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Group Theoretic Model Interactions as an Approach to 2s-1d Shell Nuclei

The Level Structure of Fluorine-20

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ABSTRACT

Based on the studies of S. Lie and E. Cartan in the theory of continuous groups of transformations, M. Moshinsky has reformulated the nuclear shell many-body problem in second-quantization language. The methods due to J.P. Elliott for simplifying the basis set of state functions by classification according to the group SU(3) are recast into the above-mentioned reformulation. The purpose is to make low-energy nuclear calculations feasible for nuclei with 4 and more particles in the 2s-1d shell and thus render the possibility of probing for SU(3) symmetries in these nuclei. A hamiltonian model consisting of pairing and quadrupole-quadrupole terms is known to approximate respectively the short- and long-ranged correlations between nucleons given by an arbitrary, reasonably shaped two-nucleon central interaction potential. The former model is generalized to include exchange effects at the long range as well as spin-orbit coupling, and is studied in detail from the viewpoint of its various group symmetries. It is then employed to calculate the low-lying levels of Fluorine-20 which show reasonable accord with the empirical level-scheme.

PROBLEM STATUS

This is an interim report on this problem; work is continuing.

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I. INTRODUCTION

Angular momentum techniques have proven extremely useful in simplifying the treatment of numerous quantum-mechanical problems in atomic, molecular, nuclear and solid state physics. These techniques, first developed in the early 1930's, involve such concepts as rotation matrices, spherical harmonics, vector-coupling, recoupling by Racah cofficients, 9-j coefficients transforming an L-S to a j-j scheme, irreducible tensor operators and state functions, etc. The validity of such techniques rests of course on the rotational invariance of many physical situations. Group theoretically speaking, the hamiltonian of a spherically symmetric problem commutes with the generators of the group of rotations in a 3-dimensions, R_3.

The fact that a large class of problems in addition possess symmetry groups larger than R_3 can be exploited by studying techniques similar to angular momentum methods but generalized to definite groups containing R_3 as a subgroup. Solving the Schrödinger equation of a many-body problem, even approximately, is a formidable if not impossible task. The matrix mechanical approach consisting essentially in setting up and diagonalizing the hamiltonian matrix is more promising and adaptable to the utilization of these higher symmetries as then the original matrix is decomposed into smaller sub-matrices. Even considering the capacity of modern electronic computors to diagonalize large matrices, the labor involved in calculating the elements of the sub-matrices is still monstrous due to the very large number of N-particle states present. It thus becomes desirable to formulate the problem in terms more easily adapted to computor languages so that the machine can do more than merely diagonalize matrices.

Consider the asymmetric top hamiltonian

$$\mathcal{H}_T = \frac{L_1^2}{2\mathcal{I}_1} + \frac{L_2^2}{2\mathcal{I}_2} + \frac{L_3^2}{2\mathcal{I}_3} \tag{1}$$

where L_1, L_2, L_3 are the operator angular momentum components in a system fixed in the body, and the $\mathcal{J}$'s the three Cartesian moments of inertia. Solving the problem for the allowable energies entails construction of the matrix $\| \mathcal{H}_T \|$ in some adequate basis, e. g., the set $|\ell m\rangle$ of eigenfunctions of operators L^2 and L_3. Thus, since $L^2|\ell m\rangle = \ell(\ell+1)|\ell m\rangle$ and $L_3|\ell m\rangle = m|\ell m\rangle$ while $L^2 = L_1^2 + L_2^2 + L_3^2$, our hamiltonian simplifies to

$$\mathcal{H}_T = \left(\frac{1}{2\mathcal{J}_2}\right) L^2 + \left(\frac{1}{2\mathcal{J}_3} - \frac{1}{2\mathcal{J}_2}\right) L_3^2 + \left(\frac{1}{2\mathcal{J}_1} - \frac{1}{2\mathcal{J}_2}\right) L_1^2 \tag{2}$$

the first two terms being diagonal in the $|\ell m\rangle$ basis set; we notice that $\mathcal{H}_T$ is <u>quadratic</u> in the non-diagonal operator L_1. Our matrix $\| \mathcal{H}_T \|$ whose rows and columns are given by all the allowed values of ℓ and m is

$$\| \langle \ell m' | \mathcal{H}_T | \ell m \rangle \| =$$

$$\left\| \left\{ \frac{\ell(\ell+1)}{2\mathcal{J}_2} + \frac{1}{2}\left(\frac{1}{\mathcal{J}_3} - \frac{1}{\mathcal{J}_2}\right) m^2 \right\} \delta_{m'm} + \frac{1}{2}\left(\frac{1}{\mathcal{J}_1} - \frac{1}{\mathcal{J}_2}\right) \langle \ell m' | L_1^2 | \ell m \rangle \right\| \tag{3}$$

where, since matrix elements between different ℓ-values vanish, the complete matrix is now decomposed into as many blocks of elements as there are different ℓ-values in the problem: the rows and columns within each block being labelled by m, where $-\ell \leq m \leq \ell$ and ℓ designates the whole block. Moreover, the complete solution of this exact calculation of energies requires only knowledge of the matrix elements

$$\langle \ell m' | L_1^2 | \ell m \rangle. \tag{4}$$

But these are definitely obtainable in closed algebraic form by simple angular momentum techniques based <u>solely</u> on the commutation relations

$$[L_1, L_2] = -i L_3 \qquad \text{(and cyclically)} \tag{5}$$

The solution is well known, but no recurrance to its <u>explicit form</u> is necessary -- only the simple relations (5) are needed to obtain

$$\langle \ell m' | L_1 | \ell m \rangle = \frac{1}{2}\sqrt{(\ell-m)(\ell+m+1)}\,\delta_{m',m+1} + \frac{1}{2}\sqrt{(\ell+m)(\ell-m+1)}\,\delta_{m',m-1} \tag{6}$$

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and thus solve (3) exactly.

In group theoretic language, the generators of infinitesimal rotations around the 1, 2 and 3 axes are L_1, L_2 and L_3 -- called simply the R_3 group generators. Commutation relations between them (5) form a Lie algebra. The Operator L^2 -- formed out of the group generators as the sum of their squares, commutes with all three generators L_1, L_2, L_3 and its eigenvalue $\ell(\ell+1)$ sufficing to characterize the R_3 transformation properties of $|\ell m\rangle$ -- is the Casimir operator of R_3 whose eigenvalue provides the classification label ℓ. The rows of each irreducible representation designated by ℓ are specified by another label m which proceeds from the R_2 (subgroup of R_3) Casimir operator L_3. The set of functions $|\ell m\rangle$ is thus said to transform irreducibly under R_3 and explicitly also under its subgroup R_2.

Moshinsky's group-theoretic interpretation of second-quantization techniques applied to the many-body problem lead to straightforward generalizations of these simple R_3 group results to the case of physical problems involving larger symmetries associated with permutations, the harmonic oscillator common potential, r-dimensional rotations and spin-isospin. The nuclear shell model problem with a spherical (inert closed-shell) core of nucleons is given by the N- extra-shell nucleon hamiltonian

$$\mathcal{H} = \sum_{i=1}^{N} \{ T_i + U_i \} + \sum_{i<j=1}^{N} V_{ij} \tag{7}$$

where $T_i = P_i^2/2m$ is the kinetic energy, U_i a central or non-central (or both) single-body interaction and V_{ij} a central two-body interaction. In the Moshinsky formulation, a single-body interaction operator is expressible as a linear combination, and a two-body interaction operator as a bilinear combination, of the generators belonging to groups of

symmetries higher than R_3. But these generators form a known commutator Lie algebra. Therefore, in principle, closed expressions for the matrix elements of any single- and two-body interaction could be obtained.

In practice, however, another approach was found both simpler and physically more meaningful: to consider the problem of allowable energies associated with a mixture of pairing and quadrupole-quadrupole interactions as a model for central two-body interaction, plus a spin-orbit single-body interaction. Contrary to the concept of a potential, these model interaction operators have a clear group-theoretic meaning in that they can be written in terms of operators which commute with the generators of various related groups and whose eigenvalues serve as irreducible representation labels. These operators are none other than the Casimir operators of the group involved. Thus the irreducible basis sets diagonalizing separately the three interactions mentioned above could be constructed by elementary algebraic techniques based on simple notions from group theory. Having chosen one of the three sets, the nondiagonal matrices of the other two interaction operators can be constructed in this base.

We chose the quadrupole-quadrupole (resembling the long-ranged part of a central two-body residual interaction) scheme for three reasons: (1) it is invariant under the group U_3, the algebraic techniques of which have been studied extensively by

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Elliott, Biedenharn, Moshinsky and others; (2) there is a close resemblance between the U_3 or SU_3 scheme and the states of the collective rotational nuclear model. There seems also to be some connection between this interaction operator and quadrupolar nuclear vibrations; and (3) classification of states by SU_3 offers the possibility suggested by Elliott of restricting, as a first approximation, the basis to the single SU_3 representation which lies lowest in energy.

These methods should be extremely powerful to carry out, within a feasible length of time, calculations on <u>families</u> of nuclei with the aim of making global studies of their diverse properties. The isotope Fluorine-20 is merely a "pilot nucleus" for our work; the fact that it is odd-odd makes it a more difficult shell model problem as such, in spite of having only four nucleons outside the doubly-magic ${}_8O_8^{16}$ core. Little is known empirically about the low-lying spins of this nucleus unambiguously, but our results on the whole are not inconsistent with known experiments to date.

II. NUCLEAR MODELS & LOW-ENERGY NUCLEAR STRUCTURE.

The nuclear shell model of Mayer & Jensen, with residual interactions between extra-closed-shell nucleons has been widely applied[1, 2] to account for such low-energy nuclear properties as level energies, spins and parities, moments, electromagnetic and β-decay transition rates and even binding energies. Its successes have been encouraging but glaring failures are evident in some respects, viz., large quadrupole moments are left unexplained.

A second approach to the problem has grown from evidence of cooperative nucleonic behavior seen in the fission process and the partial success of the liquid-drop model which seems to be the antithesis of shell structure. The liquid-drop and shell-model viewpoints were combined by Bohr & Mottelson and the Copenhagen school to propose a nuclear model allowing more generalized motion within the nucleus by the introduction of collective vibrational and rotational degrees of motion.

A third trend has been to <u>return</u> to the shell model but with specific residual interaction models that <u>simulate</u> those collective aspects to a certain extent. So much the better if these model residual interaction possess group symmetries which can be systematically exploited to reduce calculational labor.

1. Heavy Nuclei

1a. <u>Collective Behavior.</u> In 1950 Rainwater[3, 4] suggested the possibility of nuclei between magic proton and neutron numbers of acquiring equilibrium non-spherical shapes to account for observed large quadrupole moments and transition rates ---- as much as a single nucleon outside a closed-shell core having the power to polarize or deform the core by centrifugal forces. Bohr and Mottelson[5, 6] proposed (1952-3) a unified description whereby

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shell structure due to the particles was maintained alongside collective structure consisting of permanent deformations resulting in observed rotation-like spectra as well as vibrations in size and shape. They put forward a total hamiltonian

$$\mathcal{H} = H_p + H_s + H_{int} \tag{8}$$

with particle-like H_p, surface effects H_s arising from deviations from perfect sphericity and an interaction H_{int} between particle and surface effects. For a small number of extra-closed-shell nucleons one had <u>weak-coupling</u> since departure from sphericity was small enough that one could treat $H_s + H_{int}$ in a perturbation method with shell model particle states χ as zero-order functions. For many particles <u>strong-coupling</u> prevailed, that is, spheroidal shapes occurred permitting a reasonable description in terms of well-known rotating-top eigenfunctions $\mathcal{D}_{rot}$, with perturbative effects due to vibration, vibration-rotation and surface-particle couplings. The Unified Model enjoyed great success[7] in certain definite regions of the nuclide table, failing seriously in others mainly because of the difficult <u>intermediate coupling</u> situations.

Nilsson[8] (1955), using the model of a single-particle in an anisotropic harmonic oscillator common potential with axial symmetry plus a single-body spin-orbit term and a term in ℓ^2 to simulate the partial effect of a square-well, calculated for every nuclear ocillator shell the single-particle energy levels as functions of a parameter proportional directly to quadrupolar oscillator deformation and inversely to spin-orbit strength. Many applications of this simple model have been made to odd-mass nuclei, where the even

number outside closed-shells are presumed to pair off according to the shell model, leaving the one odd particle to deal with. Numerous properties were predicted approximately[9, 10] in those regions of strong or extremely weak deformation where the model is expected to apply but again, failing for intermediate cases.

Both the Bohr-Mottelson and Nilsson models have succeeded in explaining, within certain limits, the rotational band-like level structure of many strongly deformed nuclei beyond $A = 150$. These bands are designated by a quantum number K which represents the projection on the nuclear symmetry axis of the total angular momentum J composed of individual-particle j and collective R angular momenta, as shown in figure 1.

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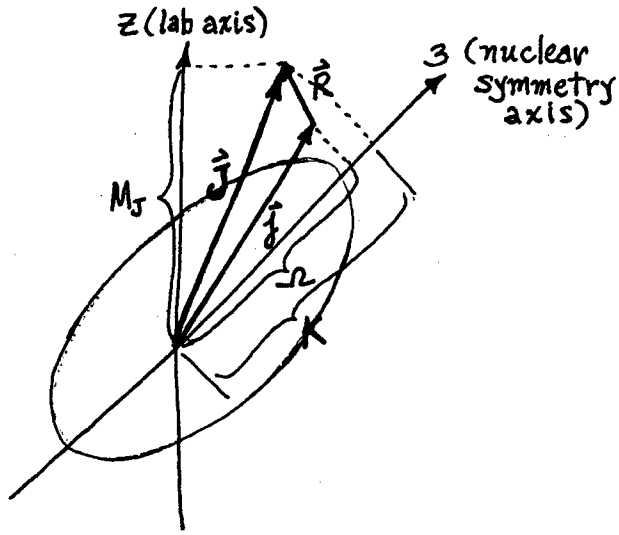

FIGURE 1

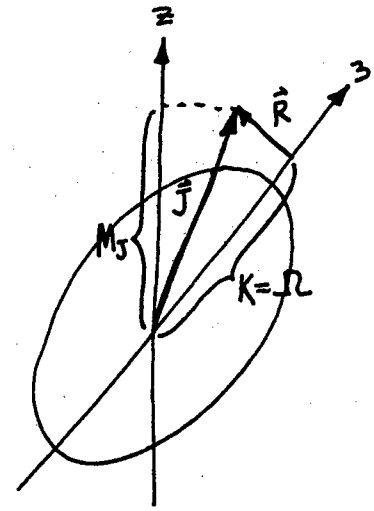

FIGURE 2

For low energies, rotations are about axes perpendicular to the nuclear symmetry axis 3 so that the total individual-particle angular momentum vector $\vec{j}$ coincides with the 3-axis, Ω becoming equal to K, and one has the situation in Figure 2. The nuclear wave function referred to <u>this axis</u> is then

$$\chi_K = \chi_\Omega = \frac{1}{\sqrt{N!}} \sum_P (-)^P \chi_{\Omega_1} \chi_{\Omega_2} \cdots \chi_{\Omega_N} , \qquad (9)$$

or the antisymmetrized product of N individual-particle wave functions.

Perhaps with the intention of having a model capable of covering a wider range of cases than the rotational models discussed above and the original Mayer-Jensen shell model, a new approach has become very popular since 1957. Elliott[10,11] Bohr and Mottelson discussed the use of long-and short-range interactions approximated respectively by a quadrupole-quadrupole (Q^2) and a pairing P force. The Q^2 force to be distinguished from a single-particle Q force of the form

$$\sum_{i=1}^{N} r_i Y_{20}(\theta_i, \varphi_i) \qquad (10)$$

used by Nilsson to deform the common oscillator well ---- it is a two-body interaction between extra-closed shell nucleons giving rise to what Mottelson called the "aligned coupling scheme" whereby the extra-shell nucleon orbits tend to align themselves along a given axis fixed in the core thus acting, effectively, as if the core itself were deformed into a (quadrupolar) ellipsoidal shape. Mottelson[12] shows that P gives rise to binding energies depending on N, while Q^2 produces effects dependent on N^2 and therefore for many particles outside the closed shells Q^2 is expected to predominate.

