NRL REPORT 4110

AN EXTENSION OF THE LUNEBERG-TYPE LENSES

J. E. Eaton

Antenna Research Branch
Radio Division I

February 16, 1953

NAVAL RESEARCH LABORATORY

WASHINGTON, D.C.

APPROVED FOR PUBLIC RELEASE
DISTRIBUTION UNLIMITED

ABSTRACT

The spherical lens first investigated by Luneberg, for which scanning throughout space without distortion is possible, has been generalized to permit the source to lie within the lens, thereby reducing the size of the path followed by the source in scanning. This, and other spherical lenses with complete spherical symmetry that simulate line sources and infinite plane reflectors, appear as special cases of an extension of the class of circularly symmetric circular lenses due to Luneberg. In general, a point source located within or on the boundary of a circular lens with variable index of refraction appears either as a virtual point source located at an arbitrary point within the lens or as a source whose energy is perfectly collimated in an arbitrary direction. Luneberg considered only the case in which the real and apparent sources were located at diametrically opposite points outside the lens.

PROBLEM STATUS

This is an interim report; work on the problem is continuing.

AUTHORIZATION

NRL Problem R09-54
RDB Project NR 509-540

Manuscript submitted January 6, 1953

UNCLASSIFIED

CONTENTS

INTRODUCTION	1
THE INTEGRAL EQUATION	5
INVERSION OF THE INTEGRAL EQUATION	7
THE INDEX OF REFRACTION	8
DETERMINATION OF THE CONSTANT	11
PLANE, CYLINDRICAL, AND SPHERICAL PHASE FRONTS	12
CONCLUSION	15

UNCLASSIFIED

AN EXTENSION OF THE LUNEBERG-TYPE LENSES

INTRODUCTION

A new catagory of lens types has become feasible in the microwave region through the development of dielectrics whose indices of refraction vary through the medium in an orderly fashion. The variation in the index has been obtained by controlling the density of the dielectric material or by loading a low-dielectric-constant material with suitably placed bits of a substance having a high dielectric constant. Two other techniques, applicable to the region between a pair of conducting plates, achieve a variation in the effective index of refraction by either altering the spacing between an essentially flat pair of plates or changing the curvature of the mean surface between a curved pair of parallel plates.

Basic theoretical work of Luneberg[1] on the optics in a variable medium has thus acquired realizations as microwave antennas. These antennas all stem from a perfectly focusing lens whose existence Luneberg exhibited. He showed that if a dielectric sphere has an index of refraction n that satisfies the relation

$$n^2 = 2 - \rho^2, \tag{1}$$

ρ being the distance from the center of the sphere normalized so that the radius is one, then all the energy entering the sphere at a point on its surface is focused in a diametrically opposite direction. Perfect scanning is then possible, for as the source moves on the surface of the sphere the resulting diffraction pattern undergoes a corresponding rotation without distortion.

The behavior of these "perfect" lenses is diagrammatized in Figure 1. Notable success has been obtained with two-dimensional lenses that couple the radiation characteristics of a line source and scanning throughout a plane without deterioration. Figure 2 is a schematic of one lens of this type designed by Peeler.[2] The plate spacing is varied with position in order to achieve the proper variation of n. Another lens (Figure 3) was first investigated by Rinehart.[3] Although two-dimensional in performance, it requires three dimensions for its construction. The surface is chosen to provide the proper path lengths for the optical rays which follow geodesics of the surface.

---

[1] Luneberg, R. K., "Mathematical Theory of Optics," Brown Univ. Lecture Notes, Providence, 1944

[2] Peeler, G. D. M., Archer, D. H., "A Two-Dimensional Microwave Luneberg Lens," NRL Report 4115 (to be published)

[3] Rinehart, R. F., "A Solution of the Rapid Scanning Problem for Radar Antennae," Journal Applied Physics, 19:860-862 (1948)

