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NAVIGATION BY MOON DOPPLER EFFECT

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CONTENTS

Abstract ii
Problem Status ii
Authorization ii

INTRODUCTION 1

DOPPLER EQUATIONS 1

MOTION OF A POINT ON EARTH RELATIVE TO THE MOON 2

CALCULATION OF LATITUDE AND LONGITUDE OF A VESSEL 3

PROBABLE ERROR ANALYSIS 5

NUMERICAL EXAMPLE 6

CONCLUSIONS 9

ACKNOWLEDGMENTS 9

REFERENCES 9

APPENDIX A - List of Parameters and their Probable Errors 10

APPENDIX B - Variation in Angular Velocity of Moon 12

APPENDIX C - Derivation of $\varepsilon_{\Delta f}$ and $\varepsilon_{\dot{\Delta} f}$ 13

APPENDIX D - Formulas and Curves for $\varepsilon_{\Delta f}$ and $\varepsilon_{\dot{\Delta} f}$ as Functions of ε_{L_R} and ε_{H_R} 18

APPENDIX E - Residual Error in H_R and L_R Assuming $\varepsilon_{\Delta f}$ and $\varepsilon_{\dot{\Delta} f}$ Equal to Zero 21

ABSTRACT

As a back-up navigation method for Polaris submarines, the feasibility of moon doppler navigation on a bistatic basis is investigated. The problem is that of finding the latitude and longitude of a vessel by measuring doppler shift and time rate of change of doppler shift in radio waves transmitted from a fixed site and reflected from the moon.

For a prescribed navigation accuracy, one must determine the necessary precision in measuring doppler frequency and time rate of change of doppler frequency. A sample calculation is made for a location on the western edge of the Norwegian Sea, and it is found that if the probable error in latitude is specified to be 0.8 naut mi and the probable error in longitude to be 1.3 naut mi, the probable error which can be allowed in doppler shift is 0.796 cps and the permissible probable error in time rate of change of doppler shift is 9.511×10^{-5} cpsps, with a transmitted frequency of 2200 Mc.

The calculated value for doppler frequency was 2459 cps, and that for doppler rate was 0.1381 cpsps. This means that doppler frequency must be measured to an accuracy of 0.0324 percent and doppler rate to an accuracy of 0.0689 percent, for this particular example. The measurement of doppler frequency seems within the realm of technical capability in the near future. The accuracy of the measurement of doppler rate is still in question. However, it appears that the rate can be computed within the required accuracy by fitting an approximate analytical expression to the set of points representing the measured doppler shift and differentiating the expression, or by using finite difference methods with the observed data to get the time derivative.

Propagation effects due to variation of index of refraction in the atmosphere have not been taken into account. These effects may be important, and should probably be the subject of investigation in the near future.

PROBLEM STATUS

This is an interim report on one phase of the problem. Work is continuing.

AUTHORIZATION

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INTRODUCTION

It is desirable that a back-up navigation system be provided for the Polaris submarine, to supplement the basic ship inertial navigation system (SINS). In this report equations are derived which show how the latitude and longitude of a vessel can be obtained from measurements of doppler shift and time rate of change of doppler shift in electromagnetic waves emanating from a source at a known fixed location and reflected from the moon. Further equations are derived which show the accuracy of measurement of these two quantities required to obtain prescribed accuracies in latitude and longitude.

DOPPLER EQUATIONS

The classical equations for the doppler shift for a moving source and for a moving observer are as follows (1):

1. For a moving source

$$f_R = \frac{f_T}{1 - \frac{v_T}{c}} \tag{1}$$

where

f_T = radiated frequency

f_R = frequency received by observer

v_T = velocity of source

c = velocity of propagation of radiation.*

Here v_T is taken as positive when the source moves toward the observer, since the observer then receives a higher frequency.

2. For a moving observer

$$f_R = f_T \left(1 + \frac{v_R}{c}\right) \tag{2}$$

where v_R = velocity of observer. Here v_R is taken as positive when the observer is moving toward the source.

*Changes in velocity of propagation through the atmosphere have not been taken into account.

3. Consequently, when both observer and source are in motion,

$$f_R = f_T \frac{1 + \frac{v_R}{c}}{1 - \frac{v_T}{c}} . \tag{3}$$

Since c is very large with respect to the other velocities involved,

$$f_R \cong f_T \left(1 + \frac{v_R}{c}\right)\left(1 + \frac{v_T}{c}\right)$$

$$\cong f_T \left(1 + \frac{v_R + v_T}{c}\right) . \tag{4}$$

Suppose that upon the earth, considered stationary, a transmitter radiates a signal of frequency f_T and this signal is reflected by the moon. If the moon is moving at a speed v_T with respect to the transmitter site, the reflected signal will have a frequency

$$F_M = f_T \left(1 + \frac{v_T}{c}\right) . \tag{5}$$

If the reflected signal is received at some point on the earth, the moon may now be considered to be a source, in motion toward the observer, and radiating at frequency F_M. We have

$$F_R = \frac{F_M}{1 - \frac{v_R}{c}} \cong F_M \left(1 + \frac{v_R}{c}\right) \tag{6}$$

or

$$F_R = f_T \left(1 + \frac{v_T}{c}\right)\left(1 + \frac{v_R}{c}\right) \tag{7}$$

where F_R is the received frequency and v_R is the speed of the moon with respect to the receiver. Then the doppler shift is

$$\Delta f = F_R - f_T . \tag{8}$$

Equations (7) and (8) combined give

$$\Delta f \cong \frac{f_T}{c} (v_R + v_T) . \tag{9}$$

MOTION OF A POINT ON EARTH RELATIVE TO THE MOON

Trotter (2) has derived a good approximation for the motion of a point on the earth relative to the moon. This equation is, for the receiving site,

$$v_R = \omega R \cos \delta \cos L_R \sin H_R \tag{10}$$

where

ω = angular velocity of the earth's rotation

R = equatorial radius of the earth

δ = declination of the moon

L_R = latitude of receiving site

H_R = hour angle of receiving site.

Similarly, for the transmitting site,

$$v_T = \omega R \cos \delta \cos L_T \sin H_T \tag{11}$$

where

L_T = latitude of transmitting site

H_T = hour angle of transmitting site.

The purpose of employing these expressions for v_R and v_T is to obtain Δf by substituting in Eq. (9). The transmitted frequency f_T is known. The value of ω is computed from the length of the sidereal day, which is given by Smart (3) as $23^h\ 56^m\ 4^s.091$.

Smart may also be consulted for the definition of the astronomical terms used. The earth's radius and the coordinates of the transmitting site are taken from Ref. 2 and reproduced in Appendix A. In addition, this reference gives values and probable errors for all the quantities involved, except L_R and H_R. The declination δ for any date and time can be obtained by consulting the American Nautical Almanac.

Strictly speaking, the value of ω used should be the difference between ω for the earth and ω for the moon, which we can call ω_E and ω_M. The value of ω_M for a circular lunar orbit is 2.661697 x 10^{-6} rad/sec, which is only 3.56 percent of ω_E. The variation in ω_M due to the ellipticity of the lunar orbit amounts to 10.9 percent of ω_M. Hence ω_M varies from 3.25 to 4.05 percent of ω_E. Therefore, ω_M is neglected (see Appendix B).