1b. Pairing Effects. Bohr, Mottelson & Pines[13] (1958) emphasized that the large spacing (gap) between the ground and first excited states of even-even nuclei as well as the mass difference between even-even and odd A nuclei may be indicative of nucleonic pairing.

The first formal use of the P plus Q^2 model seems to be due Belyaev[14] (1959) who reached very interesting conclusions regarding the effect of pairing in heavy nuclei: 1) pairing reduces the heretofore too-large predicted nuclear moments of inertia by magnitudes in much better accord with experiment. (Griffin & Rich[15] (1960); Nilsson & Prior[16] (1960) verified this admirably for 26 even-even rare earth nuclei with an average theory-to-experiment difference of only 6%). 2) Near closed shells, pairing gives rise to spherical equilibrium shapes while the Q^2 interaction low-energy vibrational modes is responsible with frequencies within observed trends. 3) The even-even nuclei gap is explained, as well as the increased level density above it. A disadvantage of Belyaev's 2nd - quantization treatment of pairing is

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that the number of particles N is not constant so that results apply to average properties in a given isotope region.

The Belyaev model was used by Kisslinger & Sorensen[17] (1960) in an extensive application to single-closed-shell heavy nuclei, for cases where P predominates over Q^2, and obtained generally encouraging results of level energies, moments and transition rates. A recent work by these authors[18] (1963) shows use of the same model with the additional treatment of quadrupolar vibrations via introduction of the phonon formalism. Reasonable agreement with empirical low-energy systematics is obtained for numerous heavy nuclei outside well-established regions of nuclear deformation.

2. Light Nuclei.

Experimental work by Litherland et al[19] (1956) on Mg^{24} strongly suggested the presence of rotational (collective) structure in the A=24,25 mass region of light nuclei. Subsequent experiments corroborated this suggestion for other nuclei in the 2S-1d shell, e.g., F^{19} and N_e^{20}.

The theoretical structure of F^{19} presented a curious situation: two apparently very different models yielded very similar results[20]. The results obtained by Elliott & Flowers[21] (1955) with a central Yukawa interaction and spin-orbit force acting on mixed shell model configurations and those of Paul[22] (1957) using the Nilsson model showed that, at least for this nucleus, the two models could not be very distinct. This embarked Elliott on a series of key researches leading to his classic 1958 papers[11]. He found that collective deformation with its associated rotational spectra is obtainable by considering particles in a harmonic oscillator

common potential and interacting with a two-body force of angular dependence P_2 (cos θ), i.e., the Q^2 force, which is diagonal in an SU_3 basis of mixed-configurational states ---- SU_3 refers to the group of unimodular unitary transformations in three dimensions. The SU_3 states referred to laboratory axes are characterized by definite orbital angular momentum L and projection M, as well as by an approximate quantum number K (appearing to be related to the rotational band quantum number K of Figure 2), the SU_3 irreducible representation $(\lambda\mu)$ and orbital premutation symmetry $[f]$. These $\Psi([f](\lambda\mu)KLM)$ are projected out of intrinsic functions $\chi([f](\lambda\mu)K)$ referred to a nuclear axis. The χ functions are classified by the subgroup U_2 in addition to SU_3 and are eigenfunctions of an anisotropic axially symmetric harmonic oscillator potential. For N=2,3,4 particles in the 2s-1d unfilled shell he found good overlaps between L-S coupled shell model wave functions and his SU_3 basis set of states corresponding to the leading (lowest-energy) representation of SU_3.

For nuclei with a few particles in the 1-p shell Kurath & Picman[23] (1959) found strong overlaps between wave functions constructed by the Elliott SU_3 technique applied to Nilsson intrinsic states and shell model intermediate coupling (jj and LS) wave functions from calculations[24] with a two-body central force in the limit of zero spin-orbit force. Similar results for nuclei at the beginning of the 2s-1d shell were found by Redlich.[25]

The Elliott SU_3 technique was applied extensively to nuclei in the 1p shell ---- with the inclusion of spin-orbit interaction ---- by Koltun[26] (1961) with results comparing favorably with earlier intermediate coupling calculations by Kurath[25] (1956).

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These techniques were extended to the 2s-1d shell by Banerjee & Levinson[27] (1963). Calculations on Mg^{24} (N=8 in 2s-1d shell) were carried out by Elliott & Harvey[78] who found a small (10 to 20%) mixing of other SU_3 representations into the leading one for this nucleus. Chacón & Moshinsky[29] (1962) calculated the low-lying levels of Ne^{20} using a competitive mixture of P and Q^2 forces and, separately, under a gaussian central potential. A remarkable resemblance between the two predicted level-schemes emerged, as well as excellent agreement for the very lowest excited levels. The Ne^{20} E2 transition lifetimes were calculated on this model by the author and co-workers[30] (1963) showing the tendency of Q^2 to deform states (decrease lifetimes) and P to produce more spherical states (increase lifetimes).

The method to be used in our work will be within the <u>third</u> approach mentioned in the beginning of this chapter. Thus, our model will comprise P, Q^2 and spin-orbit interactions whose group symmetries shall be employed to advantage. There seems to be no "a priori" reason why this interaction hamiltonian model (involving a very small number of parameters) should be restricted to certain regions of the table of nuclides.

CHAPTER

III.- THE MOSHINSKY GROUP THEORETICAL REFORMULATION OF THE NUCLEAR SHELL MANY - BODY PROBLEM.[31)]

1. Creation and Annihilation Fermi Operators.

A single particle in a state ρ is defined by a creation operator acting on a vacuum state, namely

$$b_\rho^+ |0\rangle$$

which corresponds to $\psi_\rho(\vec{r})$. An annihilation operator is given simply by the contravariant operator b^ρ where

$$b^\rho |0\rangle \equiv 0. \tag{11}$$

They obey the Fermi anti-commutation relations

$$\begin{aligned} \{b_\rho^+, b^{\rho'}\} &\equiv b_\rho^+ b^{\rho'} + b^{\rho'} b_\rho^+ = \delta_\rho^{\rho'} \\ \{b_\rho^+, b_{\rho'}^+\} &= \{b^\rho, b^{\rho'}\} = 0 \end{aligned} \tag{12a,b,c}$$

and hence, expanding the left-hand side of the second relation

$$\{b_\rho^+, b_{\rho'}^+\} = b_\rho^+ b_{\rho'}^+ + b_{\rho'}^+ b_\rho^+ = 0$$

$$b_\rho^+ b_{\rho'}^+ = -b_{\rho'}^+ b_\rho^+$$

it becomes obvious that if $\rho = \rho'$, $b_\rho^+ b_\rho^+ = 0$, i.e., the Pauli principle is satisfied: one and only one particle can be in the state ρ. For particles in a common central potential the

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state ρ is given by the assembly

$$\rho \rightarrow \nu \ell m, \sigma \tau \tag{14}$$

where ν is the principal quantum number, ℓ the orbital angular momentum, m the magnetic quantum number, σ the spin projection along an arbitrary axis and τ the isotopic spin projection.

The allowed values of ρ thus label the dimensions of a single-particle total space composed of a coordinate (or orbital) subspace ($\nu \ell m$) and a spin-isospin subspace ($\sigma\tau$). The ρ-dimensional vector defining this space is b_ρ^+. An equivalent assembly of quantum numbers for the single-particle state could be

$$\rho \rightarrow \nu \ell j m_j, \tau \tag{15}$$

where j results from coupling $\vec{\ell}$ with $\vec{1/2}$ and m_j is its projection. This choice is more appropriate for j-j coupling and here the single-particle space is decomposed into "spin-orbital" ($\nu \ell j m_j$) and "isotopic spin" sub-spaces.

2. <u>State of N-noninteracting fermions</u>.

A state of N non-interacting fermions is usually given by the normalised Slater determinant of single particle functions which is totally anti-symmetric under particle exchange:

$$\Phi \equiv \frac{1}{\sqrt{N!}} \sum_P (-)^P P \psi_{\rho_1}(1) \psi_{\rho_2}(2) \ldots \psi_{\rho_N}(N) \equiv \frac{1}{\sqrt{N!}} \begin{vmatrix} \psi_{\rho_1}(1) & \psi_{\rho_1}(2) & \cdots & \psi_{\rho_1}(N) \\ \psi_{\rho_2}(1) & \psi_{\rho_2}(2) & \cdots & \vdots \\ \vdots & & & \vdots \\ \psi_{\rho_N}(1) & \cdots & \cdots & \psi_{\rho_N}(N) \end{vmatrix} \tag{16}$$

where P is a permutation of two variables from the set 1,2,....,N. Taking another function

$$\Phi' \equiv \frac{1}{\sqrt{N!}} \sum_P (-)^P P \psi_{\rho_1'}(1) \psi_{\rho_2'}(2) \cdots \psi_{\rho_N'}(N) \qquad (17)$$

and considering the scalar product between Φ and Φ' one obtains the well-known result

$$(\Phi', \Phi) = \begin{vmatrix} \delta_{\rho_1}^{\rho_1'} & \delta_{\rho_1}^{\rho_2'} & \cdots & \delta_{\rho_1}^{\rho_N'} \\ \delta_{\rho_2}^{\rho_1'} & \delta_{\rho_2}^{\rho_2'} & \cdots & \vdots \\ \vdots & & & \vdots \\ \delta_{\rho_N}^{\rho_1'} & \cdots & \cdots & \delta_{\rho_N}^{\rho_N'} \end{vmatrix} \equiv \delta_{\rho_1 \rho_2 \cdots \rho_N}^{\rho_1' \rho_2' \cdots \rho_N'} \qquad (18)$$

which is called a "generalized delta function".

Now, since ψ_{ρ_i} corresponds to $b_{\rho_i}^{+}|0\rangle$ in second-quantization formulation it is natural to assume as the equivalent of Φ the expression

$$|\rho_1 \rho_2 \cdots \rho_N\rangle \equiv b_{\rho_1}^{+} b_{\rho_2}^{+} \cdots b_{\rho_N}^{+} |0\rangle \qquad (19)$$

which from the anti-commutation relation (12b) is clearly anti-symmetric under particle exchange and therefore also satisfies the exclusion principle. The scalar product of (19) with another state $|\rho_1' \rho_2' \cdots \rho_N'\rangle$ can easily be shown to give

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$$\langle \rho_1' \rho_2' \cdots \rho_N' | \rho_1 \rho_2 \cdots \rho_N \rangle = \langle 0 | b^{\rho_N'} \cdots b^{\rho_2'} b^{\rho_1'} b^{+}_{\rho_1} b^{+}_{\rho_2} \cdots b^{+}_{\rho_N} | 0 \rangle$$
$$= \delta^{\rho_1' \rho_2' \cdots \rho_N'}_{\rho_1 \rho_2 \cdots \rho_N} \qquad (20)$$

using the relations (12) and the fact that $b^{\rho} | 0 \rangle = 0$, and this is identical with the result (18) for (Φ', Φ).

3. <u>Single-and Two-Body Operators as Linear and Bilinear Combinations of Group Generators</u>.

An N-particle state <u>with interactions</u>, specifiable by a set of N-particle quantum numbers λ , can always be given as a <u>superposition</u> of linearly independent states of N-noninteracting-particles

$$P_\lambda = \sum_{\rho_1 \rho_2 \cdots \rho_N} B^{\lambda}_{\rho_1 \rho_2 \cdots \rho_N} \; b^{+}_{\rho_1} b^{+}_{\rho_2} \cdots b^{+}_{\rho_N} | 0 \rangle \qquad (21)$$

with arbitrary coefficients $B^{\lambda}_{\rho_1 \rho_2 \cdots \rho_N}$. These coefficients will be determined by the operator <u>set</u> $\mathcal{O}$ representing a complete set of commuting observables required to characterize the polynomial base (21) exhaustively via the eigenvalue equations

$$\mathcal{O} P_\lambda = o^\lambda P_\lambda \, . \qquad (22)$$

It is desireable therefore to obtain general operators which depend on the creation and annihilation operators b^{+}_{ρ} and $b^{\rho'}$. The most common of operators are symmetric one-and two-body

scalar operators defined as

$$\sum_{i=1}^{N} W(r_i) \text{ and } \sum_{i<j=1}^{N} V(r_{ij}) \tag{23}$$

where r_i stands for all the coordinates of the $i^{\underline{th}}$ particle and $r_{ij} \equiv |\vec{r}_i - \vec{r}_j|$. Let us call $W(r_i) \equiv W_i$ and $V(r_{ij}) \equiv V_{ij}$ and their matrix elements in the usual form are

$$\langle \eta_i | W_i | \eta_i' \rangle = \int \psi_{\eta_i}^*(i)\, W_i\, \psi_{\eta_i'}(i)\, d\tau_i$$

$$\langle \eta_i \eta_j | V_{ij} | \eta_i' \eta_j' \rangle = \iint \psi_{\eta_i}(i)\, \psi_{\eta_j}(j)\, V_{ij}\, \psi_{\eta_i'}(i)\, \psi_{\eta_j'}(j)\, d\tau_i\, d\tau_j .$$

Operators in the second-quantization formulation are usually postulated as

$$\mathcal{W} \equiv \sum_{\eta_1 \eta_1'} \langle \eta_1 | W_1 | \eta_1' \rangle\, b^+_{\eta_1} b^{\eta_1'} \tag{24}$$

$$\mathcal{V} \equiv \frac{1}{2} \sum_{\substack{\eta_1 \eta_2 \\ \eta_1' \eta_2'}} \langle \eta_1 \eta_2 | V_{12} | \eta_1' \eta_2' \rangle\, b^+_{\eta_2} b^+_{\eta_1} b^{\eta_1'} b^{\eta_2'} \tag{25}$$

(η_i being the complete set of quantum numbers needed to specify the ith single-particle state, i.e., states of the type ρ_i of the previous section). These postulates are considered valid if matrix

