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The Analysis of Variable Reluctance Transducers: The Classical Method

R. M. Moore

Transducer Branch
Sound Division

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U.S. NAVAL RESEARCH LABORATORY
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CONTENTS

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ABSTRACT

The operating equations of a variable reluctance transducer are derived in this report through the application of classical electromagnetic theory. These equations are formulated in terms of inductance and a quantity called the electrical force factor. The basic equations are shown to be nonlinear, and a set of approximate linear equations is derived for a special class of inputs.

This report and NRL Report 6089 are companion reports dealing with the derivation of the equations of variable reluctance transducers.

PROBLEM STATUS

This is an interim report on one phase of the problem; work is continuing.

AUTHORIZATION

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THE ANALYSIS OF VARIABLE RELUCTANCE TRANSDUCERS: THE CLASSICAL METHOD

THE GENERAL PROBLEM

The purpose of this report is to discuss the classical method of formulating the equations of a particular class of magnetic field transducers. The techniques which will be presented are applicable to all transducers in which the magnetic field is of external electrical origin and in which an air gap is the essential electromechanical element. Such devices can be classed under the generic name of variable reluctance (or inductance) transducers. As a preliminary to the discussion of the classical analysis method, it will be useful to review the fundamental physical relationships which are used in analyzing electromechanical devices, and the essential problems which must be overcome.

There are two basic network relations which can be used in the analysis of the electrical portion of the transducer. These are the Kirchoff node and loop laws, and they can be stated as follows:

1. Node Law: The sum of the currents entering an ideal electrical connection point which is incapable of storing charge (an electrical node) must be zero.

2. Loop Law: The sum of the voltage differences taken in a particular direction around a closed path in an electrical network (an electrical loop) must be zero.

The mathematical expressions for the node and loop laws are

$$\sum_{N} i_n = 0 \qquad\qquad \sum_{M} \Delta v_m = 0 \tag{1}$$

where

i_n = nth current entering a node N

Δv_m = mth voltage difference around a loop M

and where N and M indicate that the summations are to be carried out at a particular node or around a particular loop.

The equations of the purely electrical portion of the transducer represents only part of the problem. Another part, namely, the analysis of the purely mechanical portion of the device, can also be formulated in terms of two fundamental relations. The first of these is a form of Newton's law, and the other is a purely kinematic relation. This pair of relations can be stated as follows:

1. Force Law: The sum of the forces acting on an ideal massless connection point (a mechanical node) must be zero.

2. Velocity Law: The sum of the velocity differences taken in a particular direction around a closed path in a mechanical network (a mechanical loop) must be zero.

The mathematical versions of these statements are

$$\sum_{N} f_n = 0$$
$$\sum_{M} \Delta u_m = 0 \qquad (2)$$

where

f_n = nth force acting on a mechanical node N

Δu_m = mth velocity difference around a mechanical loop M

and where N and M have an analogous meaning to that used in Eqs. (1).

If Eqs. (1) and (2) were the only principles to be utilized in the study of electromechanical transducers, then this report would consist entirely of the applications of techniques which are quite adequately treated in the many excellent books on electrical network theory and mechanical vibration theory. Unfortunately (from the point of view of the analyst!), there are unique problems connected with the analysis of electromechanical transducers which require techniques quite different from those employed for purely electrical or purely mechanical systems. The essential problem is the determination of the mechanical to electrical effects to be included in Eqs. (1), and the electrical to mechanical effects to be included in Eqs. (2). It is with the derivation of these electromechanical effects that the major portion of this report is concerned, and the primary objective of this discussion is to develop the necessary relations in as general a manner as possible.

In summary, it can be said that the most important single aspect of any electromechanical transducer is the energy exchange process which links the purely electrical and purely mechanical portions of the device. It is this energy exchange which determines the form of the electromechanical effects which must be included in Eqs. (1) and (2). Thus the quantitative description of the energy conversion process in a variable reluctance transducer becomes the primary, and probably the most important, phase of the analysis.

The study of any transducer will consist of variations on the following steps:

1. The conception of an idealized energy exchange process having properties of as general a character as possible;

2. The derivation of the quantitative relations governing the energy conversion, which are of course based on the assumed properties of the energy exchange;

3. The insertion of the resulting electromechanical terms into the fundamental electrical and mechanical relations obtained from Eqs. (1) and (2);

4. The solution of the resulting operating equations under the external constraints of interest;

5. The physical interpretation of these solutions in the context imposed by the assumed ideal model of the transducer.

In this report attention will be focused on the energy conversion process in the variable reluctance transducer, and the resulting electromechanical effects. However, before embarking on this project it will be fruitful to review the fundamental field concepts which will be utilized in this report, and to discuss the formulation and application of the classical electromagnetic field laws.

ELECTROMAGNETIC FIELD CONCEPTS

The electromechanical effects in a variable reluctance transducer can be derived through the use of the classical theory of the electromagnetic field. This theory is formulated in terms of field vectors, and the electromagnetic laws are expressions of experimentally determined relationships. The expressions for the forces exerted on material media in an electromagnetic field are also statements of experimental observations. These force relations are formulated by assuming the existence of elemental magnetic poles on the surface of the magnetized media of interest. Normally these forces can be evaluated only after the field configuration has been determined through the use of the electromagnetic laws.

In preparation for the later use of these classical ideas, the laws and concepts which are pertinent to the analysis of variable reluctance transducers will now be reviewed and discussed. The electromagnetic relations will be presented first, and then the force laws will be outlined.

The classical electromagnetic field laws are stated in terms of the following five vector field quantities:

$\mathbf{H}$ = magnetic field strength (amperes/meter)

$\mathbf{B}$ = magnetic flux density (webers/meter^2)

$\mathbf{E}$ = electric field strength (volts/meter)

$\mathbf{D}$ = electric flux density (coulombs/meter^2)

$\mathbf{J}$ = conduction current density (amperes/meter^2)

where the units refer to the rationalized mks system.

The classical laws can be expressed in either a vector point form (commonly referred to as Maxwell's equations), or as vector integral laws. For the purposes of this report the integral form is the more convenient. There is actually a set of four such integral (or point) equations but only the two which will be directly of concern in this report will be discussed. They are:

$$\oint_{C_m} \mathbf{H} \cdot \hat{t}_m \, dl_m = \iint_{A_m} \mathbf{J} \cdot \hat{n}_m \, dA_m + \frac{d}{dt} \iint_{A_m} \mathbf{D} \cdot \hat{n}_m \, dA_m \tag{3}$$

and

$$\oint_{C_e} \mathbf{E} \cdot \hat{t}_e \, dl_e = -\frac{d}{dt} \iint_{A_e} \mathbf{B} \cdot \hat{n}_e \, dA_e \tag{4}$$

where, as is illustrated in Figs. 1 and 2,

C_m = a closed contour

$\hat{t}_m$ = unit tangent vector to C_m in the direction of integration

dl_m = incremental length on C_m

A_m = an area spanning C_m

$\hat{n}_m$ = unit normal to A_m on the positive side* of A_m

and

C_e = a closed contour

$\hat{t}_e$ = unit tangent vector to C_e in the direction of integration

dl_e = incremental length on C_e

A_e = an area spanning C_e

$\hat{n}_e$ = unit normal vector to A_e on the positive side* of A_e

dA_e = incremental area on A_e

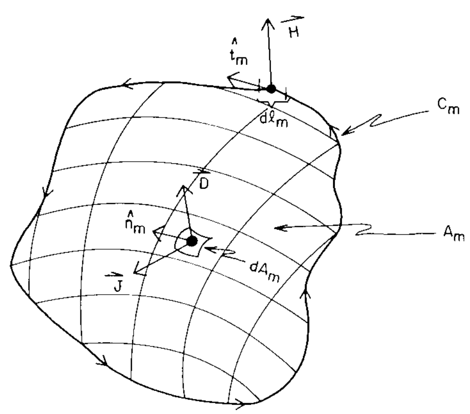

Fig. 1 - Geometry of Eq. (3)

