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FIRE CONTROL DIVISION - RADAR RESEARCH SECTION

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# ANALYSIS OF LENGTHENING OF MODULATED REPETITIVE PULSES

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By S. C. Kleene

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Approved by:

J. B. Trevor, Jr. - Head, Radar Research Section

Dr. R. M. Page
Superintendent, Fire Control Division

Rear Adm. A. H. Van Keuren, USN (Ret.)
Director, Naval Research Laboratory

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MARY L. KNIGHT
Contracting Office, Security

## ABSTRACT

It was shown in Reference (a), Appendix by Ming-Chen Wang and G. E. Uhlenbeck, that if a short pulse with repetition frequency $f_o$, periodically modulated by a signal with fundamental frequency $f_m$, is put through an ideal "box car" generator (which produces flat-topped pulses lasting until the succeeding pulse), the output frequencies are $|sf_o + qf_m|$ where s and q are integers; a formula giving the amplitudes was presented. This report contains a derivation of the result in more detail than they gave, and without the restriction to the special case of the ideal box car pulse. In the derivation, the pulse form is taken to be quite general. Both a formula and graphs are then presented for the amplitudes of the output frequencies of the pulse lengthener when the signal amplitude of the output pulse of the lengthener decays exponentially with the time constant $\alpha$, and lasts throughout the fraction $\beta$ of the interval between pulses. This reduces further to the box car case when $\alpha = 0$ and $\beta = 1$. It is noted that the frequencies $sf_o$ are ordinarily present, except in the ideal box car case. These results are intended for application to the pulse lengthener used in investigations of Doppler modulation of radar echoes.

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## INTRODUCTION

1. It was reported in Reference (a) (pp.38 ff and pp.71 ff) that if a short pulse with repetition frequency $f_o$, periodically modulated by a signal with fundamental frequency $f_m$, is put through an ideal "box car" generator,[1] the output frequencies will be $|sf_o + qf_m|$ where s and q are integers; and a formula was presented giving the amplitudes. The proof of the formula, as indicated in the appendix by Ming-Chen Wang and G. E. Uhlenbeck (pp.103 ff), is by a straightforward Fourier analysis, the details of which were not given.

2. At this Laboratory the question arose from work on Problem O-106.1R-S as to how the result would have to be modified when the output pulse does not have the ideal box car form. In this connection, interest was expressed in having a derivation of Wang and Uhlenbeck's result set down in more detail than they gave, and without the restriction to the special case which they treated. In the present report an arbitrary pulse form is presupposed in the derivation. Specialization is then made to a type of pulse form depending on parameters $\alpha$ and $\beta$, and including the ideal box car pulse for $\alpha = 0$ and $\beta = 1$.

## GENERAL CASE: ASSUMPTIONS

3. Starting with a function f(t) which represents the modulating signal, and a function g(t) which represents the unmodulated output pulse, the function F(t) which represents the output will be defined, and a Fourier expansion will be obtained for F(t). A practical set of assumptions is that f(t) and g(t) each be piecewise continuous and have a piecewise continuous first derivative.[2] In this case, f(t) shall be defined at a point of discontinuity as $\frac{1}{2}(f(t-0) + f(t+0))$, and the same convention shall be understood to apply to g(t) and F(t). However, f(t) is used only in the form of its Fourier series, and so any Fourier series which converges for all values of t can be taken as defining f(t).

## MODULATING SIGNAL AND PULSE

4. Let f(t) stand for the modulating signal amplitude,[3] where t represents time measured from the initial instant of some pulse interval, and let f(t) be periodic with the fundamental frequency $f_m$ or the period $1/f_m$. Let it be developed in the Fourier series

$$f(t) = \sum_{q=-\infty}^{\infty} c_q \, e^{2\pi i q f_m t} = \frac{a_o}{2} + \sum_{p=1}^{\infty} (a_p \cos 2\pi p f_m t + b_p \sin 2\pi p f_m t) ; \qquad (1)$$

---

[1] A box car generator ideally produces flat-topped pulses which last until the next succeeding pulse; or in more exact terms, the signal amplitude of a box car pulse is constant for the duration of the pulse interval, having as its value the signal amplitude of the modulating signal for the initial instant of the pulse interval.

[2] Compare, for example, Courant and Hilbert, Methoden der Mathematischen Physik, Volume 1, Chapter 2, Paragraph 5.

[3] The amplitudes f(t), g(t), F(t) as functions of t will be referred to as

[redacted]

where

$$c_{\pm p} = \frac{a_p}{2} \mp i\,\frac{b_p}{2}\ , \qquad c_o = \frac{a_o}{2}\ . \tag{2}$$

5. If the modulating signal were a constant signal of amplitude 1, the pulse lengthener would produce a succession of pulses, each one identical with its predecessor. Let $f_o$ be the pulse repetition frequency and $1/f_o$ the interval between pulses. Let the signal amplitude of these pulses be represented by the function $g(t)$, as shown for one pulse interval in Figure 1 (a), Plate 1.

6. When the modulating signal amplitude is $f(t)$, the output $F(t)$ of the pulse lengthener will consist of a succession of pulses, which differ from the unmodulated pulses of Paragraph 5 in that the signal amplitude $g(t)$ throughout each pulse is multiplied by the modulating signal amplitude $f(t)$ for the initial instant of the pulse interval, as shown in Figure 1 (b), Plate 1. Let the initial instant of the pulse interval in which t occurs be written $\{t\}$, i.e. $\{t\} = k/f_o$ where k is the greatest integer such that $k/f_o \leq t$. Then by the definition,[4]

$$F(t) = f(\{t\})\ g(t)\ . \tag{3}$$

DERIVATION OF A FOURIER EXPANSION FOR F(t), IN COMPLEX FORM

7. Using (1) to express $f(\{t\})$ in (3) as a series, and multiplying $g(t)$ into the series term by term,

$$F(t) = \sum_{q=-\infty}^{\infty} c_q\ e^{2\pi i q f_n \{t\}} g(t)\ . \tag{4}$$

Writing $\{t\}$ as $t - (t - \{t\})$, and factoring,

$$F(t) = \sum_{q=-\infty}^{\infty} c_q\ e^{2\pi i q f_n t}\ e^{-2\pi i q f_n (t-\{t\})}\ g(t)\ . \tag{5}$$

The expression $e^{-2\pi i q f_n (t-\{t\})}$ represents $e^{-2\pi i q f_n t}$ in the first pulse interval, and the periodic repetition of this function in other pulse intervals. Furthermore it is continuous with a continuous derivative, except at the end points of pulse intervals. Therefore the product $e^{-2\pi i q f_n (t-\{t\})}\ g(t)$ is periodic with the period $1/f_o$, and has the same continuity properties as were assumed for $g(t)$ itself (Paragraph 3). Developing it in a Fourier series,[5] where the value $e^{-2\pi i q f_n t}$

"signal amplitude" to avoid confusion with the term "amplitude" as applied to the frequency components into which the functions are analyzed.