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elements in the second-quantization scheme are identical with the corresponding ones in the usual Born-Heisenberg-Jordan scheme of matrix mechanics. Calculating the elements of $\mathcal{W}$ between non-interacting states we have

$$\langle \rho_1' \rho_2' \cdots \rho_N' | \mathcal{W} | \rho_1 \rho_2 \cdots \rho_N \rangle =$$

$$= \sum_{\eta_1 \eta_1'} \langle \eta_1 | W_1 | \eta_1' \rangle \langle 0 | b^{\rho_N'} \cdots b^{\rho_2'} b^{\rho_1'} (b^+_{\eta_1} b^{\eta_1'}) b^+_{\rho_1} b^+_{\rho_2} \cdots b^+_{\rho_N} | 0 \rangle$$

$$= \sum_{\eta_1 \eta_1'} \langle \eta_1 | W_1 | \eta_1' \rangle \langle 0 | b^{\rho_N'} \cdots b^{\rho_2'} b^{\rho_1'} (\delta_{\eta_1}^{\eta_1'} - b^{\eta_1'} b^+_{\eta_1}) b^+_{\rho_1} b^+_{\rho_2} \cdots b^+_{\rho_N} | 0 \rangle$$

having used the relations (12a) in the last step, so that

$$\langle \rho_1' \rho_2' \cdots \rho_N' | \mathcal{W} | \rho_1 \rho_2 \cdots \rho_N \rangle =$$

$$\sum_{\eta_1 \eta_1'} \langle \eta_1 | W_1 | \eta_1' \rangle \left\{ \delta_{\eta_1}^{\eta_1'} \delta_{\rho_1 \rho_2 \cdots \rho_N}^{\rho_1' \rho_2' \cdots \rho_N'} - \delta_{\eta_1 \rho_1 \rho_2 \cdots \rho_N}^{\eta_1' \rho_1' \rho_2' \cdots \rho_N'} \right\} \tag{26}$$

which is indeed the usual result (see Condon & Shortley[32]) pp. 169-74 of matrix mechanics for the matrix elements of a single-body operator between two Slater determinant states Φ' and Φ. Carrying ont a similar calculation for $\mathcal{V}$ one obtains

$$\langle \rho_1' \rho_2' \cdots \rho_N' | \mathcal{V} | \rho_1 \rho_2 \cdots \rho_N \rangle =$$

$$= \frac{1}{2}\sum_{\substack{\eta_1\eta_2\\ \eta_1'\eta_2'}} \langle \eta_1\eta_2|V_{12}|\eta_1'\eta_2'\rangle \langle 0| b^{\rho_N'}\cdots b^{\rho_2'}b^{\rho_1'}(b^+_{\eta_2}b^+_{\eta_1}b^{\eta_1'}b^{\eta_2'})b^+_{\rho_1}b^+_{\rho_2}\cdots b^+_{\rho_N}|0\rangle$$

$$= \frac{1}{2}\sum_{\substack{\eta_1\eta_2\\ \eta_1'\eta_2'}} \langle \eta_1\eta_2|V_{12}|\eta_1'\eta_2'\rangle \langle 0| b^{\rho_N'}\cdots b^{\rho_2'}b^{\rho_1'}(\delta^{\eta_1'}_{\eta_1}\delta^{\eta_2'}_{\eta_2} - \delta^{\eta_1'}_{\eta_1}b^{\eta_2'}b^+_{\eta_2}$$

$$-\delta^{\eta_1'}_{\eta_2}\delta^{\eta_2'}_{\eta_1} + \delta^{\eta_1'}_{\eta_2}b^{\eta_2'}b^+_{\eta_1} + \delta^{\eta_2'}_{\eta_1}b^{\eta_1'}b^+_{\eta_2} - \delta^{\eta_2'}_{\eta_2}b^{\eta_1'}b^+_{\eta_1} + b^{\eta_1'}b^{\eta_2'}b^+_{\eta_2}b^+_{\eta_1})\times$$

$$\times\, b^+_{\rho_1}b^+_{\rho_2}\cdots b^+_{\rho_N}|0\rangle \qquad (27)$$

where the last step is arrived at by applying the anti-commutation relations (12 a,b,c) to the expression

$$b^+_{\eta_2}b^+_{\eta_1}b^{\eta_1'}b^{\eta_2'}$$

such as to push the creation operators to the <u>right</u> and the annihilation ones to the <u>left</u>. Noting in (27) that the terms

$$\delta^{\eta_1'}_{\eta_1}\delta^{\eta_2'}_{\eta_2} - \delta^{\eta_1'}_{\eta_2}\delta^{\eta_2'}_{\eta_1} \equiv \begin{vmatrix} \delta^{\eta_1'}_{\eta_1} & \delta^{\eta_1'}_{\eta_2} \\ \delta^{\eta_2'}_{\eta_1} & \delta^{\eta_2'}_{\eta_2} \end{vmatrix} \equiv \delta^{\eta_1'\eta_2'}_{\eta_1\eta_2}$$

and utilizing the scalar product formula (20) one arrives at the

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final result that

$$\langle \rho_1'\rho_2'\cdots\rho_N' | \mathcal{V} | \rho_1\rho_2\cdots\rho_N \rangle =$$

$$\frac{1}{2}\sum_{\substack{\eta_1\eta_2\\ \eta_1'\eta_2'}} \langle \eta_1\eta_2 | V_{12} | \eta_1'\eta_2' \rangle \Big\{ \delta^{\eta_1'\eta_2'}_{\eta_1\eta_2}\, \delta^{\rho_1'\rho_2'\cdots\rho_N'}_{\rho_1\rho_2\cdots\rho_N}$$

$$-\delta^{\eta_1'}_{\eta_1}\, \delta^{\eta_2'\rho_1'\cdots\rho_N'}_{\eta_2\rho_1\cdots\rho_N} + \delta^{\eta_1'}_{\eta_2}\, \delta^{\eta_2'\rho_1'\cdots\rho_N'}_{\eta_1\rho_1\cdots\rho_N} + \delta^{\eta_2'}_{\eta_1}\, \delta^{\eta_1'\rho_1'\cdots\rho_N'}_{\eta_2\rho_1\cdots\rho_N}$$

$$-\delta^{\eta_2'}_{\eta_2}\, \delta^{\eta_1'\rho_1'\cdots\rho_N'}_{\eta_1\rho_1\cdots\rho_N} + \delta^{\eta_2'\eta_1'\rho_1'\cdots\rho_N'}_{\eta_2\eta_1\rho_1\cdots\rho_N} \Big\} \qquad (28)$$

whose properties are identical with those given in Condon & Shortley,[32] pp. 169-74 the matrix elements of a two-body operator in the usual formulation. In conclusion, therefore, one has the required identities

$$\langle \rho_1'\rho_2'\cdots\rho_N' | \mathcal{W} | \rho_1\rho_2\cdots\rho_N \rangle = \left(\Phi', \sum_{i=1}^{N} W_i\, \Phi\right) \qquad (29)$$

$$\langle \rho_1'\rho_2'\cdots\rho_N' | \mathcal{V} | \rho_1\rho_2\cdots\rho_N \rangle = \left(\Phi', \sum_{i<j=1}^{N} V_{ij}\, \Phi\right) \qquad (30)$$

Φ and Φ' being the Slater determinental states (16) and (17), thus justifying postulates (24) and (25)

Operators in the second-quantization form are used extensively in current theoretical physics. When states $|\rho_1\rho_2\cdots\rho_N\rangle$ are given in terms of creation operators $b^+_{\rho_i}$ as in (19), the usually difficulty and tedious problem of calculating matrix elements between superposed states of this type is reduced to the still tedious but intrinsically simpler task of anti-commuting b^+_ρ, $b^{\rho'}$ operators. In effect, however, even this is avoided as quicker and simpler techniques have been found for evaluating matrix elements. It may also be mentioned in passing that the second-quantization formulation is not necessarily restricted to the treatment of N fermions: a similar formulation can be derived replacing the relations (12 a,b,c) between fermi operators by <u>commutation</u> relations between boson operators for the treatment of such problems as phonon excitations in the vibrational nuclear model. Finally, we should anticipate the fact that contrary to second-quantization formulisms of the field-theory type, the total number of particles (nucleons, here) is <u>always</u> <u>conserved</u> as physical situations in low-energy nuclear physics demand.

Let us call the creation-annihilation pair $b^+_{\rho_i} b^{\rho_i'}$ by another name, for example

$$C^{\rho_i'}_{\rho_i} \equiv b^+_{\rho_i} b^{\rho_i'} \qquad (31)$$

where it is understood that this operator when acting on an arbitrary

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state destroys a single-particle state given by the quantum numbers ρ_i' and creates one given by ρ_i explicitly,

$$\mathbb{C}_{\rho_i}^{\rho_i'} |\rho_1 \rho_2 \cdots \rho_i' \cdots \rho_N\rangle = |\rho_1 \rho_2 \cdots \rho_i \cdots \rho_N\rangle \qquad (31)$$
$$= 0 \text{ (if } \rho_i' \text{ is not present)}$$

We can find the commutation relations between the $\mathbb{C}_\rho^{\rho'}$'s by using the relations (12 a,b,c):

$$\begin{aligned}
[\mathbb{C}_\rho^{\rho'}, \mathbb{C}_{\bar\rho}^{\bar\rho'}] &\equiv [b_\rho^+ b^{\rho'}, b_{\bar\rho}^+ b^{\bar\rho'}] \\
&= b_\rho^+ [b^{\rho'}, b_{\bar\rho}^+ b^{\bar\rho'}] + [b_\rho^+, b_{\bar\rho}^+ b^{\bar\rho'}] b^{\rho'} \\
&= b_\rho^+ b_{\bar\rho}^+ [b^{\rho'}, b^{\bar\rho'}] + b_\rho^+ [b^{\rho'}, b_{\bar\rho}^+] b^{\bar\rho'} \\
&\quad + b_{\bar\rho}^+ [b_\rho^+, b^{\bar\rho'}] b^{\rho'} + [b_\rho^+, b_{\bar\rho}^+] b^{\bar\rho'} b^{\rho'} \\
&= 2 b_\rho^+ b_{\bar\rho}^+ b^{\rho'} b^{\bar\rho'} + b_\rho^+ b^{\bar\rho'} \delta_{\bar\rho}^{\rho'} - 2 b_\rho^+ b_{\bar\rho}^+ b^{\rho'} b^{\bar\rho'} \\
&\quad + 2 b_{\bar\rho}^+ b_\rho^+ b^{\bar\rho'} b^{\rho'} - b_{\bar\rho}^+ b^{\rho'} \delta_\rho^{\bar\rho'} - 2 b_{\bar\rho}^+ b_\rho^+ b^{\bar\rho'} b^{\rho'} \\
&= b_\rho^+ b^{\bar\rho'} \delta_{\bar\rho}^{\rho'} - b_{\bar\rho}^+ b^{\rho'} \delta_\rho^{\bar\rho'}
\end{aligned}$$

$$\therefore [\mathbb{C}_\rho^{\rho'}, \mathbb{C}_{\bar\rho}^{\bar\rho'}] = \mathbb{C}_\rho^{\bar\rho'} \delta_{\bar\rho}^{\rho'} - \mathbb{C}_{\bar\rho}^{\rho'} \delta_\rho^{\bar\rho'}. \qquad (32)$$

As Moshinsky realized[33], this result is extremely important: the commutator of two $C_\rho^{\rho'}$'s is expressible as other $C_\rho^{\rho'}$'s. Exactly the same thing occurs with the angular momentum operators, namely,

$$[L_x, L_y] = i L_z \qquad \text{(and cyclically)}$$

are the generators for infinitesimal transformations (rotations) of the group R_3. (A better insight into the reason for calling L_x, L_y, L_z the R_3 generators for infinitesimal rotations is given in Appendix C). Moshinsky's argument (see Appendix) concludes that relations (32) indicate that the set of operators $\{C_\rho^{\rho'}\}$ forms the generators of a <u>group of unitary transformations</u>. The dimension of this group is the dimension of the space wherin the transformations take place, i.e., the possible values of the quantum number set $\rho = \nu l m, \sigma\tau$. Let the orbital sub-space $\nu l m$ be r-dimensional, the spin-isospin sub-space is 4-dimensional since $\sigma\tau = \frac{1}{2}\frac{1}{2}, \frac{1}{2}\text{-}\frac{1}{2}, \text{-}\frac{1}{2}\frac{1}{2}, \text{-}\frac{1}{2}\text{-}\frac{1}{2}$. Thus the set of $(4r)^2$ operators $\{C_\rho^{\rho'}\}$ form the generators of a $4r$-dimensional unitary group U_{4r} whose Lie Algebra[34] is given by (32).

Now, the single-body operator $\mathcal{W}$ of (24) is obviously the linear combination of U_{4r} group generators

$$\mathcal{W} = \sum_{\rho_1 \rho_1'} \langle \rho_1 | W_1 | \rho_1' \rangle C_{\rho_1}^{\rho_1'} \tag{33}$$

with relatively simple coefficients as only one-particle states are involved. For the two-body operator $\mathcal{V}$ of (25) we use (12 a,b,c) to convert

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$$
\begin{aligned}
b^{+}_{\rho_2} b^{+}_{\rho_1} b^{\rho_1'} b^{\rho_2'} &= - b^{+}_{\rho_1} b^{+}_{\rho_2} b^{\rho_1'} b^{\rho_2'} \\
&= - b^{+}_{\rho_1} (\delta^{\rho_1'}_{\rho_2} - b^{\rho_1'} b^{+}_{\rho_2}) b^{\rho_2'} \\
&= b^{+}_{\rho_1} b^{\rho_1'} b^{+}_{\rho_2} b^{\rho_2'} - \delta^{\rho_1'}_{\rho_2} b^{+}_{\rho_1} b^{\rho_2'} \\
&= \mathbb{C}^{\rho_1'}_{\rho_1} \mathbb{C}^{\rho_2'}_{\rho_2} - \delta^{\rho_1'}_{\rho_2} \mathbb{C}^{\rho_2'}_{\rho_1}
\end{aligned}
$$

so that $\mathcal{V}$ is the bilinear combination of $\mathbb{U}_{4r}$ group generators

$$
\mathcal{V} = \frac{1}{2} \sum_{\substack{\rho_1 \rho_2 \\ \rho_1' \rho_2'}} \langle \rho_1 \rho_2 | V_{12} | \rho_1' \rho_2' \rangle \left\{ \mathbb{C}^{\rho_1'}_{\rho_1} \mathbb{C}^{\rho_2'}_{\rho_2} - \delta^{\rho_1'}_{\rho_2} \mathbb{C}^{\rho_2'}_{\rho_1} \right\} \qquad (34)
$$

with coefficients depending only on two-particle states.

A large class of problems involve one-and two-body operators which are <u>independent of spin and isospin</u>. Considering this restriction, and since

$$
\rho_i \rightarrow \nu_i l_i m_i, \sigma_i \tau_i \equiv \mu_i, s_i
$$

where μ_i refers to the configuration (orbital) space quantum numbers of the $i^{\underline{th}}$ particle and s_i to spin and isospin, the coefficients in (33) and (34) will be

$$\langle \rho_1 | W_1 | \rho_1' \rangle \equiv \langle \mu_1 s_1 | W_1 | \mu_1' s_1' \rangle = \langle \mu_1 | W_1 | \mu_1' \rangle \delta_{s_1}^{s_1'}$$

$$\langle \rho_1 \rho_2 | V_{12} | \rho_1' \rho_2' \rangle = \langle \mu_1 s_1, \mu_2 s_2 | V_{12} | \mu_1' s_1', \mu_2' s_2' \rangle$$
$$= \langle \mu_1 \mu_2 | V_{12} | \mu_1' \mu_2' \rangle \delta_{s_1}^{s_1'} \delta_{s_2}^{s_2'}$$

because of the imposed independence of W_i and V_{ij} on s. Inserting these expressions in (33) and (34) one gets

$$\mathcal{W} = \sum_{\mu_1 \mu_1'} \langle \mu_1 | W_1 | \mu_1' \rangle b_{\mu_1}^{\mu_1'} \tag{35}$$

$$\mathcal{V} = \frac{1}{2} \sum_{\substack{\mu_1 \mu_2 \\ \mu_1' \mu_2'}} \langle \mu_1 \mu_2 | V_{12} | \mu_1' \mu_2' \rangle \left\{ b_{\mu_1}^{\mu_1'} b_{\mu_2}^{\mu_2'} - \delta_{\mu_2}^{\mu_1'} b_{\mu_1}^{\mu_2'} \right\} \tag{36}$$

where the new operators $b_{\mu}^{\mu'}$ are simply a contraction of the old ones over spin-isospin indices, that is .

$$b_{\mu}^{\mu'} \equiv \sum_{ss'} \delta_s^{s'} C_{\rho}^{\rho'} = \sum_{ss'} \delta_s^{s'} C_{\mu s}^{\mu' s'}$$

$$\text{or} \quad b_{\mu}^{\mu'} \equiv \sum_{s} C_{\mu s}^{\mu' s} \quad (\mu, \mu' = 1, 2, \ldots, r) \tag{37}$$

Writing in (32) μs for ρ and contracting over s we obtain the commutator algebra

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$$[\mathcal{C}_{\mu}^{\mu'}, \mathcal{C}_{\bar{\mu}}^{\bar{\mu}'}] = \mathcal{C}_{\mu}^{\bar{\mu}'}\delta_{\bar{\mu}}^{\mu'} - \mathcal{C}_{\bar{\mu}}^{\mu'}\delta_{\mu}^{\bar{\mu}'} \qquad (38)$$

which, by the argument upon (32) reveals that the r^2 operators $\mathcal{C}_{\mu}^{\mu'}$ are the generators of an r-dimensional unitary group, $\mathcal{U}_r$, of unitary transformations within the orbital sub-space of r dimensions.