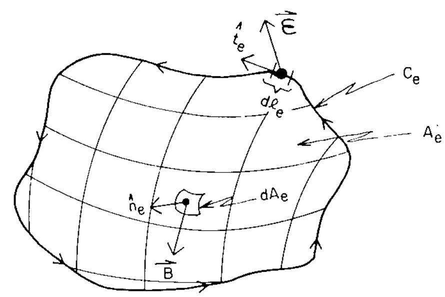

Fig. 2 - Geometry of Eq. (4)

The various line and surface integrals in Eqs. (3) and (4) are scalar quantities which are frequently used to describe an electromagnetic field. Following is a list of symbols and names commonly employed for these integrals:

*The positive sides of A_m and A_e are determined by the standard mathematical convention relating the direction of integration around the circumference of an area to the positive side of the area. This convention is illustrated in Figs. 1 and 2.

$$\oint_{C_m} \mathbf{H} \cdot \hat{t}_m \, dl_m = \text{mmf} = \underline{\text{magnetomotive}}\ \underline{\text{force}} \text{ around contour } C_m$$

$$\iint_{A_m} \mathbf{J} \cdot \hat{n}_m \, dA_m = I_c = \underline{\text{conduction}}\ \underline{\text{current}} \text{ linking contour } C_m$$

$$\iint_{A_m} \mathbf{D} \cdot \hat{n}_m \, dA_m = \psi = \underline{\text{electric}}\ \underline{\text{flux}} \text{ linking contour } C_m \tag{5}$$

$$\frac{d}{dt}\iint_{A_m} \mathbf{D} \cdot \hat{n}_m \, dA_m = I_D = \underline{\text{displacement}}\ \underline{\text{current}} \text{ linking contour } C_m$$

and

$$\oint_{C_e} \mathbf{E} \cdot \hat{t}_e \, dl_e = \text{emf} = \underline{\text{electromotive}}\ \underline{\text{force}} \text{ around contour } C_e$$

$$\iint_{A_e} \mathbf{B} \cdot \hat{n}_e \, dA_e = \lambda = \underline{\text{flux}}\ \underline{\text{linkage}} \text{ for contour } C_e. \tag{6}$$

Of these six scalars at least two, the conduction current and the flux linkage, are worth additional comment. The conduction current term I_c refers to the total conduction current piercing the surface A_m, and it may consist solely of a real physical flow of charged particles passing through A_m. However, it may also result from a multiple passage of the same charge flow through the surface spanning the contour C_m (e.g., a current carrying coil linking the contour; see Fig. 3) and under these circumstances the term to be used in Eq. (3) is not equal to the actual physical current. If the direction of the piercings of the surface are properly accounted for (e.g., by the use of positive and negative piercings), then the I_c term is the product of the net number of piercings N and the physical current i. Thus the result for a contour which is linked by an N-turn coil which is carrying current i is an expression $I_c = Ni$.

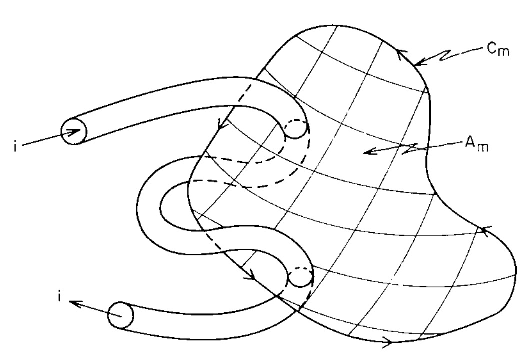

Fig. 3 – A double piercing of A_m by the same charge flow

The flux linkage scalar λ is similar to I_c in that it is the total flux linking C_e, and it may arise from the multiple passage of a single tube of magnetic flux. Thus it is important to differentiate between the concepts of magnetic flux and flux linkage, since in most cases of interest they are not equal. As an illustration of this difference it is useful to conceive of a closed and well-defined magnetic flux path around which a certain flux ϕ is flowing. This flux can be defined as

$$\phi = \iint_{A_p} \mathbf{B} \cdot \hat{n}_p \, dA_p = \underline{\text{magnetic}} \ \underline{\text{flux}} \text{ flowing through } A_p \tag{7}$$

where

A_p = cross-sectional area of the flux path (flux tube)

$\hat{n}_p$ = unit normal to A_p at the location of dA_p

dA_p = incremental surface area on A_p.

Now, suppose the contour C_e of Eq. (6) is chosen so that ϕ passes N times through the surface A_e which spans this contour (always, of course, in the same direction; see Fig. 4). Then the integral of Eq. (6) reduces to the special form

$$\lambda = \iint_{A_e} \mathbf{B} \cdot \hat{n}_e \, dA_e = N \iint_{A_p} \mathbf{B} \cdot \hat{n}_p \, dA_p = N\phi$$

and for this simple example the terms flux and flux linkage are clearly separate. Of course in practice it is difficult to obtain a perfect tube of flux which links a given contour exactly N times, and thus this example represents a very ideal and artificial situation. However, the concepts of flux and flux linkage in the general case of partial linkage (e.g., the end turns of a coil losing some of the flux due to fringing effects of the field) are of the same general physical character. That is, the flux linkage, as given in Eq. (6), usually is greater than the flux, as given in Eq. (7), due to the fact that part or all of the flux links the contour C_e more than once (pierces A_e more than once).

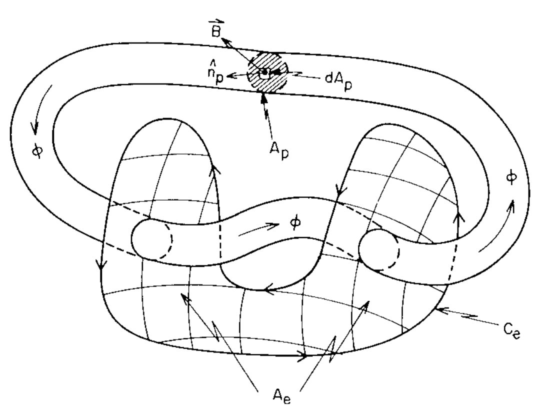

Fig. 4 – A double piercing of A_e by the same flux tube

The other scalars listed in Eqs. (5) and (6) are not likely to cause confusion, since they usually occur in an obvious fashion in practical situations. Considering these scalars as fundamental quantities which can be used to describe the field, the experimental laws embodied in Eqs. (3) and (4) can be verbally expressed in the following fashion:

Magnetomotive Force Law: The total magnetomotive force around any closed contour is equal to the total current (conduction plus displacement) which links the contour.

Electromotive Force Law: The total electromotive force around any closed contour is equal to the negative of the time rate of change of the total magnetic flux which links the contour (the flux linkage).

An analysis of a variable reluctance transducer, using the classical field approach, requires that expressions for $\mathbf{H}$ and $\mathbf{B}$ be developed which will satisfy Eqs. (3) and (4). Once these expressions have been developed, these field vectors can be used to evaluate the forces which are exerted on any material media in the field. The force function can be derived through the application of the basic force law.

The fundamental force law results from experiments performed on essentially isolated magnetic poles, and it is usually written as

$$\mathbf{F}_{12} = \frac{m_1 m_2}{4\pi\mu (r_{21})^2} \hat{r}_{21} \tag{8}$$

where

$\mathbf{F}_{12}$ = force on pole 1 due to pole 2 (newtons)

m_1 and m_2 = strengths of poles 1 and 2 (webers)

μ = permeability of the medium in which poles 1 and 2 are immersed (webers/ampere-meter)

$\mathbf{r}_{21}$ = radius vector from pole 2 to pole 1

r_{21} = magnitude of $\mathbf{r}_{21}$ (meters)

$\hat{r}_{21}$ = unit vector in the direction of $\mathbf{r}_{21}$.

Equation (8) is usually known as Coulomb's law for isolated magnetic poles.

An alternative formulation of the force on a magnetic pole is of the form

$$\mathbf{F} = m\,\mathbf{H} \tag{9}$$

where

$\mathbf{F}$ = force on pole m (newtons)

m = strength of pole (webers)

$\mathbf{H}$ = magnetic field strength at m (amperes/meter).

However, neither Eqs. (8) or (9) are in a convenient form for obtaining the force on a material body in a magnetic field. To obtain a suitable expression, the concept of a pole density, ρ, is introduced to describe the magnetic state of the body:

$\rho(x, y, z)$ = magnetic pole density (webers/meter^3).