[4] Except for $t = k/f_o$, when $F(t) = \frac{1}{2}(F(t-0) + F(t+0))$ (Paragraph 3).

[5] If the reader prefers to stay within the theory of Fourier series of real functions, the exponential factor $e^{-2\pi i q f_n (t-\{t\})}$ may first be expanded as $\cos 2\pi q f_n (t - \{t\}) - i \sin 2\pi q f_n (t - \{t\})$.

[redacted]

for $e^{-2\pi i q f_m (t - \{t\})}$ in the first pulse interval is used in expressing the coefficients, and inserting this series into (5),

$$F(t) = \sum_{q=-\infty}^{\infty} c_q \, e^{2\pi i q f_m t} \sum_{s=-\infty}^{\infty} \left( f_o \int_0^{1/f_o} e^{-2\pi i q f_m t} \, g(t) \, e^{-2\pi i s f_o t} \, dt \right) e^{2\pi i s f_o t}. \quad (6)$$

Multiplying $c_q \, e^{2\pi i q f_m t}$ into the series summed on s, and combining the exponential factors,

$$F(t) = \sum_{q=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} c_q \left( f_o \int_0^{1/f_o} g(t) \, e^{-2\pi i (s f_o + q f_m) t} \, dt \right) e^{2\pi i (s f_o + q f_m) t}. \quad (7)$$

THE EXPANSION IN REAL FORM

8. Rewriting this formula with abbreviations to save space in what follows:

$$F(t) = \sum_{q=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} C(s,q) \quad , \quad (8)$$

where

$$C(s,q) = c_q \, w_{s f_o + q f_m} \, e^{2\pi i (s f_o + q f_m) t} \quad , \quad (9)$$

and replacing $s f_o + q f_m$ by f to get an expression $w_f$ in which f is quite general,

$$w_f = f_o \int_0^{1/f_o} g(t) \, e^{-2\pi i f t} \, dt \quad . \quad (10)$$

Using $e^{\pm ix} = \cos x \pm i \sin x$ ,

$$w_{\pm f} = f_o \int_0^{1/f_o} g(t) \cos 2\pi f t \, dt \mp i \, f_o \int_0^{1/f_o} g(t) \sin 2\pi f t \, dt \quad . \quad (11)$$

9. Now let

$$A(s,q) = C(s,q) + C(-s,-q) \quad . \quad (12)$$

For $s = q = 0$,

$$\frac{1}{2} A(0,0) = C(0,0) \quad . \quad (12a)$$

To see what (8) becomes with this grouping, consider the terms C(s,q) which in (8) occur for all pairs of integers. Let s' and p denote positive integers. The pairs (s,q) may be grouped under four categories: the pair (0,0); for each s', the two pairs (s',0) and (-s',0); for each p, the two pairs (0,p) and (0,-p); and for each s' and p, the four pairs (s',p), (-s',-p), (s',-p) and (-s',p). Rearranging the summation in (8) accordingly,

[redacted]

$$F(t) = C(0,0) + \sum_{s'=1}^{\infty} \left[C(s',0) + C(-s',0)\right]$$

$$+ \sum_{p=1}^{\infty} \left\{C(0,p) + C(0,-p) + \sum_{s'=1}^{\infty} \left[C(s',p) + C(-s',-p)\right.\right.$$

$$\left.\left. + C(s',-p) + C(-s',p)\right]\right\} \quad .$$

Using (12) and (12a), and dropping the primes, this becomes

$$F(t) = \frac{1}{2} A(0,0) + \sum_{s=1}^{\infty} A(s,0) + \sum_{p=1}^{\infty} \left\{A(0,p) + \sum_{s=1}^{\infty} \left[A(s,p) + A(s,-p)\right]\right\} \quad . \qquad (13)$$

10. Write

$$c_q = |c_q| \, e^{i \arg c_q} \, , \qquad w_f = |w_f| \, e^{i \arg w_f} \quad . \qquad (14)$$

By (2) and (11),

$$|c_{-q}| = |c_q| \quad , \qquad \arg c_{-q} = -\arg c_q \quad ,$$

$$|w_{-f}| = |w_f| \quad , \qquad \arg w_{-f} = -\arg w_f \quad . \qquad (15)$$

Using (14) in (9), and the result with $e^{\pm ix} = \cos x \pm i \sin x$ in (12),

$$A(s,q) = 2 \, |c_q| \, |w_{sf_o+qf_m}| \cos \left(2\pi(sf_o + qf_m)t + \arg c_q + \arg w_{sf_o+qf_m}\right) \quad . \qquad (16)$$

For $s = q = 0$, (12a) and (9) with (2) and (10) give directly

$$\frac{1}{2} A(0,0) = c_o \, w_o = \frac{a_o}{2} \, f_o \int_0^{1/f_o} g(t) \, dt \quad . \qquad (16a)$$

For $sf_o + qf_m = 0$, (16) reduces to

$$A(s,q) = 2 \, |c_q| \, |w_o| \cos (\arg c_q + \arg w_o)$$

$$= 2 \, |c_q| \, w_o \cos (\arg c_q) \quad , \qquad (17)$$

which when $q = 0$ reduces further to (16a).

11.[6] Using (14) and for positive p setting $\arg c_{\pm p} = \pm 2\pi p f_m \delta_p \mp \pi/2$, the series (1) takes the form

[6] The formulas of this paragraph in terms of the sine are added for convenience in reducing the results to the formula of Reference (a).