CALCULATION OF LATITUDE AND LONGITUDE OF A VESSEL

Let us proceed to derive expressions for L_R and H_R based on measurements of doppler shift and time rate of change of doppler shift. For convenience, we group the necessary previous equations

$$\Delta f = \frac{f_T}{c} (v_R + v_T) \tag{9}$$

$$v_R = \omega R \cos \delta \cos L_R \sin H_R \tag{10}$$

$$v_T = \omega R \cos \delta \cos L_T \sin H_T. \tag{11}$$

Then

$$\Delta f = \frac{f_T\ \omega\ R \cos \delta}{c}\left(\cos L_R \sin H_R + \cos L_T \sin H_T\right) \tag{12}$$

and

$$\dot{\Delta f} = \frac{f_T\ \omega^2\ R \cos \delta}{c}\left(\cos L_R \cos H_R + \cos L_T \cos H_T\right). \tag{13}$$

The extra ω in Eq. (13) appears because

$$\omega = \frac{dH_R}{dt} = \frac{dH_T}{dt}. \tag{14}$$

Let

$$A = \frac{c\ \Delta\ f}{f_T\ \omega\ R \cos \delta} \tag{15}$$

$$B = \cos L_T \sin H_T \tag{16}$$

$$D = \frac{c\ \dot{\Delta f}}{f_T\ \omega^2\ R \cos \delta} \tag{17}$$

$$E = \cos L_T \cos H_T. \tag{18}$$

Then Eqs. (12) and (13) can be written

$$A - B = \cos L_R \sin H_R \tag{19}$$

$$D - E = \cos L_R \cos H_R. \tag{20}$$

Hence

$$\tan H_R = \frac{A - B}{D - E}$$

$$= \frac{\omega\ c\ \Delta\ f - f_T\ \omega^2\ R \cos \delta \cos L_T \sin H_T}{c\ \dot{\Delta f} - f_T\ \omega^2\ R \cos \delta \cos L_T \cos H_T}. \tag{21}$$

The expression for $\cos L_R$ could be obtained by squaring Eqs. (19) and (20) and adding, but a simpler expression results when Eq. (2) is used to get L_R in terms of H_R:

$$\cos L_R = \frac{D - E}{\cos H_R}. \tag{22}$$

When $\cos H_R$ and $D - E$ are negative (as in the numerical example used later) Eq. (22) can be written

$$\cos L_R = \frac{E - D}{|\cos H_R|}. \tag{23}$$

Using the identity

$$\cos L_R = \left(1 + \tan^2 H_R\right)^{-1/2} \tag{24}$$

and Eqs. (17) and (18), Eq. (23) becomes

$$\cos L_R = \left(1 + \tan^2 H_R\right)^{1/2} \left(\cos L_T \cos H_T - \frac{c\ \dot{\Delta f}}{f_T\ \omega^2\ R \cos \delta}\right). \tag{25}$$

The relation between the longitude and the local hour angle H_R of the moon is

$$\mathrm{GHA} - H_R = \text{longitude} \tag{26}$$

where GHA is the Greenwich hour angle of the moon, obtainable from the American Nautical Almanac for any particular date and time. Therefore, if we know H_R, we know the longitude of the vessel.

Since the tangent and the cosine are periodic functions, the results given by Eqs. (21) and (25) are not unique in the purely mathematical sense. However, since it may be expected that latitude is known within 90 degrees and longitude within 180 degrees, the answers are ~~unique~~ for navigational purposes. Equations (21) and (25) are quite simple in form; hence, it should not be difficult to design a computer to obtain specific numerical results.

PROBABLE ERROR ANALYSIS

It is now necessary to make a probable error analysis to find out how accurate doppler and doppler-rate measurements have to be to provide a specified accuracy in latitude and longitude. From Eq. (12),

$$\Delta f = \frac{f_T\ \omega\ R \cos \delta}{c} \left(\cos L_R \sin H_R + \cos L_T \sin H_T\right) \tag{27}$$

and, from Eq. (13),

$$\dot{\Delta f} = \frac{f_T\ \omega^2\ R \cos \delta}{c} \left(\cos L_R \cos H_R + \cos L_T \cos H_T\right). \tag{28}$$

If the probable error in a quantity x is designated as $\mathcal{E}_x$, then (2a)

$$(\mathcal{E}_{\Delta f})^2 = \left(\frac{\partial \Delta f}{\partial f_T} \mathcal{E}_{f_T}\right)^2 + \left(\frac{\partial \Delta f}{\partial c} \mathcal{E}_c\right)^2 + \left(\frac{\partial \Delta f}{\partial \omega} \mathcal{E}_\omega\right)^2 + \left(\frac{\partial \Delta f}{\partial R} \mathcal{E}_R\right)^2 + \left(\frac{\partial \Delta f}{\partial \delta} \mathcal{E}_\delta\right)^2$$

$$+ \left(\frac{\partial \Delta f}{\partial L_R} \mathcal{E}_{L_R}\right)^2 + \left(\frac{\partial \Delta f}{\partial H_R} \mathcal{E}_{H_R}\right)^2 + \left(\frac{\partial \Delta f}{\partial L_T} \mathcal{E}_{L_T}\right)^2 + \left(\frac{\partial \Delta f}{\partial H_T} \mathcal{E}_{H_T}\right)^2 \tag{29}$$

and

$$(\mathcal{E}_{\Delta \dot{f}})^2 = \left(\frac{\partial \Delta \dot{f}}{\partial f_T} \mathcal{E}_{f_T}\right)^2 + \left(\frac{\partial \Delta \dot{f}}{\partial c} \mathcal{E}_c\right)^2 + \left(\frac{\partial \Delta \dot{f}}{\partial \omega} \mathcal{E}_\omega\right)^2 + \left(\frac{\partial \Delta \dot{f}}{\partial R} \mathcal{E}_R\right)^2 + \left(\frac{\partial \Delta \dot{f}}{\partial \delta} \mathcal{E}_\delta\right)^2$$

$$+ \left(\frac{\partial \Delta \dot{f}}{\partial L_R} \mathcal{E}_{L_R}\right)^2 + \left(\frac{\partial \Delta \dot{f}}{\partial H_R} \mathcal{E}_{H_R}\right)^2 + \left(\frac{\partial \Delta \dot{f}}{\partial LT} \mathcal{E}_{L_T}\right)^2 + \left(\frac{\partial \Delta \dot{f}}{\partial H_T} \mathcal{E}_{H_T}\right)^2 . \tag{30}$$

Appendix A lists the values of the parameters and their probable errors.