This pole density arised, of course, only from magnetization of the material body and thus is analogous to bound (or polarization) charge in a dielectric, and is not similar to free charge in a conductor.

Now applying Eq. (9) to such a magnetized body we can express m as

$$m(x, y, z) = \rho(x, y, z)\, dV$$

where dV is an elemental volume. The force, $d\mathbf{F}$, on this elemental volume is then (see Fig. 5)

$$d\mathbf{F} = \mathbf{H}\, \rho\, dV$$

and the force density, or force per unit volume at each point of the body, can be expressed as

$$\mathbf{f}(x, y, z) = \frac{d\mathbf{F}}{dV} = \mathbf{H}(x, y, z)\, \rho(x, y, z).$$

The total force on any material body would then be the intergral over the volume of the body of the force density.

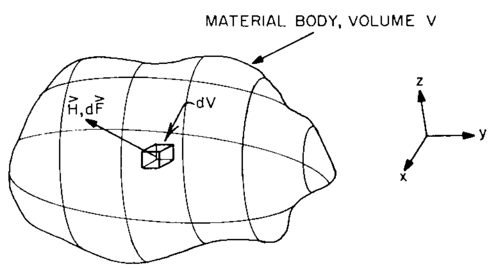

Fig. 5 – Force on an elemental volume of a material body

The magnetic pole density can be expressed in terms of the divergence of $\mathbf{H}$ at each point

$$\rho(x, y, z) = \text{div}\, \mu_0\, \mathbf{H} = \mu_0\, \text{div}\, \mathbf{H}$$

where μ_0 is the permeability of vacuum. Thus the force density vector can be written in the form

$$\mathbf{f}(x, y, z) = \mu_0\, \mathbf{H}(x, y, z)\, \text{div}\, \mathbf{H}(x, y, z) \tag{10}$$

and the total force $\mathbf{F}$ on the body of interest can be found by a volume integral of the form

$$\mathbf{F} = \int_V \mathbf{f}\, dV = \int_V \mu_0\, \mathbf{H}\, \text{div}\, \mathbf{H}\, dV$$

which is, in general, quite difficult to evaluate except for the simplest possible geometry. This volume integral can be brought into the form of a surface integral by inserting the following vector identity into the integral,

$$\mathbf{H}\, \text{div}\, \mathbf{H} = \hat{i}\, \text{div}\, H_x\, \mathbf{H} + \hat{j}\, \text{div}\, H_y\, \mathbf{H} + \hat{k}\, \text{div}\, H_z\, \mathbf{H} + \mathbf{H} \times \text{curl}\, \mathbf{H} - \frac{1}{2}\text{grad}\, (\mathbf{H} \cdot \mathbf{H})$$

where

$$\mathbf{H} = H_x \hat{i} + H_y \hat{j} + H_z \hat{k} .$$

The force integral then takes the form

$$\frac{\mathbf{F}}{\mu_0} = \hat{i} \int_V (\mathrm{div}\, H_x \mathbf{H}) dV + \hat{j} \int_V (\mathrm{div}\, H_y \mathbf{H}) dV + \hat{k} \int_V (\mathrm{div}\, H_z \mathbf{H}) dV$$

$$+ \int_V (\mathbf{H} \times \mathrm{curl}\, \mathbf{H}) dV - \frac{1}{2} \int_V \mathrm{grad}\, (\mathbf{H} \cdot \mathbf{H}) dV$$

and recognizing that $\mathrm{curl}\, \mathbf{H}$ is zero everywhere in the magnetized body for magnetostatic fields (and approximately zero for quasi-static or low-frequency fields), we can transform the above volume integrals into integrals over the surface, S, bounding the volume. The result is (see Fig. 6)

$$\mathbf{F} = \oint_S \mu_0 \left[(\mathbf{H} \cdot \hat{n})\, \mathbf{H} - \frac{1}{2} (\mathbf{H} \cdot \mathbf{H})\, \hat{n} \right] dS \tag{11}$$

where

dS = element of surface on S

$\hat{n}$ = external normal to S at dS.

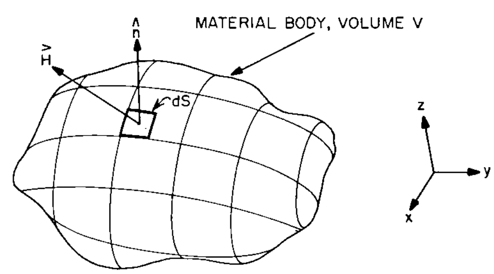

Fig. 6 – Force on material body; geometry of Eq. (11)

The form of the force given by Eq. (11) has the advantage that integrals need only be performed over a surface bounding the material body, and thus the details of the internal field are not of interest. Therefore, this formulation of the force is most convenient for general use.

Now that the concepts and laws of the classical electromagnetic field have been reviewed and discussed, they can be applied to derive the electromechanical effects in a variable reluctance transducer.

MECHANICAL TO ELECTRICAL EFFECTS

In variable reluctance transducers the displacement current term in Eq. (3) is usually negligible in comparison with the conduction current, and thus this term is normally ignored in the application of the magnetomotive force relation. This is essentially an assumption that the fields are "quasistatic" or "stationary," and it is normally only at high frequencies that the failure of this assumption will lead to an appreciable "radiation resistance" term. In excluding this term, the effects of electromagnetic radiation are considered to be unimportant, and this is usually valid for low-frequency fields which have not been specifically designed for effective radiation.

The application of Eqs. (3) and (4) usually follows a well-defined pattern. To illustrate the general procedure, the example of a current-carrying coil of N turns, which is linked by a closed magnetic path, is outlined below (refer to Fig. 7):

Step 1: A conduction current i is assumed to be flowing in the coil. The coil current i is of course not the total conduction current I_c which must be used in Eq. (3). The total current is $I_c = Ni$, where N is the number of turns of the coil.

Step 2: The mmf relation, Eq. (3), is used to determine the resulting magnetic field intensity $\mathbf{H}$ as a function of i and the position (x, y, z) in the field. Thus the magnetic field strength is obtained as a vector point function of the form $\mathbf{H}(i, x, y, z)$ which satisfies Eq. (3).

Step 3: The magnetic flux density $\mathbf{B}$ is determined from the expression for the magnetic field intensity $\mathbf{H}$. The result is a vector point function of the form $\mathbf{B}(i, x, y, z)$ which of course must include the effects of the material medium at the point (x, y, z). In other words the effects of the material must be included through the use of the appropriate permeability function in conjunction with $\mathbf{H}(i, x, y, z)$.

Step 4: The flux linkage λ is then evaluated for the contour of interest through the use of $\mathbf{B}(i, x, y, z)$ in the defining integral of Eq. (6). This leads to an evaluation of the flux linkage as a function of the current i, the geometry and material properties of the flux path, and the geometry of the chosen contour. The resulting function can be represented as $\lambda(i)$, where the explicit dependence on (x, y, z) has vanished as a result of the integration.

Step 5: The electromotive force is then completed for the contour of interest through the application of Eq. (4). The resulting emf expression is a function of the coil current i, the geometry and physical properties of the path, and the geometry of the emf contour.

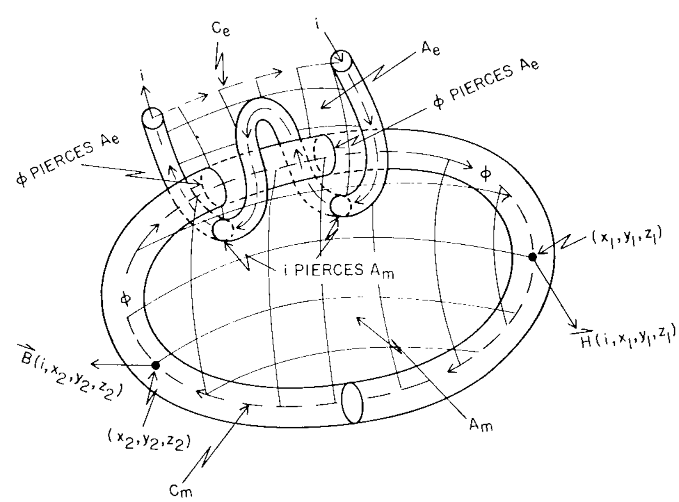

Fig. 7 - Two-turn coil linked by a flux path

The overall result of the above procedure is a relationship between the current flowing in the N-turn coil, and the emf around the chosen closed contour. Usually the contour of practical interest is the one which follows the coil of wire (see Fig. 7), and is closed through the use of either an external circuit or a field path. The result of step 5 is often used to justify the statement that the voltage across the terminals of a coil is the time rate of change of the flux linkage, usually written as $v = d\lambda/dt$, but this is only accurate if the coil has negligible resistive and capacitive effects.