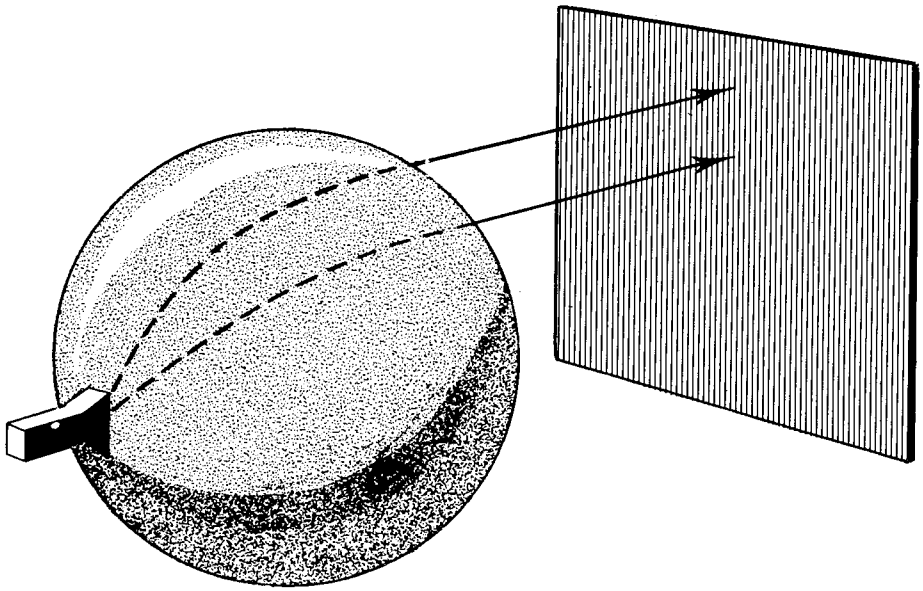

Figure 1 - The spherical lens of Luneberg

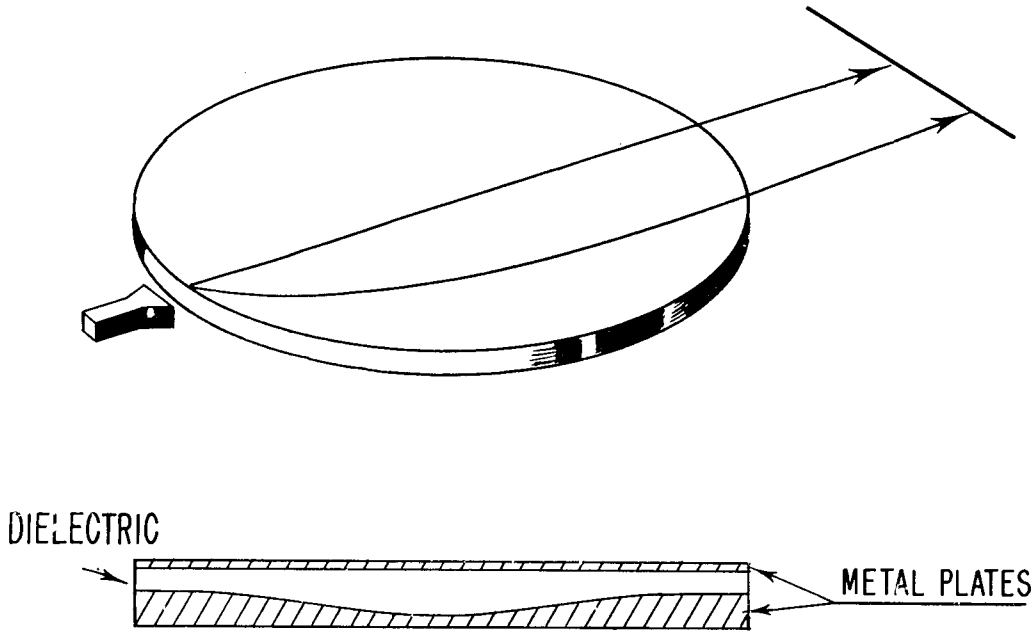

Figure 2 - A flat two-dimensional lens

UNCLASSIFIED

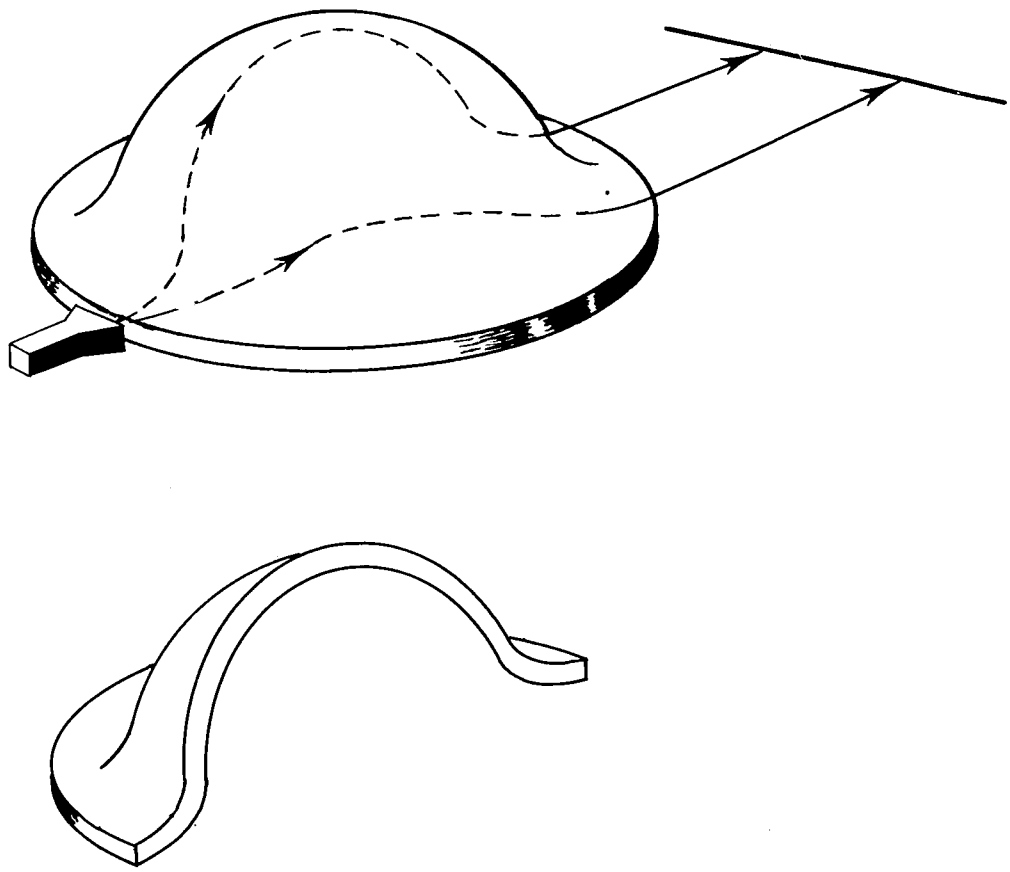

Figure 3 - A two-dimensional curved-surface lens

Mechanical complexities, however, attend any rapid scanning by such lenses due to the relatively large path the feed must follow, restricted as it is to the surface of the lens. Rinehart [4] has recognized the advantages of a reduction in size of the feed circle and has obtained that reduction by a virtual-image technique for his double-plate curved-surface version of Luneberg's lens. Although Rinehart's lens acts as a two-dimensional lens, energy in one plane only being collimated, a third dimension is essential for its construction and in particular for the technique he uses to reduce the radius of the feed circle.

A method for achieving a similar reduction in size, applicable to the general class of Luneberg-type lenses, may be of some interest and will be presented in the following pages. A formula will be obtained for the proper index of refraction of a spherical lens that focuses energy from a source inside the lens in a diametrically opposite direction. Although this result will appear as a special case of the solution of a more general problem, it is the result of greatest practical interest contained in this report. Of course, only a portion of space may be scanned since part of the lens must be removed to provide the source with freedom of motion (Figure 4). There is no such restriction for two-dimensional lenses, however, as the source may pierce the top or bottom surface of the plates.