NUMERICAL EXAMPLE

For the purpose of carrying out a sample computation, let us arbitrarily choose the location of the receiving site (Polaris vessel) at the western edge of the Norwegian Sea, at latitude $L_R = 70°$N and longitude = 10° W. The longitude is considered positive if the point considered is west of the Greenwich meridian. Thus we may write the longitude mentioned above as +10°. The relation between the longitude and the local hour angle H_R of the moon is, as mentioned before,

$$\text{GHA} - H_R = \text{longitude}. \tag{26}$$

For the sample calculation Trotter's data for July 7, 1956, at 1900 hours Universal Time are used. At this time GHA = 110°11′.1 (see Ref. 2b or consult the American Nautical Almanac). Hence we have the equation

$$110°11'.1 - H_R = +10° \tag{31}$$

or

$$H_R = 100°11'.1. \tag{32}$$

Therefore the coordinates of the chosen point are

$$L_R = 70°, \; H_R = 100°11'.1. \tag{33}$$

Let us also assume a transmitted frequency $f_T = 2200$ Mc. Basically, it is necessary to get the permissible probable error* in Δf and $\dot{\Delta} f$ corresponding to a prescribed probable error in latitude and longitude (or hour angle). It is also of interest to find the actual values of Δf and $\dot{\Delta} f$. Using the parameter values given in Appendix A and substituting in Eqs. (27) and (28), we get

$$\Delta f = 2459 \text{ cps} \tag{34}$$

and

$$\dot{\Delta} f = 0.1381 \text{ cpsps.} \tag{35}$$

For the error analysis, we use Eqs. (29) and (30) and get (see Appendix C)

$$\mathcal{E}_{\Delta f} = 0.796 \text{ cps} \tag{36}$$

and

$$\mathcal{E}_{\dot{\Delta} f} = 9.511 \times 10^{-5} \text{ cpsps} \tag{37}$$

assuming $\mathcal{E}_{f_T} = 10^{-10} f_T$, $\mathcal{E}_{L_R} = 0.8$ nautical miles, and $\mathcal{E}_{H_R} = 1.3$ nautical miles.†

Dividing $\mathcal{E}_{\Delta f}$ by Δf, we get 3.24×10^{-4}; dividing $\mathcal{E}_{\dot{\Delta} f}$ by $\dot{\Delta} f$, we get 6.89×10^{-4}. This means that doppler frequency must be measured to an accuracy of 0.0324 percent and doppler rate to 0.0689 percent.

The measurement of doppler frequency seems to be within the realm of technical capability in the reasonably near future. The accuracy of doppler-rate measurement is still in question. However, it appears that the rate can be computed within the required accuracy by fitting an approximate analytical expression to the set of points representing the measured doppler shift data and differentiating the expression, or by using finite difference methods with the observed data to get the desired time derivative. This conclusion follows from the following analysis: If we have two quantities x and y with probable errors $\mathcal{E}_x$ and $\mathcal{E}_y$, the maximum probable error in the quotient x/y will be

$$\frac{y + \mathcal{E}_y}{x - \mathcal{E}_x} - \frac{y}{x} = \frac{y}{x}\,\frac{1 + \frac{\mathcal{E}_y}{y}}{1 - \frac{\mathcal{E}_x}{x}} - \frac{y}{x} \cong \frac{y}{x}\left(1 + \frac{\mathcal{E}_y}{y}\right)\left(1 + \frac{\mathcal{E}_x}{x}\right) - \frac{y}{x} \cong \frac{y}{x}\left(\frac{\mathcal{E}_y}{y} + \frac{\mathcal{E}_x}{x}\right). \tag{38}$$

Let us select two values of time, t_1 and t_2, with corresponding doppler frequencies Δf_1 and Δf_2. We can then form the difference quotient

$$\frac{\Delta f_1 - \Delta f_2}{t_1 - t_2}$$

*Many people prefer to use the standard deviation σ. The relation between the two is

$$\text{Probable Error} = 0.6745\ \sigma.$$

†Strictly speaking, latitude and longitude are angles. When they are expressed in terms of linear measure the distances given are those which subtend the actual angular latitude and longitude at the center of the earth.

and, from Eq. (38),

$$\mathcal{E}\left(\frac{\Delta f_1 - \Delta f_2}{t_1 - t_2}\right) = \frac{\Delta f_1 - \Delta f_2}{t_1 - t_2}\left[\frac{\mathcal{E}(\Delta f_1 - \Delta f_2)}{\Delta f_1 - \Delta f_2} + \frac{\mathcal{E}(t_1 - t_2)}{t_1 - t_2}\right]. \tag{39}$$

(Here the argument notation is used instead of the subscript notation for convenience in printing.)

If the permissible probable error in Δf is, as given above, $3.24 \times 10^{-4} \times \Delta f$, then, since we are considering two doppler frequencies at two different times, the maximum probable error in their difference will be $2 \times 3.24 \times 10^{-4}(\Delta f_1 - \Delta f_2) = 6.48 \times 10^{-4}(\Delta f_1 - \Delta f_2)$.

As for errors in time measurement, let us assume (4) that we will be able to measure a given frequency to an accuracy of 1 part in 10^{10}. That is to say, let us take

$$\mathcal{E}_f = 10^{-10} f \tag{40}$$

where f = frequency in cps and $\mathcal{E}_f$ = probable error in frequency in cps. At frequency f cps, one cycle is $f^{-1} = t$ seconds long. Then, by the previously used error analysis method,

$$(\mathcal{E}_t)^2 = (\mathcal{E}_{f^{-1}})^2 = \left(\frac{dt}{df}\,\mathcal{E}_f\right)^2 = \left(\frac{df^{-1}}{df}\,\mathcal{E}_f\right)^2 = f^{-4}\,\mathcal{E}_f^{\,2} \tag{41}$$

or

$$\mathcal{E}_t = f^{-2}\,\mathcal{E}_f = f^{-2} \times 10^{-10} f = 10^{-10} f^{-1} = 10^{-10} t. \tag{42}$$

This is to say that the probable error in frequency, divided by frequency, is the same as the probable error in time, divided by time. Hence the maximum value of the probable error $\mathcal{E}(t_1 - t_2)$ in Eq. (39) is $2 \times 10^{-10}(t_1 - t_2)$. If the time interval is sufficiently small, then the difference quotient is approximately equal to the derivative:

$$\mathcal{E}_{\dot{\Delta f}} \cong \dot{\Delta f}\,(6.48 \times 10^{-4} + 2 \times 10^{-10}). \tag{43}$$

The second term in the parentheses may be considered negligible in comparison with the first. Then, since $\dot{\Delta f} = 0.1381$ cpsps the calculated error is

$$\mathcal{E}_{\dot{\Delta f}} = 8.95 \times 10^{-5} \text{ cpsps}. \tag{44}$$

Since the prescribed error is of the same order ($\mathcal{E}_{\dot{\Delta f}} = 9.511 \times 10^{-5}$ cpsps) it would seem that the desired accuracies are attainable. Curves have been plotted showing how $\mathcal{E}_{\Delta f}$ and $\mathcal{E}_{\dot{\Delta f}}$ vary as $\mathcal{E}_{L_R}$ and $\mathcal{E}_{H_R}$ are varied (see Appendix D, Figs. D1 - D4).

It is of interest to calculate the residual errors in latitude and hour angle when we assume that $\mathcal{E}_{\Delta f}$ and $\mathcal{E}_{\dot{\Delta f}}$ are zero. Using the set of parameters previously assumed, we obtain (see Appendix E)

$$\left(\mathcal{E}_{H_R}\right)_0 = 0.616 \text{ naut mi} \tag{45}$$

$$\left(\mathcal{E}_{L_R}\right)_0 = 0.277 \text{ naut mi}. \tag{46}$$

where the zero subscripts indicate that these are residual errors.

These results show that there would be a finite probable error in latitude and longitude, even if Δf and $\Delta \dot{f}$ were to be measured exactly; this is because of unavoidable errors in measuring other physical quantities such as c, ω, R, and so forth. The calculated errors shown in Eqs. (45) and (46), however, are smaller than the navigation error requirement, and therefore will not be limiting errors.

