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A NUMERICAL PROCEDURE FOR SHOCK AND FOURIER ANALYSIS

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ABSTRACT

This paper describes and derives a numerical integration technique presently being used at NRL for problems in structural dynamics. The method is capable of computing damped and undamped shock spectra, Fourier sine, cosine, and regular Fourier integrals (frequency response from transient response), and the inversion of the Fourier sine and cosine integrals (transient response from frequency response). It can also be used to calculate Fourier series coefficients, and for the numerical solution of nonlinear equations. Several examples are worked out in detail and some others calculated by NAREC (NRL's digital computer) are shown to present some idea of the precision of the method. This report does not assume a sophisticated mathematical background and uses only those techniques which are available to undergraduate students in engineering.

PROBLEM STATUS

This is an interim report on one phase of the problem; work is continuing.

AUTHORIZATION

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A NUMERICAL PROCEDURE FOR SHOCK AND FOURIER ANALYSIS

INTRODUCTION

In the calculation of the response of linear elastic structures to transient forces and foundation motions it is often necessary or convenient to compute the response of a set of single degree of freedom oscillators to the impressed transient. Normal mode theory, for example, defines (1-3) a set of single degree of freedom systems whose superposed responses become the structural response. It will be shown in a later section that the solution of certain Duhamel integrals (dealing with the undamped single degree of freedom mechanical system) leads to a means of evaluating Fourier series coefficients and to the representation of the Fourier transform.

Quite often only a graphical record of the impressed transient is known or a Fourier spectrum of the solution of the differential equations is known and the problem is to calculate the structural response (4,5). This is usually accomplished by electrical analog techniques, some general numerical integration method (6,7), or by graphical or semi-graphical solutions (1,8,9). It is quite legitimate therefore to ask "Why present another method?" The answer would be that although many procedures are available, most are designed to show how a special technique, an iteration, an integration by replacement of the original differential equation by one of finite differences, a Maclaurin series expansion with associated assumptions in regard to the behavior of the derivative of the acceleration, or an analogy is used to solve the problem. It was thought that the individuals involved in solving these problems might profit both timewise in the calculation of the response, and by an increased understanding of the problem, if a simple, powerful, yet precise technique which closely followed an easily understood physical argument was presented.

In this method the differential equation of motion for the oscillator is not replaced in the classical sense by one of finite differences, nor is any iteration required in the linear case to obtain a solution, even though damping is present. The technique is simply a logical extension of ones previously presented (1) and has been used extensively at the U.S. Naval Research Laboratory since the author proposed it in 1956.

This report deals primarily with the undamped response of the oscillator to a foundation motion when the velocity of the base is known. The numerical integration method is also presented with an example for this case with linear damping, and the equations for the damped or undamped response to applied forces or foundation accelerations are also presented. The nonlinear cases are only briefly mentioned, as they will be the subject of future reports. Some Fourier series coefficients are calculated and two inverse Fourier transform problems are worked out.

SYMBOLS

A dot over a symbol indicates differentiation with respect to time.

A_n n^{th} Fourier cosine coefficient

$\bar{A}_n$ n^{th} Fourier cosine coefficient for half range expansion

B_n n^{th} Fourier sine coefficient

$\bar{B}_n$ n^{th} Fourier sine coefficient for half range expansion

$D(\dot{x})$	nonlinear dissipative damping term
$\bar{D}(\dot{x})$	$D(\dot{x})$ with linear term removed
$F(T)$	force as a time function
F_n	force at time denoted by subscript
$F_c(\omega)$	Fourier cosine transform value at ω.
$F_s(\omega)$	Fourier sine transform value at ω.
K	linear spring constant
L	a dummy variable
M	mass
N	total number of data points
$R(x)$	nonlinear resistance function
$\bar{R}(x)$	nonlinear resistance function with linear term absent
S_n	first forward difference at n
S_n^2	second forward difference at n
S_{n-1}^2	$S_n - S_{n-1}$
T	time
T_o	time at end of input
V	step change of velocity of value V
X	relative displacement
Y	absolute displacement
Z	foundation displacement
a, b	an angle
c	linear damping term in $c\dot{X}$
d_i	deviation or error at i
f	frequency, cycles/time
$f(t)$	a function of time
$f(t+0)$	right-hand limit of $f(t)$
$f(t-0)$	left-hand limit of $f(t)$
$g(r), g(s)$	functions of r, s, etc.
h	an increment of time
p	damped natural frequency, rad/time
t	time
α	ratio of damping to critical linear damping
Δ_n	negative of average value of a function
π	3.14159 - - - - - - -
θ	an angle
λ	a frequency, rad/time
ω	undamped natural frequency of linear system, rad/time

THE SIMPLE OSCILLATOR

The simple oscillator of Fig. 1 has the following general differential equation of motion:

$$M\ddot{Y} + D(\dot{X}) + R(X) = F(t) \tag{1}$$

where $X = (Y - Z)$, $D(\dot{X})$ is a dissipative term, and $R(X)$ is a restoring force term. Equation (1) may be written in the form

$$M\ddot{Y} + C(\dot{Y} - \dot{Z}) + K(Y - Z) + \overline{D}(\dot{X}) + \overline{R}(X) = F(t) \tag{2}$$

where $\overline{D}(\dot{X})$ and $\overline{R}(X)$ are the nonlinear terms of the dissipative and restoring forces. If the system is linear, Eq. (2) reduces to

$$M\ddot{Y} + C(\dot{Y} - \dot{Z}) + K(Y - Z) = F(t). \tag{3}$$

Assuming that $C < 2\sqrt{KM}$ (i.e., $C <$ critical), and using the notation

$$\omega^2 = K/M, \quad \alpha = \frac{C}{2M\omega}, \quad P = \omega\sqrt{1 - \alpha^2}$$

Eq. (3) may be written as

$$\ddot{Y} + 2\alpha\omega\dot{Y} + \omega^2 Y = 2\alpha\omega\dot{Z} + \omega^2 Z + \frac{F(t)}{M}. \tag{4}$$

There are two cases that will be the subject of this report: there is an applied force, but no foundation motion; and there is foundation motion, but no applied force. Combinations of solution for the linear system can of course be determined by superposition.

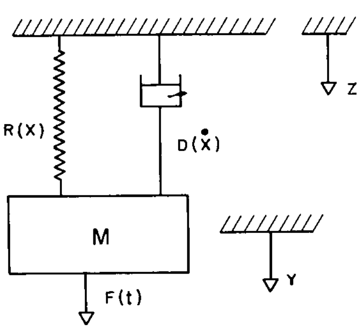

Fig. 1 - The simple oscillator

For relative motion (X) in the case of foundation motion with no applied force, Eq. (4) reduces to

$$\ddot{X} + 2\alpha\omega\dot{X} + \omega^2 X = -\ddot{Z} \tag{5a}$$

for the damped case, and

$$\ddot{X} + \omega^2 X = -\ddot{Z} \tag{5b}$$

for the undamped case. For applied force and no foundation motion they become

$$\ddot{X} + 2\alpha\,\omega\,\dot{X} + \omega^2 X = \frac{F(t)}{M} \tag{5c}$$

and

$$\ddot{X} + \omega^2 X = \frac{F(t)}{M} \tag{5d}$$

since $X = Y - Z$, and $Z \equiv 0$.

DERIVATION OF METHOD

Undamped Response

In order to shorten this paper only the numerical integration equations for a linear oscillator responding to a foundation motion will be derived in detail. The problem considered is to obtain the relative response if a record of the velocity of the foundation is known as a function of time.

The solution of Eq. (5b) and its derivative can be shown to be (1)

$$X = X_o \cos \omega t + \frac{\dot{X}_0}{\omega} \sin \omega t - \frac{1}{\omega}\int_0^t \ddot{Z}(T) \sin \omega (t - T)\, dT \tag{6a}$$

and

$$\frac{\dot{X}}{\omega} = -X_o \sin \omega t + \frac{\dot{X}_0}{\omega} \cos \omega t - \frac{1}{\omega}\int_0^t \ddot{Z}(T) \cos \omega (t - T)\, dT. \tag{6b}$$

These are linear equations, so if the solution were known at $t = t_1$, the values of displacement and relative velocity could be considered to be "initial values" for time redefined to begin there. Therefore if the Duhamel integrals could be integrated with good precision over a time interval, $\Delta t_1 = h$, say, then these new values of X and $\dot{X}/\omega$ at $t = t_1 + \Delta t_1$ could be used as "initial values" to continue the problem.