[4] Rinehart, R. F., "A Family of Designs for Rapid Scanning Radar Antennas," Proc. I.R.E., 40:686-688 (1952)

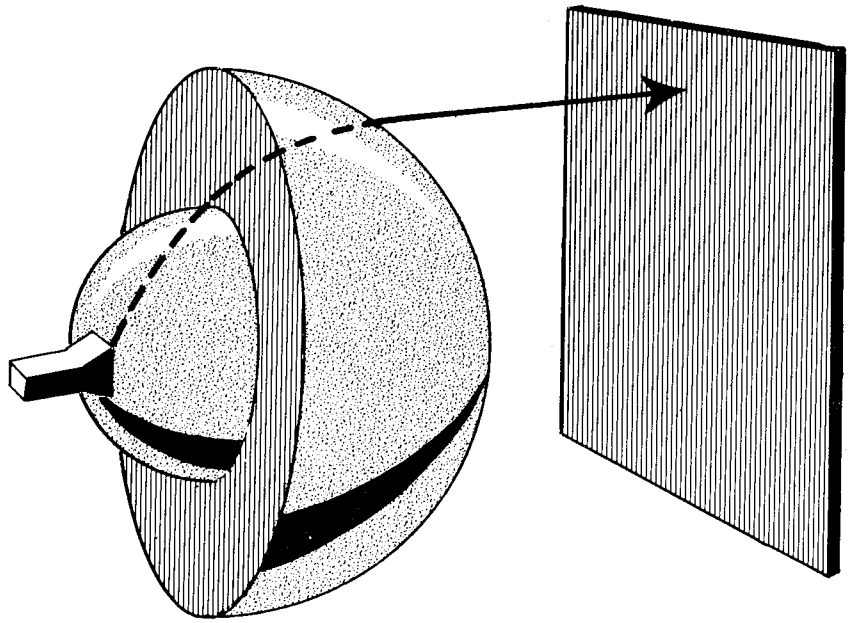

Figure 4 - Spherical lens with interior source

The problem was framed in greater generality to complement the formulation originally considered by Luneberg. It may be recalled that Luneberg showed that in any spherically symmetric medium the path followed by any optical ray lies in a plane. Thus the study of a spherical lens excited by a point source reduces to a two-dimensional problem provided that the index of refraction depends only on the distance from the center. A plane section of the sphere containing both the point of excitation and the center determines a circular lens with radial-varying index whose performance as an optical system in two-dimensional space completely describes the spatial performance of the spherical lens. The line through the center and the source is an axis of symmetry of each lens and a phase front formed by the spherical lens is generated by the corresponding phase front of the circular lens under rotation about the axis. The problem considered by Luneberg was the characterization of a circular lens such that rays from a point outside the lens, or on its circumference, would form contracting circular or linear phase fronts. The center of the phase front was to lie outside the lens, on the opposite side from the source, and on the line passing through the center of the lens and the source. The linear phase front was obtained as a limiting case of circular phase fronts. (Luneberg's description was not phrased in terms of phase fronts.)

The extensions to be made in the present paper permit the source to lie within the lens for the generation of linear phase fronts or the center of circular phase fronts to lie within the lens when excited by a source on the boundary of the lens. The center of the circular phase fronts is not required to lie on the line through the source and the center of the lens, nor is there a similar restriction on the orientation of the linear phase front. This implies that unless a suitably directive source is used, there is a second set of phase fronts present for the two-dimensional lens so placed to preserve the symmetry of the lens-source system. The corresponding three-dimensional lens would generate toroidal phase fronts of little physical interest save for the special cases of planes, spheres, and cylinders.

The methods used to obtain a solution are somewhat different from those of Luneberg. Attention is directed to the phase fronts desired rather than to the paths followed by optical rays. An integral equation expresses the condition that the optical path length from the

UNCLASSIFIED

source to the desired phase front is the same for all rays. A solution of the integral equation yields the proper index of refraction to insure that the desired phase front is indeed a true phase front. The orientation of the phase front is determined by the path followed by a particular ray. The three special cases noted above are then examined in greater detail.

THE INTEGRAL EQUATION

Suppose, for convenience, that the circular lens to be examined has unit radius. Let a polar coordinate system be imposed with the pole at the center of the circle and the source lying on the negative polar axis at P_0 a distance ρ_0 from the pole (Figure 5). In the general discussion ρ_0 will not exceed one and, for circular phase fronts, only the case $\rho_0 = 1$ will be considered fully. The index of refraction n is a function of ρ alone, being independent of the angular coordinate ϑ. It is assumed that n is continuous at all points except for isolated singularities and has the value one outside the circle. Let P_3 be the point where an optical ray from P_0 emerges from the lens and denote by τ the acute angle between the ray and the radial line at P_3.

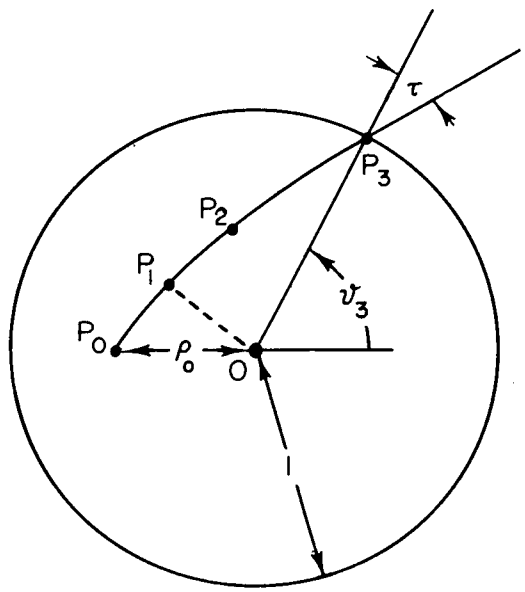

Figure 5 - The geometry of the circular lens

If circular phase fronts are to be generated, let $\overline{P}\,(\bar{\rho},\bar{\vartheta})$ be their center with $\bar{\rho} \leqslant 1$ (Figure 6). Consider the particular phase front that lies on a portion of the circle that is an exterior tangent to the lens at, say, T. Let Q be the point on it pierced by the ray from P_3. Simple trigonometry shows the distance $\overline{P}P_3$ to be $\cos\tau \pm \sqrt{\cos^2\tau - 1 + \bar{\rho}^2}$. The ambiguity in sign may be resolved by specifying the portion of the circle on which the phase front is to lie. If it is on part of the semicircle RTS, whose midpoint is T, the positive sign is chosen, while the complementary semicircle RT'S corresponds to the negative sign before the radical. The distance P_3Q, that is, the optical length from P_3 to Q, since in this region n = 1, is

$$P_3Q = 1 + \bar{\rho} - \cos\tau \mp \sqrt{\cos^2\tau - 1 + \bar{\rho}^2} \qquad (2)$$

The optical length from P_0 to P_3 may be expressed as

$$P_0 P_3 = \int_{P_0}^{P_3} n(\rho)\, ds, \tag{3}$$

the line integral being evaluated along an extremal.