In analyzing field configurations for which the flux can be considered to exist in well-defined tubes of flux, the five-step analysis outlined can be shortened by compressing steps 2, 3, and 4 into a single operation. This is done by introducing a quantity which is referred to as the "reluctance" of the flux path. Such a procedure has the attraction of suppressing entirely the explicit use of the vector field quantities and vector integrals. Of course this method is based on the five-step analysis presented above, and the use of the reluctance function presupposes a knowledge of the pertinent properties of the magnetic flux tube.

As an illustration of the use of the reluctance parameter, the example of a closed and well-defined magnetic flux path will be analyzed (refer to Fig. 8). If the geometry of the path is described by the symbols

A_p = cross-sectional area of the flux tube

C_p = contour consisting of a closed path around the flux tube

then the flux in the tube can be written from Eq, (7) as

$$\phi = \iint_{A_p} \mathbf{B} \cdot \hat{n}_p \, dA_p$$

where

$\hat{n}_p$ = unit normal to A_p

dA_p = incremental area on A_p.

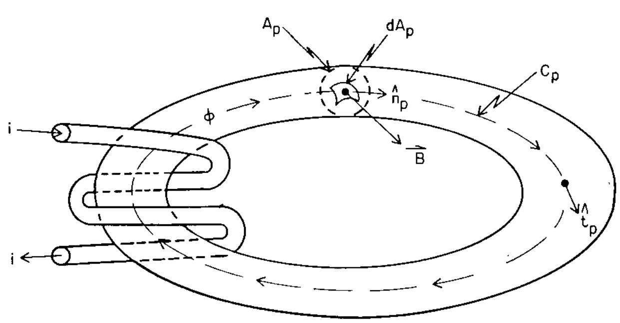

Fig. 8 – Geometry of a flux tube and a coil

If this flux tube is linked by an N-turn coil which is carrying current i, then the mmf around the closed tube is

$$\text{mmf} = \oint_{C_p} \mathbf{H} \cdot \hat{t}_p \, dl_p = Ni$$

where

$$\hat{t}_p = \text{unit tangent vector to } C_p$$

$$dl_p = \text{incremental length on } C_p .$$

The reluctance of a flux tube is defined as

$$R = \frac{\oint_{C_I} \mathbf{H} \cdot \hat{t}_p \, dl_p}{\iint_{A_p} \mathbf{B} \cdot \hat{n}_p \, dA_p} = \frac{\text{mmf}}{\phi} \tag{12}$$

where in general the reluctance will depend on the properties and geometry of the medium in the flux tube, as well as the flux magnitude (i.e., possible nonlinear permeability effects).

If this reluctance relation is rearranged in terms of the scalars, mmf, and ϕ, it becomes

$$\phi = \frac{\text{mmf}}{R}$$

and the flux linkage of the N-turn coil can be written

$$\lambda = N\,\phi = \frac{N}{R}\,\text{mmf} .$$

Inserting the special expression for the mmf (mmf = Ni) the result is

$$\lambda = \frac{N^2}{R} i \tag{13}$$

which expresses the flux linkage in terms of the turns of the coil, reluctance of the flux path, and the coil current.

Thus steps 2 to 4 of the general analysis have been accomplished through the introduction of the reluctance function defined in Eq. (12), and this has resulted in a purely algebraic operation as presented in Eq. (13). The rub in all of this is, of course, that the definition of the reluctance, Eq. (12), has actually absorbed all of the vector operations outlined in the original five-step analysis. Although it may appear, in light of this statement, that the reluctance function is merely a subterfuge to conceal the vector character of the laws of electromagnetics, it is in reality a very useful and convenient function.

The frequent application of the reluctance function in practical situations is primarily a result of two circumstances. First, the reluctance can be approximately computed through a simple procedure for a large class of paths which are of interest; and second, the reluctance function conveniently groups together the geometrical and material properties of the magnetic flux tube in a single parameter.

The approximate method of determining the reluctance of a flux tube is presented in Appendix A, together with the resulting approximate formulas. The existence and use of such formulas should not be taken to imply that the concept of reluctance is itself in any way a limited or faulty concept. On the contrary, if a closed flux tube exists for which every closed contour around the tube has the same current linking it (the same mmf), then the reluctance of the tube is given exactly by Eq. (12). The approximation arises as a practical necessity associated with the difficulties of evaluating Eq. (12) for any reasonable physical situation.

In using the concept of reluctance for complex flux paths, it becomes expedient either to utilize the approximate results of Appendix A or to resort to an experimental determination of the reluctance function. The experimental approach is the preferred method when any but the grossest effects in variable reluctance transducers are under study. All subsequent use of the term reluctance in this report will imply the exact reluctance function defined by Eq. (12), and determined by experiment, unless specifically stated otherwise.

The concept of a reluctance function has served to group all of the geometrical and material properties of a flux path into a single descriptive quantity. When such a flux path is associated with an isolated coil of wire (see Fig. 8), it becomes desirable to consolidate the description of the overall interaction between the current-carrying coil and the electromagnetic field. This consolidation is accomplished by introducing a quantity called the self-inductance (or simply the inductance) of the coil.

The inductance of a coil linked only by flux generated by its own current (an isolated coil) is defined as

$$L(i) = \frac{\lambda(i)}{i} \tag{14}$$

where

$$\lambda(i) = \text{flux linkage of the coil}$$

and where, in general, L depends on the properties and geometry of the flux path, as well as the explicitly shown dependence on the current (which allows for possible nonlinearity of the permeability).

As an illustration, the inductance parameter will now be found for the simple example of a well-defined flux path linking an N-turn coil (see Fig. 8). This permits the use of Eq. (13) for the flux linkage. The result is

$$L(i) = \frac{N^2}{R(i)} \tag{15}$$

where the possible nonlinear permeability effects in the magnetic path have been included by representing the reluctance as a function of current. It should be emphasized that Eq. (15) is an approximate expression based on the idealized model of the N-turn coil linked perfectly by a flux tube described by R, whereas Eq. (14) is the definition of inductance. Since this exact quantity can be determined experimentally, the concept of inductance is not at all approximate. In this report the term inductance will imply the exact quantity, defined by Eq. (14), unless otherwise stated.

The voltage difference which occurs in a coil due to the interaction between the coil current and the electromagnetic field can be written using either the reluctance or the inductance as the basic descriptor of the coil current and field interaction. Both possible forms are given for completeness:

$$v = L\frac{di}{dt} + i\,\frac{dL}{dt} \tag{16}$$

$$v = \frac{N^2}{R}\left(\frac{di}{dt} - \frac{i}{R}\frac{dR}{dt}\right)$$

where

v = voltage difference arising in the coil due to electromagnetic interaction.

Only the inductance form Eq. (16), will be utilized in the remainder of this development.

The inductance form, Eq. (16), can be used to determine the mechanical to electrical effects which occur in a variable inductance (or reluctance) device. If the electromechanical interaction results from a change in the length of the air gap, or gaps, in the transducer, the inductance will depend on this air gap length. If, in addition, it is assumed that the permeability of every portion of the flux path is independent of the flux or current level, then the inductance is not a function of ϕ or i. This is, in essence, an assumption that the entire flux path is magnetically linear. Under this assumption the inductance parameter has the functional form $L(x)$, where x is a geometrical variable representing air gap displacement.