[redacted]

$$f(t) = c_o + \sum_{p=1}^{\infty} 2\,|c_p|\cos(2\pi p f_m t + \arg c_p)$$

$$= c_o + \sum_{p=1}^{\infty} 2\,|c_p|\sin 2\pi p f_m(t + \delta_p) \quad . \tag{18}$$

Then (16) becomes, for positive p,

$$A(s, \pm p) = \pm 2|c_p|\;|w_{sf_o \pm pf_m}|\sin(2\pi(sf_o \pm pf_m)t \pm 2\pi p f_m \delta_p + \arg w_{sf_o \pm pf_m}), \tag{19}$$

which for s = 0 (p positive) reduces to

$$A(0,p) = 2|c_p|\;|w_{pf_m}|\sin(2\pi p f_m t + 2\pi p f_m \delta_p + \arg w_{pf_m}) \quad . \tag{19a}$$

For q = 0, (16) reduces as follows:

$$\begin{aligned} A(s,0) &= 2|c_o|\;|w_{sf_o}|\cos(2\pi sf_o t + \arg c_o + \arg w_{sf_o}) \\ &= 2\,c_o\,|w_{sf_o}|\cos(2\pi sf_o t + \arg w_{sf_o}) \\ &= 2\,c_o\,|w_{sf_o}|\sin(2\pi sf_o t + \pi/2 + \arg w_{sf_o}) \quad . \end{aligned} \tag{20}$$

RELATIVE AMPLITUDE AND RELATIVE PHASE

12. By (18) the factor $2|c_p| = 2|c_q|$ in the above formulas is the amplitude of the frequency $pf_m = |q|f_m$ in the modulating function f(t), except for q = 0, when it is twice that amplitude. However, for q = 0 and s = 0, only half of A(s,q) is used in (13). Thus by (13) and (16), for any particular s and q, $|w_{sf_o+qf_m}|$ is the ratio of the amplitude of the term in (13) for that s and q, which has the frequency $sf_o + qf_m$, to the amplitude of the frequency $pf_m = |q|f_m$ in f(t), except for q = 0 and s ≠ 0 when it is half that ratio.[7] The quantity $|w_{sf_o+qf_m}|$ will accordingly be called the <u>relative amplitude</u> of the output frequency $sf_o + pf_m$ for that s and q.[8]

13. The angle arg $c_q$ is the phase of the frequency $pf_m = |q|f_m$ in f(t) considered as a cosine, taken with the algebraic sign of q. Hence by (16), arg $w_{sf_o+qf_m}$ is the <u>relative phase</u> of the frequency $sf_o + qf_m$ in the output term for that s and q.

A SPECIAL CASE

14. Specialization is now made of the pulse function g(t) to the following, where $0 < \beta \leq 1$:

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[7] This exception will be clarified in connection with the graphical treatment, Paragraph 24.

[8] When the same frequency $|sf_o + qf_m|$ is given by more than one term of the Fourier expansion (13), phase relationships have to be taken into account to get the total amplitude of this frequency in the output (see Paragraph 26 ff).

$$g(t) = \begin{cases} e^{-\alpha t} & \text{when } 0 < t < \beta/f_o \ , \\ 0 & \text{when } \beta/f_o < t < 1/f_o \ . \end{cases} \tag{21}$$

The pulse is shown in Figure 2 (a), Plate 1.

15. For purposes of illustration, the modulating function $f(t)$ may be taken as follows, where $S_o, \epsilon \geqslant 0$:

$$f(t) = S_o(1 + \epsilon \sin 2\pi f_m(t + \delta)). \tag{22}$$

In this case the only frequencies $pf_m$ present in the modulating function are those for $p = 0$ and $p = 1$; $c_o = S_o$, $2|c_1| = \epsilon S_o$, $\delta_1 = \delta$, and (13) reduces to

$$F(t) = \frac{1}{2} A(0,0) + \sum_{s=1}^{\infty} A(s,0) + A(0,1) + \sum_{s=1}^{\infty} \left[A(s,1) + A(s,-1)\right] \tag{23}$$

However, the discussion will be carried out so that its application when there are more frequencies present in the modulating function will be evident. With the modulating function chosen for illustration, the modulating function and output are as shown in Figure 2 (b), Plate 1.

16. Using (21) in (10), the interval of integration can be taken as $(0, \beta/f_o)$, since the integrand is 0 in the remainder of the indicated interval. Integrating, if $\alpha$ and $f$ are not both 0,

$$w_f = \beta \frac{1 - e^{-\alpha\beta/f_o - 2\pi i \beta f/f_o}}{\alpha\beta/f_o + 2\pi i \beta f/f_o} \ . \tag{24}$$

Then, as may be verified by substituting in (14),[9]

$$|w_f| = \beta \left\{ \frac{1 + e^{-2\alpha\beta/f_o} - 2e^{-\alpha\beta/f_o} \cos 2\pi\beta f/f_o}{(\alpha\beta/f_o)^2 + (2\pi\beta f/f_o)^2} \right\}^{\frac{1}{2}} , \tag{25}$$

and

$$\arg w_f = \sin^{-1} \frac{e^{-\alpha\beta/f_o} \sin 2\pi\beta f/f_o}{(1 + e^{-2\alpha\beta/f_o} - 2e^{-\alpha\beta/f_o} \cos 2\pi\beta f/f_o)^{\frac{1}{2}}} \tag{26}$$

$$-\tan^{-1} ((2\pi\beta f/f_o)/(\alpha\beta/f_o)) \ ,$$

where for $\alpha > 0$ the principal values of the inverse functions are to be used.

---

[9] The formulas are easily discovered and verified by using the representation of (24) in the complex plane, applying in the case of the numerator of (24) the law of cosines and law of sines for a triangle with sides 1 and $e^{-\alpha\beta/f_o}$ and included angle $2\pi\beta f/f_o$.