Consider a portion of a velocity record as shown in Fig. 2. For convenience the record is divided in a number of regions by equally spaced time increments, $h = \Delta t$.* Assume that X and $\dot{X}/\omega$ are known for $t = nh$, i.e., X_n and $\dot{X}_n/\omega$. The problem resolves itself to finding the values of $t = (n+1)h$. If X_n and $\dot{X}_n/\omega$ are considered to be "initial values" and time redefined as starting at this point only, the integrals need be evaluated at the new $t = h$. Suppose it was desired to use the average foundation acceleration over the increment as the approximation to the true foundation acceleration; then

$$\text{Average foundation acceleration} = \frac{1}{h}\int_{t_n}^{t_{n+1}} \ddot{Z}(t)\, dt = \frac{\dot{Z}_{n+1} - \dot{Z}_n}{h} \tag{7}$$

by the mean value theorem. Over this region then (using the redefined t) the equation for the foundation velocity becomes

$$\dot{Z}(T) = \dot{Z}_n + \frac{S_n\, t}{h} \tag{8}$$

*This is an unnecessary, but convenient restriction. It makes the calculations less cumbersome and machine programming easier.

where $S_n = \dot{Z}_{n+1} - \dot{Z}_n$. S_n is in reality a first forward difference and Eq. (8) in effect replaces the curved foundation velocity function in this region by a straight line. If greater precision is desired an assumption can be made about the rate of change of the foundation acceleration, such as, let $\dddot{Z}$ be constant over the increment. A parabola passed through the points $\dot{Z}_{n-1}$, $\dot{Z}_n$, and $\dot{Z}_{n+1}$, or one passed through $\dot{Z}_n, \dot{Z}_{n+1}, \dot{Z}_{n+2}$, will satisfy this requirement over the increment. Therefore

$$\dot{Z}(t) = \dot{Z}_n + \frac{S_n t}{h} + \frac{S^2_{n-1}}{2} \left(\frac{t^2}{h^2} - \frac{t}{h}\right) \tag{9}$$

or

$$\dot{Z}(t) = \dot{Z}_n + \frac{S_n t}{h} + \frac{S^2_n}{2} \left(\frac{t^2}{h^2} - \frac{t}{h}\right) \tag{10}$$

where

$$S^2_{n-1} = S_n - S_{n-1} = \dot{Z}_{n+1} - 2\dot{Z}_n + \dot{Z}_{n-1}$$

$$S^2_n = S_{n+1} - S_n = \dot{Z}_{n+2} - 2\dot{Z}_{n+1} + \dot{Z}_n .$$

Note that S^2 is not the square of S but is the second forward difference. In both equations when $t = h$, $\dot{Z}(h) = \dot{Z}_{n+1}$, so the curve is not in this respect like a Taylor or Maclaurin series expansion which has been truncated. It is generally more convenient to use Eq. (9) in the numerical problem, because after the solution is started it uses a rearward point, and no difficulties are encountered at the end of the forcing function.

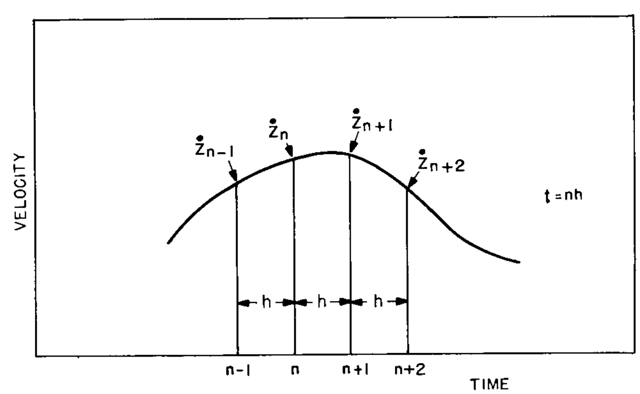

Fig. 2 - Portion of a velocity record

Over the time interval from $\dot{Z}_n$ to $\dot{Z}_{n+1}$ the equation of the curve has now been defined, so the integration and evaluation at h can be accomplished. If the point by point numerical integration equations are then scaled by ω for convenience in number handling, the results are

$$X_{n+1}\,\omega = X_n\,\omega \cos \omega h + \dot{X}_n \sin \omega h - \frac{S_n (1 - \cos \omega h)}{\omega h}$$

$$- S^2_{n-1} \left(\frac{1 + \cos \omega h}{2\omega h} - \frac{\sin \omega h}{\omega^2 h^2}\right) \tag{11a}$$

and

$$\dot{X}_{n+1} = - X_n \omega \sin \omega h + \dot{X}_n \cos \omega h - \frac{S_n \sin \omega h}{\omega h}$$

$$- S^2_{n-1} \left(\frac{1 - \cos \omega h}{\omega^2 h^2} - \frac{\sin \omega h}{2 \omega h} \right) . \tag{11b}$$

Two reasons for using equally spaced time increments are apparent immediately: (a) it is comparatively easy to use the parabolic approximation of the forcing function with equally spaced points; (b) for any frequency the sines and cosines need only be calculated once, and hence the trigometeric functions merely become numerical constants multiplying $X\omega$, $\dot{X}$, S, and S^2. Of course if the parabolic approximation to the forcing function is not desired, the S^2_{n-1} terms can be ignored, or higher order approximations to the input can be considered. The natural expansion functions for the response of an oscillator are sines and cosines (or, in the damped case, damped sines and cosines) and the results are of this form.

There still remains the problem of starting the solution. At the beginning of the input Eqs. (6) are still valid. It is only necessary to account for the initial conditions, if they are different from zero, and for the second difference term, S^2_{0-1}, which does not exist for the first step of the solution. Assume the general initial conditions that at the beginning of the forcing function (in reality at the left-hand limit of $(t - 0)$ as $t \to 0$, that is, as $t \to 0$ from the negative direction) that $X = d$, $\dot{X} = v$, and there is a step change in foundation velocity V. Then the starting equations become

$$X_1 \omega = d \omega \cos \omega h + (v - V) \sin \omega h - \frac{S_o (1 - \cos \omega h)}{\omega h}$$

$$- S^2_o \left(\frac{1 - \cos \omega h}{2 \omega h} - \frac{\sin \omega h}{\omega^2 h^2} \right) \tag{12a}$$

and

$$\dot{X}_1 = - d \omega \sin \omega h + (v - V) \cos \omega h - S_o \frac{\sin \omega h}{\omega h}$$

$$- S^2_o \left(\frac{1 - \cos \omega h}{\omega^2 h^2} - \frac{\sin \omega h}{2 \omega h} \right) . \tag{12b}$$

Since during the integration each pair of points $X\omega$ and $\dot{X}$ are considered to be new "initial conditions" for the next points, these equations also can be used to handle finite discontinuities in the foundation velocity, at any time.

The step by step numerical integration equations exactly satisfy the original differential equation over any increment which is a straight line (S^2_{n-1} made zero) or a parabolic arc. Therefore, consider the problem of the response of a linear oscillator to a velocity forcing function which has at most a finite number of finite discontinuities and the segments between these discontinuities are a combination of straight lines and second order parabolas. The proper use of these numerical integration equations will exactly solve the problem at each calculated point with the exception of number round-off error, of course. The numerical integration equations are stable and exact for this case regardless of increment size. For segments which are higher order curves, the differential equation is exactly satisfied up to the second difference in the forcing function. The closer the approximating curve lies to the true one, the more precise the answer. The error in this technique lies in the closeness of fit of the approximating segmental curves to the forcing function; no error is inherent due to increment size alone, in the linear case.

Damped Response

The differential equation of motion to be solved for the damped case is

$$\ddot{X} + 2\alpha\omega\dot{X} + \omega^2 X = -\ddot{Z} \tag{5}$$

where again it is assumed the foundation velocity is the known forcing function. This differential equation has the solution (1)

$$X = X(0)\ e^{-\alpha\omega t}\left(\cos pt + \frac{\alpha}{\sqrt{1-\alpha^2}}\sin pt\right) + \frac{\dot{X}(0)\ e^{-\alpha\omega t}\sin pt}{p} - \frac{1}{p}\int_0^t \ddot{Z}(T)\,e^{-\alpha\omega[t-T]}\sin p(t-T)\,dT\,. \tag{13}$$

If the same procedure is used to approximate the forcing function as in the damped case then the step by step numerical integration equations (scaled by ω for convenience) become

$$\begin{aligned}X_{n+1}\,\omega = {} & X_n\,\omega e^{-\alpha\omega h}\left(\cos ph + \frac{\alpha}{\sqrt{1-\alpha^2}}\sin ph\right) + \frac{\dot{X}_n\ e^{-\alpha\omega h}\sin ph}{\sqrt{1-\alpha^2}} \\ & - \frac{S_n}{\omega h}\left[1 - e^{-\alpha\omega h}\left(\cos ph + \frac{\alpha}{\sqrt{1-\alpha^2}}\sin ph\right)\right] \\ & - \frac{S^1_{n-1}}{\omega h}\left\{\frac{1}{2} - \frac{2\alpha}{\omega h} + e^{-\alpha\omega h}\left[\left(\frac{1}{2} + \frac{2\alpha}{\omega h}\right)\cos ph - \left(\frac{1-2\alpha^2}{\omega h} - \frac{\alpha}{2}\right)\frac{\sin ph}{\sqrt{1-\alpha^2}}\right]\right\}\end{aligned} \tag{14a}$$

and

$$\begin{aligned}X_{n+1} = {} & -X_n\,\omega\,\frac{e^{-\alpha\omega h}\sin ph}{\sqrt{1-\alpha^2}} + \dot{X}_n\ e^{-\alpha\omega h}\left(\cos ph - \frac{\alpha}{\sqrt{1-\alpha^2}}\sin ph\right) \\ & - \frac{S_n}{\omega h}\,\frac{e^{-\alpha\omega h}\sin ph}{\sqrt{1-\alpha^2}} - \frac{S^2_{n-1}}{\omega h}\left\{\frac{1}{\omega h} - e^{-\alpha\omega h}\left[\frac{\cos ph}{\omega h} + \left(\frac{\alpha}{\omega h} + \frac{1}{2}\right)\frac{\sin ph}{\sqrt{1-\alpha^2}}\right]\right\}.\end{aligned} \tag{14b}$$

The procedure for obtaining the starting equations is the same as the undamped case.