Thus from Eq. (16) the voltage difference arising in the coil due to the mechanical-magnetic-electrical interaction can be expressed as

$$v = L(x)\frac{di}{dt} + i\frac{dL(x)}{dx}\frac{dx}{dt} \tag{17}$$

where the first term is in the same form as the usual electromagnetic effect in a coil but in this case is an electromechanical term because L is a function of x. The second term is a completely new form not encountered in standard electrical analysis and is also an electromechanical term, which in general depends on i, x, and the time derivative of x.

If the coil under consideration has neglibible capacitive effects at the frequency of operation, then the electrical loop law of Eqs. (1) yields

$$L(x)\frac{di}{dt} + i\frac{dL(x)}{dx}\frac{dx}{dt} + R_w i = v_e(x) \tag{18}$$

where

R_w = resistance of the wire in the coil

$v_e(x)$ = external voltage applied across the terminals of the coil

as the basic electrical equation of the device. Note that both of the electromechanical terms of Eq. (18) are nonlinear forms in the variables i and x. Thus Eq. (18) is a nonlinear differential equation in i and x.

This equation is the first of the two operating equations of a variable reluctance transducer. It has been derived through the applications of the classical concepts of electromagnetic field theory. Discussion of the consequences of its nonlinear form will be deferred until the electrical to mechanical effects in a variable reluctance device have been considered.

ELECTRICAL TO MECHANICAL EFFECTS

The classical expressions for the electrical to mechanical effects in a variable reluctance device follow from the application of the force law formulated in Eq. (11). In order to apply this equation it is necessary to determine the $\mathbf{H}$ field on some convenient surface bounding the body of interest. This can be done through the application of the mmf law, Eq. (3), as was discussed in the preceding development of mechanical to electrical effects. After finding $\mathbf{H}$, the application of the force integral is straightforward.

As an illustration of the general procedure, the force on a material body which is part of a magnetic circuit will now be analyzed (see Fig. 9). This flux tube is assumed to be linked by an N-turn coil which is carrying a current i. The derivation of the force acting on the material body is outlined below in two steps.

Step 1: The magnetic field intensity $\mathbf{H}$ in the air gaps is determined as a function of current i and position in the field (x, y, z). The result is the vector point function $\mathbf{H}(x, y, z)$, which is the same as that obtained by steps 1 and 2 of the preceding mechanical to electrical analysis.

Step 2: The force integral given by Eq. (11) is applied to the surface consisting of the side surface A_3, closed by surfaces A_1 and A_2 (which are everywhere normal to the $\mathbf{H}$ field). The force law then reduces to three integrals (see Fig. 9),

$$\mathbf{F} = \iint_{A_1} \mu_0 \left[(\mathbf{H} \cdot \hat{n}_1)\,\mathbf{H} - \frac{1}{2}(\mathbf{H} \cdot \mathbf{H})\,\hat{n}_1\right] dA_1$$

$$+ \iint_{A_2} \mu_0 \left[(\mathbf{H} \cdot \hat{n}_2)\,\mathbf{H} - \frac{1}{2}(\mathbf{H} \cdot \mathbf{H})\,\hat{n}_2\right] dA_2$$

$$+ \iint_{A_3} \mu_0 \left[(\mathbf{H} \cdot \hat{n}_3)\,\mathbf{H} - \frac{1}{2}(\mathbf{H} \cdot \mathbf{H})\,\hat{n}_3\right] dA_3 .$$

The third integral is identically zero, since all of the flux, and thus the lines of $\mathbf{H}$, are confined to the flux tube (by definition). In the first integral $\mathbf{H}$ and $\hat{n}_1$ are antiparallel, and in the second $\mathbf{H}$ and $\hat{n}_2$ are parallel, and thus the integral for the force reduces to

$$\mathbf{F} = \iint_{A_1} \left(\frac{1}{2}\mu_0 |\mathbf{H}|^2\, \hat{n}_1\right) dA_1 + \iint_{A_2} \left(\frac{1}{2}\mu_0 |\mathbf{H}|^2\, \hat{n}_2\right) dA_2 .$$

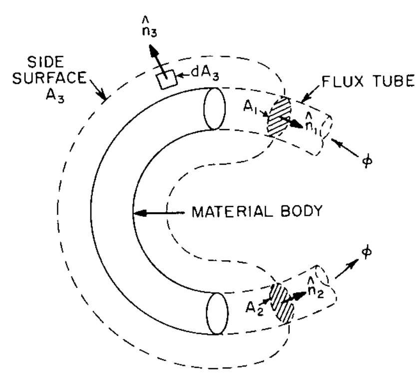

Fig. 9 – Material body in flux tube

The total force is thus given in terms of integrals over the two air gaps. Obviously any change in the geometry of the air gaps (due for example to the motion of the material body) will change the force. This fact can be explicitly recognized by adopting x as a general geometrical variable specifying the configuration of the air gaps and then expressing the force in the functional form $\mathbf{F}(i, x)$. In this fashion the dependence of the force on both current and geometry is indicated in a concise notation.

It is useful to consolidate the magnetic to mechanical characteristics of the air gaps into a single parameter. This can be done by defining the "vector magnetic force factor" associated with the air gaps as

$$\mathbf{J}(i, x) = \frac{\mathbf{F}(i, x)}{\phi(i, x)}$$

or

$$\mathbf{J}(i, x) = \frac{\iint_{A_1} \frac{1}{2}\mu_0 |\mathbf{H}|^2 \, \hat{n}_1 \, dA_1 + \iint_{A_2} \frac{1}{2}\mu_0 |\mathbf{H}|^2 \, \hat{n}_2 \, dA_2}{\iint_{A_\rho} \mathbf{B} \cdot \hat{n}_\rho \, dA_\rho} \tag{19}$$

where

$\mathbf{J}(i, x)$ = vector magnetic force factor

$\phi(i, x)$ = flux through flux tube

A_ρ = cross section of flux tube

$\hat{n}_\rho$ = unit normal to A_ρ at dA_ρ

dA_ρ = incremental area on A_ρ.

Using this parameter the force on the body of interest can be expressed as

$$\mathbf{F}(i, x) = \mathbf{J}(i, x)\, \phi(i, x)$$

and the vector magnetic force factor $\mathbf{J}$ serves as a describing function for the air gaps in the same fashion as the reluctance serves to describe a flux tube. That is it concentrates all of the geometrical and magnetic properties of the air gap into a single parameter insofar as the magnetic to mechanical effects are concerned.

The next logical step is to define a parameter which will describe the overall interaction from the coil current to the force. This can be done by defining the "vector electrical force factor" as

$$\mathbf{G}(i, x) = \frac{\mathbf{F}(i, x)}{i}$$

or

$$\mathbf{G}(i, x) = \frac{\iint_{A_1} \frac{1}{2}\mu_0 |\mathbf{H}|^2 \hat{n}_1 \, dA_1 + \iint_{A_2} \frac{1}{2}\mu_0 |\mathbf{H}|^2 \hat{n}_2 \, dA_2}{i}. \tag{20}$$

Since $\mathbf{F}$ and ϕ have been previously defined to be

$$\mathbf{F}(i, x) = \mathbf{J}(i, x)\, \phi(i, x)$$

$$\phi(i, x) = \frac{Ni}{R(i, x)}$$

the expression for $\mathbf{G}(i, x)$ becomes

$$\mathbf{G}(i, x) = \frac{N}{R(i, x)} \mathbf{J}(i, x) \tag{21}$$

and the force can be written

$$\mathbf{F}(i, x) = \mathbf{G}(i, x)\, i \tag{22}$$

where all of the geometrical and magnetic properties of the air gaps and the overall flux tube are absorbed into the single parameter $\mathbf{G}(i, x)$.

In the special case where the material body of interest is constrained to translation along a fixed line (see Fig. 10) the above set of vector quantities can be replaced by scalars. Thus a scalar force, $F(i, x)$, can be defined as

$$F(i, x) = \hat{t} \cdot \mathbf{F}(i, x) \tag{23}$$

where

$\hat{t}$ = unit tangent vector to line of translation.