[redacted]

Integrating when $\alpha$ and f are both 0,

$$w_f = |w_f| = \beta \quad \text{and} \quad \arg w_f = 0 \quad . \tag{27}$$

17. When $\alpha = 0$, but $f \neq 0$, the formulas simplify as follows:

$$w_f = \beta \frac{1 - e^{-2\pi i \beta f/f_o}}{2\pi i \beta f/f_o} \quad \frac{\beta}{\pi\beta f/f_o} \frac{e^{\pi i \beta f/f_o} - e^{-\pi i \beta f/f_o}}{2i} e^{-\pi i \beta f/f_o}$$

$$= \beta \frac{\sin \pi\beta f/f_o}{\pi\beta f/f_o} \; e^{-\pi i \beta f/f_o} \quad , \tag{28}$$

$$|w_f| = \beta \left| \frac{\sin \pi\beta f/f_o}{\pi\beta f/f_o} \right| , \qquad \arg w_f = - \pi\beta f/f_o + \arg \frac{\sin \pi\beta f/f_o}{\pi\beta f/f_o} \quad .$$

18. Substituting from (25) and (26) (or from (28) if $\alpha = 0$) into (19), (19a) and (20), and from (27) into (16a), and using the results in (23) with the values of $c_o$, $2|c_1|$ and $\delta_1$ given there, the Fourier expansion of F(t) can be obtained explicitly; and it is easily verified that the terms simplify to those of the formula of Reference (a) for the case that $\alpha = 0$ and $\beta = 1$ which gives the ideal box car pulse. In the unsimplified case when $\beta < 1$ and $\alpha > 0$, the general term A(s,q) of (23) is:

$$A(s,q) = 2|c_q|\beta \left\{ \frac{1 + e^{-2\alpha\beta/f_o} - 2e^{-\alpha\beta/f_o} \cos 2\pi\beta \frac{sf_o + qf_m}{f_o}}{\left(\frac{\alpha\beta}{f_o}\right)^2 + \left(2\pi\beta \frac{sf_o + qf_m}{f_o}\right)^2} \right\}^{\frac{1}{2}}$$

$$\cos \left\{ 2\pi(sf_o + qf_m)\, t + \arg c_q + \sin^{-1} \frac{e^{-\alpha\beta/f_o} \sin 2\pi\beta \frac{sf_o + qf_m}{f_o}}{\left(1 + e^{-2\alpha\beta/f_o} - 2e^{-\alpha\beta/f_o} \cos 2\pi\beta \frac{sf_o + qf_m}{f_o}\right)^{\frac{1}{2}}} \right.$$

$$\left. - \tan^{-1} \frac{2\pi\beta \frac{sf_o + qf_m}{f_o}}{\frac{\alpha\beta}{f_o}} \right. \tag{29}$$

where

$$c_o = |c_o| = S_o \quad , \qquad \arg c_o = 0 \quad ,$$

$$|c_1| = \frac{\epsilon S_o}{2} \quad , \qquad \arg c_{\pm 1} = \pm 2\pi f_m \delta \mp \frac{\pi}{2} \quad .$$

Use of this special case is made in the study of a practical pulse lengthener in Reference (b). However, at least in the unsimplified case, a graphical presentation of the results is more illuminating than the rather unwieldy formulas.

[redacted]

## THE FREQUENCY NOMOGRAPH

19. A nomographic method of exhibiting the output frequencies $|sf_o \pm pf_m|$ was given in Reference (a). Thus in Plate 2, corresponding to each of the frequencies $pf_m$ for $p = 0, 1, \ldots$ which is present in the modulating function, let a horizontal line be drawn (shown dotted for $p = 1$) with the ordinate $pf_m$. The abscissas of the intersections of this line with the lines of the frequency diagram then give all the possible output frequencies associated with the modulating frequency $pf_m$. This is clear, since the equations of the lines of the diagram are $|f| = |sf_o \pm pf_m|$, if $|f|$ is considered as a continuously varying abscissa and $pf_m$ as a continuously varying ordinate. In this connection it is helpful to think of the lines $|f| = |sf_o \pm pf_m|$ as coming from lines $f = sf_o + qf_m$ where $p = |q|$ by two reflections, first in the axis of abscissas when absolute value signs are applied to q, and then in the axis of ordinates when absolute value signs are further applied to f. The small figure at the upper right in Plate 2 identifies the signs of q and f along the three parts into which the line $f = sf_o + qf_m$ is broken by the reflections.

20. To illustrate with the modulating function of Paragraph 15, $pf_m$ is then restricted to the two values $0 \cdot f_m$ and $1 \cdot f_m$. For $p = 0$, the horizontal line is the base line, and the corresponding output frequencies are 0 and $sf_o$ ($s = 1, 2, \ldots$). For $p = 1$, the horizontal line is as shown in Plate 2, and gives the additional output frequencies $f_m$, $sf_o + f_m$, $|sf_o - f_m|$. (Dotted lines are drawn to help in recognizing where these frequencies fall on the base scale, but they are labeled only at the intersections where they originate, omitting absolute value signs when superfluous.)

## THE RELATIVE AMPLITUDE CURVE

21. It is next proposed to draw a curve which will give the amplitudes of each of these frequencies in their terms $A(s,q)$ in the Fourier expansion (13),[10] in relation to the amplitudes of the modulating frequencies from which they originate. From the discussion of Paragraph 12, only one curve is required for any particular pulse function $g(t)$, and this curve is simply $|w_f|$ as a function of $|f|$, where $w_f$ is defined from $g(t)$ by the integral (10). The family of these curves is now presented for the family of pulse functions depending on parameters $\alpha$ and $\beta$ which was introduced in Paragraph 14.

22. In the computation, a number of fixed values of $\alpha\beta/f_o$ were chosen, and $|w_f|$ in multiples of $\beta$ was calculated as a function of $|f|$ in multiples of $f_o/\beta$ by (25).[11] The resulting curves are plotted in Plate 3. The scales would become numerical if the values of $\beta$ and $f_o$ were specified.

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[10] For $s = q = 0$, the term is not $A(s,q)$ but $\frac{1}{2}A(s,q)$. This term and any others with $f = sf_o + qf_m = 0$ may be considered exceptions to the main discussion, and their treatment is postponed to Paragraph 28.

[11] The labor of computation is not great, as for each $\alpha\beta/f_o$, the numerator of (25) requires to be calculated only for one half cycle of $2\pi\beta f/f_o$; thereafter values of the numerator can be used over. The calculation becomes especially simple if done graphically from the triangle mentioned in the footnote to Paragraph 16. For $\alpha\beta/f_o = 0$, $|w_f|/\beta$ is taken from $(\sin x)/x$ tables, by (28).