From the analysis of this section, an analogy with the asymmetric top and its allowable energies of the Introduction is thus evident: the hamiltonian $\mathcal{H}_T$ in (1) is quadratic in the R_3 generators while here a nuclear hamiltonian $\mathcal{H}$ with group symmetry greater than R_3, composed of both $\mathcal{W}$ and $\mathcal{V}$ types of interactions, is expressible as linear and bilinear combinations of the generators of a larger group, in general, $\mathbb{U}_{4r}$. To evaluate the $\|\mathcal{H}_T\|$ matrix of (3) in the $|\ell m\rangle$ basis for an irreducible representation of $R_3 \supset R_2$ only the matrix elements $\langle \ell m' | L_1 | \ell m \rangle$ were needed. In our nuclear problem, to obtain the $\|\mathcal{H}\|$ matrix we would need the matrix elements of the symmetry group generators. Although these are known for a unitary group of arbitrary dimension,[75] other approaches will be necessary as solving the problem via the top analogy imposes great difficulties.

Our approach will be to construct a (truncated) basis which transforms irreducibly according to the various symmetry groups of a chosen portion of the model hamiltonian. To accomplish this, one begins with the problem of <u>classifying</u> the pertinent N-particle states with appropriate quantum numbers. However, a brief sketch of the treatment of this problem in R_3 will be most useful.

4.- Spherical Symmetry and the R_3 Group.

Consider first how one obtains in group theoretic language a basis for an irreducible representation of the group of three-dimensional rotations R_3 and the labelling of these states. The generators of infinitesimal rotations around the x,y and z axes are respectively L_x, L_y and L_z, where

$$
\begin{aligned}
L_x &= (\vec{r}\times\vec{p})_x = y p_z - z p_y = \frac{1}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \\
L_y &= (\vec{r}\times\vec{p})_y = z p_x - x p_z = \frac{1}{i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \\
L_z &= (\vec{r}\times\vec{p})_z = x p_y - y p_x = \frac{1}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right).
\end{aligned}
\tag{39}
$$

In spherical components one has the three generators

$$
\begin{aligned}
L_+ &\equiv L_x + i L_y \\
L_0 &\equiv L_z \\
L_- &\equiv L_x - i L_y
\end{aligned}
\tag{40}
$$

which by $[L_x, L_y] = i L_z$ (and cyclically) obey the Lie algebra

$$[L_0, L_\pm] = \pm L_\pm \quad , \quad [L_+, L_-] = 2L_0 \tag{41}$$

of which only L_0 is hermitean since

$$(L_\pm)^\dagger = L_\mp \quad \text{and} \quad L_0^\dagger = L_0 \, ,$$

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L_x, L_y, L_z being hermitean. Rotations are <u>linear transformations</u> and therefore the basis functions should be linearly independent homogeneous polynomials in the spherical space-components x_+, x_0 and x_- of fixed degree, say h, which are solutions of the Euler equation

$$\vec{r}\cdot\nabla\, P(x_+, x_0, x_-) = h\, P(x_+, x_0, x_-). \tag{43}$$

These solutions however form a basis for a <u>reducible</u> representation. To decompose this set into <u>irreducible</u> sub-sets a further restriction on P must be imposed, namely

$$L_0 P = m P \tag{44}$$

using the hermitean operator L_0 which commutes with the previous operator $\vec{r}\cdot\nabla$ so that it can be employed simultaneously with $\vec{r}\cdot\nabla$ to further caracterize the polynomial by the integer m. The <u>weight</u> of the polynomial P is defined by m. Since L_+ and L_- also commute with $\vec{r}\cdot\nabla$ the polynomials $P' = L_+P$ and $P'' = L_-P$ both satisfy (43). What weights do P' and P" have? Using the relations (41) one has

$$\begin{aligned} L_0 P' = L_0 L_+ P &= [L_0, L_+] P + L_+ L_0 P \\ &= L_+ P + m L_+ P \\ &= (m+1) P'. \end{aligned}$$

$$L_0 P'' = L_0 L_- P = [L_0, L_-] P + L_- L_0 P$$
$$= -L_- P + m L_- P$$
$$= (m-1) P''.$$

The operator L_+ thus <u>raises</u> the weight of an irreducible basis P by one unit and L_- <u>lowers</u> the weight by one unit. L_+ and L_- are therefore called the <u>raising</u> and <u>lowering</u> <u>operators</u> of the group R_3, while L_0 is the <u>weight operator</u>.

Among the solutions of (43) there are some polynomials of <u>highest weight</u> $\mathbb{P}$ given by the three equations

$$\vec{r}\cdot\nabla \mathbb{P} = h \mathbb{P} \tag{45}$$

$$L_0 \mathbb{P} = \ell \mathbb{P} \tag{46}$$

and

$$L_+ \mathbb{P} = 0. \tag{47}$$

From (45) we take the general form

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$$\mathbb{P}(x_+, x_0, x_-) = \sum_{\alpha\beta\gamma} C_{\alpha\beta\gamma}\, x_+^{\alpha}\, x_-^{\beta}\, x_0^{\gamma} \qquad (\alpha+\beta+\gamma=h),$$

in spherical components which, like (40), are $x_\pm = x \pm iy$, $x_0 = z$. Applying the restriction (46) with $L_0 = x_+ \partial/\partial x_+ - x_- \partial/\partial x_-$ gives the relation $\beta = \alpha - \ell$ and (47) with $L_+ = -(x_+ \partial/\partial x_0 + x_0 \partial/\partial x_-)$ gives a recurrence relation between the remaining coefficients so that

$$\mathbb{P}(x_+, x_0, x_-) = A_0\, x_+^{\ell} \sum_{\alpha=0}^{\frac{1}{2}(h-\ell)} \binom{\frac{h-\ell}{2}}{\alpha} x_0^{2\left(\frac{h-\ell}{2}-\alpha\right)} (-2x_+x_-)^{\alpha}$$

with A_0 an arbitrary constant and $\binom{a}{b}$ the binomial coefficients finally yields the solution to (45,46,47) as

$$\mathbb{P}_h = A_0\, x_+^{\ell}\, (x_0^2 - 2x_+x_-)^{\frac{h-\ell}{2}} \tag{48}$$

with

$$h = \ell, \ell-2, \ell-4, \ldots, \begin{matrix}1\\0\end{matrix} \tag{49}$$

But the term $(x_0^2 - 2x_+x_-)$ is simply $r^2 = x^2 + y^2 + z^2$ so that one has

$$\mathbb{P}_{h\ell} = A_0\, x_+^{\ell}\, r^{h-\ell} \quad \left(h = \ell, \ell-2, \ell-4, \ldots, \begin{matrix}1\\0\end{matrix}\right) \tag{51}$$

which (apart from an invariant $r^{h-\ell}$ and a numerical constant) is the familiar solid spherical harmonic of maximum projection

$$\mathbb{P}_{h\ell} \sim r^{h-\ell}\, \mathcal{Y}_{\ell\ell}(\theta,\varphi)$$
$$\mathcal{Y}_{\ell\ell}(r,\theta,\varphi) \equiv r^{\ell}\, Y_{\ell\ell}(\theta,\varphi). \tag{52a,b}$$

To summarize, the three equations

$$\vec{r}\cdot\nabla \mathbb{P} = h\,\mathbb{P}, \qquad L_0 \mathbb{P} = \ell\,\mathbb{P}, \qquad L_+ \mathbb{P} = 0$$

ensure solutions (51), or (52), which for the simplest case $h=\ell$ (maximum representation) then $\mathbb{P}_\ell \sim \mathcal{Y}_{\ell\ell}(\theta,\varphi)$ is the maximum weight polynomial of the basis for the irreducible representation labeled by ℓ. The full basis can be generated by successive application of the lowering operator L_-, giving us the rows m of the represention ℓ thusly:

$$P_{\ell m} = \sqrt{\frac{(\ell+m)!}{(\ell-m)!\,(2\ell)!}}\,(L_-)^{\ell-m}\,\mathbb{P}_\ell = r^\ell\, Y_{\ell m}(\theta,\varphi) \tag{53}$$

$$P_{\ell m} = \mathcal{Y}_{\ell m}(r,\theta,\varphi)$$

where the radical in front of the lowering operation $(L_-)^{\ell-m}$ is for normalization[35] and m takes the $(2\ell+1)$ allowed values

$$-\ell \leq m \leq \ell \tag{54}$$

which are unique for a given ℓ-values, i.e., the set of numbers m labelling the rows is multiplicity free. The polynomial basis $P_{\ell m}$ is clearly an eigenfunction of the commuting operators L^2 and L_z,

$$L^2 P_{\ell m} = \ell(\ell+1)\,P_{\ell m}$$

$$L_z P_{\ell m} = m\,P_{\ell m} \tag{55a,b}$$

and the operator L^2 is referred to as the Casimir operator[36, 37, 38] of R_3 as 1) it is constructed from the R_3 generators by

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$$L^2 = L_x^2 + L_y^2 + L_z^2 = \sum_q (-)^q L_q L_{-q} \qquad (q = +, -, 0) \qquad (56)$$

2) it commutes with them, namely $[L^2, L_q] = 0$ and 3) it suffices to characterize each irreducible representation of R_3 by the label ℓ. The L_z operator generates rotations around the z-axis and is thus the single generator of the group R_2 of axial rotations. Hence, the polynomial basis $P_{\ell m}$ is said to be explicitly reduced with respect to the chaim of groups

$$R_3 > \begin{pmatrix} R_2 & 0 \\ 0 & 1 \end{pmatrix} \qquad (57)$$

the 1 in the lower-right hand corner coming from the fact that axial rotations here are about the z-axis.

Exactly analogous techniques have been developed[39] for unitary group irreducible bases which are needed to construct wave functions for problems with symmetries larger than rotational.

5.- <u>Supermultiplet Classification of States by Unitary Groups. Permutational Symmetry</u>.

The group of unitary transformations within the total orbital-spin-isospin space of $4r$ dimensions is U_{4r} whose $(4r)^2$ generators are $C_\rho^{\rho'} \equiv b_\rho^+ b^{\rho'}$, ρ designating the components, which obey the Lie commutator algebra (32). Our transformations are linear, so the polynomial basis must be a set of linearly independent homogeneous polynomials P of degree N in the b_ρ^+. The set more-

over corresponds to the completely antisymmetric representation $[1^N 0^{4r-N}]$ of $\mathbb{U}_{4r}$. The scalar product between two such polynomials is defined as

$$(P_1, P_2) \equiv \langle 0 | P_1^+ P_2 | 0 \rangle \tag{58}$$

where in P_1^+, all b_ρ^+ are replaced by b^ρ, and thus evaluation of (58) depends only on the properties (12 a,b,c) and (11). The creation and annihilation operators themselves transform as

$$\mathbb{U}_{4r}: \quad \begin{aligned} b'^+_{\rho'} &= \sum_\rho \mathbb{U}^\rho_{\rho'} b^+_\rho \\ b'^{\rho'} &= \sum_\rho (\mathbb{U}^\rho_{\rho'})^* b^\rho \end{aligned} \qquad (\rho = 1, 2, \ldots, 4r) \tag{59}$$

the $\mathbb{U}^\rho_{\rho'}$ being elements of a unitary matrix.

As already seen in (37), contraction of $C_\rho^{\rho'} \equiv C_{\mu s}^{\mu' s'}$ over the index s gives an operator set $\{ \mathcal{E}_\mu^{\mu'} \}$ with Lie algebra (38) and therefore constitute the generators of a group $\mathcal{U}_r$ of unitary transformations in the orbital space of r dimensions, i.e.,

$$\mathcal{U}_r: \quad \begin{aligned} b'^+_{\mu' s} &= \sum_\mu \mathcal{U}^\mu_{\mu'} b^+_{\mu s} \\ b'^{\mu' s} &= \sum_\mu (\mathcal{U}^\mu_{\mu'})^* b^{\mu s}. \end{aligned} \qquad \begin{pmatrix} \mu = 1, 2, \ldots, r \\ s = 1, 2, 3, 4 \end{pmatrix} \tag{60}$$

Moreover, we can form the 4^2=16 operators

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$$C_s^{s'} \equiv \sum_\mu \delta_\mu^{\mu'} C_{\mu s}^{\mu' s'} = \sum_\mu C_{\mu s}^{\mu s'} \quad (s, s' = 1, 2, 3, 4) \qquad (61)$$

by contraction over the index μ , and see inmediately from (32), contracted in the same manner, that

$$[C_s^{s'}, C_{s''}^{s'''}] = C_s^{s'''} \delta_{s''}^{s} - C_{s''}^{s'} \delta_s^{s'''} \qquad (62)$$

is a Lie algebra identical with (38) and (32) so that the operator set $\{C_s^{s'}\}$ form the generators of a group U_4 of transformations

$$U_4 : \quad \begin{aligned} b'^{+}_{\mu s'} &= \sum_s U_{s'}^{s} b^{+}_{\mu s} \\ b'^{\mu s'} &= \sum_s (U_{s'}^{s})^{*} b^{\mu s}. \end{aligned}$$

The generators of $\mathcal{U}_r$ commute with those of U_4 :

$$[\mathcal{C}_\mu^{\mu'}, C_s^{s'}] = [\sum_{\bar{s}} C_{\mu\bar{s}}^{\mu'\bar{s}}, \sum_{\bar{\mu}} C_{\bar{\mu}s}^{\bar{\mu}s'}]$$

$$= \sum_{\bar{s},\bar{\mu}} [C_{\mu\bar{s}}^{\mu'\bar{s}}, C_{\bar{\mu}s}^{\bar{\mu}s'}]$$

$$= \sum_{\bar{s},\bar{\mu}} \left\{ C_{\mu\bar{s}}^{\bar{\mu}s'} \delta_{\bar{\mu}}^{\mu'} \delta_{s}^{\bar{s}} - C_{\bar{\mu}s}^{\mu'\bar{s}} \delta_{\mu}^{\bar{\mu}} \delta_{\bar{s}}^{s'} \right.$$

$$= \sum_{\bar{s},\bar{\mu}} \left\{ C_{\mu s}^{\mu's'} - C_{\mu s}^{\mu's'} \right\}$$

$$= 0. \tag{64}$$

recalling that $\delta_{\rho}^{\rho'} \equiv \delta_{\mu s}^{\mu's'} = \delta_{\mu}^{\mu'} \delta_{s}^{s'}$. Hence, the group $\mathbb{U}_{4r}$ of matrices $\| \mathbb{U}_{\mu's'}^{\mu s} \|$ contains the product sub-group $\mathcal{U}_r \times U_4$ of matrices $\| \mathcal{U}_{\mu'}^{\mu} U_{s'}^{s} \|$ and this contains separately the sub-groups $\mathcal{U}_r$ with $\| \mathcal{U}_{\mu'}^{\mu} \delta_{s'}^{s} \|$ and U_4 with $\| \delta_{\mu'}^{\mu} U_{s'}^{s} \|$. In shorthand:

$$\underset{\substack{\uparrow \\ \text{(orbital-spin-isospin)}}}{\mathbb{U}_{4r}} \supset \mathcal{U}_r \times U_4 \left\{ \begin{array}{l} \supset \mathcal{U}_r \longleftarrow \text{(orbital)} \\ \supset U_4 \longleftarrow \text{(spin-isospin)} \end{array} \right. \tag{65}$$

the names in parentheses referring to the single-particle transformation space.