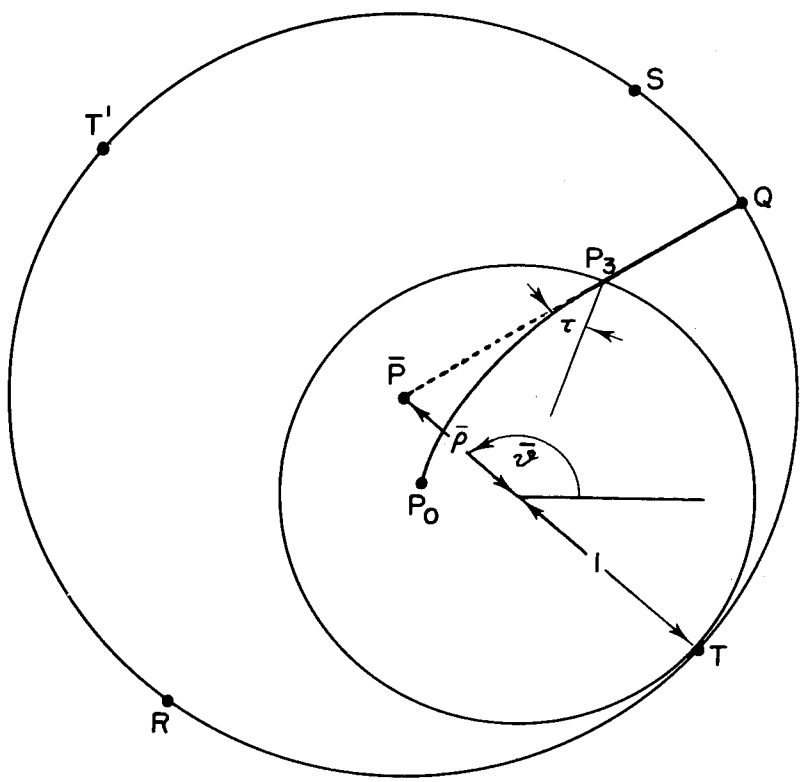

Figure 6 - Circular phase fronts

The optical length from P_0 to the desired phase front must be the same for all permissible rays and if it is the same, the desired phase front will be the true phase front for those rays. An interpretation of this requirement in terms of the path lengths given in Equations (2) and (3) leads to the integral equation

$$\int_{P_0}^{P_3} n(\rho)\, ds = \frac{k\pi}{2} + \cos\tau \pm \sqrt{\cos^2\tau - 1 + \bar{\rho}^2} \tag{4}$$

The constant $k\pi/2$ (written in that form for later convenience) has absorbed both the constant value of the total path length and the constant $1 + \bar{\rho}$ of Equation (2).

A similar integral equation may be developed for the circular lens generating linear phase fronts (Figure 7). In this case, Equation (2) becomes

$$P_3 Q = 1 - \cos\tau \tag{5}$$

UNCLASSIFIED

while Equation (3) is formally the same. The integral equation is then

$$\int_{P_0}^{P_3} n(\rho)\, ds = \frac{k\pi}{2} + \cos\tau . \tag{6}$$

The same methods may be used for solving Equations (4) and (6) and so, for simplicity, they will be treated simultaneously as

$$\int_{P_0}^{P_3} n(\rho)\, ds = \frac{k\pi}{2} + \cos\tau + \eta \sqrt{\cos^2\tau - 1 + \bar{\rho}^2} . \tag{7}$$

The number η may have the value +1, -1, or 0, the first two corresponding to circular phase fronts and the last to linear phase fronts.

INVERSION OF THE INTEGRAL EQUATION

It is well known in the calculus of variations that an extremal of the integral of Equation (7) (whose integrand is without explicit dependence on ϑ) is a solution of

$$n\rho = \sin\tau \sqrt{1 + \left(\frac{\rho'}{\rho}\right)^2} \tag{8}$$

that passes through P_0. The appearance of $\sin\tau$ in Equation (8) insures that one of the two solutions through P_0 also passes through P_3 in the proper direction, the other solution being its symmetrical image with respect to the line $\vartheta = 0$. Of course, implicit in the derivation of Equation (8) is the assumption that ρ and ρ' are well-behaved functions of ϑ, the prime denoting differentiation with respect to ϑ. At an extremum of ρ, $\rho' = 0$ and the ray is perpendicular to a radial line. From the continuity of n and the circular symmetry of the lens, it is immediately apparent that the path of the ray and its extension through P_0 is symmetric with respect to this radial line. If the path had more than one extremum and hence more than one line of symmetry, it would be impossible for the ray to leave the lens.