A Word of Caution

Often it is assumed that the damped and undamped natural frequencies are equal. If such an assumption is made the second difference (S^2_{n-1}) should be dropped from the calculation because the error in the exponential and trigonometric coefficient of this term in the X equation can become larger than the true value of the coefficient. The abridged equations become

$$\begin{aligned}X_{n+1}\,\omega = {} & X_n\,\omega e^{-\alpha\omega h}(\cos\omega h + \alpha\sin\omega h) + \dot{X}_n\ e^{-\alpha\omega h}\sin\omega h \\ & - \frac{S_n}{\omega h}\left\{1 - e^{-\alpha\omega h}(\cos\omega h + \alpha\sin\omega h)\right\}\end{aligned} \tag{15a}$$

$$\begin{aligned}\dot{X}_{n+1} = {} & -X_n\,\omega\,e^{-\alpha\omega h}\sin\omega h + \dot{X}_n\ e^{-\alpha\omega h}(\cos\omega h - \alpha\sin\omega h) \\ & - \frac{S_n}{\omega h}\ e^{-\alpha\omega h}\sin\omega h\,.\end{aligned} \tag{15b}$$

SAMPLE PROBLEMS

Two simple problems were chosen to show sample computations. Known solutions exist, so that comparison with theoretical values is possible. Consider the following velocity forcing function: Let

$$\dot{Z} = 0, \quad \text{for} \quad t \leq 0$$

$$\dot{Z} = \dot{Z}_0 \sin \lambda\, t, \quad \text{for} \quad 0 \leq t \leq 4\pi/\lambda$$

$$\dot{Z} = 0, \quad \text{for} \quad t \geq 4\pi/\lambda$$

and assume the initial conditions are such that the mass is at rest. The values of the chosen parameters are $\omega = 600$ rad/sec, $\lambda = 1200$ rad/sec, $h = \pi/7200$ sec, $f_\omega = 95.5$ cps, $f = 191.0$ cps, $\dot{Z}_o = 90$ in./sec, and $\alpha = 0.1 = 10\%$ of critical (when damping present).

Table 1 is the listing of the velocity forcing function for 24 increments needed for the integration. The values of the sines, cosines, and exponentials used in these examples were taken to six decimal places, but the actual numerical integration was held to three decimal places in the answers.

Table 1
Velocity Forcing Function

n	$\dot{Z}_n$	S_n	S^2_{n-1}
0	0	+45.000 000	-12.057 714*
1	+45.000 000	+32.942 286	-12.057 714
2	+77.942 286	+12.057 714	-20.884 572
3	+90.000 000	-12.057 714	-24.115 428
4	+77.942 286	-32.942 286	-20.884 572
5	+45.000 000	-45.000 000	-12.057 714
6	0	-45.000 000	0
7	-45.000 000	-32.942 286	+12.057 714
8	-77.942 286	-12.057 714	+20.884 572
9	-90.000 000	+12.057 714	+24.115 428
10	-77.942 286	+32.942 286	+20.884 572
11	-45.000 000	+45.000 000	+12.057 714
12	0	+45.000 000	0
13	+45.000 000	+32.942 286	-12.057 714
14	+77.942 286	+12.057 714	-20.884 572
15	+90.000 000	-12.057 714	-24.115 428
16	+77.942 286	-32.942 286	-20.884 572
17	+45.000 000	-45.000 000	-12.057 714
18	0	-45.000 000	0
19	-45.000 000	-32.942 286	+12.057 714
20	-77.942 286	-12.057 714	+20.884 572
21	-90.000 000	+12.057 714	+24.115 428
22	-77.942 286	+32.942 286	+20.884 572
23	-45.000 000	+45.000 000	+12.057 714
24	0		

*This is really S^2_0.

Undamped Response

The numerical integration equations for the undamped response become

$$X_{n+1}\,\omega = 0.965926\, X_n\,\omega + 0.258819\, \dot{X}_n - 0.130154\, S_n + 0.021593\, S^2_{n-1} \qquad (16a)$$

$$\dot{X}_{n+1} = -0.258819\, X_n\,\omega + 0.965926\, \dot{X}_n - 0.988616\, S_n - 0.002843\, S^2_{n-1} \qquad (16b)$$

Table 2 was prepared and the integration carried out in it. The theoretical $X\omega$ is included to show the precision of the solution. The forcing function was defined for every 30 degrees (on its own sine wave) and the response was therefore calculated at a time which corresponds to an angle of 15 degrees.

Table 2
Undamped Response Computations

n	0.965926 $X_n\omega$	0.258819 $\dot{X}_n$	-0.130154 S_n	+0.021593 S^2_{n-1}	$X_{n+1}\omega$	Theoretical $X_{n+1}\omega$	-0.258819 $X_n\omega$	+0.965926 $\dot{X}_n$	-0.988616 S_n	-0.002843 S^2_{n-1}	$\dot{X}_{n+1}$
0	0	0	-5.857	-0.260	-6.117	-5.994	0	0	-44.488	+0.034	-44.454
1	-5.909	-11.506	-4.288	-0.260	-21.963	-21.962	+1.583	-42.939	-32.567	+0.034	-73.889
2	-21.215	-19.124	-1.569	-0.451	-42.359	-42.427	+5.684	-71.371	-11.920	+0.059	-77.548
3	-40.916	-20.071	+1.569	-0.521	-59.939	-60.000	+10.963	-74.906	+11.920	+0.069	-51.954
4	-57.897	-13.447	+4.288	-0.451	-67.507	-67.491	+15.513	-50.184	+32.567	+0.059	-2.045
5	-65.207	-0.529	+5.857	-0.260	-60.139	-60.000	+17.472	-1.975	+44.488	+0.034	+60.019
6	-58.090	+15.534	+5.857	0	-36.699	-36.433	+15.565	+57.974	+44.488	0	+118.027
7	-35.449	+30.548	+4.288	+0.260	-0.353	0	+9.498	+114.005	+32.567	-0.034	+156.036
8	-0.341	+40.385	+1.569	+0.451	+42.064	+42.427	+0.091	+150.719	+11.920	-0.059	+162.671
9	+40.631	+42.102	-1.569	+0.521	+81.685	+81.962	-10.887	+157.128	-11.920	-0.069	+134.252
10	+78.902	+34.747	-4.288	+0.451	+109.812	+109.918	-21.142	+129.677	-32.567	-0.059	+75.909
11	+106.070	+19.647	-5.857	+0.260	+120.120	+120.000	-28.421	+73.322	-44.488	-0.034	+0.379
12	+116.027	+0.098	-5.857	0	+110.268	+109.918	-31.089	+0.366	-44.488	0	-75.211
13	+106.511	-19.466	-4.288	-0.260	+82.497	+81.962	-28.539	-72.648	-32.567	+0.034	-133.720
14	+79.686	-34.609	-1.569	-0.451	+43.057	+42.427	-21.352	-129.164	-11.920	+0.059	-162.377
15	+41.590	-42.026	+1.569	-0.521	+0.612	0	-11.144	-156.844	+11.920	+0.069	-155.999
16	+0.591	-40.376	+4.288	-0.451	-35.948	-36.433	-0.158	-150.683	+32.567	+0.059	-118.215
17	-34.723	-30.596	+5.857	-0.260	-59.722	-60.000	+9.304	-114.187	+44.488	+0.034	-60.361
18	-57.687	-15.623	+5.857	0	-67.453	-67.491	+15.457	-58.304	+44.488	0	+1.641
19	-65.155	+0.425	+4.288	+0.260	-60.182	-60.000	+17.458	+1.585	+32.567	-0.034	+51.576
20	-58.131	+13.349	+1.569	+0.451	-42.762	-42.427	+15.576	+49.819	+11.920	-0.059	+77.256
21	-41.305	+19.995	-1.569	+0.521	-22.358	-21.962	+11.068	+74.624	-11.920	-0.069	+73.703
22	-21.596	+19.076	-4.288	+0.451	-6.357	-5.994	+5.787	+71.192	-32.567	-0.059	+44.353
23	-6.140	+11.479	-5.857	+0.260	-0.258	0	+1.645	+42.842	-44.488	-0.034	-0.035

Damped Response

For this problem the numerical equivalent of Eqs. (14) become

$$X_{n+1}\,\omega = 0.966512\, X_n\,\omega + 0.252160\, \dot{X}_n - 0.127915\, S_n + 0.021036\, S^2_{n-1} \qquad (17a)$$

$$\dot{X}_{n+1} = -0.252160\, X_n\,\omega + 0.916080\, \dot{X}_n - 0.963180\, S_n - 0.007006\, S^2_{n-1} \qquad (17b)$$

The results and comparisons with the theoretical solution are shown in Table 3.