In addition, expressions for the scalar magnetic and electrical force factors are

$$\begin{aligned} J(i, x) &= \hat{t} \cdot \mathbf{J}(i, x) \\ G(i, x) &= \hat{t} \cdot \mathbf{G}(i, x) = \frac{NJ(i, x)}{R(i, x)} \end{aligned} \tag{24}$$

where

$J(i, x)$ = scalar magnetic force factor

$G(i, x)$ = scalar electrical force factor.

Thus the scalar force can be written

$$\begin{aligned} F(i, x) &= J(i, x)\, \phi(i, x) \\ &= G(i, x)\, i. \end{aligned} \tag{25}$$

The scalar force F is that component of the vector force $\mathbf{F}$ which is not balanced by the constraint force acting to confine the motion of the material body to translation along the given fixed line.

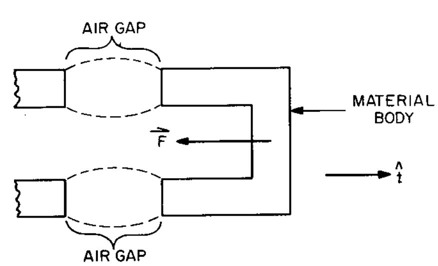

Fig. 10 – Constrained material body

The vector magnetic force factor and the vector electrical force factor are obviously very convenient descriptors of the magnetic to mechanical and the electrical to mechanical properties of a variable reluctance transducer. However, they are useful in practical situations only if an easy and standard procedure exists by which they can be computed. Fortunately, it is possible to derive a simple, approximate expression for $J(i, x)$ for the class of air gaps which are of usual practical interest. This approximate calculation, and its results, are presented in Appendix B.

Furthermore, it is always possible to obtain $J(i, x)$ through experimental measurement, and thus an exact determination of this parameter is possible. As in the case of reluctance, the approximation for $J(i, x)$ is not due to any inexactness in the concept but is due to the mathematical difficulties encountered in the application of the exact definition of Eq. (19) to practical situations. In this report the use of the magnetic and electrical gap force factor parameters will imply their exact forms unless otherwise stated.

The overall mechanical equation of a variable reluctance transducer can now be derived. If the transducer of interest has a mechanical configuration which is constrained to translation and can be considered to be composed of a concentrated linear mass, spring, and viscous damper, then the application of the force law (Newton's law) of Eq. (2) will yield

$$M\ddot{x} + B\dot{x} + Kx + G(i, x)\, i = f_e(t) \tag{26}$$

where the scalar force from Eq. (25) has been used and where

$f_e(t)$ = applied external force positive in a direction to widen the air gap

M = mass of the mechanical system

B = linear viscous damping coefficient of the mechanical system

K = linear spring constant of the mechanical system.

Note that the electrical to mechanical term of Eq. (26) is nonlinear in i and x, and thus this equation is a nonlinear differential equation in the dependent variables i and x.

This mechanical equation is the second of the two operating equations for the variable reluctance transducer. It has been derived through the applications of the force laws of classical electromagnetics. Together with the electrical equation, Eq. (18), it serves to describe the characteristics of the variable reluctance transducer. Now that the mechanical equation has been derived, the consequences of the form of this pair of equations can be discussed.

DISCUSSION OF THE TRANSDUCER EQUATIONS

In the foregoing sections the overall equations of a variable reluctance transducer have been derived from the concepts and laws of classical electromagnetic field theory. The operating equations are given by Eqs. (18) and (26) and they are repeated for reference purposes:

$$\begin{aligned} L(x)\frac{di}{dt} + R_W\, i + i\,\frac{dL}{dx}\frac{dx}{dt} &= v_e(t) \\ M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx + G(i, x)\, i &= f_e(t) \end{aligned} \tag{27}$$

These are a set of simultaneous nonlinear differential equations in i and x, with $v_e(t)$ and $f_e(t)$ as the specified independent inputs. No general solution method exists for such equations, and they must be solved by series, numerical, or graphical techniques. Thus, if $v_e(t)$ and $f_e(t)$ are the inputs to the transducer, the development of the transducer characteristics becomes stalemated by purely mathematical difficulties.

There are some alternate forms of these equations that can be investigated. For example, if the current $i(t)$ is considered to be the independent electrical input, then the equations can be rearranged in the form

$$v_e(t) - i\frac{dL}{dx}\frac{dx}{dt} - \frac{di}{dt}L(x) = R_w\, i(t) \tag{28}$$

$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx + i\,G(i,\, x) = f_e(t)$$

where v_e and x are now the dependent variables and $i(t)$ and $f_e(t)$ are the inputs.

It does not appear at first glance that any advantage results from this rearrangement, but if $L(x)$ and $G(i, x)$ are linear forms in x, then an advantage does accrue. Under these circumstances, the equations become a set of simultaneous linear equations, in dependent variables v_e and x, with time varying coefficients. It should be noted that the condition on $L(x)$ and $G(i, x)$ is only that they be linear in x, and no restrictive condition has been placed on their behavior with respect to i. The possible existence of such a set of linear, nonconstant-coefficient equations reduces somewhat the mathematical complexities of the transducer problem. However, except for certain special forms of such equations, the solution methods are still formidable.

One other rearrangement of these equations can be made. Suppose that the geometrical variable, $x(t)$, is considered to be the independent mechanical input. The operating equations can be rewritten in the form

$$v_e(t) = L(x)\frac{di}{dt} + R_w\, i + i\frac{dL}{dx}\frac{dx}{dt} \tag{29}$$

$$f_e(t) = M\frac{dx^2}{dt^2} + B\frac{dx}{dt} + Kx + G(i,\, x)\, i$$

which is a set of linear equations for v_e and f_e. That is, given $i(t)$ and $x(t)$ as independently specified functions of time, the values of $v_e(t)$ and $f_e(t)$ can be obtained exactly from Eq. (29). In the usual practical case, where $v_e(t)$ and $f_e(t)$ are actually the only variables which can be specified independently, the use of Eq. (29) is actually equivalent to assuming (guessing!) the possible forms of $i(t)$ and $x(t)$ and determining how closely they satisfy Eq. (27) written in the form of Eq. (29).

In Appendixes A and B, some approximate expressions for the reluctance and gap force factor parameters have been obtained. If the geometrical variable associated with each air gap is a varying length, then the general displacement variable x represents this variable length. If the remainder of the flux path is in a single magnetic material of constant cross-sectional area, then the reluctance of the flux tube can be obtained from the approximate formulas given by Eqs. (A1) and (A2) in Appendix A:

$$R(x) = \frac{l}{\mu A} + \frac{2x}{\mu_0 A_0} \tag{30}$$

where the factor of two accounts for two air gaps of equal length, and

l = length of the flux path in the magnetic material

x = length of the flux path in each of the equal length air gaps

μ = permeability of the magnetic material

μ_0 = permeability of the air gap

A = area of the flux path in the magnetic material

A_0 = area of the flux path in the air gap.

For such a transducer, the magnetic force factor of the air gap can be obtained from the approximate formula of Eq. (B2):

$$J(i, x) = \frac{\phi(i, x)}{\mu_0 A_0}.$$

The electrical force factor can be evaluated from Eq. (24):

$$G(i, x) = N \frac{J(i, x)}{R(i, x)} = \frac{N}{\mu_0 A_0} \frac{\phi(i, x)}{R(i, x)}.$$

But,

$$\phi(i, x) = \frac{\text{mmf}}{R(x)} = \frac{Ni}{R(x)}.$$

Thus,

$$G(i, x) = \frac{N^2}{\mu_0 A_0} \frac{i}{R^2(i, x)} \tag{31}$$

where the value of $R(i, x)$ from Eq. (30) is to be used in Eq. (31). It is obvious from the above that $G(i, x)$ is not linearly dependent on x even for this approximate evaluation. Thus Eq. (28) will not become a linear, time-varying-coefficient equation, even as a first-order approximation.

The consequence of this development is that the form of Eq. (27) requires that a set of simultaneous, nonlinear differential equations be solved if exact results are required. Now that this conclusion has been reached, the possibility of an approximate linearized form of these equations will be considered.