CONCLUSIONS

It appears, on the basis of the analysis involved in this report, that moon doppler navigation is feasible. The required accuracy of doppler frequency measurement is probably attainable in the near future. By curve-fitting or finite difference methods, it is believed that doppler rate can be calculated within the required accuracy. It must be noted that there has been no consideration in this report of propagation variation in the atmosphere.

There are two approaches which could be considered in an attempt to make the method more immediately feasible. One is to relax the requirements on navigational accuracy. According to the curves in Appendix D, a large relaxation in required navigation accuracy would result in relatively small increases in allowable doppler and doppler rate error.

The other way of improving the method is to improve electronic techniques to the point where the required accuracies can be more reliably attained. It appears that this is the direction in which further efforts should be made.

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   a. p. 22
   b. p. 19

3. Smart, W. M., "Foundations of Astronomy," London:Longmans, Green and Co., 1947 (4th impression)

4. NRL Memorandum Report 745, "NRL Research and Development Program for the Fleet Ballistic Missile: Summary of NRL Activity During August and September 1957," pp. 23 et seq., Sept. 30, 1957

Appendix A

LIST OF PARAMETERS AND THEIR PROBABLE ERRORS*

$c = 2.997893 \times 10^{8}$ meters/second (velocity of propagation)

$\varepsilon_c = 5 \times 10^{2}$ meters/second (probable error in c)

$\omega = 7.292116 \times 10^{-5}$ rad/sec (angular velocity of earth with respect to moon)

$\varepsilon_\omega = 1 \times 10^{-8}$ rad/sec (probable error in ω)

$R = 6.378260 \times 10^{6}$ meters (equatorial radius of earth)

$\varepsilon_R = 1 \times 10^{2}$ meters (probable error in R)

$\delta = 20^\circ\ 19'$ (declination of moon, July 7, 1956, at 1900 UT)

$\varepsilon_\delta = 4 \times 10^{-7}$ radians (probable error in δ)

$\sin \delta = 0.34721$

$\cos \delta = 0.93779$

$L_T = 38.545$ degrees $= 0.673$ radians (latitude of transmitter)

$\varepsilon_{L_T} = 5 \times 10^{-7}$ radians (probable error in L_T)

$\sin L_T = 0.62303$

$\cos L_T = 0.78212$

$H_T = 31.914$ degrees $= 0.557$ radians (local hour angle of moon at transmitter)

$\varepsilon_{H_T} = 1 \times 10^{-6}$ radians (probable error in H_T)

$\sin H_T = 0.55194$

$\cos H_T = 0.83389$

$L_R = 70$ degrees

$\varepsilon_{L_R} = 0.8$ naut mi $= 2.323 \times 10^{-4}$ radians†

$\sin L_R = 0.93969$

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*Most of these values are taken from B. E. Trotter, "Navigation Systems," NRL Report 5005 (Secret Report, Unclassified Title), pp. 19 and 20, Oct. 22, 1957

†The conversion of nautical miles to radians is derived in Appendix C.

$$\cos L_R = 0.34202$$

$$H_R = 100^\circ 11'.1$$

$$\varepsilon_{H_R} = 1.3 \text{ naut mi} = 1.104 \times 10^{-3} \text{ radians}^\dagger$$

$$\sin H_R = 0.98425$$

$$\cos H_R = -0.17683$$

$$f_T = 2200 \text{ Mc}$$

$$= 2.200 \times 10^9 \text{ cps}$$

$$\varepsilon_{f_T} = 10^{-10} f_T = 0.22 \text{ cps.}$$

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†The conversion of nautical miles to radians is derived in Appendix C.

Appendix B

VARIATION IN ANGULAR VELOCITY OF MOON

The angular velocity ω_M of the moon is proportional to $1/r^2$, where r is the radius of the orbit at a particular time. As given by Smart,* the eccentricity e of the moon's orbit is 0.0549. From the same source† we find that the mean geocentric radius of the orbit, which is the same as the semimajor axis a, is 3.84400×10^8 km.

Using the relation

$$a(1 + e) > r > a(1 - e) , \tag{B1}$$

and the given figures, it is found that the mean value of ω_M is 2.661697×10^{-6} rad/sec, which is 3.6501 percent of the earth's angular velocity ω_E. The value of ω_M varies by 10.9 percent from the mean value, which means that its value is 3.6501 ± 0.3979 percent of the earth's angular velocity. That is to say, ω_M varies from 3.25 to 4.05 percent of ω_E.

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*W. M. Smart, "Foundations of Astronomy," Longmans, Green and Co., p. 176, 1947 (4th impression)

†p. 258

Appendix C

DERIVATION OF $\mathcal{E}_{\Delta f}$ AND $\mathcal{E}_{\dot{\Delta f}}$

The doppler shift is

$$\Delta f = \frac{f_T \omega R}{c} \cos \delta \left(\cos L_R \sin H_R + \cos L_T \sin H_T\right). \tag{C1}$$

For the parameters assumed (Appendix A), we get

$$\Delta f = 2459 \text{ cps}. \tag{C2}$$

Differentiating Δf with respect to time, we get

$$\dot{\Delta f} = \frac{f_T \omega^2 R}{c} \cos \delta \left(\cos L_R \cos H_R + \cos L_T \cos H_T\right). \tag{C3}$$

The numerical result is

$$\dot{\Delta f} = 0.1381 \text{ cpsps}. \tag{C4}$$

First $\mathcal{E}_{\Delta f}$ is calculated using the equation

$$(\mathcal{E}_{\Delta f})^2 = \left(\frac{\partial \Delta f}{\partial f_T} \mathcal{E}_{f_T}\right)^2 + \left(\frac{\partial \Delta f}{\partial c} \mathcal{E}_c\right)^2 + \left(\frac{\partial \Delta f}{\partial \omega} \mathcal{E}_\omega\right)^2 + \left(\frac{\partial \Delta f}{\partial R} \mathcal{E}_R\right)^2 + \left(\frac{\partial \Delta f}{\partial \delta} \mathcal{E}_\delta\right)^2$$

$$+ \left(\frac{\partial \Delta f}{\partial L_R} \mathcal{E}_{L_R}\right)^2 + \left(\frac{\partial \Delta f}{\partial H_R} \mathcal{E}_{H_R}\right)^2 + \left(\frac{\partial \Delta f}{\partial L_T} \mathcal{E}_{L_T}\right)^2 + \left(\frac{\partial \Delta f}{\partial H_T} \mathcal{E}_{H_T}\right)^2 . \tag{C5}$$

All the quantities on the right-hand side are known, except $\mathcal{E}_{f_T}$, $\mathcal{E}_{L_R}$, and $\mathcal{E}_{H_R}$. For $\mathcal{E}_{f_T}$ we use a value of $10^{-10} f_T$, on the basis of assumptions as to future advances in electronic techniques and equipment.* The errors in latitude and hour angle are established on the basis of whatever desired navigation accuracy is specified. The values used here are

$$\mathcal{E}_{f_T} = 10^{-10} f_T \tag{C6}$$

$$\mathcal{E}_{L_R} = 0.8 \text{ naut mi} \times \frac{1852}{R} \tag{C7}$$

$$\mathcal{E}_{H_R} = 1.3 \text{ naut mi} \times \frac{1852}{R \cos L_R} \tag{C8}$$

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*NRL Memo Report 745, "NRL Research and Development Program for the Fleet Ballistic Missile: Summary of NRL Activity During August and September 1957," pp. 23 et seq., Sept. 30, 1957 where R is the earth's equatorial radius in meters and the figure 1852 is the number of meters in one nautical mile. Using a transmitted frequency f_T of 2200 x 10^6 cps and R = 6.378 x 10^6 meters, we get

$$\mathcal{E}_{f_T} = 0.22 \text{ cps} \tag{C9}$$

$$\mathcal{E}_{L_R} = 2.323 \times 10^{-4} \text{ radians} \tag{C10}$$

$$\mathcal{E}_{H_R} = 1.104 \times 10^{-3} \text{ radians.} \tag{C11}$$

The other errors are given in Appendix A.