[redacted]

23. These curves may be used in conjunction with the frequency nomograph to associate the relative amplitudes with the frequencies pictorially. Suppose for illustration that $\beta = 0.75$ and $\alpha\beta/f_o = 0.5$, and let the modulating function be the one of Paragraph 15. Using the value $\beta = 0.75$ to locate $f_o$ at 0.75 of the distance from the origin to the point labeled $f_o/\beta$ on Plate 3, let a frequency nomograph (Plate 2) be made up to this scale. Then selecting proper curve of Plate 3 for $\alpha\beta/f_o = 0.5$, imagine Plate 3 positioned over the nomograph in a plane perpendicular to the nomograph, first along the base line for the output frequencies associated with the modulating frequency $pf_m = 0 \cdot f_m$ $(p = 0)$, and then along the horizontal line at the modulating frequency $pf_m = 1 \cdot f_m$ $(p = 1)$ for the output frequencies associated with the modulating frequency $1 \cdot f_m$, as shown in Plate 4. Lines erected to the relative amplitude curve at the nomograph points previously found as representing the output frequencies (Paragraph 20) represent the relative amplitudes associated with those frequencies in the terms $A(s,q)$ of the Fourier expansion obtained for the output. The absolute amplitudes are obtained by multiplying the heights of the lines on the $p = 0$ panel by $S_o$ and those on the $p = 1$ panel by $\epsilon S_o$.

24. On the $p = 0$ panel, the relative amplitudes are shown twice (by a double line), except for $|f| = 0$, indicating that in each of these cases the absolute amplitude is twice the height of the line times $S_o$. This is necessary from the formulas (compare Paragraph 12). It becomes intelligible why this is so, if it is supposed that the panel had been positioned for a very small ordinate $pf_m$. There would then be, for each $sf_o$ $(s = 1, 2, \ldots)$ two lines erected at nearby frequencies which would come together at the frequency $sf_o$ when $pf_m$ approaches 0.

25. It is to be emphasized that what has been obtained are the amplitudes of particular terms $A(s,q)$ in the Fourier Expansion obtained for $F(t)$. It can, however, happen that more than one of these terms have the same frequency (i.e., $|sf_o + qf_m| = |s'f_o + q'f_m|$ where $s \neq s'$ or $q \neq q'$), and in such a case the total amplitude of the frequency in $F(t)$ will depend on the phases with which the several terms combine. Also, the terms for frequency $|sf_o + qf_m| = 0$ constitute something of an exception, and some remarks are called for to prevent misinterpretation of the relative amplitude lines in their case. These details are treated next (Paragraphs 26 - 28).

## COINCIDENT OUTPUT FREQUENCIES

26. When the modulating function $f(t)$ contains a frequency $pf_m$ with $p > 0$ which is an exact multiple of $f_o/2$, the horizontal line for this $p$ will pass through double points of the frequency nomograph where the lines forming the nomograph cross. As in the case of Paragraph 24, double lines may be erected; but unlike that case, the two lines of a pair belong to distinct terms $A(s,q)$ of (13) which will ordinarily have different phases.[12] Two or more terms of (13) with the same $|sf_o + qf_m|$ may also occur with different values of $p = |q|$; that is, when the horizontal lines corresponding to two or more modulating frequencies intersect the lines of the nomograph in points having the same abscissas.

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[12] In the case of Paragraph 24, the two nearby frequencies have a phase difference which vanishes as they come together, as is to be expected since they are represented by nearby points of the line $f = sf_o + qf_m$ before the double reflection (Paragraph 19). In the case of the present paragraph, the points of the two intersecting nomograph lines are distant points of the line $f = sf_o + qf_m$, which have been brought into superposition by the double reflection.

[redacted]

27. By Paragraph 13, the phase of any term $A(s,q)$ may be found from the phase of the modulating frequency $pf_m = |q|f_m$ to which it corresponds by adding the relative phase angle $\arg w_{sf_o+qf_m}$. For this purpose the algebraic signs of q and $f = sf_o + qf_m$ cannot be disregarded, as they were in the case of amplitude (compare (15)). The signs of q and f for any point of a line of the frequency nomograph can be read off the supplementary diagram in Plate 2 (Paragraph 19). The relative phase $\arg w_f$ is $\arg w_{|f|}$ with the sign of f. If a need to determine the phases of the terms $A(s,q)$ for the pulse functions of Paragraph 14 should arise, the family of the curves for $\arg w_{|f|}$ as a function of $|f|$ could be obtained from (26) in the same manner as the family for $|w_f| = |w_{|f|}|$ as a function of $|f|$ was obtained from (25) (Paragraph 22), and could be plotted to accompany Plate 3.

CONSTANT (0 FREQUENCY) TERMS

28. The relative amplitude line erected at the origin represents $|w_o|$. Multiplying it by the constant term $c_o$ of $f(t)$ and applying the algebraic sign of $w_o$ gives the term $\frac{1}{2}A(0,0)$ of $F(t)$ (compare (16a)).[13] If any one of the modulating frequencies $pf_m$ with $p > 0$ is an exact multiple of $f_o$, it gives rise to an additional constant term. The relative amplitude line erected for this term multiplied by the amplitude of the modulating frequency gives a quantity which is a maximum possible absolute value of the term, which may not actually be realized. Such a constant term may be thought of as originating from a sinusoidal fluctuation, with that relative amplitude, which as $|f|$ went to 0 has been frozen in some particular phase (compare (17)).

AMPLITUDE NULLS

29. It is noted that for $\alpha = 0$ and $\beta = 1$, the frequencies $sf_o$ ($s = 1, 2, \ldots$) are absent. This appears since the curve of Plate 3 for $\alpha\beta/f_o = 0$ has nulls at the frequencies $sf_o/\beta$ (compare (28)), and with $\beta = 1$, the $sf_o$ frequency points from which the curve is entered coincide with these nulls. The amount by which these frequencies reappear if $\beta$ decreases from 1, so that the points $sf_o$ no longer coincide with the nulls, is indicated by the steepness of the curve at those points.[14] If $\alpha \neq 0$, so that $\alpha\beta/f_o \neq 0$, the curves no longer have nulls, but for small values of $|\alpha|$ they have pronounced minima occurring approximately at the points $sf_o/\beta$ with approximately the values $\beta|1 - e^{-\alpha\beta/f_o}|/((\alpha\beta/f_o)^2 + (2\pi s)^2)^{\frac{1}{2}}$ (compare (25)). These considerations are of interest for designing pulse lengtheners in the effort to eliminate pulse repetition frequencies.

ACKNOWLEDGMENTS

30. Thanks are due to Mrs. Jeanne Schwartz for preparing the plates and editing this report, and to Lt. Bertram Yood, USNR, for valuable criticisms.

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13 For any $g(t)$ of Paragraph 14, $w_o$ is positive.