Let us now define the irreducible representations connected with this chain of groups. The r operators b_{μ}^{μ} and 4 operators C_{s}^{s} are hermitean since

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$$(\mathcal{C}_\mu^{\ \mu})^+ \equiv \sum_s (b^+_{\mu s}\, b^{\mu s})^+ = \sum_s b^+_{\mu s}\, b^{\mu s} = \mathcal{C}_\mu^{\ \mu}$$

$$(C_s^{\ s})^+ \equiv \sum_\mu (b^+_{\mu s}\, b^{\mu s})^+ = \sum_\mu b^+_{\mu s}\, b^{\mu s} = C_s^{\ s} \qquad (66)$$

while $\mathcal{C}_\mu^{\ \mu'}$ ($\mu \neq \mu'$) and $C_s^{\ s'}$ ($s \neq s'$) have the hermiticity properties

$$(\mathcal{C}_\mu^{\ \mu'})^+ = \mathcal{C}_{\mu'}^{\ \mu} \ ; \ (C_s^{\ s'})^+ = C_{s'}^{\ s} .$$

In analogy with the R_3 hermitean operator L_0 one can define, as in (44), in $\mathcal{U}_r$ and simultaneously in U_4 , the weight of the set of homogeneous polynomials P of degree N by

$$\mathcal{C}_\mu^{\ \mu} P = w_\mu P \qquad (\mu = 1, 2, \ldots, r) \qquad (67)$$

$$C_s^{\ s} P = \omega_s P \qquad (s = 1, 2, 3, 4) \qquad (68)$$

since the operator sets commute mutually, so that the weight of P in $\mathcal{U}_r$ is the set of numbers $[w_1 w_2 \cdots w_r]$, and in U_4 it is $[\omega_1 \omega_2 \omega_3 \omega_4]$. From (67), the integer w_μ is the degree of P in $b^+_{\mu s}$ with respect to the index μ while, from (68), the integer ω_s gives the degree with respect to s. <u>Example</u>:

$$\underline{N=5}: \quad P = b^+_{11}\, b^+_{12}\, b^+_{22}\, b^+_{32}\, b^+_{44}$$

has $\mathcal{U}_r$ weight $[w_1 w_2 \cdots w_r] = [2111\,00\cdots 0]$ and U_4 weight

$[\omega_1\omega_2\omega_3\omega_4] = [1301]$. The sum of individual degrees

$$\sum_{\mu=1}^{r} w_\mu = \sum_{s=1}^{4} \omega_s = N$$

is the total number of particles, which in turn is the eigenvalue of the number operator

$$\mathcal{N} = \sum_{\mu=1}^{r} \mathcal{C}_\mu^{\ \mu} = \sum_{s=1}^{4} C_s^{\ s} \qquad (71)$$

$$\mathcal{N} P = N P$$

A polynomial P of weight $[w_1 w_2 \cdots w_r]$ in $\mathcal{U}_r$ is of <u>higher</u> <u>weight</u> than P' of weight $[w_1' w_2' \cdots w_r']$ in $\mathcal{U}_r$ if in the set of differences

$$[w_1 - w_1', w_2 - w_2', \cdots, w_r - w_r'] \qquad (72)$$

the first non-zero number is positive. Likewise, the same criterion for higher and lower weights in $\mathcal{U}_4$ applies.

Consider now the remaining operators of $\mathcal{U}_r$, i.e., $\mathcal{C}_\mu^{\ \mu'}$ $(\mu \neq \mu')$ and of $\mathcal{U}_4$, i.e., $C_s^{\ s'}$ $(s \neq s')$. Take the two polynomial classes P' and P'' as

$$P' = \mathcal{C}_\mu^{\ \mu'} P \qquad (\mu < \mu')$$
$$P'' = \mathcal{C}_{\mu'}^{\ \mu} P$$

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Using (11), (12 a,b,c) and (67), one can find their weights:

$$\mathcal{E}_\mu^{\mu} P' = \mathcal{E}_\mu^{\mu} \mathcal{E}_\mu^{\mu'} P = [\mathcal{E}_\mu^{\mu}, \mathcal{E}_\mu^{\mu'}] P + \mathcal{E}_\mu^{\mu'} \mathcal{E}_\mu^{\mu} P$$

$$= \mathcal{E}_\mu^{\mu'} P + w_\mu \mathcal{E}_\mu^{\mu'} P = (w_\mu + 1) P'$$

$$\mathcal{E}_{\mu'}^{\mu'} P' = \mathcal{E}_{\mu'}^{\mu'} \mathcal{E}_\mu^{\mu'} P = [\mathcal{E}_{\mu'}^{\mu'}, \mathcal{E}_\mu^{\mu'}] P + \mathcal{E}_\mu^{\mu'} \mathcal{E}_{\mu'}^{\mu'} P$$

$$= -\mathcal{E}_\mu^{\mu'} P + w_{\mu'} \mathcal{E}_\mu^{\mu'} P = (w_{\mu'} - 1) P'$$

$$\mathcal{E}_\mu^{\mu} P'' = (w_\mu - 1) P'' \qquad \mathcal{E}_{\mu'}^{\mu'} P'' = (w_{\mu'} + 1) P''$$

so that P' has weight $[w_1, \ldots, w_\mu + 1, \ldots, w_{\mu'} - 1, \ldots, w_r]$ and P" weighs $[w_1, \ldots, w_\mu - 1, \ldots, w_{\mu'} + 1, \ldots, w_r]$ which are respectively higher and lower than the weight $[w_1, w_2, \ldots, w_\mu, \ldots, w_{\mu'}, \ldots, w_r]$ of P. Thus in analogy to L_+ and L_- of R_3, $\mathcal{E}_\mu^{\mu'} (\mu < \mu')$ are the <u>raising</u> and $\mathcal{E}_\mu^{\mu'} (\mu > \mu')$ the <u>lowering</u> operators of the group $\mathcal{U}_r$. In the same manner, one can see that $C_s^{s'}$ $(s < s')$ and $C_s^{s'}$ $(s > s')$ are the raising and lowering operators of $\mathcal{U}_4$. Thus

$$\mathcal{U}_r: \quad \{\mathcal{C}_\mu^{\mu'}\} \equiv \begin{array}{c} \text{WEIGHT OPERATORS} \quad \text{RAISING} \\ \left\{ \begin{array}{ccccc} \mathcal{C}_1^1 & \mathcal{C}_1^2 & \mathcal{C}_1^3 & \cdots & \mathcal{C}_1^r \\ \mathcal{C}_2^1 & \mathcal{C}_2^2 & \mathcal{C}_2^3 & \cdots & \\ \mathcal{C}_3^1 & \mathcal{C}_3^2 & \mathcal{C}_3^3 & \cdots & \\ \vdots & & & & \vdots \\ \mathcal{C}_r^1 & \mathcal{C}_r^2 & \cdots & & \mathcal{C}_r^r \end{array} \right\} \\ \text{LOWERING} \end{array} \tag{72}$$

$(\mu, \mu' = 1, 2, \ldots, r)$

and a similar array for $\{C_s^{s'}\}$ of U_4 clearly show the three categories of raising, lowering and weight operators present in unitary groups which is the generalization for these groups of the similar result obtained previously for R_3 .

Among the set of linearly independent polynomials P which satisfy equations (67) and (68) thus being characterized by both $[w_1 w_2 \cdots w_r]$ and $[\omega_1 \omega_2 \omega_3 \omega_4]$, there are some, say $\mathbb{P}$, of highest weight in both $\mathcal{U}_r$ and U_4 given by

$$\mathcal{C}_\mu^\mu \mathbb{P} = h_\mu \mathbb{P} \quad , \quad \mathcal{C}_\mu^{\mu'} \mathbb{P} = 0 \quad \begin{pmatrix} \mu, \mu' = 1, 2, \ldots, r \\ \mu < \mu' \end{pmatrix} \tag{73a, b}$$

and simultaneously by

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$$C_s^s \mathbb{P} = v_s \mathbb{P} \quad , \quad C_s^{s'} \mathbb{P} = 0 \quad \begin{pmatrix} s,s' = 1,2,3,4 \\ s<s' \end{pmatrix} \qquad (74a,b)$$

the <u>highest weight polynomial set</u> $\mathbb{P}$ now being characterized by $[h_1 h_2 \cdots h_r] \times \{v_1 v_2 v_3 v_4\}$, which is the representation label of the direct product group $\mathcal{U}_r \times \mathcal{V}_4$. The set of equations (73) and (74) for $\mathcal{U}_r \times \mathcal{V}_4$ is analogous to the set (46) and (47) for R_3 : they <u>select</u> the maximum weight functions $\mathbb{P}$ out of a larger set P. Should one further insist on an analogy here of the Euler equation (45) treated in the discussion of R_3 , we can cite

$$\sum_\rho C_\rho^\rho \mathbb{P} = N \mathbb{P}$$

which like (45) gives the total degree of $\mathbb{P}$.

We recall from (53) of our discussion of the rotation group that the <u>complete base</u> $P_{\ell m}$, corresponding to irreducible representation ℓ of R_3 , where the row index m is given by - $-\ell \leq m \leq \ell$, can be generated from the maximum weight and h= ℓ polynomial $\mathbb{P}_\ell \sim x_+^\ell$ of (51) by successive applications of the R_3 lowering generator L_- .

In the case of $\mathcal{U}_r \times \mathcal{V}_4$, it is possible also to <u>generate the full basis</u> ---- labeled by the appropriate rows to be discussed further on ---- via application of operators which are functions of the lowering generators of this group. The representation $[h_1 h_2 \cdots h_r] \times \{v_1 v_2 v_3 v_4\}$ of $\mathcal{U}_r \times \mathcal{V}_4$ is thus definitely <u>irreducible</u>. Moreover, this irreducible representation is unique (See Appendix B -).

The direct product group $\mathcal{U}_r \times \mathcal{V}_4$ contains $\mathcal{U}_r$ and $\mathcal{V}_4$

as subroups. The highest weight term of an irreducible basis for $\mathcal{U}_r$ and $\mathcal{V}_4$, labelled respectively by $[h_1 h_2 \cdots h_r]$ and $\{v_1 v_2 v_3 v_4\}$, obey restrictions analogous to (49) for the irreducible representations of R_3. We recall that (49) followed simply from the fact that $\mathbb{P}$ was a homogeneous polynomial in x_+, x_0 and x_-. In the present case, if one considers the non-negative scalar product

$$
\begin{aligned}
(\mathcal{C}_\mu^{\mu'}\mathbb{P}, \mathcal{C}_\mu^{\mu'}\mathbb{P}) &= \langle 0|\, \mathbb{P}^+ \mathcal{C}_{\mu'}^{\mu} \mathcal{C}_\mu^{\mu'} \mathbb{P}\, |0\rangle \qquad (\mu > \mu') \\
&= \langle 0|\, \mathbb{P}^+ [\mathcal{C}_{\mu'}^{\mu}, \mathcal{C}_\mu^{\mu'}] \mathbb{P}\, |0\rangle \\
&= \langle 0|\, \mathbb{P}^+ (\mathcal{C}_{\mu'}^{\mu'} - \mathcal{C}_\mu^{\mu}) \mathbb{P}\, |0\rangle \\
&= (h_{\mu'} - h_\mu)(\mathbb{P}, \mathbb{P}) \geqslant 0
\end{aligned}
\tag{75}
$$

$$\therefore \quad \underline{h_\mu \geqslant h_{\mu'}} \quad \text{for } (\mu < \mu').$$

where in step 2 we used (73b), in 3 (73a) and in the last step the fact $(\mathbb{P}, \mathbb{P}) \geqslant 0$. Similar results are obtained from the scalar product $(C_s^{s'}\mathbb{P}, C_s^{s'}\mathbb{P})$ for $s > s'$. Furthermore, from (71) one obtains $\sum_{\mu=1}^{r} h_\mu = \sum_{s=1}^{4} v_s = N$. In conclusion, we have the following conditions:

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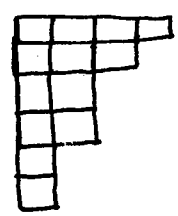
$$\mathcal{U}_r: \quad \begin{array}{l} h_1 \geqslant h_2 \geqslant h_3 \geqslant \cdots \geqslant h_r \geqslant 0 \\ h_1 + h_2 + h_3 + \cdots + h_r = N \end{array} \tag{76}$$

$$U_4: \quad \begin{array}{l} V_1 \geqslant V_2 \geqslant V_3 \geqslant V_4 \geqslant 0 \\ V_1 + V_2 + V_3 + V_4 = N \end{array} \tag{77}$$

which are precisely the requirements on the well-known Young diagrams (or partitions, or patterns) giving the permutation symmetry of an N-particle function.

The unique highest weight function of an arbitrary orbital permutation symmetry and the corresponding (conjugate) spin-isospin permutation symmetry can be constructed at once. For instance, take the 13 particle orbital symmetry given by the partition

$$\mathcal{U}_r: \quad [h_1 h_2 \cdots h_6] = [432211] = \tag{78}$$



where obviously 4+3+2+2+1+1 = 13. A polynomial function $\mathbb{P}(b^+_{\mu s})$ with this $\mathcal{U}_r$ irreducible representation and of maximum weight in $\mathcal{U}_r$ is simply

$$\mathbb{P}(b^{+}_{\mu s}) = \begin{aligned} & b^{+}_{11}\, b^{+}_{12}\, b^{+}_{13}\, b^{+}_{14} \\ & \times b^{+}_{21}\, b^{+}_{22}\, b^{+}_{23} \\ & \times b^{+}_{31}\, b^{+}_{32} \\ & \times b^{+}_{41}\, b^{+}_{42} \\ & \times b^{+}_{51} \\ & \times b^{+}_{61} \end{aligned} \qquad (79)$$

which is clearly totally anti-symmetric with respect to interchange of any pair of orbital-spin-isospin states. Note that the arrangement of b^{+}'s purposively follows the Young diagram (78). The degrees h_μ of $\mathbb{P}$ with respect to the components μ in $b^{+}_{\mu s}$ is $h_1 = 4$ so that <u>four</u> states with $\mu = 1$ are constructed; $h_2 = 3$ and <u>three</u> states with $\mu = 2$ are constructed, and so forth. The second index of each b^{+} operator ---- denoting the spin-isospin state ---- is placed in numerical order <u>in each row</u>. (If two of these indices coincide the whole expression vanishes as from (12b) any $b^{+}_{\mu s}\, b^{+}_{\mu s} = 0$, i.e., the Pauli principle). Naturally therefore as $1 \leqslant s \leqslant 4$, there are $\leqslant$ four blocks in any given row. By the same taken, there are $\leqslant r$ (=number of single nucleon orbital states available) rows in the whole expression.

Mental application of $\mathcal{C}^{\mu}_{\mu}$ ($\mu = 1,2,\ldots.6$) shows that

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indeed $[h_1 h_2 h_3 h_4 h_5 h_6] = [4\ 3\ 2\ 2\ 1\ 1]$. The effect of any $b_\mu^{\mu'}$ ($\mu < \mu'$) is zero as in some column a factor $b^+_{\mu s} b^+_{\mu s} = 0$ would then occur, thus (78) is of maximum weight in $\mathcal{U}_r$.