LINEARIZATION OF THE TRANSDUCER EQUATIONS

The nonlinearity of the transducer equations, Eqs. (27), greatly compounds the problem of determining the general operating characteristics of variable reluctance devices. In essence, it is necessary to determine the output of the transducer for each individual input. As a result of this disadvantage, it is of interest to determine whether there is any general class of inputs for which the operating equations could be approximated in a linear form. There is one such class of inputs, and this class will now be considered.

If the inputs are of the form

$$v_e(t) = V_e + v_e^*(t)$$
$$f_e(t) = F_e + f_e^*(t) \tag{32}$$

where

V_e = constant component of $v_e(t)$

F_e = constant component of $f_e(t)$

and

v_e^* = variation of $v_e(t)$ around V_e

f_e^* = variation of $f_e(t)$ around F_e

then each response of the transducer will in general consist of similar terms. Thus, the responses will be

$$i(t) = I_0 + i^*(t)$$
$$\lambda(t) = \Lambda_0 + \lambda^*(t) \tag{33}$$
$$x(t) = X_0 + x^*(t)$$
$$F(t) = F_0 + f^*(t)$$

where I_0, Λ_0, X_0, and F_0 are the constant components of i, λ, x, and F respectively, and i^*, λ^*, x^*, and f^* are the variations of i, λ, x, and F around I_0, Λ_0, X_0, and F_0 respectively.

The operating equations can be rewritten in a more fundamental form as

$$R_w i + \frac{d\lambda(i, x)}{dt} = v_e(t)$$
$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx + F(i, x) = f_e(t) \tag{34}$$

and if the relations of Eqs. (32) and (33) are inserted it is possible to separate these operating equations into two pairs of equations. The first pair involves only the constant components of the inputs and responses and is

$$R_w I_0 = V_e$$
$$K X_0 + F_0 = F_e \tag{35}$$

and the second pair involves only the variational components of the inputs and responses and is given by

$$R_w i^* + \frac{d\lambda^*}{dt} = v_e{}^*$$

$$M \frac{d^2 x^*}{dt^2} + B \frac{dx^*}{dt} + Kx^* + f^* = f_e{}^*. \tag{36}$$

The first pair of equations, Eqs. (35), constitute the bias or polarizing equations, and the second pair is the variational or small signal set of equations. In the original set of equations, Eqs. (34), the nonlinearity arose from the λ and F terms, which are nonlinear functions of i and x. Thus it is necessary to obtain expressions for Λ_0, λ^*, F_0, and f^* in terms of these original nonlinear expressions. This can be done through the use of a Taylor series expansion of λ and x around the point (I_0, X_0), and the results are

$$\Lambda_0 = \lambda(I_0, X_0)$$

$$F_0 = F(I_0, X_0)$$

and the approximate relations

$$\lambda^* \approx \left.\frac{\partial\lambda}{\partial i}\right|_0 i^* + \left.\frac{\partial\lambda}{\partial x}\right|_0 x^*$$

$$f^* \approx \left.\frac{\partial F}{\partial i}\right|_0 i^* + \left.\frac{\partial F}{\partial x}\right|_0 x^* \tag{37}$$

where $\lambda(i, x)$ and $F(i, x)$ are the exact nonlinear expressions for the flux linkage and gap force, and the zero subscript on the partial derivatives denotes evaluation at the point (I_0, X_0).

The approximate form for λ^* and f^* is more nearly correct the smaller i^* and x^* become, and these forms of f^* and λ^* constitute the essential approximations in the linearization process. Before inserting these relations into Eqs. (36) it will be convenient to consider the evaluated partial derivatives as incremental parameters. Thus,

$l^* = \left.\frac{\partial\lambda}{\partial i}\right|_0$ = incremental inductance around (I_0, X_0)

$g^*_{em} = \left.\frac{\partial\lambda}{\partial x}\right|_0$ = incremental mechanical to electrical coupling parameter around (I_0, X_0)

$g^*_{me} = \left.\frac{\partial F}{\partial i}\right|_0$ = incremental electrical to mechanical coupling parameter around (I_0, X_0)

$h^* = \left.\frac{\partial F}{\partial x}\right|_0$ = incremental stiffness around (I_0, X_0)

and the variational equations can be rewritten as

$$\lambda^* = l^* i^* + g^*_{em} x^*$$

$$f^* = g^*_{me} i^* + h^* x^*. \tag{38}$$

If Eqs. (38) are now inserted into Eqs. (36), the result is a set of linearized operating equations for the variational components of input and output. These equations are

$$l^* \frac{di^*}{dt} + R_w i^* + g^*_{em} \frac{dx^*}{dt} = v_e^*$$

$$g^*_{me} i^* + M \frac{d^2x^*}{dt^2} + B \frac{dx^*}{dt} + Kx^* + h^*x^* = f_e^* \tag{39}$$

and they can be solved by standard linear analysis techniques.

An indication of the form to be expected for l^*, g^*_{em}, g^*_{me}, and h^* can be obtained by using the approximations developed in Appendixes A and B. Thus λ and F can be written in the approximate forms

$$\lambda(i, x) = L(x)i = \frac{N^2}{R(x)} i$$

$$F(i, x) = G(i, x)\, i = \frac{N^2}{\mu_0 A} \frac{i^2}{R^2(x)}$$

where

$$R(x) = \frac{l}{\mu A} + \frac{2x}{\mu_0 A_0}$$

and if these are inserted into the defining relations for the incremental parameters, the results are

$$l^* = \frac{N^2}{R(X_0)}$$

$$g^*_{em} = -\frac{2N^2}{\mu_0 A_0} \frac{I_0}{R^2(X_0)}$$

$$g^*_{me} = \frac{2N^2}{\mu_0 A_0} \frac{I_0}{R^2(X_0)}$$

$$h^* = -\frac{4N^2}{\mu_0{}^2 A_0{}^2} \frac{I_0{}^2}{R^3(X_0)} .$$

Thus for this particular set of approximations the electromechanical coupling parameters g^*_{em} and g^*_{me} differ only by a minus sign. In actuality, this result is completely general and can be derived from energy considerations. However, such an energy treatment would be incompatible with the remainder of this report and thus will not be discussed here.

SUMMARY

The concepts and laws of the classical electromagnetic field can be used to analyze the energy conversion process in a variable reluctance transducer. The results obtained are stated in Eqs. (17) and (25). These expressions are based upon the introduction of the inductance and electrical gap force factor as fundamental descriptors of the electromechanical interaction. The usefulness of these relations in analysis depends heavily on the ease and accuracy with which the parameters can be determined.

When these electromechanical relations are inserted in the fundamental electrical and mechanical laws, expressed by Eqs. (1) and (2), the two overall equations of the single field transducer are obtained, and these are stated in Eqs. (27). They are found to be nonlinear for the general case, as well as for the specific approximations developed in Appendixes A and B.

The operating equations can be approximately linearized, provided that the inputs consist of constant components plus a "small" variational component. The results of such a linearization is given in Eqs. (35) and (39), and the set of linearized variational equations, Eqs. (39), can be analyzed by the standard methods of linear analysis.

PROLOGUE TO NRL REPORT 6089

The classical derivation of the electromechanical effects, and overall equations, of a variable reluctance transducer has one serious deficiency. This is the circumstance that the development of the force and voltage difference (or flux linkage) expressions are the result of two apparently unrelated and complex methods involving considerable vector manipulation. As a result, it is not at all apparent that there is any fundamental interrelation between these effects, other than their obvious mutual dependence on the state and geometry of the field. However, exactly the converse is the case, and there does exist a very simple and striking interdependence between these electromechanical terms.

In order to develop this relationship, it is necessary to consider the energy stored in the magnetic field as the fundamental descriptor of the electromechanical interaction. Once this step has been made, a completely consistent analysis of the energy conversion process can be developed in which the interdependence of the electromechanical effects explicitly appears. In this new scheme the inductance and electrical gap stiffness are not primary descriptors of the energy conversion process. This energy method is developed in a separate report.*

*R. M. Moore, "The Analysis of Variable Reluctance Transducers: The Energy Method," NRL Report 6089, June 1964.

APPENDIX A

APPROXIMATE CALCULATION OF RELUCTANCE

The approximate method of calculating reluctance is based on a series of assumptions concerning the direction and magnitude of the field vectors $\mathbf{H}$ and $\mathbf{B}$. The final results are given in terms of "effective values" of the various geometrical and material properties of the flux tube under discussion.