Only those rays will be considered along which ρ assumes an extreme, necessarily minimum value at, say, P_1 (Figure 5). Let P_2 be the image of P_0 in the line OP_1. Since OP_1 is a line of symmetry of the ray,

$$\int_{P_0}^{P_1} n(\rho)\, ds = \int_{P_1}^{P_2} n(\rho)\, ds. \tag{9}$$

Equation (7) may then be written as

$$2\int_{P_1}^{P_2} n(\rho)\, ds + \int_{P_2}^{P_3} n(\rho)\, ds = \frac{k\pi}{2} + \cos\tau + \eta \sqrt{\cos^2\tau - 1 + \bar{\rho}^2} . \tag{10}$$

If only those lenses are sought for which $\rho\, n(\rho)$ is a properly monotonic function of ρ, then Equation (10) may be converted to an integral equation of the type considered by

Abel, namely,

$$\int_0^t \frac{f(v) + g(v)}{\sqrt{t - v}}\, dv = \frac{k\pi}{2} + \sqrt{t} + \eta\sqrt{t - 1 + \bar{\rho}^2}\ , \tag{11}$$

through the transformations

$$t = \cos^2 \tau\ , \tag{12}$$

$$v = 1 - n^2\rho^2\ , \tag{13}$$

$$f(v)\, dv + n^2\rho\, d\rho = 0\ , \tag{14}$$

$$g(v) = \begin{cases} f(v),\ 0 \leqslant \rho \leqslant \rho_0 \\ 0,\ \rho_0 < \rho \leqslant 1. \end{cases} \tag{15}$$

One way to invert Equation (11) is to regard it as a convolution. The Laplace transform of each member yields

$$\sqrt{\frac{\pi}{s}}\, L:[f(v) + g(v)] = \frac{k\pi}{2s} + \frac{1}{2s}\sqrt{\frac{\pi}{s}} + \eta\, \frac{1}{2s}\sqrt{\frac{\pi}{s}}\, e^{-(1-\bar{\rho}^2)s} \tag{16}$$

The inverse transform then shows

$$f(v) + g(v) = \frac{k}{2\sqrt{v}} + \frac{1}{2} + \frac{\eta}{2} S(v) \tag{17}$$

where S(v) is the step function

$$S(v) = \begin{cases} 0,\ v < 1 - \bar{\rho}^2 \\ 1,\ v > 1 - \bar{\rho}^2 \end{cases} \tag{18}$$

THE INDEX OF REFRACTION

The discontinuities of Equations (15) and (17) introduce some complexities for circular phase fronts. For this reason and also to achieve some simplification in identifying k with the orientation of the phase front, a solution will be carried through when $\eta \neq 0$ only if $\rho_0 = 1$. This implies that for circular phase fronts the source is to lie on the circumference of the lens. With this restriction, Equations (15), (17), and (18) may be combined to give for circular phase fronts ($\eta \neq 0$)

$$f(v) = \begin{cases} \dfrac{k}{4\sqrt{v}} + \dfrac{1+\eta}{4}\ ,\ 0 \leqslant n\rho \leqslant \rho\ ; \\ \dfrac{k}{4\sqrt{v}} + \dfrac{1}{4}\ , \qquad \bar{\rho} < n\rho \leqslant 1; \end{cases} \tag{19}$$

and for linear phase fronts ($\eta = 0$)

$$f(v) = \begin{cases} \dfrac{k}{4\sqrt{v}} + \dfrac{1}{4}\ , \qquad 0 \leqslant \rho \leqslant \rho_0\ ; \\ \dfrac{k}{2\sqrt{v}} + \dfrac{1}{2}\ , \qquad \rho_0 < \rho \leqslant 1. \end{cases} \tag{20}$$

UNCLASSIFIED

Equation (14) may be rewritten as

$$\frac{f(v)\,dv}{1-v} + \frac{d\rho}{\rho} = 0. \tag{21}$$

Its solutions for the various forms of f(v) given in Equations (19) and (20) are

$$f(v) = \frac{k}{4\sqrt{v}}, \quad a_1 = \rho^2\,[F(n\rho)]^k; \tag{22}$$

$$f(v) = \frac{k}{4\sqrt{v}} + \frac{1}{4}, \quad a_2 n = \rho\,[F(n\rho)]^k; \tag{23}$$

$$f(v) = \frac{k}{4\sqrt{v}} + \frac{1}{2}, \quad a_3 n^2 = [F(n\rho)]^k; \tag{24}$$

$$f(v) = \frac{k}{2\sqrt{v}} + \frac{1}{2}, \quad a_4 n = [F(n\rho)]^k; \tag{25}$$

where

$$F(n\rho) = \frac{1 + \sqrt{1 - n^2\rho^2}}{n\rho} \tag{26}$$

The constants a_i must be chosen to insure the continity of n.

A convenient parameterization of n and ρ may be introduced by

$$n\rho = \sin\phi \tag{27}$$

for with ϕ so defined

$$F(n\rho) = \cot\frac{\phi}{2}. \tag{28}$$

The indices of refraction then take the forms discussed below. For convenience there is also noted the relation between k and $\bar{\vartheta}$, the angular orientation of the phase front. The derivation of this relation appears in the following section.

Case 1. $\eta = 1$. A phase front forms part of the semicircle RTS (Figure 6) whose center is at $(\bar{\rho},\bar{\vartheta})$, $0 \leqslant \bar{\rho} \leqslant 1$. The source is at $(1,\pi)$.

$$\left.\begin{aligned} \rho &= \sin\phi\sqrt{\frac{\tan^k(\phi/2)}{\bar{\rho}}}, \\ n &= \sqrt{\bar{\rho}\cot^k(\phi/2)}, \end{aligned}\right\} 0 \leqslant \phi \leqslant \arcsin\bar{\rho}; \tag{29}$$

$$\left.\begin{aligned} \rho &= \sqrt{\sin\phi\tan^k(\phi/2)}, \\ n &= \sqrt{\sin\phi\cot^k(\phi/2)}, \end{aligned}\right\} \arcsin\bar{\rho} \leqslant \phi \leqslant \pi/2; \tag{30}$$

$$k = 2 - \frac{2}{\pi}\bar{\vartheta}. \tag{31}$$

Case 2. $\eta = -1$. A phase front forms part of the semicircle RT′S (Figure 6) whose center is at (ρ,ϑ), $0 \leqslant \rho \leqslant 1$. The source is at $(1,\pi)$.