14 For special values of s and the other quantities, a frequency $sf_o$ might be absent because of coincidence with $sf_o/\beta$ for a different value of s; or conceivably due to complete phase cancellation among terms of the same frequency for several values of p (Paragraph 26).

[redacted]

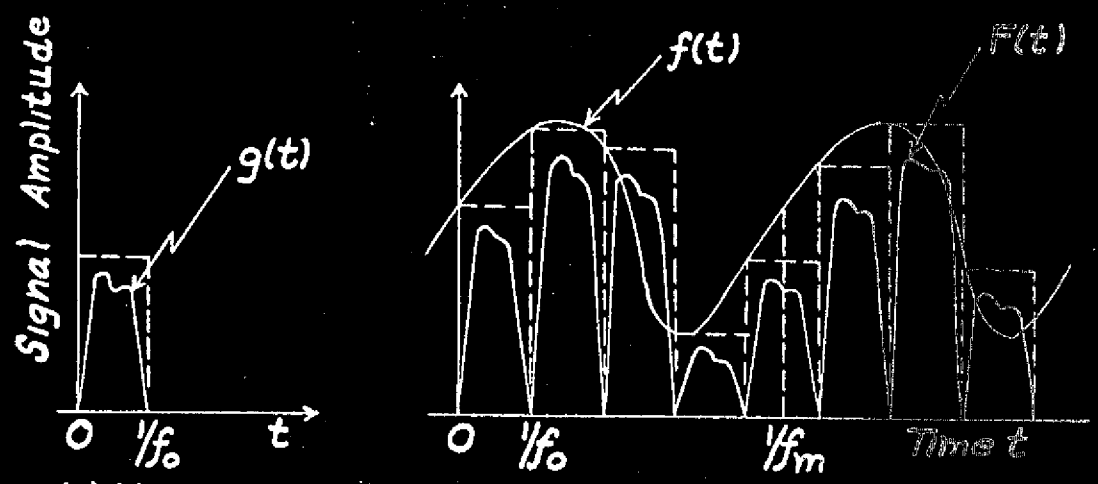
REFERENCES

31. (a) Radiation Laboratory Report No. S-10, "Detection of Propeller and Sambo Modulations," J. L. Lawson, Editor, dated 16 May 1944.

(b) NRL Report No. R-2561, "Accuracy of Range Tracking Through Window Aided by Echo Modulation with the Radar Mark 12," dated 13 July 1945.

[redacted]

## GENERAL CASE

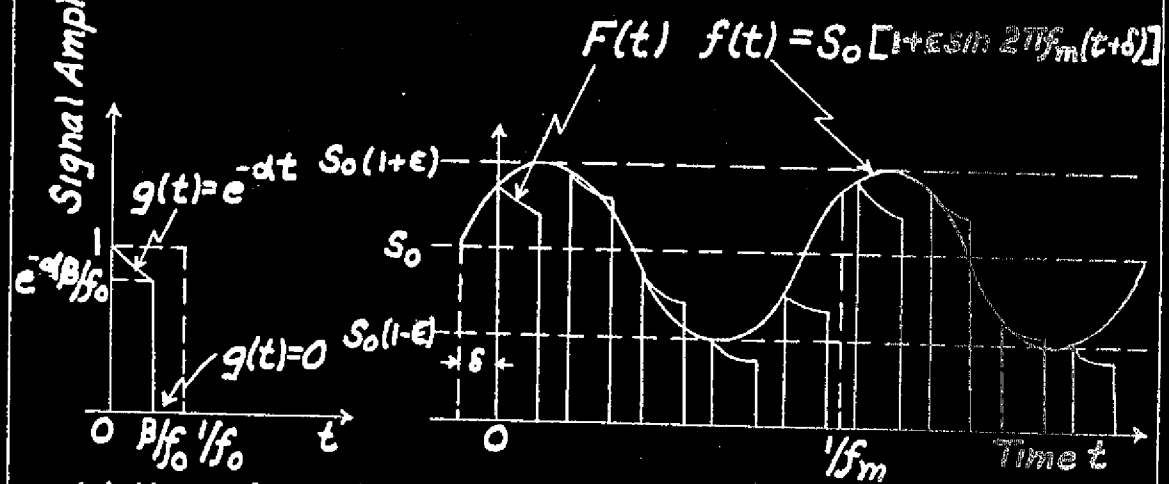



(a) Unmodulated Pulse g(t).

(b) Modulating Signal f(t), and Output F(t).

FIG. 1

## SPECIAL CASE



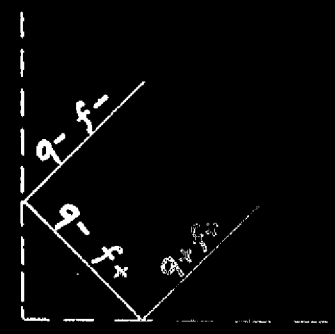

(a) Unmodulated Pulse g(t).

(b) Modulating Signal f(t), and Output F(t).