Table 3
Damped Response

n	0.966512 $X_n\omega$	0.252160 $\dot{X}_n$	-0.127915 S_n	0.021036 S^2_{n-1}	$X_{n+1}\omega$	Theoretical $X_{n+1}\omega$	-0.252160 $X_n\omega$	0.916080 $\dot{X}_n$	-0.963180 S_n	-0.007006 S^2_{n-1}	$\dot{X}_{n+1}$
0	0	0	-5.756	-0.254	-6.010	-5.890	0	0	-43.343	+0.084	-43.259
1	-5.809	-10.908	-4.214	-0.254	-21.185	-21.190	+1.515	-39.629	-31.729	+0.084	-69.759
2	-20.476	-17.590	-1.542	-0.439	-40.047	-40.187	+5.342	-63.905	-11.614	+0.146	-70.031
3	-38.706	-17.659	+1.542	-0.507	-55.330	-55.391	+10.098	-64.154	+11.614	+0.169	-42.273
4	-53.477	-10.660	+4.214	-0.439	-60.362	-60.344	+13.952	-38.725	+31.729	+0.146	+7.102
5	-58.341	+1.791	+5.756	-0.254	-51.048	-50.909	+15.221	+6.506	+43.343	+0.084	+65.154
6	-49.339	+16.429	+5.756	0	-27.154	-26.896	+12.872	+59.686	+43.343	0	+115.901
7	-26.245	+29.226	+4.213	+0.254	+7.448	+7.779	+6.847	+106.175	+31.729	-0.084	+144.667
8	+7.199	+36.479	+1.542	+0.439	+45.659	+45.985	-1.878	+132.527	+11.614	-0.146	+142.117
9	+44.130	+35.836	-1.542	+0.507	+78.931	+79.163	-11.513	+130.191	-11.614	-0.169	+106.895
10	+76.288	+26.954	-4.214	+0.439	+99.467	+99.528	-19.903	+97.924	-31.729	-0.146	+46.146
11	+96.136	+11.636	-5.756	+0.254	+102.270	+102.119	-25.082	+42.273	-43.343	-0.084	-26.236
12	+98.845	-6.616	-5.756	0	+86.473	+86.117	-25.788	-24.034	-43.343	0	-93.165
13	+83.577	-23.492	-4.214	-0.254	+55.617	+55.113	-21.805	-85.347	-31.729	+0.084	-138.797
14	+53.754	-34.999	-1.542	-0.439	+16.774	+16.214	-14.024	-127.149	-11.613	+0.146	-152.640
15	+16.212	-38.490	+1.542	-0.507	-21.243	-21.749	-4.230	-139.830	+11.613	+0.169	-132.278
16	-20.532	-33.355	+4.214	-0.439	-50.112	-50.469	+5.357	-121.177	+31.729	+0.146	-83.945
17	-48.434	-21.168	+5.756	-0.254	-64.100	-64.245	+12.636	-76.900	+43.343	+0.084	-20.837
18	-61.953	-5.254	+5.756	0	-61.451	-61.373	+16.163	-19.088	+43.343	0	+40.418
19	-59.393	+10.192	+4.214	+0.254	-44.733	-44.741	+15.495	+37.026	+31.729	-0.084	+84.166
20	-43.235	+21.223	+1.542	+0.439	-20.031	-19.662	+11.280	+77.103	+11.614	-0.146	+99.851
21	-19.360	+25.178	-1.542	+0.507	+4.783	+5.159	+5.051	+91.472	-11.614	-0.169	+84.704
22	+4.623	+21.368	-4.214	+0.439	+22.216	+25.509	-1.206	+77.629	-31.729	-0.146	+44.548
23	+21.472	+11.233	-5.756	+0.254	+27.203	+27.206	-5.602	+40.810	-43.343	-0.084	+8.219

Increment Size

The size of increment for shock spectrum calculations is restricted if the method is used without modification. Since this is a point by point technique, if $h = \Delta t$ is chosen too large for the high frequencies, it may turn out that values near the peak are missed. This results in a value which is too low. In the Fourier portion of the calculations (to be discussed later) the only restriction on increment size is that the arcs approximating the forcing function must lie close to the true curve, because the method must compute the end point anyway. A good rule of thumb is to note if a set of trapezoids at a chosen increment size would be a good approximation to the curve.

For forcing functions which are a combination of straight lines, are parabolic arcs, or are "sufficiently smooth," the numerical integration equations themselves can be used for interpolation purposes* without changing the approximation to the forcing function.

The analyst should be careful however not to use the values at $n + 1/2$ when integrating between $n + 1$ and $n + 2$. This technique should be particularly useful when hunting for maximum values and has no justification or need in Fourier spectrum type of calculations.

The NRL digital computer NAREC has been coded to use this technique (without interpolation) in the case of the velocity inputs and some results are shown later.

*Caution: This is not a normal linear or quadratic interpolation. See Appendix A.

FOURIER ANALYSIS

Fourier Integrals

Let a function $f(t)$ be real and satisfy the Dirichlet conditions (10), and let

$$\int_{-\infty}^{+\infty} |f(t)|\,dt \tag{18}$$

exist. Then

$$1/2\left[f(t+0)+f(t-0)\right] = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{+\infty} f(s)\cos\omega(s-t)\,ds\,d\omega. \tag{19}$$

This is known as the Fourier Integral Theorem (10-12). The function on the left merely indicates that the double integral converges to the average value of the left- and right-hand limits at a finite discontinuity of $f(t)$. For the remainder of this paper this will be written as $f(t)$, but this point should be remembered.

If $f(t)$ is assumed to be either an even function $(f(-t) = f(t))$ or an odd function $(f(-t) = -f(t))$, simplifications are possible. They result in

$$f(t) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{+\infty} f(s)\cos\omega s\cos\omega t\,ds\,d\omega$$

for $f(t)$ even and

$$f(t) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{+\infty} f(s)\sin\omega s\sin\omega t\,ds\,d\omega$$

for $f(s)$ odd.

It is convenient to have the previous Fourier sine and cosine integrals as transform pairs. The cosine transform is,

$$F_c(\omega) = \int_0^{\infty} f(s)\cos\omega s\,ds \tag{20a}$$

where

$$f(t) = \frac{2}{\pi}\int_0^{\infty} F_c(\omega)\cos\omega t\,d\omega \tag{20b}$$

and the sine transform is

$$F_s(\omega) = \int_0^{\infty} f(s)\sin\omega s\,ds \tag{21a}$$

where

$$f(t) = \frac{2}{\pi}\int_0^{\infty} F_s(\omega)\sin\omega t\,dt. \tag{21b}$$

Examination of Eq. (20b) will show that if $t = 0$

$$f(o) = \frac{2}{\pi} \int_0^\infty F_c(\omega)\, d\omega.$$

This fact has application when the inverse transform of a function is to be obtained.