$$\left.\begin{aligned} \rho &= \sqrt{\bar{\rho}\tan^k(\phi/2)}\,, \\ n &= \sin\phi\sqrt{\frac{\cot^k(\phi/2)}{\bar{\rho}}}\,, \end{aligned}\right\} \quad 0 \leqslant \phi \leqslant \arcsin\rho\,; \tag{32}$$

$$\left.\begin{aligned} \rho &= \sqrt{\sin\phi\tan^k(\phi/2)}\,, \\ n &= \sqrt{\sin\phi\cot^k(\phi/2)}, \end{aligned}\right\} \quad \arcsin\bar{\rho} \leqslant \phi \leqslant \pi/2\,; \tag{33}$$

$$k = 2 - \frac{2}{\pi}\,\bar{\vartheta}\,. \tag{34}$$

Case 3. $\eta = 0$. A phase front forms part of the line tangent to the lens at $(1, \bar{\vartheta})$ (Figure 7). The source lies at (ρ_0, π), $0 < \rho_0 \leqslant 1$.

$$\left.\begin{aligned} \rho &= \sqrt{\rho_0\sin\phi\tan^k(\phi/2)}, \\ n &= \sqrt{\frac{\sin\phi\cot^k(\phi/2)}{\rho_0}}\,, \end{aligned}\right\} \quad 0 \leqslant \rho \leqslant \rho_0\,; \tag{35}$$

$$\left.\begin{aligned} \rho &= \sin\phi\tan^k(\phi/2)\,, \\ n &= \cot^k(\phi/2)\,, \end{aligned}\right\} \quad \rho_0 \leqslant \rho \leqslant 1\,; \tag{36}$$

$$k = 1 - \frac{2}{\pi}\,\bar{\vartheta}\,. \tag{37}$$

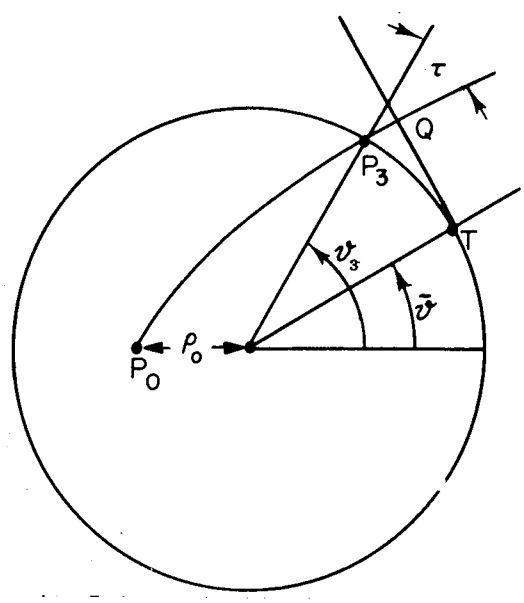

Figure 7 - Linear phase fronts

In each case, not all the rays leaving the source generate the phase front stated. If the region about the source be divided into quadrants as indicated in Figure 8 with $\psi = \arcsin\bar{\rho}$ for Cases 1 and 2 and $\psi = \pi/2$ for Case 3, only those rays leaving the source

UNCLASSIFIED

in the first quadrant contribute to the particular phase front considered unless $\bar{\vartheta}$ is 0 or π when rays from the fourth quadrant also are effective. Fourth quadrant rays form a phase front symmetrically placed with respect to the axis of the lens-source system (the line $\vartheta = 0$). Along rays emerging in the other two quadrants, either Equations (2) and (8) are incompatible in the real field or ρ does not reach a minimum value as required in Equation (9); hence the preceding analysis does not predict their behavior.

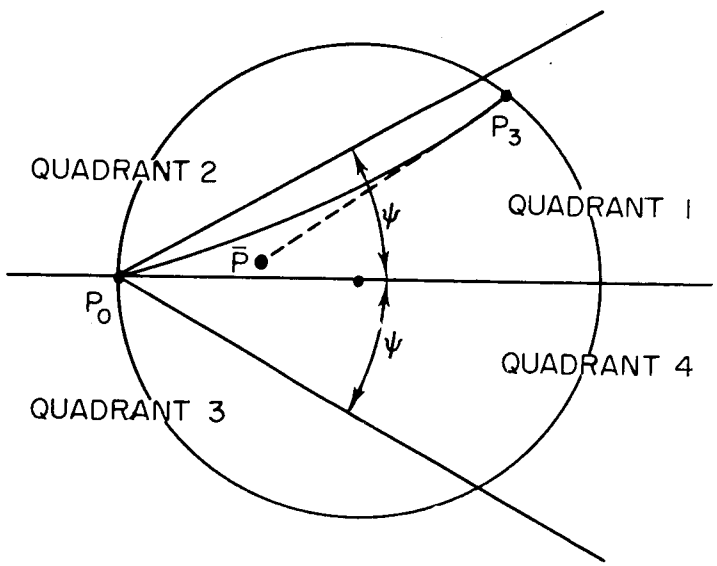

Figure 8 - Division of the rays into four quadrants

DETERMINATION OF THE CONSTANT

The value of k is most readily determined by an examination of a marginal ray. For Case 3, this is a ray leaving the source in a direction perpendicular to the polar axis. For the first two cases, it is a ray tangent to the circular interface between the two regions of the lens. When the parameter ϕ is introduced into Equation (8), the differential equation for the rays, there are slight differences that separate the circular and linear phase-front cases and it may be clearer to consider the two instances separately.

Cases 1 and 2. The ray lies entirely in the outer region where n is determined by Equations (30) or (33). Equation (8) becomes

$$2\, d\vartheta = - \frac{\sin \tau (\cot \phi + k \csc \phi)}{\sqrt{\sin^2 \phi - \sin^2 \tau}}\, d\phi . \tag{38}$$

The minus sign selects the marginal ray that leaves the source in the first quadrant. At the point of tangency of the interface, $\rho' = 0$ and Equation (8) implies that $\phi = \tau$ at that point. But the point of tangency is the midpoint of the ray (Figure 9), and there $\vartheta = \pi/2 + \vartheta_3/2$.