We now proceed to list each term on the right-hand side of Eq. (C5), together with its numerical value:

$$\left(\frac{\partial \Delta f}{\partial f_T} \mathcal{E}_{f_T}\right)^2 = \left[\frac{\omega R}{c} \cos \delta \left(\cos L_R \sin H_R + \cos L_T \sin H_T\right) \mathcal{E}_{f_T}\right]^2$$

$$= 6.048 \times 10^{-14} \tag{C12}$$

$$\left(\frac{\partial \Delta f}{\partial c} \mathcal{E}_c\right)^2 = \left[-\frac{f_T \omega R}{c^2} \cos \delta \left(\cos L_R \sin H_R + \cos L_T \sin H_T\right) \mathcal{E}_c\right]^2$$

$$= 1.682 \times 10^{-5} \tag{C13}$$

$$\left(\frac{\partial \Delta f}{\partial \omega} \mathcal{E}_\omega\right)^2 = \left[\frac{f_T R}{c} \cos \delta \left(\cos L_R \sin H_R + \cos L_T \sin H_T\right) \mathcal{E}_\omega\right]^2$$

$$= 1.137 \times 10^{-1} \tag{C14}$$

$$\left(\frac{\partial \Delta f}{\partial R} \mathcal{E}_R\right)^2 = \left[\frac{f_T \omega}{c} \cos \delta \left(\cos L_R \sin H_R + \cos L_T \sin H_T\right) \mathcal{E}_R\right]^2$$

$$= 1.486 \times 10^{-3} \tag{C15}$$

$$\left(\frac{\partial \Delta f}{\partial \delta} \mathcal{E}_\delta\right)^2 = \left[-\frac{f_T \omega R}{c} \sin \delta \left(\cos L_R \sin H_R + \cos L_T \sin H_T\right) \mathcal{E}_\delta\right]^2$$

$$= 1.326 \times 10^{-7} \tag{C16}$$

$$\left(\frac{\partial\Delta f}{\partial L_R}\,\mathcal{E}_{L_R}\right)^2 = \left[-\frac{f_T\omega R}{c}\cos\delta\left(\sin L_R\,\sin H_R\right)\mathcal{E}_{L_R}\right]^2$$

$$= 4.729 \times 10^{-1} \tag{C17}$$

$$\left(\frac{\partial\Delta f}{\partial H_R}\,\mathcal{E}_{H_R}\right)^2 = \left[\frac{f_T\omega R}{c}\cos\delta\left(\cos L_R\,\cos H_R\right)\mathcal{E}_{H_R}\right]^2$$

$$= 4.568 \times 10^{-2} \tag{C18}$$

$$\left(\frac{\partial\Delta f}{\partial L_T}\,\mathcal{E}_{L_T}\right)^2 = \left[\frac{f_T\omega R}{c}\cos\delta\left(-\sin L_T\,\sin H_T\right)\mathcal{E}_{L_T}\right]^2$$

$$= 3.029 \times 10^{-7} \tag{C19}$$

$$\left(\frac{\partial\Delta f}{\partial H_T}\,\mathcal{E}_{H_T}\right)^2 = \left[\frac{f_T\omega R}{c}\cos\delta\left(\cos L_T\,\cos H_T\right)\mathcal{E}_{H_T}\right]^2$$

$$= 4.358 \times 10^{-6}. \tag{C20}$$

Adding these terms, we get

$$(\mathcal{E}_{\Delta f})^2 = 0.6338\ (\text{cps})^2 \tag{C21}$$

and

$$\mathcal{E}_{\Delta f} = 0.796\ \text{cps}. \tag{C22}$$

Now $\mathcal{E}_{\Delta\dot{f}}$ is calculated using the equation

$$(\mathcal{E}_{\Delta\dot{f}})^2 = \left(\frac{\partial\Delta\dot{f}}{\partial f_T}\,\mathcal{E}_{f_T}\right)^2 + \left(\frac{\partial\Delta\dot{f}}{\partial c}\,\mathcal{E}_c\right)^2 + \left(\frac{\partial\Delta\dot{f}}{\partial\omega}\,\mathcal{E}_\omega\right)^2 + \left(\frac{\partial\Delta\dot{f}}{\partial R}\,\mathcal{E}_R\right)^2 + \left(\frac{\partial\Delta\dot{f}}{\partial\delta}\,\mathcal{E}_\delta\right)^2$$

$$+ \left(\frac{\partial\Delta\dot{f}}{\partial L_R}\,\mathcal{E}_{L_R}\right)^2 + \left(\frac{\partial\Delta\dot{f}}{\partial H_R}\,\mathcal{E}_{H_R}\right)^2 + \left(\frac{\partial\Delta\dot{f}}{\partial L_T}\,\mathcal{E}_{L_T}\right)^2 + \left(\frac{\partial\Delta\dot{f}}{\partial H_T}\,\mathcal{E}_{H_T}\right)^2. \tag{C23}$$

Proceeding as before,

$$\left(\frac{\partial\Delta\dot{f}}{\partial f_T}\,\mathcal{E}_{f_T}\right)^2 = \left[\frac{\omega^2 R}{c}\cos\delta\left(\cos L_R\,\cos H_R + \cos L_T\,\cos H_T\right)\mathcal{E}_{f_T}\right]^2$$

$$= 1.908 \times 10^{-22} \tag{C24}$$

$$\left(\frac{\partial \Delta \dot{f}}{\partial c}\ \mathcal{E}_c\right)^2 = \left[-\frac{f_T \omega^2 R}{c^2} \cos \delta \left(\cos L_R \cos H_R + \cos L_T \cos H_T\right) \mathcal{E}_c\right]^2$$

$$= 5.308 \times 10^{-14} \tag{C25}$$

$$\left(\frac{\partial \Delta \dot{f}}{\partial \omega}\ \mathcal{E}_\omega\right)^2 = \left[\frac{2\ f_T \omega R}{c} \cos \delta \left(\cos L_R \cos H_R + \cos L_T \cos H_T\right) \mathcal{E}_\omega\right]^2$$

$$= 1.435 \times 10^{-9} \tag{C26}$$

$$\left(\frac{\partial \Delta \dot{f}}{\partial R}\ \mathcal{E}_R\right)^2 = \left[\frac{f_T \omega^2}{c} \cos \delta \left(\cos L_R \cos H_R + \cos L_T \cos H_T\right) \mathcal{E}_R\right]^2$$

$$= 4.691 \times 10^{-12} \tag{C27}$$

$$\left(\frac{\partial \Delta \dot{f}}{\partial \delta}\ \mathcal{E}_\delta\right)^2 = \left[-\frac{f_T \omega^2 R}{c} \sin \delta \left(\cos L_R \cos H_R + \cos L_T \cos H_T\right) \mathcal{E}_\delta\right]^2$$