Fourier Series

It is often convenient to expand an arbitrary function in the range from $t = 0$ to $t = T_o$ in a trigonometeric series of the form

$$f(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos \frac{2n\pi t}{T_0} + \sum_{n=1}^{\infty} B_n \sin \frac{2n\pi t}{T_0}. \tag{22}$$

This is the full range expansion form of the Fourier series and the coefficients are found by means of the formulas

$$A_0 = \frac{2}{T_0} \int_0^{T_0} f(t)\, dt, \tag{23}$$

$$A_n = \frac{2}{T_0} \int_0^{T_0} f(t) \cos \frac{2n\pi t}{T_0}\, dt \tag{24}$$

and

$$B_n = \frac{2}{T_0} \int_0^{T_0} f(t) \sin \frac{2n\pi t}{T_0}\, dt. \tag{25}$$

Half Range Expansions

However, it may become convenient to expand in a series of only sines or cosines. For example

$$f(t) = \sum_{n=1}^{\infty} \bar{B}_n \sin \frac{n\pi t}{T_0} \tag{26}$$

or

$$f(t) = \frac{\bar{A}_0}{2} + \sum_{n=1}^{\infty} \bar{A}_n \cos \frac{n\pi t}{T_0}. \tag{27}$$

These coefficients are given by

$$\bar{B}_n = \frac{2}{T_0} \int_0^{T_0} f(t) \sin \frac{n\pi t}{T_0}\, dt \tag{28}$$

$$\bar{A}_0 = \frac{2}{T_0} \int_0^{T_0} f(t)\, dt \tag{29}$$

and

$$\bar{A}_n = \frac{2}{T_0} \int_0^{T_0} f(t) \cos \frac{n\pi t}{T_0}\, dt. \tag{30}$$

Parseval's Theorems

Of the many theorems associated with Fourier analysis some of the most useful are known as Parseval's theorems. These are presented without proof. For the Fourier integrals,

$$\int_0^{\infty} f^2(t)\, dt = \frac{2}{\pi} \int_0^{\infty} F_c^2(\omega)\, d\omega = \frac{2}{\pi} \int_0^{\infty} F_s^2(\omega)\, d\omega = \frac{1}{\pi} \int_0^{\infty} F(2)\, d\omega.$$

For Fourier series,

$$\int_0^{T_0} f^2(t)\, dt = \frac{T_0}{2} \left[\frac{A_0^2}{2} + \sum_{n=1}^{\infty} (A_n^2 + B_n^2) \right].$$

And for the half range expansions,

$$\int_0^{T_0} f^2(t)\, dt = \frac{T_0}{2} \left[\frac{\bar{A}_0^2}{2} + \sum_{n=1}^{\infty} \bar{A}_n^2 \right] = \frac{T_0}{2} \sum_{n=1}^{\infty} \bar{B}_n^2 .$$

FOURIER ANALYSIS AND THE LINEAR OSCILLATOR

The integrals of the Fourier transform pairs, and for calculation of the Fourier series coefficients are all of the form

$$\int_0^{L} g(r) \sin ar\, dr \tag{31a}$$

or

$$\int_0^{L} g(r) \cos ar\, dr. \tag{31b}$$

Now sine and cosine functions are obviously the natural expansion functions when dealing with the solution of an undamped linear oscillator. The question to be answered is then, "Is it possible and efficient to use the undamped response of a linear oscillator to an applied transient to calculate Fourier spectra, Fourier series coefficients, and inverse Fourier transforms?" The answer is, "Yes."

Fourier Transform

To examine the problem in detail it is now time to return to Eq. (6a), which is repeated for convenience,

$$X = X_o \cos \omega t + \frac{\dot{X}_0}{\omega} \sin \omega t - \frac{1}{\omega} \int_0^t \ddot{Z}(T) \sin \omega(t - T)\, dT. \tag{6a}$$

If the Duhamel integral is integrated by parts and then the mass is assumed at rest initially, there results

$$X = - \int_0^t \dot{Z}(T) \cos \omega(t - T)\, dT. \tag{32}$$

Use of the trigonometrical substitution

$$\cos (a - b) = \cos a \cos b + \sin a \sin b$$

yields

$$X = - \cos \omega t \int_0^t \dot{Z}(T) \cos \omega t\, dT - \sin \omega t \int_0^t \dot{Z}(T) \sin \omega T\, dT. \tag{33}$$

Both of these integrals can be associated with the Fourier transform pairs for time equal to t, so the equation may be written

$$X = \sqrt{F_c(\omega)^2 + F_s(\omega)^2} \cos (\omega t - \theta)$$

where

$$\theta = \tan^{-1} \frac{F_s}{F_c} = \sin^{-1} \frac{F_s}{\sqrt{F_c^2 + F_s^2}} = \cos^{-1} \frac{F_c}{\sqrt{F_c^2 + F_s^2}} .$$

The quantity $(F_c^2 + F_s^2)^{1/2}$ is usually called the Fourier spectrum magnitude value at ω, and θ is the phase angle associated with this vibration. The maximum value of this equation, evaluated after the shock motion is over, yields the interesting result that the "after" or "residual" shock spectrum is in reality the Fourier spectrum of the foundation velocity.

Equation (33) and its derivative give

$$X = - \cos \omega t \int_0^t \dot{Z}(T) \cos \omega T\, dT - \sin \omega t \int_0^t \dot{Z}(T) \sin \omega T\, dT . \tag{34a}$$

and

$$\frac{(\dot{X} + \dot{Z})}{\omega} = \sin \omega t \int_0^t \dot{Z}(T) \cos \omega T\, dT - \cos \omega t \int_0^t \dot{Z}(T) \sin \omega T\, dT. \tag{34b}$$

If X, $\dot{X}$, and $\dot{Z}$ are known for a particular time T_0 (in this case, say, the end of the transient), then simultaneous solution of these equations gives the integral values. They are

$$\int_0^{T_0} \dot{Z}(T) \cos \omega T \, dT = -X(T_0) \cos \omega T_0 + \frac{\dot{X}(T_0) + \dot{Z}(T_0)}{\omega} \sin \omega T_0 \tag{35a}$$

and

$$\int_0^{T_0} \dot{Z}(T) \sin \omega T \, dT = -\frac{\dot{X}(T_0) + \dot{Z}(T_0)}{\omega} \cos \omega T_0 - X(T_0) \sin \omega T_0 . \tag{35b}$$

However, these are precisely $F_c(\omega)$ and $F_s(\omega)$ as defined by Eqs. (20) and (21), for a function which ends in finite time and are of the form of Eqs. (31).

Fourier Series Coefficients

To find Fourier series coefficients for the full range expansion let $\omega_n = 2n\pi/T_0$, and evaluate the response X and $\dot{X}$ at T_0 for each n desired.

Then

$$A_n = \frac{2}{T_0} \int_0^{T_0} \dot{Z}(T) \cos \frac{2n\pi T}{T_0} \, dT = -\frac{2X}{T_0}$$

and

$$B_n = \frac{2}{T_0} \int_0^{T_0} \dot{Z}(T) \sin \frac{2n\pi T}{T_0} \, dT = -\frac{2}{T_0} \frac{(\dot{X} + \dot{Z})}{\omega_n} .$$

For the half range expansions let $\omega_n = n\pi/T_0$. The coefficients become

$$\bar{A}_n = -\frac{2}{T_0} (-1)^n X$$

and

$$\bar{B}_n = -\frac{2}{T_0} (-1)^n \frac{(\dot{X} + \dot{Z})}{\omega_n}$$

where the X and $\dot{X}$ have again been evaluated for each n at $t = T_0$. This is an automatic consequence of Eqs. (35).

Inverse Fourier Transform

With the exception of a constant multiplier, the transform and its inverse as defined here are symmetric. Therefore, if one is willing to stop the ω integration short of infinity, an approximation to the inverse transform can be found by the same methods. However some unusual scaling is involved, especially if a machine program has been used which scales the X and $\dot{X}/\omega$ by ω. When the direct transform of an arbitrary function is found, Fourier sine and cosine spectra are scaled by ω with the proposed numerical integration equations. (This was so that more convenient numbers could be handled but is not a restriction as they could be de-scaled.) Consider the Fourier cosine transform only, because the same approach applies to the sine transform. Suppose a machine program calculated ωF_c as a function of frequency f. However, for the input let $g(T)$ in reality be $g(\omega)$ and attempt to obtain the inverse transform. The machine at each frequency f calculates

$$2\pi f \overline{F}_c = 2\pi f \int_0^L g(\omega) \cos 2\pi f \, \omega \, d\omega$$

whereas

$$g(t) = \frac{2}{\pi} \int_0^L g(\omega) \cos \omega t \, d\omega$$

is desired. If the integral was multiplied by $1/\pi^2 f$ for each f, then the coefficient of the integral would be correct. Time t in the second equation must correspond to $2\pi f$ in the first, so $t = 2\pi f$. An example showing the application of the numerical integration equations to the inverse transform problem is shown later.

There are of course many examples of the use of Fourier transforms in the literature. Two of the interesting ones are the calculation of frequency response knowing the time response to impulse, and the inverse of this, namely, the calculation of the time response for impulse knowing the frequency (steady-state) characteristics. To demonstrate this consider the Duhamel integral

$$g(\omega) = -\int_0^t f(T) \cos \omega (t - T) \, dT.$$

By successive changes of variable, this may be written as

$$g(\omega) = -\int_0^t f(t - T) \cos \omega T \, dT.$$

If $f(T)$ was the response to impulse the cosine term could be considered to be the driving function and the response to steady-state vibration can be found for large t.