At P_3, $\vartheta = \vartheta_3$ and $\phi = \pi/2$. An integration of Equation (38) between these pairs of limits shows

$$\pi - \vartheta_3 = (k + 1)\,\pi/2 - \tau. \tag{39}$$

Since $\bar{\rho} = \sin\tau$, it follws that $\bar{\vartheta} = \vartheta_3 + \pi/2 - \tau$ and Equations (31) and (34) are thus immediate consequences of Equation (39).

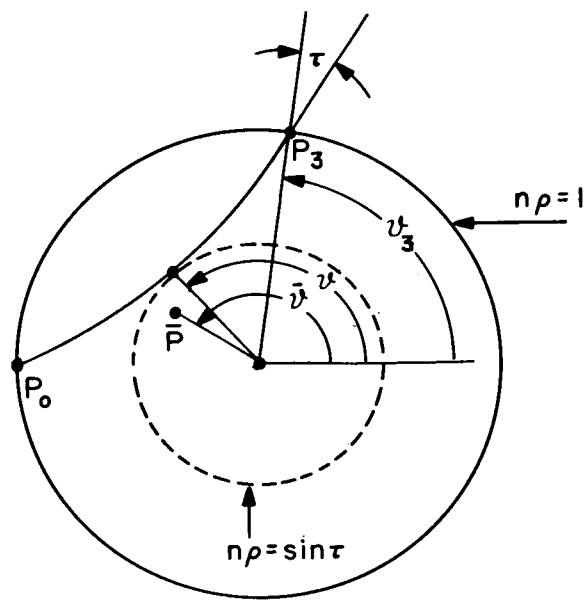

Figure 9 - The marginal ray for circular phase fronts

Case 3. Here again the ray lies entirely in the outer region where n is determined by Equation (36). Equation (8) is transformed into a differential equation identical with Equation (38) if the number 2 on the left side is replaced by a unit multiplier. When $\vartheta = \pi$, $\rho' = 0$ and hence from Equation (8), $\phi = \tau$. At P_3, $\vartheta = \vartheta_3$ and $\phi = \pi/2$. An integration of the modified differential equation shows

$$\pi - \vartheta_3 = (k + 1)\,\pi/2 - \tau. \tag{40}$$

The inclination of the normal to the phase front is $\bar{\vartheta} = \vartheta_3 - \tau$ (Figure 7), and Equation (37) follows directly from Equation (40).

PLANE, CYLINDRICAL, AND SPHERICAL PHASE FRONTS

As has been previously observed, the performance of a spherical lens may be determined from its circular analogue. The three-dimensional phase fronts in general lie on toroidal surfaces whose axis passes through the source and the center of the lens, and whose generators are the two-dimensional phase fronts already considered. There are three spherical lenses whose phase fronts are of especial physical interest and for which n may be expressed in simple algebraic terms.

In Case 3 if k = 1, then $\vartheta = 0$ and the linear phase front is perpendicular to the axis of symmetry (Figure 10). The phase front of the corresponding spherical lens lies on a

UNCLASSIFIED

plane. Equations (35) and (36) may be replaced by

$$n = \sqrt{\frac{2\rho_0 - \rho^2}{\rho_0^2}}, \quad 0 \leqslant \rho \leqslant \rho_0 ; \tag{41}$$

$$n = \sqrt{\frac{2 - \rho}{\rho}}, \quad \rho_0 \leqslant \rho \leqslant 1 . \tag{42}$$

This reduces to Equation (1), the limiting case considered by Luneberg, when $\rho_0 = 1$. An integration of Equation (8) using these values of n shows that the paths followed by the rays lie on the ellipses

$$\rho^2 = \frac{\rho_0 \sin^2 \tau}{1 - \cos\tau \cos(2\vartheta - \vartheta_2)}, \quad 0 \leqslant \rho \leqslant \rho_0 ; \tag{43}$$

and

$$\rho = \frac{\sin^2 \tau}{1 - \cos\tau \cos\vartheta}, \quad \rho_0 \leqslant \rho \leqslant 1. \tag{44}$$

The value of ϑ_2 is chosen to insure continuity at $\rho = \rho_0$ and is given by

$$\cos\vartheta_2 = \frac{\rho_0 - \sin^2 \tau}{\rho_0 \cos\tau} \tag{45}$$

The circular aperture of the lens is determined by the points where the marginal rays pierce the plane phase front. Its diameter is

$$D = 2\sqrt{2\rho_0 - \rho_0^2} \tag{46}$$

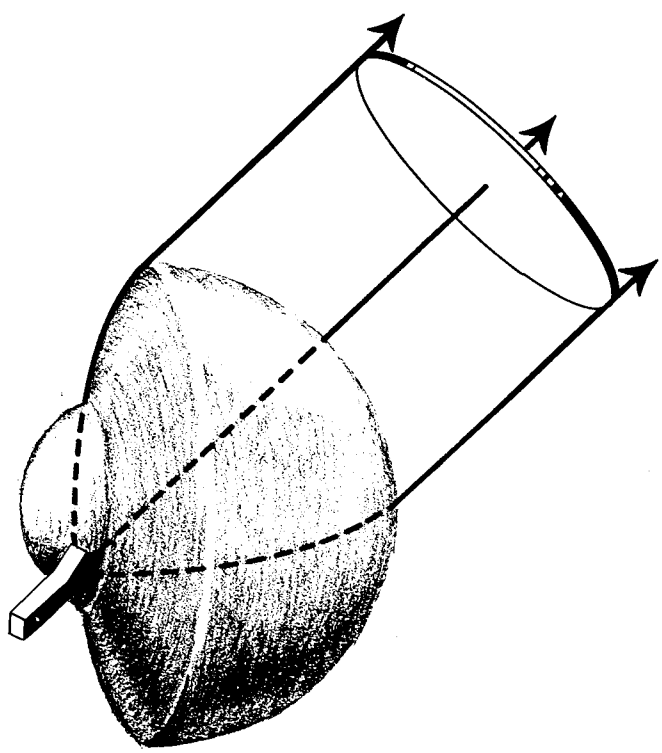

Figure 10 - Perfectly focusing, spherically symmetric lens capable of scanning one-seventh of space ($\pm 45^0$). The radius of the feed path is one-third the diameter of the aperture.