$$= 4.185 \times 10^{-16} \tag{C28}$$

$$\left(\frac{\partial \Delta \dot{f}}{\partial L_R}\ \mathcal{E}_{L_R}\right)^2 = \left[-\frac{f_T \omega^2 R}{c} \cos \delta \left(\sin L_R \cos H_R\right) \mathcal{E}_{L_R}\right]^2$$

$$= 8.118 \times 10^{-11} \tag{C29}$$

$$\left(\frac{\partial \Delta \dot{f}}{\partial H_R}\ \mathcal{E}_{H_R}\right)^2 = \left[-\frac{f_T \omega^2 R}{c} \cos \delta \left(\cos L_R \sin H_R\right) \mathcal{E}_{H_R}\right]^2$$

$$= 7.525 \times 10^{-9} \tag{C30}$$

$$\left(\frac{\partial \Delta \dot{f}}{\partial L_T}\ \mathcal{E}_{L_T}\right)^2 = \left[-\frac{f_T \omega^2 R}{c} \cos \delta \left(\sin L_T \cos H_T\right) \mathcal{E}_{L_T}\right]^2$$

$$= 3.676 \times 10^{-15} \tag{C31}$$

$$\left(\frac{\partial \Delta \dot{f}}{\partial H_T}\ \mathcal{E}_{H_T}\right)^2 = \left[-\frac{f_T \omega^2 R}{c} \cos \delta \left(\cos L_T \sin H_T\right) \mathcal{E}_{H_T}\right]^2$$

$$= 1.015 \times 10^{-14} \tag{C32}$$

Adding these terms, we get

$$\left(\mathcal{E}_{\Delta\dot{f}}\right)^2 = 9.046 \times 10^{-9} \text{ (cpsps)}^2 \tag{C33}$$

and

$$\mathcal{E}_{\Delta\dot{f}} = 9.511 \times 10^{-5} \text{ cpsps.} \tag{C34}$$

Appendix D

FORMULAS AND CURVES FOR $\mathcal{E}_{\Delta f}$ AND $\mathcal{E}_{\Delta \dot{f}}$ AS FUNCTIONS OF $\mathcal{E}_{L_R}$ AND $\mathcal{E}_{H_R}$

In order to find out how the requirements imposed on the accuracy of frequency measurement and frequency rate measurement vary as different prescribed latitude and longitude accuracies are assumed, we take the equations for $\mathcal{E}_{\Delta f}$ and $\mathcal{E}_{\Delta \dot{f}}$ as stated in Appendix B, hold everything at its calculated value except the pertinent independent variable (either $\mathcal{E}_{L_R}$ or $\mathcal{E}_{H_R}$), and plot the four resulting curves.

To be more specific: if we hold $\mathcal{E}_{H_R}$ constant and allow $\mathcal{E}_{L_R}$ to vary, we get

$$\mathcal{E}_{\Delta f} = \left(0.161 + 8.764 \times 10^6 \, \mathcal{E}_{L_R}^{\,2}\right)^{1/2} \tag{D1}$$

and if we hold $\mathcal{E}_{L_R}$ constant and allow $\mathcal{E}_{H_R}$ to vary, we get

$$\mathcal{E}_{\Delta f} = \left(0.588 + 3.748 \times 10^4 \, \mathcal{E}_{H_R}^{\,2}\right)^{1/2}. \tag{D2}$$

Similarly, for $\mathcal{E}_{\Delta \dot{f}}$ we get

$$\mathcal{E}_{\Delta \dot{f}} = \left(8.965 \times 10^{-9} + 1.504 \times 10^{-3} \, \mathcal{E}_{L_R}^{\,2}\right)^{1/2} \tag{D3}$$

and

$$\mathcal{E}_{\Delta \dot{f}} = \left(1.521 \times 10^{-9} + 6.174 \times 10^{-3} \, \mathcal{E}_{H_R}^{\,2}\right)^{1/2} . \tag{D4}$$

The resulting curves are shown in Figs. D1 through D4. It is evident from the curves that a large relaxation in required navigation accuracy would give relatively small increases in allowable doppler and doppler rate accuracy.

The "x" on each curve indicates the original specified navigation error.

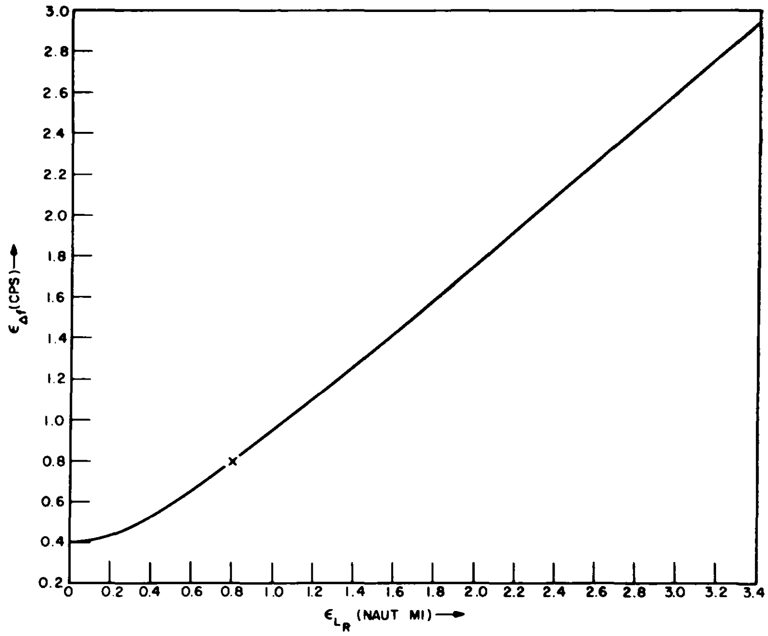

Fig. D1 - Probable error $\mathcal{E}_{\Delta f}$ in doppler frequency vs prescribed probable error $\mathcal{E}_{L_R}$ in latitude

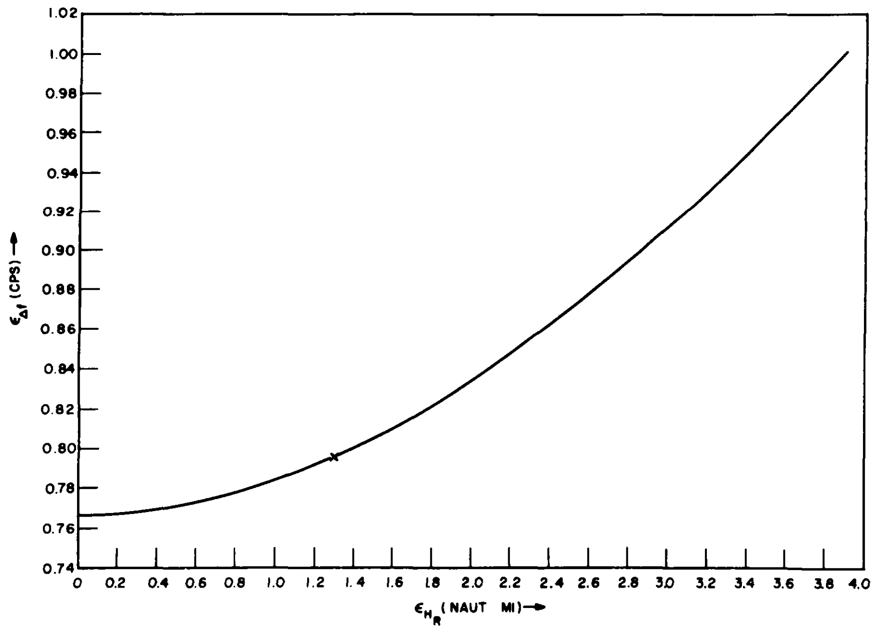

Fig. D2 - Probable error $\mathcal{E}_{\Delta f}$ in doppler frequency vs prescribed probable error $\mathcal{E}_{H_R}$ in longitude