Again, mental application of $C_s^{s'}$ (s= 1,2,3,4) shows that $[V_1 V_2 V_3 V_4]$, the degrees of $\mathbb{P}$ in S=1,2,3,4, is $[V_1 V_2 V_3 V_4] = [6421]$, which satisfy (77), and furthermore is the <u>conjugate representation</u> of $[h_1 h_2 h_3 h_4 h_5 h_6]$ as

$$\widetilde{[h_1 h_2 h_3 h_4 h_5 h_6]} = \widetilde{[4\,3\,2\,2\,1\,1]} = [6421] = [V_1 V_2 V_3 V_4] \qquad (80)$$

as can be seen by reflecting the diagram (78) about its principal diagonal. (This was to be expected since the irreducible representation $[h_1 h_2 \cdots h_r] \times \{V_1 V_2 V_3 V_4\}$ of $\mathcal{U}_r \times \mathcal{U}_4$ is contained in the irreducible representation $[1^N 0^{4r-N}]$ of $\mathbb{U}_{4r}$). Moreover, (79) satisfies $C_s^{s'} \mathbb{P} = 0$, for $s < s'$, since the effect of any of these $C_s^{s'}$ would create pairs of the form $b^+_{\mu s} b^+_{\mu s} = 0$.

The state (79) is the analogue for the $\mathcal{U}_r \times \mathcal{U}_4$ group of (51). Extremely powerful and simple is the technique of constructing <u>any</u> $\mathbb{P}^{[h_1 h_2 \cdots h_r] \times \{V_1 V_2 V_3 V_4\}}$ of maximum weight. Lowering operations as in (53) for obtaining the <u>complete</u> bases $P_{\ell m}$, irreducible under $R_3 \supset R_2$, with $-\ell \leq m \leq \ell$ and of dimension ($2\ell+1$), also have their analogue in unitary group theory.

The polynomial set $P_{\ell m}$ transforms irreducibly under R_3 according to the well-known "rotation matrices"[40]

$$P'_{\ell m}(x,y,z) = \sum_{m'} \mathcal{D}^{(\ell)}_{m m'} \; P_{\ell m'}(x,y,z). \qquad (81)$$

The <u>row</u> of the irreducible representation $\mathcal{D}^{(\ell)}$ of dimension

$(2\ell+1)$ by $(2\ell+1)$ is given by m which, apart from being the polynomial <u>weight</u> in R_3, is also the <u>irreducible representation</u> of R_2 which is contained in R_3 as a subgroup, i.e., $P_{\ell m}$ transforms irreducibly under the chain of group transformations designated as

$$R_3 \supset \begin{pmatrix} R_2 & 0 \\ 0 & 1 \end{pmatrix}. \tag{82}$$

Similarly, the <u>rows</u> (and columns) that would specify the <u>full basis</u> of a polynomial set $P^{[h_1 h_2 \cdots h_r]}(b^+_{\mu s})$ transforming irreducibly under $\mathcal{U}_r$ could come from sub-groups of $\mathcal{U}_r$. Not <u>any</u> chain of sub-groups, however, would provide a <u>complete</u> classification of the polynomial set. For example, the chains

$$\mathcal{U}_6 \supset \mathcal{U}_3, \quad U_4 \supset U_2 \times U_2, \qquad \mathcal{U}_6 \supset R_6 \tag{83}$$

among other physically important cases[37,41] are known as "non-simply reducible" because for a given irreducible representation of the large group there may be <u>repeated</u> sub-group irreducible representations thus requiring additional labels[39] (quantum numbers, physical or not) to distinguish these multiplicities. However, the so-called "canonical chain"[39]

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$$\mathcal{U}_r \supset \begin{pmatrix} \mathcal{U}_{r-1} & 0 \\ 0 & 1 \end{pmatrix} \supset \begin{pmatrix} \mathcal{U}_{r-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \supset \cdots \supset \begin{pmatrix} \mathcal{U}_1 & & & \\ & 1 & & 0 \\ & & 1 & \\ 0 & & & \ddots \\ & & & & 1 \end{pmatrix} \tag{84}$$

<u>unequivocally specifies the rows of a $\mathcal{U}_r$ irreducible representation</u>.

The "canonical chain" for the spin-isospin group $\mathcal{U}_4$ would hence be

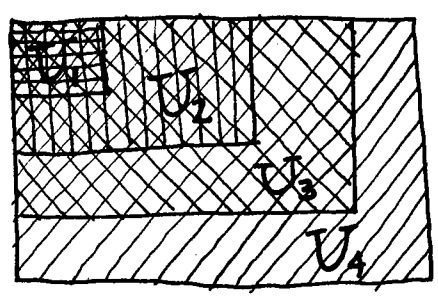
$$\mathcal{U}_4 \supset \begin{pmatrix} \mathcal{U}_3 & 0 \\ 0 & 1 \end{pmatrix} \supset \begin{pmatrix} \mathcal{U}_2 & 0 & 0 \\ 0 & 1 & \\ 0 & & 1 \end{pmatrix} \supset \begin{pmatrix} \mathcal{U}_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{85}$$

and as this is smaller than $\mathcal{U}_r$ in general, let us illustrate the sub-group generators. The 4^2 generators of $\mathcal{U}_4$ are

$$\mathcal{U}_4: \quad \{C_s^{s'}\} \equiv \left\{ \begin{array}{ccc|c} C_1^1 & C_1^2 & C_1^3 & C_1^4 \\ C_2^1 & C_2^2 & C_2^3 & C_2^4 \\ C_3^1 & C_3^2 & C_3^3 & C_3^4 \\ \hline C_4^1 & C_4^2 & C_4^3 & C_4^4 \end{array} \right\} \tag{86}$$

with Lie algebra $[C_s^{s'}, C_{s''}^{s'''}] = C_s^{s'''}\delta_{s''}^{s'} - C_{s''}^{s'}\delta_s^{s'''}$ $(s,s'=1,2,3,4)$, namely the commutator of any two generators never gives other than U_4 generators. The U_3 generators can be chosen as the 3^2 operators in the upper-left hand corner above the solid lines. They satisfy the <u>same</u> Lie algebra as U_4, but with $s,s'=1,2,3$ ---- as can be quickly verified. The U_2 generators $C_1^1, C_2^2, C_2^1, C_1^2$ also obey the same algebra, but with $s,s'=1,2$. Finally, C_1^1 is the sole generator of U_1 and the commutator $[C_1^1, C_1^1] = 0$ also gives nothing outside U_1. An illustrative diagram showing the group chain (85) in sofar as the 4^2 generators $C_s^{s'}$ $(s,s'$ 1,2,3,4) are concerned is:

U_4: $\{C_s^{s'}\} \longrightarrow$



Generalizing, $\mathcal{U}_r$ will from (84) have ($r-1$) sub-groups $\mathcal{U}_p$ (p=1,2,...,r-1) each with p^2 generators $\mathcal{C}_\mu^{\mu'}$ $(\mu,\mu'=1,2,\cdots,p)$. Moreover, the equations

$$\mathcal{C}_\mu^{\mu} P = h_{\mu_p} P \qquad \mathcal{C}_\mu^{\mu'} P = 0 \tag{87}$$

$$(\mu=1,2,\cdots,p) \quad (\mu<\mu') \quad (p=1,2,\cdots,r-1)$$

define the irreducible representation label $[h_{1p}\, h_{2p} \cdots h_{pp}]$ of the sub-group $\mathcal{U}_p \subset \mathcal{U}_r$. Likewise,

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$$C_s^s P = v_{sg} P \qquad C_s^{s'} P = 0 \tag{88}$$

$$(s=1,2,\dots,g) \quad (g=1,2,3) \quad (s<s')$$

define the irreducible representation $\{v_{1g} v_{2g} \cdots v_{gg}\}$ of the subgroup $U_g \subset U_4$.

Hence, the generic state forming a complete basis for irreducible representations of the groups in the chain

$$\mathbb{U}_{4r} \supset \mathcal{U}_r \times U_4 \supset \begin{cases} \mathcal{U}_r \supset \begin{pmatrix} \mathcal{U}_{r-1} & 0 \\ 0 & 1 \end{pmatrix} \supset \cdots \supset \begin{pmatrix} \mathcal{U}_1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} & (89a) \\ U_4 \supset \begin{pmatrix} U_3 & 0 \\ 0 & 1 \end{pmatrix} \supset \begin{pmatrix} U_2 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix} \supset \begin{pmatrix} U_1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} & (89b) \end{cases}$$

could be designated with a notation used by Gel'fand & Zetlin[42] as:

$$P^{[h_{kp}]\times\{v_{fg}\}}(b^+_{\mu s}) \equiv |h_{kp}; v_{fg}\rangle \equiv$$

$$\left.\begin{array}{c} h_{1r}\, h_{2r} \;\cdots\cdots\; h_{rr} \\ h_{1r-1}\, h_{2r-1} \cdots\cdots h_{r-1\,r-1} \\ \cdots\cdots \\ h_{12}\;\; h_{22} \\ h_{11} \end{array} \;;\; \begin{array}{c} V_{14}\;\; V_{24}\;\; V_{34}\;\; V_{44} \\ V_{12}\;\; V_{22}\;\; V_{32} \\ V_{13}\;\; V_{23} \\ V_{14} \end{array}\right\rangle \qquad (90)$$

with $p \leq k$ =1,2,...,r and $g \leq f$ =1,2,3,4,
which, for the chain (89), is the analogue of (53) for the simple chain (57). In the latter, one had the restriction that $-\ell \leq m \leq \ell$. An analogous restriction holds in (89), for both $\mathcal{U}_r$ and U_4 chains: Let Π_N be the symmetric (or permutation) group of N objects. As $\Pi_N > \Pi_{N-1}$, the former having irreducible representations (Young patterns) $[f_1 f_2 \cdots f_N]$ and the latter $[f_1' f_2' \cdots f_{N-1}']$, then

$$f_1 \geqslant f_1' \geqslant f_2 \geqslant f_2' \geqslant \cdots \geqslant f_{N-1}' \geqslant f_N \geqslant 0 \qquad (91)$$

(see Weyl, ref. 43). Because of the intimate relation between

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the Π_N and $\mathcal{U}_r$ groups[44], namely, that they possess the same irreducible representation labels, one deduces from (91) the following inequalities for $\mathcal{U}_r$ and its canonical chain subgroups:

$$
\begin{array}{c}
h_{1r} \geqslant h_{1r-1} \geqslant h_{2r} \geqslant h_{2r-1} \geqslant \cdots \geqslant h_{r-1\,r-1} \geqslant h_{rr} \geqslant 0 \\
h_{1r-1} \geqslant h_{1r-2} \geqslant \cdots\cdots \geqslant h_{r-2\,r-2} \geqslant h_{r-1\,r-1} \geqslant 0 \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
h_{12} \geqslant h_{11} \geqslant h_{22} \geqslant 0
\end{array}
$$

or, to use the same arrangement as in the generic state (90),

$$
\begin{array}{ccccccccccc}
h_{1r} & & h_{2r} & & h_{3r} & \cdot & \cdot & \cdot & \cdot & h_{rr} & \geqslant 0 \\
\geqslant & \geqslant & & \geqslant & & \geqslant & & & & & \\
 & h_{1r-1} & & h_{2r-1} & & h_{3r-1} & \cdots & & & h_{r-1\,r-1} & \geqslant 0 \\
 & & \geqslant & & \geqslant & & & & & & \\
 & & \cdot & \cdot & \cdot & \cdot & \cdot & & & \geqslant 0 & \\
 & & & \geqslant & h_{12} & & h_{22} & & & \geqslant 0 & \\
 & & & & & \geqslant & h_{11} & & & &
\end{array}
\qquad (92)
$$

For $\mathcal{U}_4$ and its sub-groups, the same reasoning in (91) gives the requirements

$$\begin{array}{ccccccc}
V_{14} & & V_{24} & & V_{34} & & V_{44} \geqslant 0 \\
\geqslant & \geqslant & \geqslant & \geqslant & \geqslant & \geqslant & \\
 & V_{13} & & V_{23} & & V_{33} \geqslant 0 & \\
 & \geqslant & \geqslant & \geqslant & \geqslant & & \\
 & & V_{12} & & V_{22} \geqslant 0 & & \\
 & & \geqslant & \geqslant & & & \\
 & & & V_{11} & & &
\end{array} \qquad (93)$$

The polynomial $\mathbb{P}^{[h_1 h_2 \cdots h_r] \times \{V_1 V_2 V_3 V_4\}}$ of <u>highest weight</u> in both $\mathcal{U}_r$ and $\mathcal{U}_4$ must satisfy both (73) and (74). Now, the conditions (73b) and (74b), i.e.,

$$\mathcal{C}_{\mu}^{\mu'} \mathbb{P} = 0 \quad (\mu < \mu'), \quad C_s^{s'} \mathbb{P} = 0 \quad (s < s')$$

ensure that $\mathbb{P}$ is automatically of <u>highest weight in all the canonical chain subgroups of $\mathcal{U}_r$ and $\mathcal{U}_4$</u>.

Concretely, if the representation $[h_{kr}]$ (k=1,2,...,r) of $\mathcal{U}_r$ is of highest weight, the sub-group representations $[h_{kp}]$ ($k \leqslant p$=1,2,..,r-1) take the maximum value compatible with (92) <u>and</u> if the $\mathcal{U}_4$ representation $\{V_{f4}\}$ (f=1,2,3,4) is of highest weight, its sub-group representations $\{V_{fg}\}$ ($f \leqslant g$=1,2,3) acquire the maximum values allowed by (93). This result will be very significant in future developments, but for the moment we note that in the explicit Gel'fand-Zetlin notation the solution of (73) and (74) (a)

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can consequently be symbolized in the simple form

$$\mathbb{P} \equiv \mathbb{P}^{[h_{1r}\, h_{2r} \cdots h_{rr}] \times \{V_{14}\, V_{24}\, V_{34}\, V_{44}\}} = \tag{94}$$

$$\left| \begin{array}{ccccc} h_{1r} & h_{2r} & h_{3r} & \cdots\cdots & h_{rr} \\ & h_{1r} & h_{2r} & h_{3r} \cdots & h_{r-1\,r} \\ & & \cdot & \cdot & \cdot \\ & & & h_{1r} & h_{2r} \\ & & & & h_{1r} \end{array} \right. ; \left. \begin{array}{cccc} V_{14} & V_{24} & V_{34} & V_{44} \\ & V_{14} & V_{24} & V_{34} \\ & & V_{14} & V_{24} \\ & & & V_{14} \end{array} \right\rangle$$

where the h's along any diagonal parallel to the lowest one are equal among themselves. The same holds for the v's.