A cylindrical phase front (Figure 11) may be obtained from Case 3 if k = 0, for then $\bar{\vartheta} = \pi/2$. There is no loss in generality in taking $\rho_0 = 1$, for Equations (36) reduce to n = 1. Equations (35) become

$$n = \rho. \tag{47}$$

The paths followed by the rays lie on the equilateral hyperbolas

$$\rho^2 = \sin \tau \csc (\tau \pm 2\vartheta). \tag{48}$$

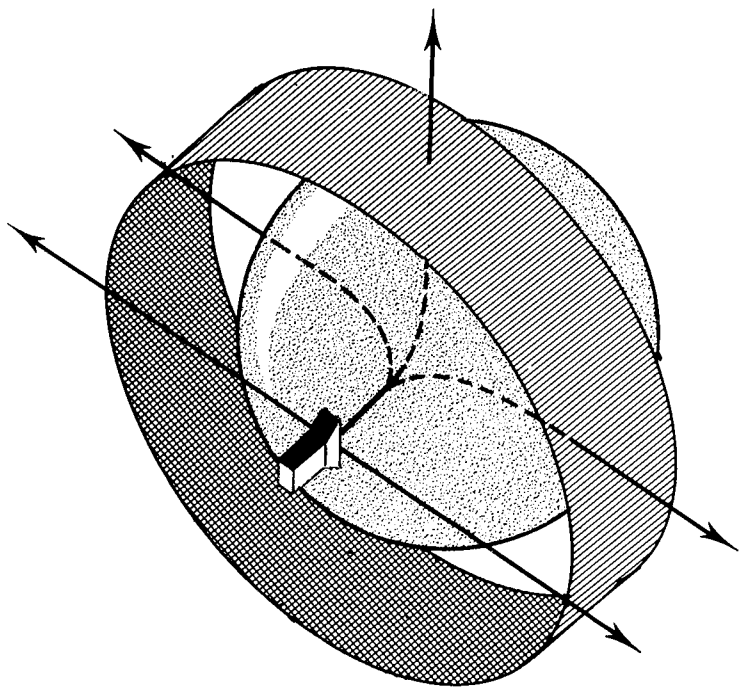

Figure 11 - Spherical lens that converts a point source into an apparent line source

A lens that essentially reverses the direction of the rays from a point source while maintaining spherical phase fronts (Figure 12) may be obtained from Case 1 with $\bar{\vartheta} = 0$. If, for simplicity, $\bar{\rho} = 1$ then from Equations (29)

$$n = \sqrt{\frac{2-\rho}{\rho}} \tag{49}$$

[cf. Equation (42)]. The paths followed by the rays lie on the ellipses

$$\rho = \frac{\sin^2 \tau}{1 + \cos \tau \cos (\vartheta \pm \tau)}. \tag{50}$$

UNCLASSIFIED

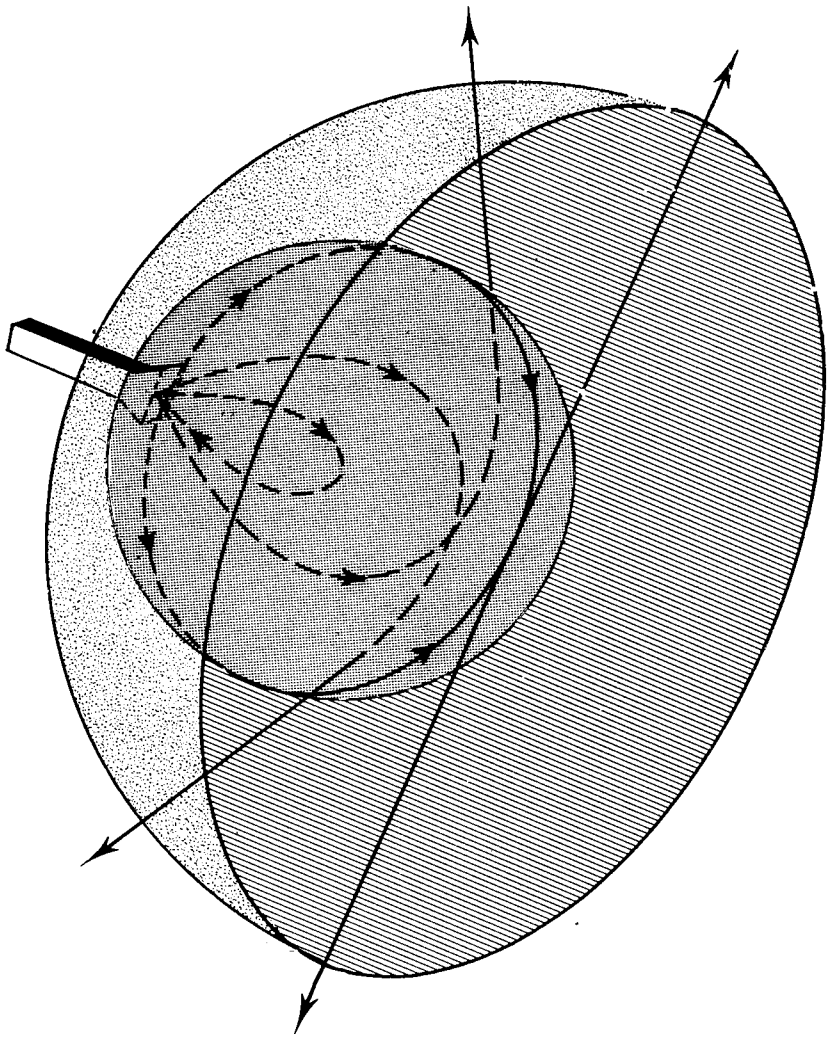

Figure 12 - Spherical lens that acts as an infinite reflecting plane

CONCLUSION

The methods developed in the preceding discussion may be applied to arbitrary normal congruences if a suitable phase front of the congruence is selected and the phase front is properly identified. For such generality, however, the determination of the orientation of the phase front is accomplished differently.

* * *

DPPO PRNC. WASH., D.C.