APPLICATION TO NONLINEAR PROBLEMS

The numerical integration equations as derived here can be applied to nonlinear single degree of freedom systems and to two degree of freedom linear and nonlinear systems. This has been done and will be the subject of a later NRL report. To merely indicate the method of solution consider a nonlinear oscillator responding to an applied force. The equation of motion might be written as

$$\ddot{X} + 2\alpha\omega\dot{X} + \omega^2 X = g(t) - \overline{D}(\dot{X}) - \overline{R}(X). \tag{2a}$$

Now if over a short time increment the quantity on the right-hand side was considered to be $-\Delta_n$ which is defined as the negative of the average value over this increment then this equation may be written as

$$\ddot{X} + 2\alpha\omega\dot{X} + \omega^2 X = -\Delta_n.$$

This over this short time interval is a second-order linear differential equation with constant coefficients and the solution is well known. Since the average value of Δ_n is required, iteration can be used with the proper numerical integration equations as derived here, and a step by step evaluation of the response is possible.

The undamped response equations could also be used by putting the $\dot{x}$ term on the right-hand side. This then becomes the numerical equivalent of the phase plane graphical method.

EXAMPLES OF MACHINE COMPUTATIONS

Fourier Series Coefficients

The 20th to the 30th Fourier cosine series (halfrange expansion) coefficients of the function $y = (\pi/4)\,|\sin 2\pi t\,|$ are used to illustrate this application. The arch of the sine curve was divided into 4, 6, 8, 10, and 12 increments (increment sizes 1/8, 1/12, 1/16, 1/20, and 1/24 the size of a sine period) and for each of these increment sizes the response of an undamped linear oscillator was used to calculate the coefficients at 2-cps increments from 40 to 60 cps. The input was given to six significant figures. Table 4 shows how many periods of the responding system correspond to the increment size used. For example, at 60 cps when 10 increments were used ($n = 10$) the ratio of the increment size (1/20) to the period of the responding oscillator (1/60) is 3.0.

Table 4
Number of Periods of the Responding System
Corresponding to the Increment Size Used

Oscillator Frequency (cps)	Ratio of Increment Size to Oscillator Period When the Number of Increments Used is n				
	n = 4	n = 6	n = 8	n = 10	n = 12
40	5.000	3.333	2.500	2.000	1.167
42	5.250	3.500	2.625	2.100	1.750
44	5.500	3.667	2.750	2.200	1.833
46	5.750	3.833	2.875	2.300	1.917
48	6.000	4.000	3.000	2.400	2.000
50	6.250	4.167	3.125	2.500	2.083
52	6.500	4.333	3.250	2.600	2.167
54	6.750	4.500	3.375	2.700	2.225
56	7.000	4.667	3.500	2.800	2.333
58	7.250	4.833	3.625	2.900	2.417
60	7.500	5.000	3.750	3.000	2.500

Table 5 contains the results of the computation. The values of each of the coefficients have been multiplied by minus 10^6. The numbers have been rounded to the sixth decimal place to correspond to the six decimal place input. To show the precision of the method the errors in percent of the calculated value when compared to the exact theoretical value are given. For example, at 60 cps, with $n = 10$, $\bar{A}_{30} = 0.000287$ (calculated), $\bar{A}_{30} = 0.000278$ (exact theory), and Error = 3.2%.

This was using an increment size which is three times the responding oscillator's period. Therefore thirty or more coefficients can be calculated using only a few input points.

For those forcing functions which are exactly satisfied by the approximations, the only error present is the normal round-off error.

Table 5
Comparison of Calculated and Theoretical Values (Multiplied by 10^6) for Fourier Coefficients

Frequency (cps)	Theoretical Value	Computed $n = 4$	Percent Error	Computed $n = 6$	Percent Error	Computed $n = 8$	Percent Error	Computed $n = 10$	Percent Error	Computed $n = 12$	Percent Error
40	625	728	16.5	660	5.6	663	6.1	645	3.2	634	1.4
42	567	693	22.2	580	2.3	581	2.5	608	7.2	580	2.3
44	517	544	5.2	547	5.8	543	5.0	539	4.3	532	2.9
46	473	574	21.4	534	12.9	509	7.6	484	2.3	487	3.0
48	434	505	16.4	470	8.3	455	4.8	439	1.1	444	2.3
50	400	489	22.3	458	14.5	439	9.8	403	0.8	422	5.5
52	370	390	5.4	391	5.7	389	5.1	375	1.4	383	3.5
54	343	417	21.6	351	2.3	351	2.3	352	2.6	351	2.3
56	319	371	16.3	337	5.6	323	1.3	332	4.1	323	1.3
58	297	363	22.2	336	13.1	305	2.7	312	5.1	300	1.0
60	278	293	5.4	301	8.3	292	5.3	287	3.2	279	0.4

Inverse Fourier Transforms

To demonstrate that this method can be used to calculate the inverse Fourier transform without the use of special tables the following two cases were worked out on the digital computer NAREC.

Case I – The Fourier transform of a step function of heigth one-half unit, lasting two seconds of time, and returning to zero with a step change of one-half unit was used as an input to the shock record program. (This program prints out the scaled Fourier transform, the Fourier sine and cosine transform, as well as $+X\omega(\text{max})$ and $-X\omega(\text{max})$.) Of course the integration could not be carried out to infinite frequency so a cutoff frequency of approximately 83 cps was decided upon and about 100 output points were used to define the function, up to time equal to π seconds. The error in the resulting transformation was so small that a graphical plot of the original function superposed upon the results of the inverse transformation will not show the precision of the method. Table 6 shows the average algebraic error, the average absolute error, and the root mean square error.

Case II – As possibly a more severe test case, it was decided to try a damped sine wave. The procedure was to let the NAREC calculate the original function for about four complete cycles, defining it at 100 points. This digital record was then used as an input to the shock record program to calculate the Fourier sine and cosine transforms at intervals of 2 cps from 0 cps to 300 cps. The sine and cosine transforms were then used to calculate the original time function again by using them as inputs to the shock record program. The original time function was

$$y = e^{-\alpha\omega t} \sin pt$$

where $\alpha = 0.2$, $\omega = 300$, ($f \approx 47$ cps), $p = \omega\sqrt{1 - \alpha^2} = 293.938^+$, and the maximum value of y turned out to be about 0.736. Figure 3 shows the results of this computation. Agreement would have been better if a higher cutoff frequency had been chosen, say 500 cps or 750 cps, and if more closely spaced points had been used to define the original input function.

Table 6
Numerical Results for Case I

Error Computation*	Inverse Fourier Cosine Transformation	Inverse Fourier Sine Transformation
Average Error $= \frac{\sum_i d_i}{N}$	0.23×10^{-3}	0.14×10^{-3}
Average Absolute Error $= \frac{\sum_i \lvert d_i \rvert}{N}$	0.95×10^{-3}	1.33×10^{-3}
Root Mean Square Error $= \frac{\sqrt{\sum_i d_i^2}}{N}$	0.18×10^{-3}	0.28×10^{-3}

*d_i is the deviation at point i and N is the total number of data points (101 for the cosine transform, and 94 for the sine transform).

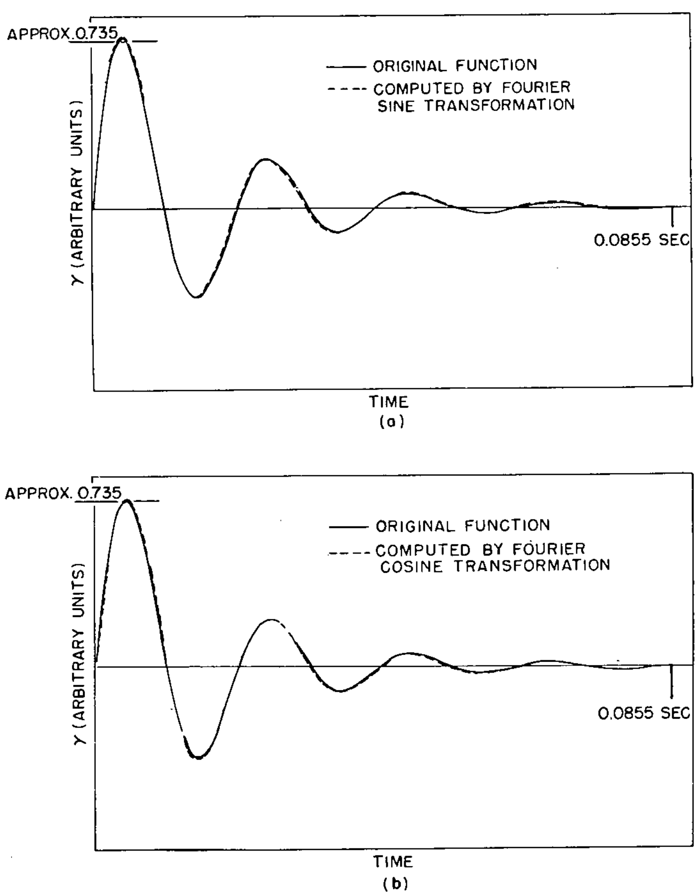

Fig. 3 - Comparison of results for Case II, a damped sine wave

Damped Shock Spectra

To show the effect of damping upon shock spectra an actual foundation velocity record (Fig. 4) was used. It was divided into 760 increments and shock spectra for damping ratios of $\alpha = 0$, $\alpha = 0.01$, and $\alpha = 0.10$ were calculated and are shown in Fig. 5. (Curves for $\alpha = 0.001$ and $\alpha = 0.0001$ were also obtained but are not shown for reasons of clarity.) Figure 6 is a plot of the effect of damping upon the peak value and the effect upon the shock spectrum value useful in stress analysis (4) (that value corresponding to that fixed base natural frequency of the structure).