Equations (67) and (68) define the weights in $\mathcal{U}_r$ and $\mathcal{U}_4$ of a given polynomial state such as (90) ---- weights being nothing more than the eigenvalues of $\mathcal{C}_\mu^{\mu}$ (μ=1,2,...,r) in the case of $\mathcal{U}_r$ and, of C_S^S (S=1,2,3,4) for $\mathcal{U}_4$. From relations (38) it is easy to see that the sum of operators

$$\sum_{\mu=1}^{p} \mathcal{C}_\mu^{\mu} = \text{Trace (of } \mathcal{U}_p \text{ weight generators)}$$

conmutes with <u>all</u> the generators of $\mathcal{U}_p$ <u>and</u> its subgroups $\mathcal{U}_{p-1}$, $\mathcal{U}_{p-2}$,..., $\mathcal{U}_1$. Thus the eigenvalue of this sum is the

same for arbitrary state (90) as for state (94) of maximun weight in $\mathcal{U}_p$. For the latter, the eigenvalues of $\mathcal{C}_\mu^{\mu}$ (μ=1,2,...,p) are simply $h_{\mu p}$ so that

$$\sum_{\mu=1}^{p} \mathcal{C}_\mu^{\mu} \quad \text{has eigenvalue} \quad (h_{1p}+h_{2p}+\cdots+h_{pp})$$

for any dimension p. Hence, $\mathcal{C}_\mu^{\mu}$ has eigenvalue

$$w_\mu = (h_{1\mu}+h_{2\mu}+\cdots+h_{\mu\mu})-(h_{1\mu-1}+h_{2\mu-1}+\cdots+h_{\mu-1\mu-1}). \quad (94a)$$

Nagel & Moshinsky[45] have recently constructed operators $\mathbb{L}_r^k$ polynomial in the lowering <u>generators</u> $\mathcal{C}_\mu^{\mu'}$ ($\mu'<\mu$ =1,2,..,r) of a unitary group $\mathcal{U}_r$ for any r . These <u>lowering operators</u> $\mathbb{L}_r^k$ decrease by one unit the kth representation label h_{kr-1} of the subgroup $\mathcal{U}_{r-1}$, keeping the representation of $\mathcal{U}_{r-1}$ in maximum weight. Now, an N-particle totally antisymmetric $\mathbb{U}_{4r}$ representation $[1^N\, 0^{4r-N}]$ will contain <u>several</u>, say x, $\mathcal{U}_r$ representations of highest weight. (See examples in Appendix A). <u>Neglect</u> for the moment the spin-isospin part of (94) ----- which is irrelevant for spin-isospin independent interactions and which is equivalent to assuming maximum spin and isospin projections M_s=S, M_T=T, for any S and T, in the calculation of matrix elements of a function only of coordinates. Successive application of $\mathbb{L}_r^k$ (for k=1,2,...,r-1) on each of the x inmediately constructible states, like (79),

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$$\left| \begin{array}{ccccccc} h_{1r} & h_{2r} & \cdots & h_{kr} & \cdots & h_{rr} \\ & h_{1r} & h_{2r} \cdots & h_{kr} & \cdots & h_{r-1\,r} \\ & & \cdot & \cdot & \cdot\cdot & \\ & & & h_{1r} & h_{2r} & \\ & & & & h_{1r} & \end{array} \right\rangle , \qquad (95)$$

will give the x sets of kets with <u>all</u> the representations of $\mathcal{U}_{r-1}$ in maximum weight contained in the x original $\mathcal{U}_r$ representations of the type $[h_{1r}\, h_{2r} \cdots h_{rr}]$ and which satisfy the inequalities of the first two lines in (92). Next, by similar applications of $\mathbb{I}^k_{r-1}$ (for k=1,2,...,r-2) one could generate all the states associated with all $\mathcal{U}_{r-2}$ representation contained under <u>each</u> $\mathcal{U}_{r-1}$ representation of maximum weight generated in the previous step, and which satisfied the inequalities of the 2<u>nd</u> and 3<u>rd</u> rows of (92). Continuing this process until one obtained all the representations contain in the $\mathcal{U}_2$ representations, according now to the inequalities of the last two rows of (92), the full set of linearly independent polynomial functions transforming irreducibly with respect to the canonical $\mathcal{U}_r$ chain of subgroups (89a) would be derived.

Removing now the restriction of spin-isospin independence, the complete basis with regard to the chain (89b) can also be constructed, though with more physical operators,[46] as will be seen later.

One would therefore have <u>all</u> the N-particle totally anti-

symmetric states compatible with $[1^N\ 0^{4r-N}]$ <u>but</u> in a "canonical" (or "mathematical") chain beginning with $\mathcal{U}_r$, i.e.,

$$|h_{1r}h_{2r}\cdots h_{rr}, h_{1r-1}h_{2r-1}\cdots h_{r-1r-1}, \cdots h_{12}h_{22}, h_{11}; \{V_{14}V_{24}V_{34}V_{44}\}\beta S M_S T M_T\rangle \quad (96)$$

where β stands for a pair of hermitean operator eigenvalues needed to fully distinguish multiple occurances of (ST) contained in the U_4 representation $\{V_{14}V_{24}V_{34}V_{44}\} = [\overbrace{h_{1r}h_{2r}\cdots h_{rr}}]$. However, this basis (still non-physical in the orbital part) could be transformed to a scheme with definite total orbital angular momentum L and projection M_L with the aid of the matrix $\|L^2\|$ calculated in the "canonical chain" basis (89a). The resulting complete set of totally antisymmetric N-particle states after this transformation would be:

$$|[f]\gamma L M_L; [\tilde{f}]\beta S M_S T M_T\rangle \quad (97)$$

$$\left\{\begin{array}{l} [f] \equiv [h_{1r}h_{2r}\cdots h_{rr}] \quad (\text{Young Partition}) \\ [\tilde{f}] \equiv [V_{14}V_{24}V_{34}V_{44}] = [\overbrace{h_{1r}h_{2r}\cdots h_{rr}}] \end{array}\right\} \quad (98)$$

provided the operator L_z were diagonal in the original basis (96), and this can be chosen to be so. Here, γ designates the set of quantum numbers required to distinguish multiple L-values contained in a given $[f]$ representation. The (physical) chain of groups under which the complete set (97) transforms irreducibly is thus

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$$
\begin{array}{ccccccccc}
 & & & & \mathcal{U}_r & \supset & R_3 & \supset & R_2(\text{orbital}) \\
 & & & & \uparrow & & \uparrow & & \uparrow \\
 & & & & [f] & (\gamma) & L & & M_L \\
\mathbb{U}_{4r} & \supset & \mathcal{U}_r \times U_4 & \supset & & & & & \\
\uparrow & & \uparrow & & & & & & \\
[1^N 0^{4r-N}] & & [f]\times\{\tilde{f}\} & & & & & & \\
 & & & & U_4 & \supset & U_2 \times U_2 & \supset & R_2(\text{spin}) \times R_2(\text{isospin}) \\
 & & & & \uparrow & & \uparrow \quad \uparrow & & \uparrow \qquad \uparrow \\
 & & & & \{\tilde{f}\} & (\beta) & S \quad T & & M_S \qquad M_T
\end{array}
\tag{99}
$$

where the quantum number labels of the ket (97) are given underneath the chain; those with <u>arrows</u> pointing to a specific group are the labels for the irreducible representations of <u>that group</u>. The subchain $R_3 \supset R_2$ represents the special case of spherical and axial symmetries discussed in Section 4. The group U_4 of transformations in spin-isospin space contains the subgroup $U_2 \times U_2$ of <u>separate</u> unitary transformations in 2-dimensional spinor and isospinor spaces. Each SU_2 sub-group, as is well-known[40], is <u>homomorphic</u> to a group R_3(spin or isospin) of rotations in the (spin or isospin) space of three-dimensions spanned by the components

$$\begin{pmatrix} S_x, S_y, S_z \\ \text{or} \\ T_\zeta, T_\eta, T_\xi \end{pmatrix}$$

, which in turn contains the subgroup $R_2\begin{pmatrix}\text{spin} \\ \text{or} \\ \text{isospin}\end{pmatrix}$ of rotations around the $\begin{pmatrix} z \\ \text{or} \\ \xi \end{pmatrix}$ axis. <u>Example</u>:

$$SU_2(\text{spin}) \approx \underset{\substack{\uparrow \\ S}}{R_3(\text{spin})} \supset \underset{\substack{\uparrow \\ M_S}}{R_2(\text{spin})} \qquad (100)$$

The label β comes from hermitean operators formed from the generators of the U_4 group which are to completely characterize the rows in $U_4 > U_2 \times U_2$. The label γ stands for irreducible representation labels associated with subgroups contained between $\mathcal{U}_r$ and R_3 <u>and</u> also with hermitean operators which may be necessary to completely characterize the rows between two succeeding subgroups of the chain.

The problem of deducing the irreducible representations of R_3 contained in a given one of $\mathcal{U}_r$ is a simple one. The same is true of the irreducible representations of $U_2 \times U_2$ contained in a given U_4 representation. In the former, one obtains the L-structure of a given N-particle Young diagram and in the latter, the S and T values contained in the conjugate N-particle young diagram are derived. For a given L , the different terms

$$^{2S+1,\,2T+1}L$$

arising from a representation $\{v_1 v_2 v_3 v_4\}$ are called "supermultiplets". Jahn[41)] discusses the p and d orbits ($\mathcal{U}_3$ and $\mathcal{U}_5$, respectively) and the (S,T)-structure for up to N=10 particles. For some illustrative examples, see Appendix **A**.

The generic state (97) can be L-S coupled to total angular

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momentum J by the usual Clebsch-Gordan coefficient to give states

$$|[f]\gamma L\,;\,[\tilde{f}]\beta S, TM_T\,;\,JM_J\rangle =$$

$$\sum_{M_L M_S} \langle LSM_L M_S | JM_J\rangle \, |[f]\gamma L M_L\,;\,[\tilde{f}]\beta S M_S, TM_T\rangle. \qquad (101)$$

Now, the total number of linearly independent anti-symmetric ortibal-spin-isospin N-particle states is given by the simple result from statistical theory

$$\binom{4r}{N} = \frac{(4r)!}{(4r-N)!\;N!} \qquad (102)$$

where $4r$ is the number of single-particle states available. In the nuclear 2s-1d shell, for instance, $r = \sum_k (2\ell_k+1) = 1 + (2\bullet 2+1) = 6$ orbital states. The number of orbital-spin-isospin single-particle states in this shell is 6X4 = 24. Then, for N = 1,2,3,4,5 particles there are

N	# of totally anti-symmetric N-particle states
1	24
2	276
3	2,024
4	10,626
5	42,504

etc!

TABLE III.5.1

Obviously, one cannot hope to solve an N-body nuclear shell problem ---- even with a model hamiltonian ---- without making drastic reductions on the number of states to be considered as pertinent. One such criterion, first used by Wigner[47], is based on the fact that the attractivity of nuclear forces will favor the most symmetrical orbital configurations as lowest in energy. (This being the exact opposite of Hund's rule in atomic structure where forces are repulsive). For example, restricting 3-particle states in the 2s-1d shell

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U_{24}	$\supset$	$\mathcal{U}_6 \times U_4$	#
[111]	$\supset$	[3]x{111}	224
	$\supset$	[21]x{21}	1400
	$\supset$	[111]x{3}	400
		TOTAL:	2,024 states

(103)

to the symmetric [3] partition of $\mathcal{U}_6$ would reduce the original number of states 2,024 to only 224. The U_4 representation [111] is equivalent to [1] with (S,T) = ($\frac{1}{2}$,$\frac{1}{2}$) (See Appendix A) so that 2T+1 = 2, corresponding to $M_T = \pm 1/2$. For light nuclei coulomb effects are negligible so that one need only deal with <u>half</u> the 224 states, leaving 112. Moreover, our hamiltonian would certainly commute with $\vec{J} = \vec{L} + \vec{S}$ so that instead of dealing with ($2J+1$) values of M_J for a given J one could limit the calculation to $M_J = J$. The 112 states are thus reduced to 17 states. This is <u>still</u> a large number resulting in still large matrices to be calculated and even further restrictions will be warranted. For this we turn to the simple harmonic oscillator well as a model for the nuclear common potential.

6.- Harmonic Oscillator Symmetry and the U_3 group.

6a. Single Particle.

High-energy electron scattering from nuclei[48] indicate that while the nuclear common potential of heavy nuclei resemble a flat bottom shape with tapering edges of the Saxon-Woods well, that of light nuclei up to the Mg region resemble a harmonic oscillator parabolic well shape. The harmonic oscillator is simple to deal with analytically because of its group-symmetry properties. Let us therefore assume a hamiltontian

$$H = \sum_{i=1}^{N} H_i^{osc} + \sum_{i=1}^{N} H_i^{MISC} + \sum_{i<j=1}^{N} V(r_{ij}) \qquad (104)$$

$$H_i^{osc} = \frac{p_i^2}{2m} + \frac{1}{2} m\omega^2 r_i^2 \qquad (105)$$

including any miscellaneous single-body interactions H_i^{MISC} that may arise, viz. spin-orbit coupling, as well as residual inter-nucleon interactions $\sum_{i<j} V(r_{ij})$. The allowed energies and angular momenta a single particle are known to be given by

$$H_i^{osc} \psi_{nlm}(\vec{r}) = (\nu + \tfrac{3}{2})\hbar\omega \, \psi_{nlm}(\vec{r})$$

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$$L_i^2 \, \psi_{n\ell m}(\vec{r}) = \ell(\ell+1)\, \psi_{n\ell m}(\vec{r}) \qquad (106\ a, b, c)$$

$$L_{iz} \psi_{n\ell m}(\vec{r}) = m\, \psi_{n\ell m}(\vec{r})$$

$$\psi_{n\ell m}(\vec{r}) = \sqrt{\frac{2(n!)}{\Gamma(n+\ell+\frac{3}{2})}}\; e^{-r^2/2}\, L_n^{\ell+1/2}(r^2)\, r^\ell\, Y_{\ell m}(\theta,\varphi)$$

and with $\nu \equiv 2n+\ell$ being the principal quantum number, n the radial quantum number in the Laguerre polynomial $L_n^{\ell+1/2}(r^2)$, ℓ and m the angular momentum and its projection. Since n is a non-negative integer,

$$\ell = \nu, \nu-2, \nu-4, \ldots, \begin{matrix}1\\0\end{matrix} \qquad (107)$$

and in general several orbitals (given ℓ) appear degenerate in a given shell (specified by ν). The energy difference between shells is $\hbar\omega$. The number of single-particle orbital states available within a given shell is

$$r = \sum_{\ell=0 \text{ or } 1}^{\nu} (2\ell+1) = \tfrac{1}{2}(\nu+1)(\nu+2) \qquad (108)$$

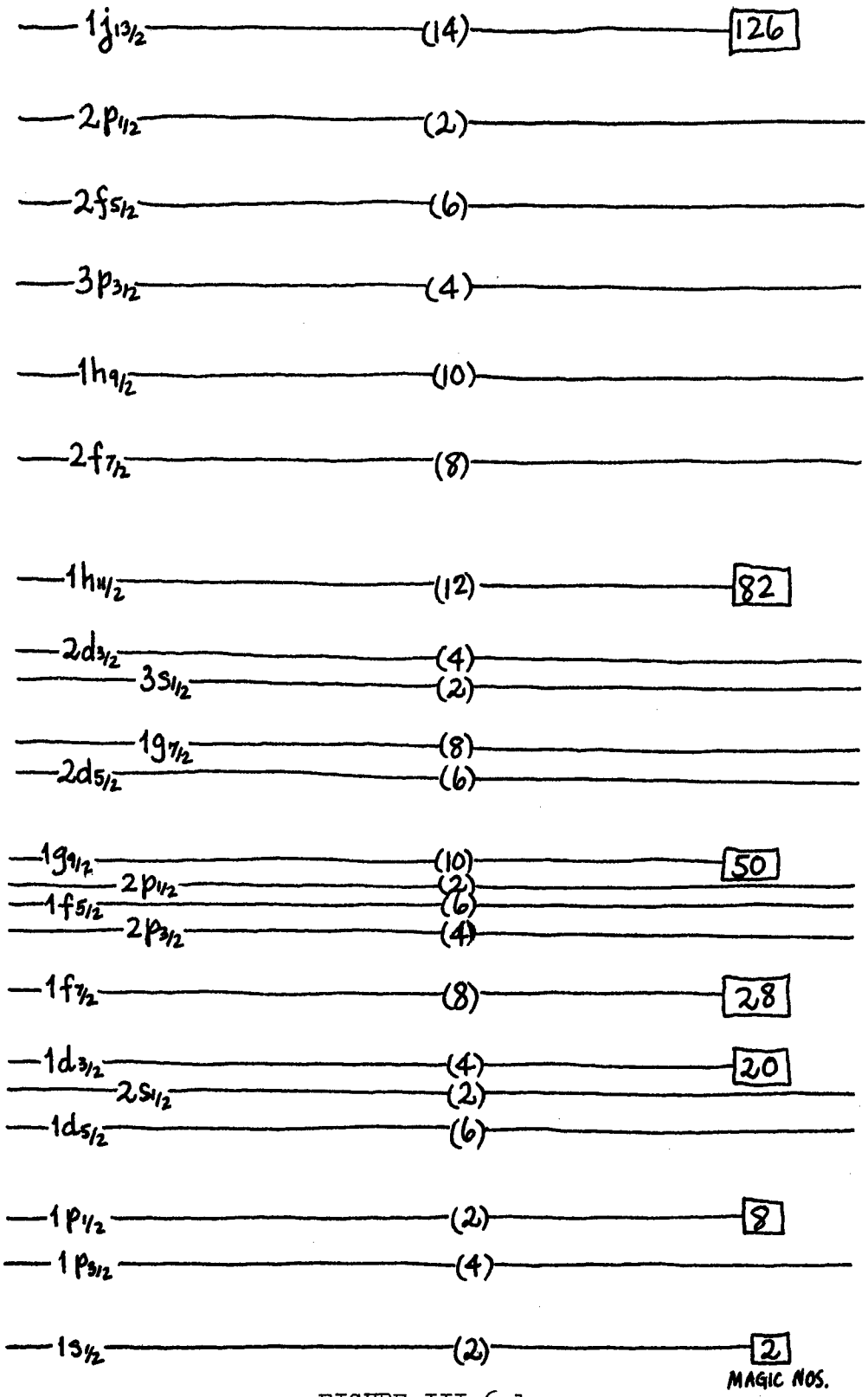

FIGURE III.6.1
Doublet-splittings for neutrons.