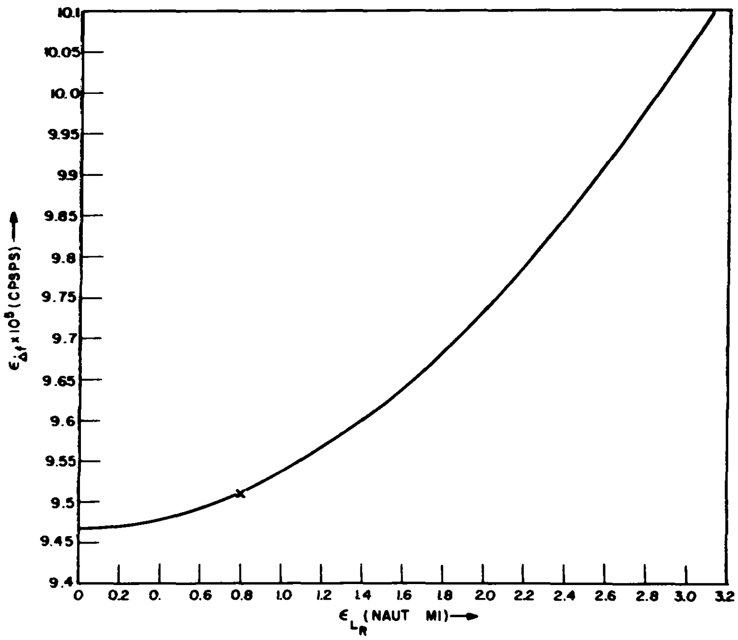

Fig. D3 - Probable error $\mathcal{E}_{\Delta \dot{f}}$ in doppler rate vs prescribed probable error $\mathcal{E}_{L_R}$ in latitude

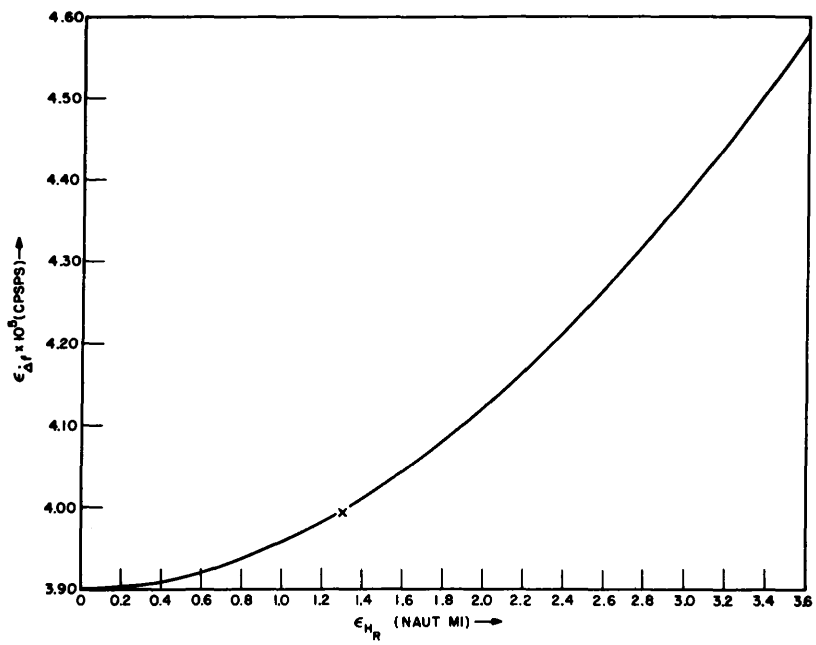

Fig. D4 - Probable error $\mathcal{E}_{\Delta \dot{f}}$ in doppler rate vs prescribed probable error $\mathcal{E}_{H_R}$ in longitude

Appendix E

RESIDUAL ERROR IN H_R AND L_R ASSUMING $\mathcal{E}_{\Delta f}$ AND $\mathcal{E}_{\dot{\Delta f}}$ EQUAL TO ZERO

The following notation is used for the sake of conciseness:

$$p = \tan H_R \tag{E1}$$

$$a = \omega\, c\, \Delta f - f_T\, \omega^2 R \cos\delta \cos L_T \sin H_T \tag{E2}$$

$$b = c\, \dot{\Delta f} - f_T \omega^2 R \cos\delta \cos L_T \cos H_T\ . \tag{E3}$$

Hence

$$p = \frac{a}{b}\ . \tag{E4}$$

If q_i is any independent variable, then

$$\mathcal{E}_{H_R} = \left[\sum_{i=1}^{n}\left(\frac{\partial H_R}{\partial q_i}\mathcal{E}_{q_i}\right)^2\right]^{1/2}\ . \tag{E5}$$

Assuming perfect measurements; i.e., $\mathcal{E}_{\Delta f}$ and $\mathcal{E}_{\dot{\Delta f}}$ equal to zero, there will remain seven independent variables to be considered: f_T, c, ω, R, δ, L_T, and H_T. Therefore $n = 7$ in Eq. (E5).

To find the required partial derivatives, the relation

$$\frac{\partial H_R}{\partial q_i} = \frac{\partial(\arctan p)}{\partial q_i} = \frac{1}{1+p^2}\ \frac{\partial p}{\partial q_i} \tag{E6}$$

is used. It is necessary first to evaluate $p = \tan H_R$, so that the common factor $1/(1+p^2)$ may be known. If the data in Appendix A and the previously computed values of Δf and $\dot{\Delta f}$ are used, the result is

$$p = -\,5.55906. \tag{E7*}$$

Then

$$\frac{1}{1+p^2} = 3.134 \times 10^{-2}. \tag{E8}$$

*The quantity $p = \tan H_R$ can also be obtained by using $\sin H_R$ and $\cos H_R$ from Appendix A. The values are only slightly different.