The calculation is based on the assumed existence of a clearly defined, and closed, flux tube having the same magnetomotive force around every closed contour which is included in the flux tube. It is desired to evaluate approximately the following definition of reluctance for this flux tube:

$$R = \frac{\oint_{C_p} \mathbf{H} \cdot \hat{t}_p \, dl_p}{\iint_{A_p} \mathbf{B} \cdot \hat{n}_p \, dA_p} = \frac{mmf_{C_p}}{\phi_{A_p}}. \tag{12}$$

The calculation method consists of four steps, as given in the following sections.

STEP 1

If the cross-sectional area of the flux tube varies from point to point, and/or if the material media through which the flux passes changes appreciably, then the closed path is conceptually subdivided into a number of connected tubes. Each of these K sections is then characterized by the "effective" geometrical properties,

A_k = effective cross section of the kth section

l_k = effective length of the kth section and by the "effective" material property

μ_k = permeability of the kth section

where the use of a scalar permeability implies the assumption of isotropic magnetic properties for all material in the flux tube.

The use of the scalar permeability assumes that the equation relating the field intensity and flux density is of the form $\mathbf{B}_k = \mu_k \mathbf{H}_k$ for each section of the tube.

STEP 2

A flux ϕ is assumed to be flowing through the magnetic path, Eq. (7) is used to evaluate the flux density $\mathbf{B}_k$ in each section of the tube. It is assumed that $\mathbf{B}_k$ is everywhere normal to the cross-sectional area A_k, and that $\mathbf{B}$ is constant in magnitude over this area. Thus the flux can be evaluated approximately as

$$\phi = \iint_{A_k} \mathbf{B}_k \cdot \hat{n}_k \, dA_k \approx B_k A_k$$

where B_k is the "effective" magnitude of $\mathbf{B}_k$ across A_k and can therefore be expressed in terms of ϕ as $B_k = \phi/A_k$.

Using the permeability μ_k, the "effective" field intensity magnitude H_k can be expressed as a function of ϕ:

$$H_k = \frac{B_k}{\mu_k} \approx \frac{\phi}{\mu_k A_k}.$$

The result of this series of approximations is an expression for the "effective" field intensity H_k in terms of the flux ϕ in the flux tube. An unstated assumption in the above calculation is that there is no leakage flux from the path, and such an assumption is always implied in the concept of a flux tube.

STEP 3

In the definition of the reluctance function, Eq. (12), the only remaining obstacle is the evaluation of the mmf around the flux tube. Applying the defining relation for mmf from Eq. (5), together with the assumption that H_k is everywhere tangent to the path of integration, the result is

$$\mathrm{mmf} = \oint_{C_p} \mathbf{H} \cdot \hat{t}_p \, dl_p \approx \sum_{k=1}^{k=K} H_k l_k$$

where

C_p = closed contour in flux tube

$\hat{t}_p$ = unit tangent to C_p

dl_p = incremental length on C_p.

This result is stated in terms of average field intensity H_k. Inserting the value for the H_k from the previous step the result is

$$\mathrm{mmf} = \sum_{k=1}^{K} \frac{\phi}{\mu_k A_k} l_k = \phi \sum_{k=1}^{K} \frac{l_k}{\mu_k A_k}$$

since ϕ is independent of k.

STEP 4

Inserting the expression for the mmf in Eq. (12) the result is

$$R = \frac{\phi \sum_{k=1}^{K} \frac{l_k}{\mu_k A_k}}{\phi} = \sum_{k=1}^{K} \frac{l_k}{\mu_k A_k}. \tag{A1}$$

Now, if each of the terms in the summation is considered to represent the reluctance of a section of the tube:

$$R_k = \frac{l_k}{\mu_k A_k} = \text{reluctance of kth section of tube} \tag{A2}$$

then the reluctance expression can be written in the standard form

$$R = \sum_{k=1}^{K} R_k . \tag{A3}$$

APPENDIX B

APPROXIMATE CALCULATION OF THE MAGNETIC FORCE FACTOR

The approximate method of calculating the magnetic force factor of the air gaps is based on idealizations concerning the direction and magnitude of the field vectors **H** and **B**. The final result is given in terms of various effective values for the geometrical and magnetic properties of the flux path under consideration.

It is desired to evaluate the following integral definition of the "vector magnetic force factor:"

$$J(i, x) = \frac{\iint_{A_1} \frac{1}{2}\mu_0 |\mathbf{H}|^2 \hat{n}_1 \, dA_1 + \iint_{A_2} \frac{1}{2}\mu_0 |\mathbf{H}|^2 \hat{n}_2 \, dA_2}{\iint_{A_\rho} \mathbf{B} \cdot \hat{n}_\rho \, dA_\rho} \tag{19}$$

where the areas A_1 and A_2 will be chosen to span the air gaps and A_ρ is any arbitrary cross section of the flux path. The special geometry which will be considered for this approximate evaluation is illustrated in Fig. 11. The directions of **H** relative to $\hat{n}_1$ and $\hat{n}_2$ are the same as those in Fig. 9, which was used to obtain the magnetic force factor equation given by Eq. (19).

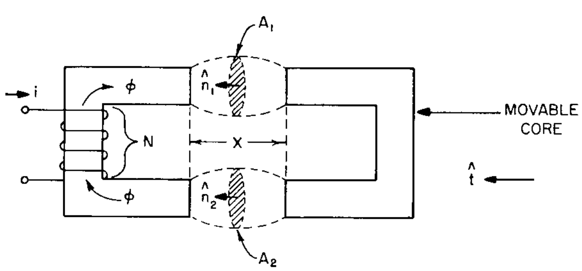

Fig. 11 – Geometry for Eq. (19)

The approximate calculation method consists of four steps, as outlined below.

STEP 1

The air gaps are assumed to be characterized by the geometrical properties

$$A = A_1 = A_2 = \text{effective area of air gap}$$

$$x = \text{length of air gap}$$

and by the magnetic property

μ_0 = permeability of air gap (assumed to be equal to free space)

which means that the **B** and **H** fields in the air gaps are related by

$$\mathbf{B} = \mu_0 \mathbf{H}.$$

STEP 2

The "effective" magnitude of the flux density **B** is evaluated in terms of ϕ using Eq. (7). It is assumed that **B** is everywhere normal to A_1 (or A_2) and constant in magnitude across the cross section. Thus the flux can be evaluated approximately as

$$\phi = \iint_A \mathbf{B} \cdot \hat{n}\, dA \approx BA$$

where

B = "effective" magnitude of **B** over A_1 (or A_2).

Thus B can be expressed as

$$B = \frac{\phi}{A}.$$

STEP 3

The "effective" magnitude of **H** over the areas A_1 and A_2 can then be expressed as

$$H = \frac{\phi}{\mu_0 A}$$

where

H = "effective" magnitude of **H** over A_1 and A_2.

This value for the magnitude of **H** can then be inserted into the integrals of Eq. (19) to give

$$\iint_{A_1} \tfrac{1}{2}\mu_0 |\mathbf{H}|^2 \hat{n}_1\, dA_1 = \frac{1}{2}\frac{\phi^2}{\mu_0 A}\hat{n}_1$$

$$\iint_{A_2} \tfrac{1}{2}\mu_0 |\mathbf{H}|^2 \hat{n}_2\, dA_2 = \frac{1}{2}\frac{\phi^2}{\mu_0 A}\hat{n}_2$$

and since $\hat{n}_1$ and $\hat{n}_2$ are parallel to the unit tangent vector $\hat{t}$ of the line of translation, the numerator of Eq. (19) becomes

$$\frac{\phi^2}{\mu_0 A}\hat{t}.$$

STEP 4

The denominator of Eq. (19) is equal to ϕ by definition, and thus the magnetic force factor becomes

$$\mathrm{J}(i, x) = \frac{\phi}{\mu_0 A} \hat{t} \tag{B1}$$

and from the definition of the "scalar magnetic force factor" we obtain

$$J(i, x) = \frac{\phi}{\mu_0 A}. \tag{B2}$$