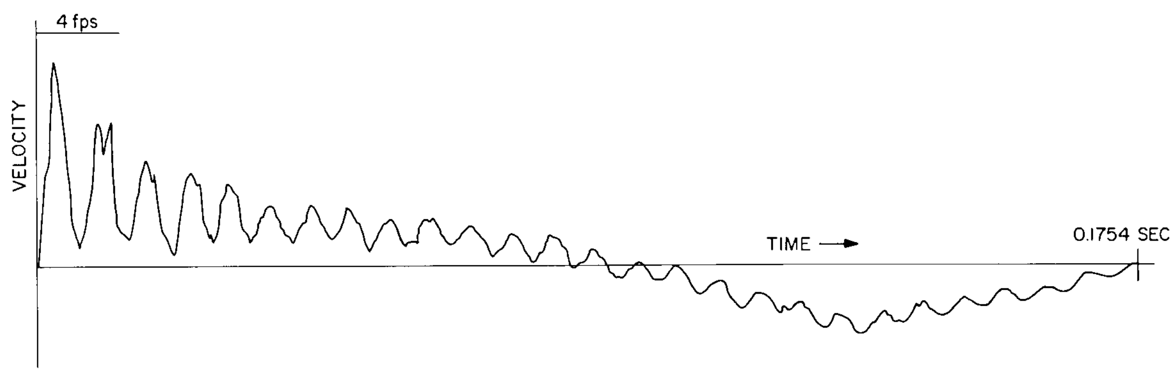

Fig. 4 - Foundation velocity

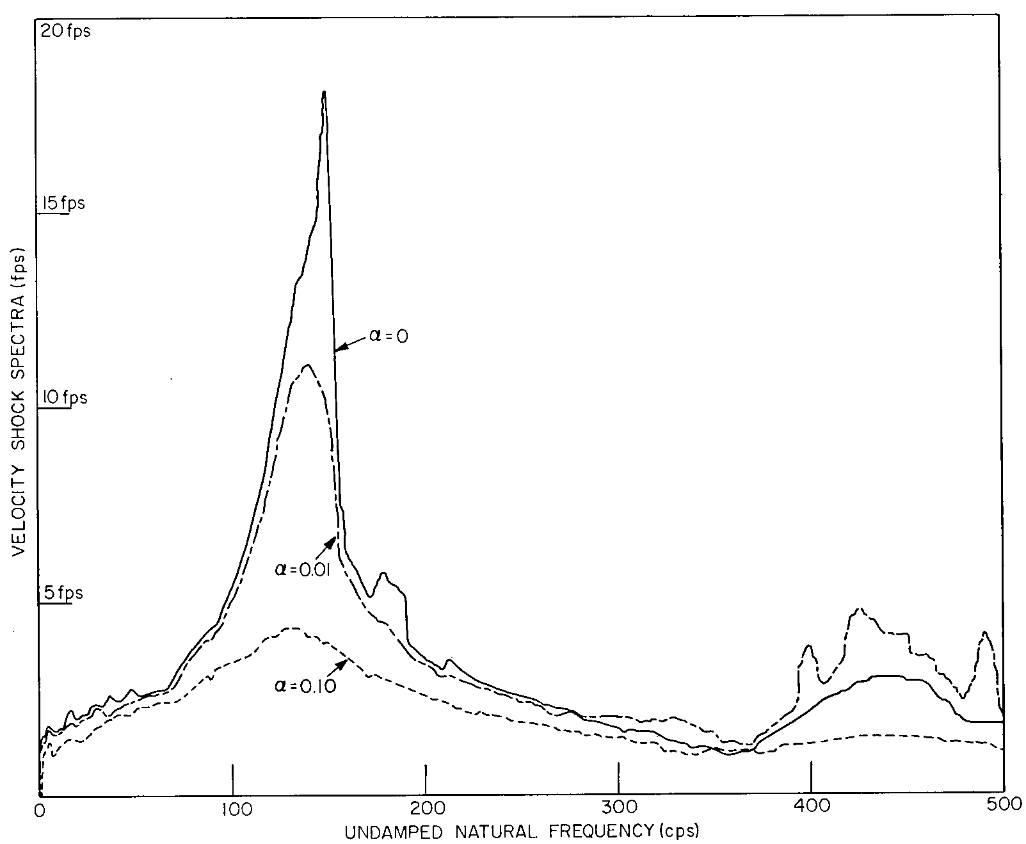

Fig. 5 - Undamped and damped shock spectra

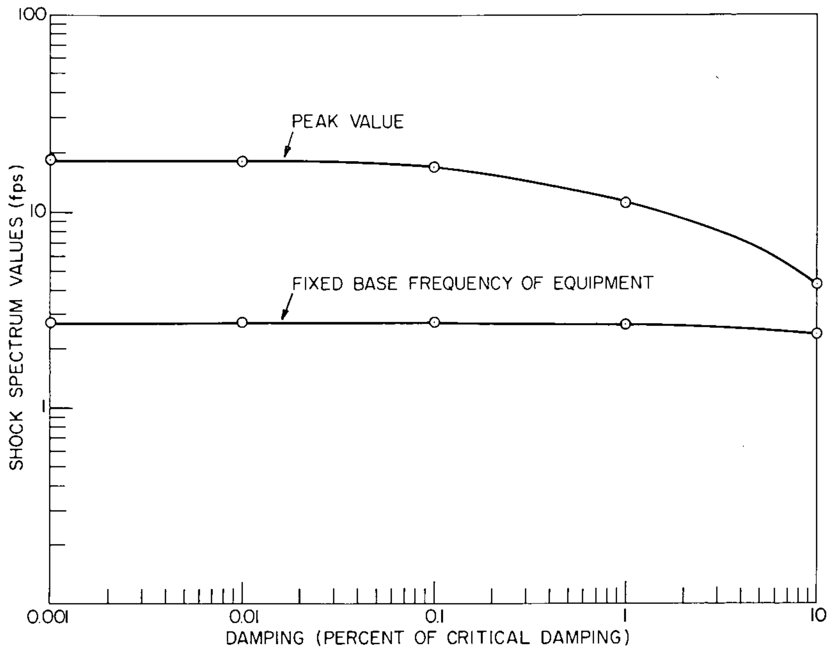

Fig. 6 - Effect of damping upon the shock spectrum values of interest

SUMMARY AND CONCLUSIONS

An easily understood, yet precise, numerical integration method has been derived which allows not only shock calculations to be made but also Fourier analysis. This technique, like all others, has two kinds of error present: inherent and round off. For initial value problems, and problems with forcing functions which have a finite number of finite discontinuities and can be exactly described by a set of straight lines or parabolic arcs, the method (as handled by a desk operator) has no inherent error. For continuous curves the differential equation is exactly satisfied up to the second difference of the forcing function. The inherent error lies in the closeness of fit rather than in any built-in ones. This method has its own advantages and disadvantages which might be partially listed as follows.

1. High precision when using second differences since error depends upon "closeness" of fit.

2. Response including a linear damping assumption is not much more difficult to compute than response with no damping because the exponentials, sines, and cosines must be calculated only once per frequency, if the interval is kept constant.

3. There are no pseudoequations assumed for the original differential equation.

4. There are no iterations required for linear systems.

5. The numerical solution for a theoretically solvable problem is more rapid than the exact solution because the original values of the trigonometric and exponential functions are used throughout rather than being calculated anew for each point.

6. As has been indicated and will be shown in a future report, the method can be applied to nonlinear one and two degree of freedom systems.

7. As with any step by step method it tends to become tedious for a large number of increments when using a desk calculator but is conveniently coded for an electronic computer.

8. Each new value depends on a pair of previously calculated ones, so mistakes are carried along.

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APPENDIX A

OTHER FORMS OF NUMERICAL INTEGRATION EQUATIONS: INTERPOLATION

The numerical integration equations are presented in this appendix in their time form. That is to say they have not been evaluated for $t = h$, so that they may be used to interpolate for $t < h$. To obtain the normal step by step equations let $t = h$ and simplify. The input (forcing function) is defined in the same manner as in the derivation.