Now

$$\frac{\partial p}{\partial q_i} = \frac{1}{b^2}\left(b\,\frac{\partial a}{\partial q_i} - a\,\frac{\partial b}{\partial q_i}\right) \tag{E9}$$

from Eq. (E4). The numerical results for the seven required terms are

$$\frac{\partial p}{\partial f_T} = 3.046 \times 10^{-8} \tag{E10}$$

$$\left(\frac{\partial p}{\partial f_T}\,\mathcal{E}_{f_T}\right)^2 = 4.492 \times 10^{-17} \tag{E11}$$

$$\frac{\partial p}{\partial c} = -\,2.236 \times 10^{-7} \tag{E12}$$

$$\left(\frac{\partial p}{\partial c}\,\mathcal{E}_c\right)^2 = 1.249 \times 10^{-8} \tag{E13}$$

$$\frac{\partial p}{\partial \omega} = 1.664 \times 10^{6} \tag{E14}$$

$$\left(\frac{\partial p}{\partial \omega}\,\mathcal{E}_\omega\right)^2 = 2.769 \times 10^{-4} \tag{E15}$$

$$\frac{\partial p}{\partial R} = 1.051 \times 10^{-5} \tag{E16}$$

$$\left(\frac{\partial p}{\partial R}\,\mathcal{E}_R\right)^2 = 1.104 \times 10^{-6} \tag{E17}$$

$$\frac{\partial p}{\partial \delta} = -\,2.481 \times 10^{1} \tag{E18}$$

$$\left(\frac{\partial p}{\partial \delta}\,\mathcal{E}_\delta\right)^2 = 9.851 \times 10^{-11} \tag{E19}$$

$$\frac{\partial p}{\partial L_T} = -\,5.339 \times 10^{1} \tag{E20}$$

$$\left(\frac{\partial p}{\partial L_T}\,\mathcal{E}_{L_T}\right)^2 = 7.125 \times 10^{-10} \tag{E21}$$

$$\frac{\partial p}{\partial H_T} = -\ 2.887 \times 10^{1} \tag{E22}$$

$$\left(\frac{\partial p}{\partial H_T}\ \mathcal{E}_{H_T}\right)^2 = 8.332 \times 10^{-10}. \tag{E23}$$

Summing the squared terms gives

$$\sum_{i=1}^{7}\left(\frac{\partial p}{\partial q_i}\ \mathcal{E}q_i\right)^2 = 2.780 \times 10^{-4}. \tag{E24}$$

Extracting the square root of Eq. (E24) and multiplying by the value of $1/(1 + p^2)$ from Eq. (E8) yields

$$\left(\mathcal{E}_{H_R}\right)_0 = 5.227 \times 10^{-4} \text{ radians,} \tag{E25}$$

where the zero subscript refers to the fact that this is the residual error. This value is well within the prescribed maximum probable error

$$\mathcal{E}_{H_R} = 1.104 \times 10^{-3} \text{ radians.} \tag{C11}$$

In nautical miles

$$\left(\mathcal{E}_{H_R}\right)_0 = 0.616 \text{ naut mi} \tag{E26}$$

compared with (see Appendix A)

$$\mathcal{E}_{H_R} = 1.3 \text{ naut mi.} \tag{E27}$$

Consider now the equation

$$\cos L_R = \left(1 + \tan^2 H_R\right)^{1/2}\left(\cos L_T \cos H_T - \frac{c\ \Delta\dot{f}}{f_T \omega^2 R \cos \delta}\right). \tag{25}$$

It is convenient to introduce the notation

$$r = \cos L_R \tag{E28}$$

$$s = \cos L_T \cos H_T - \frac{c\ \Delta\dot{f}}{f_T\ \omega^2\ R \cos \delta}. \tag{E29}$$

Then Eq. (25) can be written

$$r = s\left(1 + p^2\right)^{1/2}. \tag{E30}$$

If q_i is any of the independent variables, then

$$\frac{\partial L_R}{\partial q_i} = \frac{\partial(\arccos r)}{\partial q_i} = -\ (1 - r^2)^{-1/2}\ \frac{\partial r}{\partial q_i}. \tag{E31}$$

It is necessary to calculate $r = s(1 + p^2)^{1/2}$ and from that to get $(1 - r^2)^{-1/2}$. The results are

$$s = 6.05421 \times 10^{-2} \tag{E32}$$*

$$(1 + p^2)^{1/2} = 5.64829. \tag{E33}$$

Hence

$$r = 3.41959 \times 10^{-1} \tag{E34}$$

$$(1 - r^2)^{-1/2} = 1.06417. \tag{E35}$$

From Eq. (E30)

$$\frac{\partial r}{\partial q_i} = (1 + p^2)^{1/2} \frac{\partial s}{\partial q_i} + sp(1 + p^2)^{-1/2} \frac{\partial p}{\partial q_i}. \tag{E36}$$

The only quantity which one has to compute is $\partial s/\partial q_i$, since all the other quantities are known; $\partial p/\partial q_i$ and p are obtained from the above H_R calculations. Putting in the numerical values, we get

$$\frac{\partial r}{\partial q_i} = 5.64829 \frac{\partial s}{\partial q_i} -5.95858 \times 10^{-2} \frac{\partial p}{\partial q_i} . \tag{E37}$$

Evaluation of the seven terms yields

$$\frac{\partial r}{\partial f_T} = -2.961 \times 10^{-10} \tag{E38}$$

$$\left(\frac{\partial r}{\partial f_T} \mathcal{E}_{f_T}\right)^2 = 4.244 \times 10^{-21} \tag{E39}$$

$$\frac{\partial r}{\partial c} = 2.173 \times 10^{-9} \tag{E40}$$

$$\left(\frac{\partial r}{\partial c} \mathcal{E}_c\right)^2 = 1.180 \times 10^{-12} \tag{E41}$$

$$\frac{\partial r}{\partial \omega} = -7.499 \times 10^{3} \tag{E42}$$

$$\left(\frac{\partial r}{\partial \omega} \mathcal{E}_\omega\right)^2 = 5.623 \times 10^{-9} \tag{E43}$$

*Strictly speaking, $(1 + p^2)^{1/2} = \sec H_R$, which is negative. This introduces a negative sign into both terms of Eq. (E36). Since each $\partial r/\partial q_i$ is to be squared, the positive $(1 + p^2)^{1/2}$ was used.

$$\frac{\partial r}{\partial R} = -1.022 \times 10^{-7} \tag{E44}$$

$$\left(\frac{\partial r}{\partial R}\,\mathcal{E}_R\right)^2 = 1.044 \times 10^{-10} \tag{E45}$$

$$\frac{\partial r}{\partial \delta} = 2.412 \times 10^{-1} \tag{E46}$$

$$\left(\frac{\partial r}{\partial \delta}\,\mathcal{E}_\delta\right)^2 = 9.309 \times 10^{-15} \tag{E47}$$

$$\frac{\partial r}{\partial L_T} = 2.466 \times 10^{-1} \tag{E48}$$

$$\left(\frac{\partial r}{\partial L_T}\,\mathcal{E}_{L_T}\right)^2 = 1.520 \times 10^{-14} \tag{E49}$$

$$\frac{\partial r}{\partial H_T} = -7.183 \times 10^{-1} \tag{E50}$$

$$\left(\frac{\partial r}{\partial H_T}\,\mathcal{E}_{H_T}\right)^2 = 5.159 \times 10^{-13}\ . \tag{E51}$$

Summation of the squared terms gives

$$\sum_{i=1}^{7}\left(\frac{\partial r}{\partial q_i}\,\mathcal{E}q_i\right)^2 = 7.236 \times 10^{-9}\,. \tag{E52}$$

Extraction of the square root and multiplication by $(1 - r^2)^{-1/2} = 1.06417$ gives

$$\left(\mathcal{E}_{L_R}\right)_0 = 8.506 \times 10^{-5} \text{ radians} \tag{E53}$$

in absolute value. This may be compared with the prescribed value

$$\mathcal{E}_{L_R} = 2.323 \times 10^{-4} \text{ radians.} \tag{C10}$$

In nautical miles

$$\left(\mathcal{E}_{L_R}\right)_0 = 0.293 \text{ naut mi} \tag{E54}$$

compared with (see Appendix A)

$$\mathcal{E}_{L_R} = 0.8 \text{ naut mi.} \tag{E55}$$

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