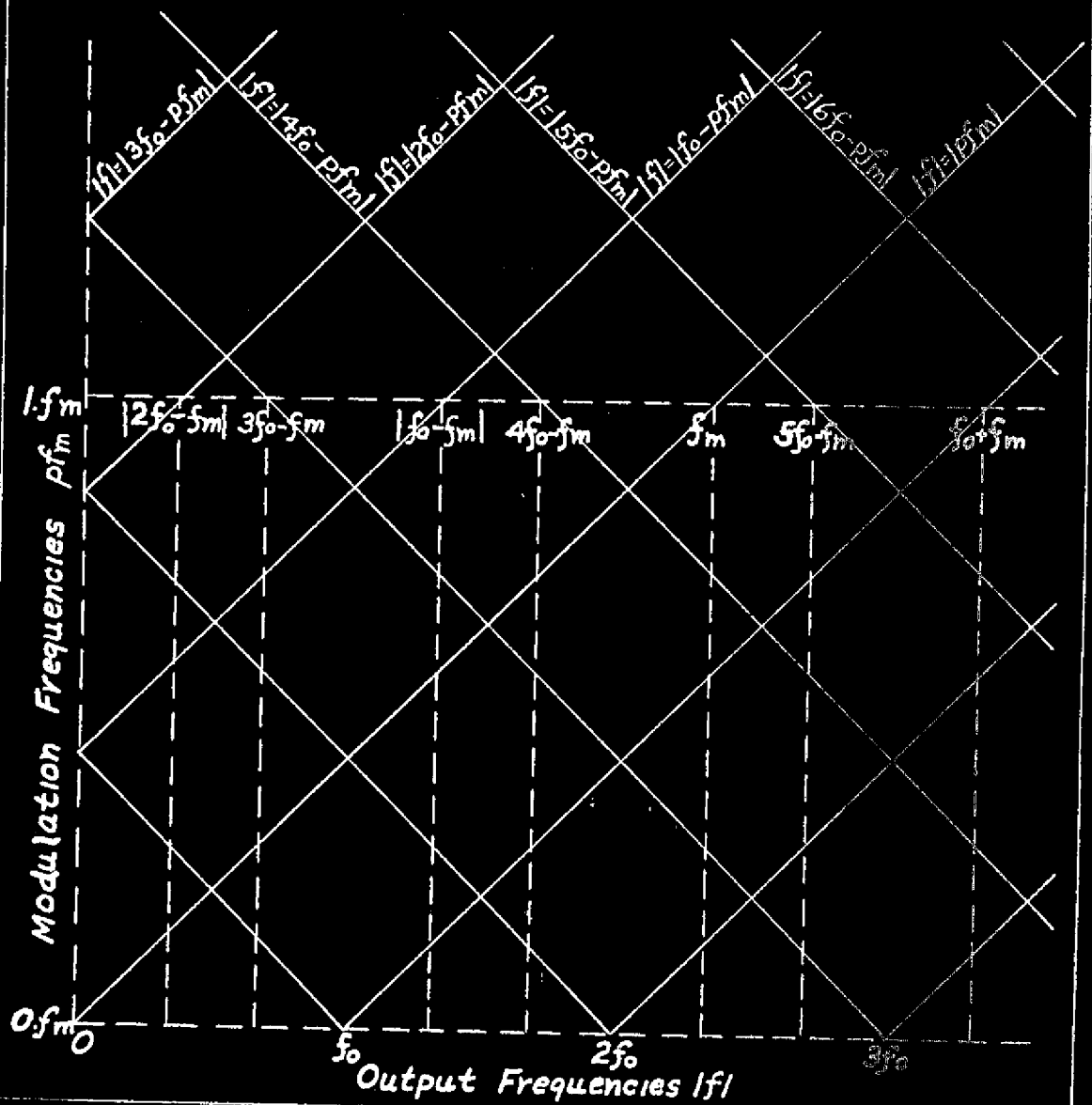
FIG. 2

| | | |
|---|---|---|
| | U.S.N.L. FIRE CONTROL DIVISION | PLATE 1 |
| | 26 APR. 45 R-2555 | |

# RELATIONSHIP BETWEEN INPUT AND OUTPUT FREQUENCIES




Signs of q and f




| | | |
|---|---|---|
| [illegible] | N.D.L. FIRE CONTROL DIVISION<br>J.R.S. 28 APR. 45 R-2555 | PLATE 2 |

