^2}{d\theta^2}\left(\frac{R}{R_o}\right) = -k^2\cos^2\theta\,\frac{\sin\beta}{\beta} - \csc^2\theta\cos\beta - \cot^2\theta\cos\beta$$

$$+ \cot^2\theta\,\frac{\sin\beta}{\beta} + \csc^2\theta\,\frac{\sin\beta}{\beta}. \tag{B11}$$

Factoring out $\sin\beta$ and $\cos\beta$,

$$\frac{d^2}{d\theta^2}\left(\frac{R}{R_o}\right) = \left(\frac{\csc^2\theta + \cot^2\theta - k^2\cos^2\theta}{\beta}\right)\sin\beta$$

$$- (\csc^2\theta + \cot^2\theta)\cos\beta. \tag{B12}$$

Replacing $\csc^2\theta$ with $(1 + \cot^2\theta)$ yields

$$\frac{d^2}{d\theta^2}\left(\frac{R}{R_o}\right) = \left(\frac{1 + 2\cot^2\theta - k^2\cos^2\theta}{\beta}\right)\sin\beta$$

$$- (1 + 2\cot^2\theta)\cos\beta. \tag{B13}$$

Substituting in (B13) the previous expressions for k and β, we have the final expression:

$$\frac{d^2}{d\theta^2}\left(\frac{R}{R_o}\right) = \left(\frac{1 + 2\cot^2\theta - (\pi^2\ell^2/\lambda^2)\cos^2\theta}{(\pi\ell/\lambda)\sin\theta}\right)\sin\left[(\pi\ell/\lambda)\sin\theta\right]$$

$$- (1 + 2\cot^2\theta)\cos\left[(\pi\ell/\lambda)\sin\theta\right]. \tag{B14}$$