FOUNDATION VELOCITY FORCING FUNCTION

The undamped equations are

$$X_{n+1}\,\omega = X_n\,\omega\cos\omega t + \dot{X}_n\sin\omega t - S_n\,\frac{(1-\cos\omega t)}{\omega h}$$
$$-\frac{S^2_{n-1}}{\omega h}\left(\frac{t}{h} - \frac{1}{2} + \frac{\cos\omega t}{2} - \frac{\sin\omega t}{\omega h}\right)$$

and

$$\dot{X}_{n+1} = -X_n\,\omega\sin\omega t + \dot{X}_n\cos\omega t - \frac{S_n\sin\omega t}{\omega h}$$
$$-\frac{S^2_{n-1}}{\omega h}\left(\frac{1}{\omega h} - \frac{\cos\omega t}{\omega h} - \frac{\sin\omega t}{2}\right).$$

The damped equations are

$$X_{n+1}\,\omega = X_n\,\omega\, e^{-\alpha\omega t}\left(\cos pt + \frac{\alpha}{\sqrt{1-\alpha^2}}\sin pt\right) + \frac{\dot{X}_n\, e^{-\alpha\omega t}\sin pt}{\sqrt{1-\alpha^2}}$$
$$-\frac{S_n}{\omega h}\left[1 - e^{-\alpha\omega t}\left(\cos pt + \frac{\alpha}{\sqrt{1-\alpha^2}}\sin pt\right)\right]$$
$$-\frac{S^2_{n-1}}{\omega h}\left\{\frac{t}{h} - \frac{1}{2} - \frac{2\alpha}{\omega h} + \left[\frac{\alpha}{2} - \frac{(1-2\alpha^2)}{\omega h}\right]\frac{e^{-\alpha\omega t}\sin pt}{\sqrt{1-\alpha^2}}\right.$$
$$\left. + \left(\frac{1}{2} + \frac{2\alpha}{\omega h}\right)e^{-\alpha\omega t}\cos pt\right\}$$

and

$$\dot{X}_{n+1} = -\frac{X_n\,\omega\, e^{-\alpha\omega t}\sin pt}{\sqrt{1-\alpha^2}} + \dot{X}_n\, e^{-\alpha\omega t}\left(\cos pt - \frac{\alpha}{\sqrt{1-\alpha^2}}\sin pt\right)$$
$$-\frac{S_n}{ph}\,e^{-\alpha\omega t}\sin pt - \frac{S^2_{n-1}}{\omega h}\left\{\frac{1}{\omega h} - e^{-\alpha\omega t}\left[\cos pt\right.\right.$$
$$\left.\left. + \left(\frac{\alpha}{\omega h} + \frac{1}{2}\right)\frac{\sin pt}{\sqrt{1-\alpha^2}}\right]\right\}.$$

APPLIED FORCES

From the nth to the (n+1)th increment F(T) is defined as

$$F(T) = F_n + \frac{S_n T}{h} + \frac{S_{n-1}^2}{2}\left(\frac{T^2}{h^2} - \frac{T}{h}\right).$$

The undamped equations are

$$X_{n+1} = X_n \cos \omega t + \frac{\dot{X}_n}{\omega} \sin \omega t + \frac{F_n}{K}(1 - \cos \omega t)$$

$$+ \frac{S_n}{K}\left(\frac{t}{h} - \frac{\sin \omega t}{\omega h}\right)$$

$$+ \frac{S_{n-1}^2}{2K}\left[\frac{t^2}{h^2} - \frac{t}{h} - \frac{2(1 - \cos \omega t)}{\omega^2 h^2} + \frac{\sin \omega t}{\omega h}\right]$$

and

$$\frac{\dot{X}_{n+1}}{\omega} = -X_n \omega \sin \omega t + \dot{X}_n \cos \omega t + \frac{F_n}{K} \sin \omega t$$

$$+ \frac{S_n}{K}\left(\frac{1 - \cos \omega t}{\omega h}\right)$$

$$+ \frac{S_{n-1}^2}{2K}\left(\frac{2t}{\omega h^2} - \frac{1}{\omega h} + \frac{\cos \omega t}{\omega h} - \frac{2 \sin \omega t}{\omega^2 h^2}\right).$$

The damped equations are

$$X_{n+1} = X_n e^{-\alpha \omega t}\left(\cos pt + \frac{\alpha}{\sqrt{1 - \alpha^2}} \sin pt\right) + \frac{\dot{X}_n}{p} e^{-\alpha \omega t} \sin pt$$

$$+ \frac{F_n}{K}\left[1 - e^{-\alpha \omega t}\left(\cos pt + \frac{\alpha}{\sqrt{1 - \alpha^2}} \sin pt\right)\right]$$

$$+ \frac{S_n}{K}\left[\frac{t}{h} - \frac{2\alpha}{\omega h}\left(1 - e^{-\alpha \omega t} \cos pt\right) - \frac{(1 - 2\alpha^2) e^{-\alpha \omega t} \sin pt}{ph}\right]$$

$$+ \frac{S_{n-1}^2}{2K}\left\{\frac{t^2}{h^2} - \frac{t}{h} - \frac{4\alpha t}{\omega h^2} - \left[\frac{2(1 - 4\alpha^2)}{\omega^2 h^2} - \frac{2\alpha}{\omega h}\right](1 - e^{-\alpha \omega t} \cos pt)\right.$$

$$\left. + \left[\frac{1 - 2\alpha^2}{\omega h} + \frac{2\alpha(3 - 4\alpha^2)}{\omega^2 h^2}\right] \frac{e^{-\alpha \omega t} \sin pt}{\sqrt{1 - \alpha^2}}\right\}$$

and

$$\frac{\dot{X}_{n+1}}{\omega} = -\frac{X_n\, e^{-\alpha\omega t} \sin pt}{\sqrt{1-\alpha^2}} + \frac{\dot{X}_n}{\omega} e^{-\alpha\omega t}\left(\cos pt - \frac{\alpha}{\sqrt{1-\alpha^2}} \sin pt\right)$$

$$+ \frac{F_n}{K} \frac{e^{-\alpha\omega t} \sin pt}{\sqrt{1-\alpha^2}} + \frac{S_n}{K}\left[\frac{1}{\omega h} - \frac{e^{-\alpha\omega t}}{\omega h}\left(\cos pt + \frac{\alpha}{\sqrt{1-\alpha^2}} \sin pt\right)\right]$$

$$+ \frac{S_{n-1}^2}{2K}\left\{\frac{2t}{\omega h^2} - \left(\frac{1}{\omega h} + \frac{4\alpha}{\omega^2 h^2}\right)\left(1 - e^{-\alpha\omega t} \cos pt\right)\right.$$

$$\left. - \left[\frac{2(1-2\alpha^2)}{\omega^2 h^2} - \frac{\alpha}{\omega h}\right] \frac{e^{-\alpha\omega t} \sin pt}{\sqrt{1-\alpha^2}}\right\}.$$

FOUNDATION ACCELERATION

By means of the similarity between Eqs. (5a,b) and (5c,d) it is only necessary to change the following in the numerical integration equations for applied forces to obtain those for foundation acceleration. Let

$$F_n = -M\, \ddot{Z}_n$$

so

$$\frac{F_n}{K} = -\frac{\ddot{Z}_n}{\omega^2},\ \frac{S_n}{K} = -\frac{S_n}{\omega^2},\ \text{and}\ \frac{S_{n-1}^2}{K} = -\frac{S_{n-1}^2}{\omega^2}.$$

APPENDIX B

ANOTHER SET OF EQUATIONS FOR FOURIER ANALYSIS

The solution of the differential equation with the mass initially at rest, and the foundation acceleration as the driving function can also be used to calculate the Fourier Integrals. Consider

$$X = -\frac{1}{\omega}\int_0^t \ddot{Z}(T)\sin\omega(t-T)\,dT$$

and

$$\dot{X} = -\int_0^t \ddot{Z}(T)\cos\omega(t-T)\,dT.$$

These may be expanded to

$$X\omega = -\sin\omega t\int_0^t \ddot{Z}(T)\cos\omega T\,dT + \cos\omega t\int_0^t \ddot{Z}(T)\sin\omega T\,dT$$

and

$$\dot{X} = -\cos\omega t\int_0^t \ddot{Z}(T)\cos\omega T\,dT - \sin\omega t\int_0^t \ddot{Z}(T)\sin\omega T\,dT.$$

Simultaneous solution yields

$$\int_0^t \ddot{Z}(T)\cos\omega T\,dT = -X\omega\sin\omega t - \dot{X}\cos\omega t$$

and

$$\int_0^t \ddot{Z}(T)\sin\omega T\,dT = X\omega\cos\omega t - \dot{X}\sin\omega t.$$