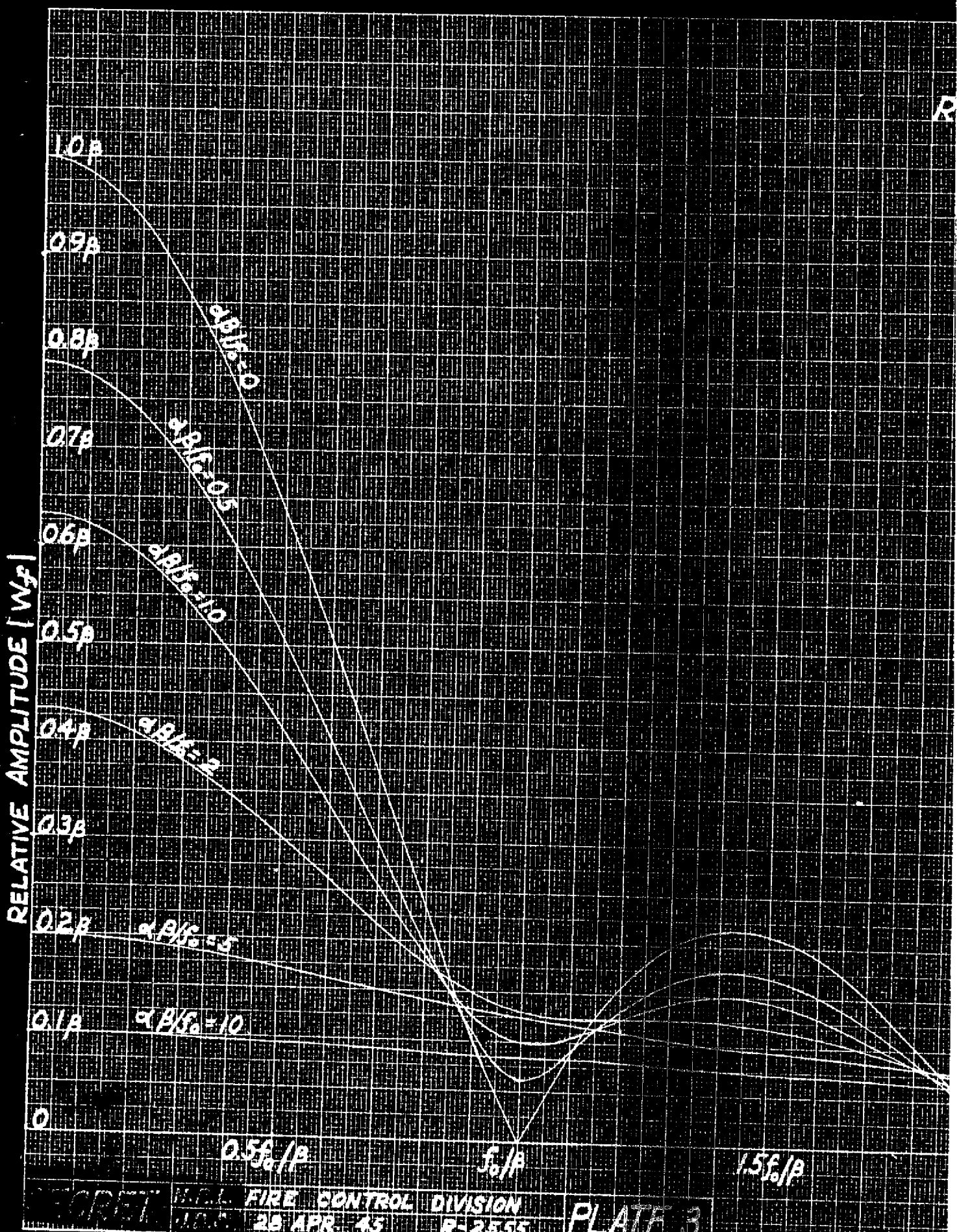



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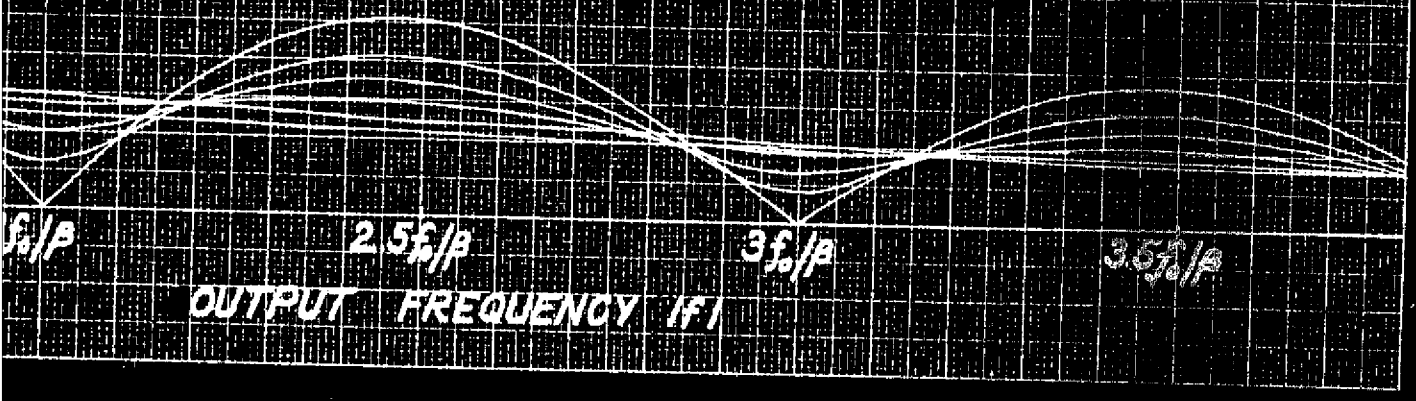
PLATE 3

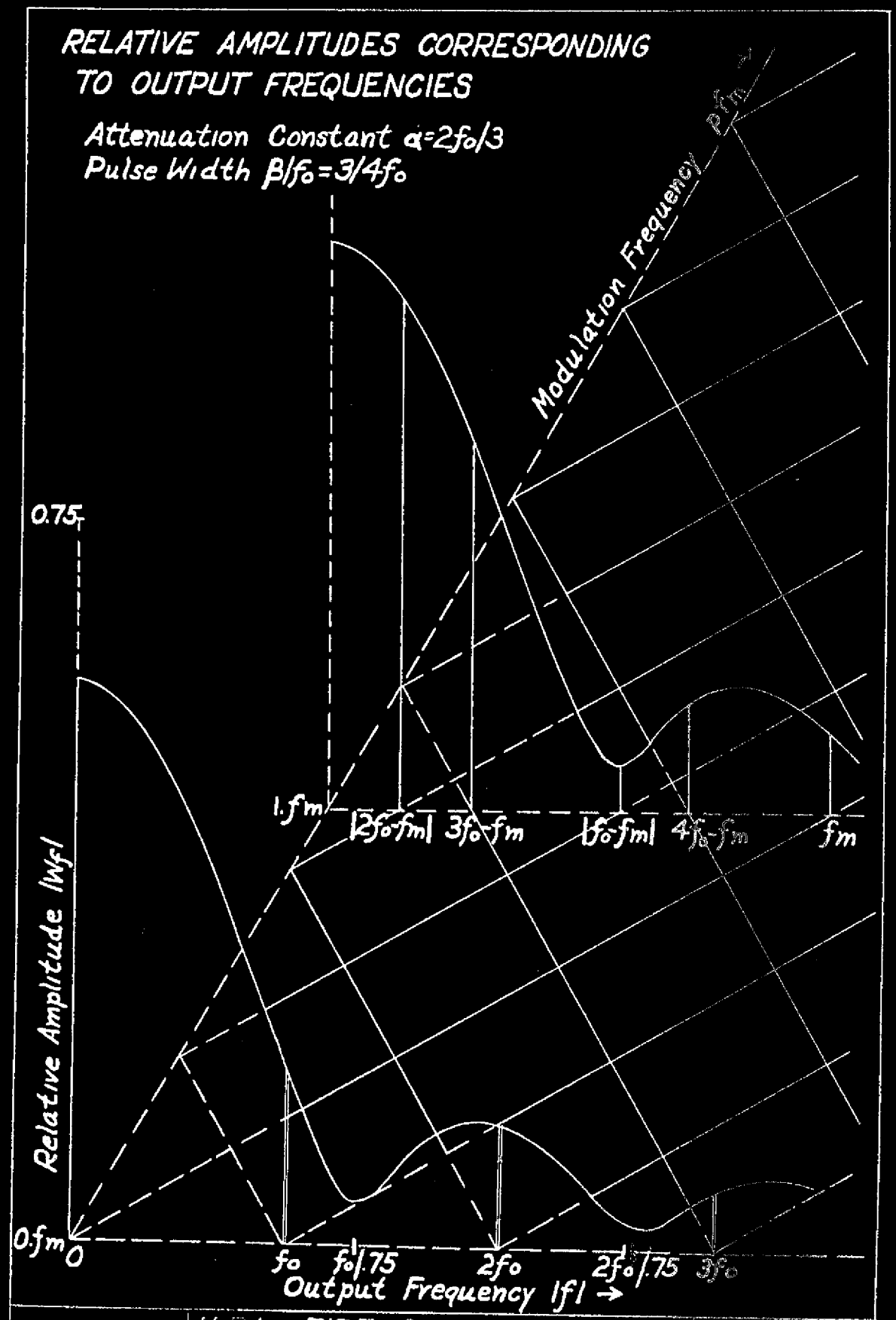
ATIVE AMPLITUDE V.S. OUTPUT FREQUENCY

$f_0$ = Generator Frequency

$\alpha$ = Attenuation Constant

$\beta$ = Fractional Pulse Width



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