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so that $2r$ neutrons or protons fill a given shell. Thus for $\nu = 0$, one has 2. For $\nu = 1$, there are 6; for $\nu = 2$, there are 12; for $\nu = 3$, 20, for $\nu = 4$, 30, etc. Hence, this model based solely on the harmonic oscillator well predicts the following neutron (or proton) numbers at each closure:

ν	0	1	2	3	4	5	...
#	2	8	20	40	70	112	...

(109)

But, the <u>empirical</u> "magic numbers" at which nuclei show special stability in many regards are rather 2,8,(14),20,(28),50,82 and 126 ---- numbers in parenthesis referring to less pronounced stabilities. Thus only the first three fully-magic numbers 2,8 and 20 are predicted in the scheme (109). By introducing a strong, attractive spin-orbit term to the oscillator hamiltonian Goeppert-Mayer[49] and independently Haxel, Jensen & Suess[50] predicted the magic and semi-magic numbers correctly, thanks to the ensuing doublet splitting (See Figure III.6.1).

The single-particle oscillator hamiltonian (105) and solutions (106) can, as is well-known, be formulated in terms of creation and annihilation boson operators.

Taking $m = \hbar = \omega = 1$, then

$$H_i^{osc} = \frac{1}{2}(p^2 + r^2)_i$$
$$= \frac{1}{2}\sum_{q=1,\bar{1},0}(p_q p^q + x_q x^q)_i \qquad (110)$$

where $x_{\pm 1} = \mp\frac{1}{\sqrt{2}}(x \pm iy)$, $x_0 = z$; $p_{\pm 1} = \mp\frac{1}{\sqrt{2}}(p_x \pm ip_y)$, $p_0 = p_z$ (111)

are the spherical components of vectors $\vec{r}$ and $\vec{p}$, the scalar products $\vec{p}\cdot\vec{p}$ and $\vec{r}\cdot\vec{r}$ in (110) being given with the usual metric, namely

$$\begin{aligned} \vec{r}\cdot\vec{r} &= r^2 = \sum_q (-)^q x_q x_{-q} = \sum_q x_q x^q \\ \vec{p}\cdot\vec{p} &= p^2 = \sum_q (-)^q p_q p_{-q} = \sum_q p_q p^q \end{aligned} \qquad (112)$$

Furthermore, one defines the creation a_q^+ and annihilation operators as the linear combinations

$$a_q^+ \equiv \frac{1}{\sqrt{2}}(x_q - i p_q) \quad \text{and} \quad a^q \equiv \frac{1}{\sqrt{2}}(x^q + i p^q) \qquad (113)$$
$$\text{or} \quad a_q = \frac{1}{\sqrt{2}}(x_q + i p_q)$$

and since

$$\begin{aligned} [p_q, x^{q'}] = [p^q, x_{q'}] &= \frac{1}{i}\delta_q^{q'} \qquad (\hbar = 1) \\ [p_q, p^{q'}] = [x_q, x^{q'}] &= 0 \end{aligned} \qquad (114)$$

one obtains the relations

$$\begin{aligned} [a^q, a_{q'}^+] &= \delta_q^{q'} \\ [a^q, a^{q'}] = [a_q^+, a_{q'}^+] &= 0 \end{aligned} \qquad (115a, b)$$

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indeed the commutation relations for boson operators. Raising and lowering of the indices is governed by the metric in (112) so that

$$a^{+q} = (-)^q a^+_{-q}, \qquad a_q = (-)^q a^{-q}, \qquad (a^q)^+ = a^+_q . \quad (116)$$

Suppressing the particle index i in (110), it follows from (112) and (113) that

$$H^{osc} = \sum_q a^+_q a^q + \frac{3}{2} \equiv \sum_q c^q_q + \frac{3}{2} . \qquad (117)$$

Similary, it can be sown that

$$L_q = (\vec{r} \times \vec{p})_q = \sum_{q'q''} \epsilon_{qq'q''} \; a^{+q'} a^{q''}$$

$$= \sum_{q'q''} (-)^{q'} \epsilon_{q-q'q''} \, a^+_{q'} a^{q''}$$

$$L_q = \sum_{q'q''} (-)^{q'} \epsilon_{q-q'q''} \; C^{q''}_{q'} \qquad (118)$$

the $\epsilon_{qq'q''}$ being the completely antisymmetric tensor, $\epsilon_{qq'q''} = 1$ if $q\,q'q''$ are cyclic in order $1\,1\,0$, $\epsilon_{qq'q''} = -1$ if not cyclic and $\epsilon_{qq'q''} = 0$ if any two indices repeat. In the step previous to the last in (118) the index q' was lowered according to rules (116).

As early as 1940, it was noted[51] that a harmonic oscillator potential remained invariant under the transformations of the three-dimensional unitary group U_3 . In (117) and (118) we defined the single-particle operators

$$C_q^{q'} \equiv a_q^+ a^{q'} \quad (q, q' = 1, \bar{1}, 0) \tag{119}$$

which, upon using relations (115a, b), can be seen to obey the commutation relations

$$[C_q^{q'}, C_{q''}^{q'''}] = C_q^{q'''} \delta_{q''}^{q'} - C_{q''}^{q'} \delta_q^{q'''} \tag{120}$$

which ---- by the same arguments associated with (32) ---- stand for the Lie commutator algebra of the $3^2 = 9$ generators for infinitesimal transformations of the group U_3 . That this is the <u>symmetry group</u> of H^{osc} of (117) follows from the fact that H commutes with <u>all</u> the generators of U_3 , that is

$$\begin{aligned} [H^{osc}, C_{q'}^{q''}] &= \sum_q [C_q^q, C_{q'}^{q''}] \\ &= \sum_q \{ C_q^{q''} \delta_{q'}^q - C_{q'}^q \delta_q^{q''} \} \\ &= 0. \end{aligned}$$

Again, as in (72) the set $\{ C_q^{q'} \}$ of U_3 generators can be subdivided into three classes: <u>lowering</u> $C_{\bar{1}}^1$, $C_0^{\bar{1}}$, C_0^1 ; <u>weight-giving</u> C_1^1 , $C_{\bar{1}}^{\bar{1}}$, C_0^0 (in that order), and <u>raising</u> $C_1^{\bar{1}}$, $C_{\bar{1}}^0$ and C_1^0 . Moreover, R_3 being a subgroup of U_3 , the generators of the former (L_q with $q = 1, \bar{1}, 0$) should be expressible as linear combinations of those of U_3 and from (118) this is indeed the case. Operators L^2 and $L_z = L_0$ can be simultaneously diagonalized along with H^{osc} since

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$$[H^{osc}, L^2] = [H^{osc}, \sum_q (-)^q L_q L_{-q}] = 0$$

$$[H^{osc}, L_0] = 0 \qquad (122)$$

$$[L^2, L_0] = 0.$$

Now, an arbitrary single-particle harmonic oscillator state can be written as

$$|n_1 n_{\bar{1}} n_0\rangle = \frac{(a_1^+)^{n_1} (a_{\bar{1}}^+)^{n_{\bar{1}}} (a_0^+)^{n_0}}{\sqrt{n_1!\ n_{\bar{1}}!\ n_0!}} |\bar{0}\rangle \qquad (123)$$

in terms of the numbers n_q of oscillator quanta along the three spherical "directions" $q = 1, \bar{1}, 0$. The state $|\bar{0}\rangle$ is the oscillator ground state (no excitation) and the radical provides normalization. There being $r = \frac{1}{2}(\nu+1)(\nu+2)$ different pssibilities for the triplet ($n_1\ n_{\bar{1}}\ n_0$) where $n_1 + n_{\bar{1}} + n_0 = \nu$ always, the vector (123) spans the r-dimensional orbital subspace of our problem. From (115a) one notices that the effect of a^q acting on any homogeneous polynomial P(a_q^+) is equivalent to a partial derivative:

$$a^q P(a_q^+) = \frac{\partial}{\partial a_q^+} P(a_q^+). \tag{124}$$

The effect of H^{osc} and L_0 upon $|n_1 n_{\bar{1}} n_0\rangle$ is thus immediate:

$$H^{osc}|n_1 n_{\bar{1}} n_0\rangle = \left\{\sum_q a_q^+ \frac{\partial}{\partial a_q^+} + \frac{3}{2}\right\} \frac{(a_1^+)^{n_1}(a_{\bar{1}}^+)^{n_{\bar{1}}}(a_0^+)^{n_0}}{\sqrt{n_1!\, n_{\bar{1}}!\, n_0!}}|\bar{0}\rangle$$

$$= \left(n_1 + n_{\bar{1}} + n_0 + \frac{3}{2}\right)|n_1 n_{\bar{1}} n_0\rangle$$

$$= \left(\nu + \frac{3}{2}\right)|n_1 n_{\bar{1}} n_0\rangle \tag{125}$$

$$L_0|n_1 n_{\bar{1}} n_0\rangle = (c_1^1 - c_{\bar{1}}^{\bar{1}})|n_1 n_{\bar{1}} n_0\rangle$$

$$= (n_1 - n_{\bar{1}})|n_1 n_{\bar{1}} n_0\rangle$$

$$\equiv m|n_1 n_{\bar{1}} n_0\rangle \tag{126}$$

These results are identical with (106a,c) in the sense that the energy and the angular momentum projection are diagonal. In the present case, L is <u>not</u> diagonal as it was in the old sets $\psi_{nlm}(\bar{r})$ which forms a basis for an irreducible representation according to the chain of single-particle transformation groups

$$U_3 \supset R_3 \supset R_2 \tag{127}$$

with representation labels ν, ℓ and m, respectively. Dealing with this <u>physical</u> chain is more difficult at present than working

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with the canonical (or mathematical) chain

$$U_3 \supset U_2 \supset U_1 \tag{128}$$

for which the basis $|n_1 n_{\bar{1}} n_0\rangle$ is most appropiate. Specifically, the group, its weight-giving operators and the corresponding representation labels are simply

$$U_3: \quad C_1^1, C_{\bar{1}}^{\bar{1}}, C_0^0 \longrightarrow (n_1 n_{\bar{1}} n_0)$$
$$U_2: \quad C_1^1, C_{\bar{1}}^{\bar{1}} \longrightarrow (n_1 n_{\bar{1}})$$
$$U_1: \quad C_1^1 \longrightarrow (n_1)$$

and, in the Gel'fand notation, a general state forming the basis transforming irreducibly according to (128) is

$$|n_1 n_{\bar{1}} n_0\rangle = \left| \begin{matrix} n_1 & n_{\bar{1}} & n_0 \\ & n_1 & n_{\bar{1}} \\ & & n_1 \end{matrix} \right\rangle . \tag{129}$$

6b. Many Particles.

The operators $C_q^{q'}$ defined in (119) are single-particle operators. We are now interested in dealing with the N-particle operator

$$\sum_{i=1}^{N} C_q^{q'}(i).$$

Being spin and isospin independent we can use formula (35) to cast (130) into second-quantization formulism:

$$C_q^{q'} = \sum_{\mu\mu'} \langle \mu' | C_q^{q'} | \mu \rangle \, \mathcal{C}_{\mu'}^{\mu} \tag{131}$$

recalling from (37) that

$$\mathcal{C}_{\mu'}^{\mu} = \sum_s b^+_{\mu' s} \, b^{\mu s} \qquad \begin{pmatrix} \mu, \mu' = 1, 2, \ldots, r \\ s \equiv \sigma\tau = 1, 2, 3, 4 \end{pmatrix} \tag{132}$$

are the group generators of $\mathcal{U}_r$, the group of transformations in orbital space, and where μ stood for the single-particle quantum numbers $\nu l m$, to be replaced here by the new set $n_1 n_\tau n_0$.

Thus,

$$C_q^{q'} = \sum_{n_1 n_\tau n_0} \sum_{n_1' n_\tau' n_0'} \langle n_1' n_\tau' n_0' | C_q^{q'} | n_1 n_\tau n_0 \rangle \, \mathcal{C}_{n_1' n_\tau' n_0'}^{n_1 n_\tau n_0} . \tag{133}$$

The coefficients in this expansion are very easily found from definitions (119) and (124) whose effect on the normalized oscillator state (123) is seen to give the simple result (see Appendix G for details):

$$C_q^{q'} = \sum_{n_1 n_\tau n_0} \sqrt{n_{q'} (n_q - \delta_{qq'} + 1)} \; \mathcal{C}_{n_1 + \delta_{q1} - \delta_{q'1},\, n_\tau + \delta_{q\tau} - \delta_{q'\tau},\, n_0 + \delta_{q0} - \delta_{q'0}}^{n_1 \qquad n_\tau \qquad n_0} \tag{134}$$

These 9 operators, from (131) and relations (120), obey the commutation relations

$$[C_q^{q'}, C_{q''}^{q'''}] = \sum_{\mu\mu'} \langle \mu' | [C_q^{q'}, C_{q''}^{q'''}] | \mu \rangle \, \mathcal{C}_{\mu'}^{\mu}$$

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$$=\left(\sum_{\mu\mu'}\langle\mu'|\, C_q^{q'''}\,|\mu\rangle\, b_{\mu'}^{\mu}\right)\delta_{q''}^{q'}-\left(\sum_{\mu\mu'}\langle\mu'|\, C_{q''}^{q'}\,|\mu\rangle\, b_{\mu'}^{\mu}\right)\delta_{q}^{q'''}$$

$$[C_q^{q'}, C_{q''}^{q'''}] = C_q^{q'''}\delta_{q''}^{q'} - C_{q''}^{q'}\delta_q^{q'''} \qquad (135)$$

which compose the Lie algebra of U_3. For the particular case of the 2s-1d shell

$$n_1+n_{\bar{1}}+n_0=\nu=\ 2\ \text{quanta}; \qquad r=\tfrac{1}{2}(\nu+1)(\nu+2)=6$$
$$\mathcal{U}_r=\mathcal{U}_6\supset U_3. \qquad (136)$$

Expansion (134) of the generators of the subgroup U_3 in terms of those of the group $\mathcal{U}_6$ can be explicitly given if we first take the convention of enumerating the 6 states $\mu=(n_1 n_{\bar{1}} n_0)$ in order of decreasing weights (in the sense of 72) and calling $q=1,\bar{1},0\rightarrow 1,2,3$ for convenience:

μ	$n_1 n_{\bar{1}} n_0$
1	2 0 0
2	1 1 0
3	1 0 1
4	0 2 0
5	0 1 1
6	0 0 2

and

$$C_1^1 \rightarrow C_1^1$$
$$C_{\bar{1}}^{\bar{1}} \rightarrow C_2^2 \qquad (137)$$
$$C_0^0 \rightarrow C_3^3$$
$$C_1^0 \rightarrow C_1^3$$
etc

Therefore, we have the set of U_3 generators $\{C_q^{q'}\}$ expressed as:

$$\text{RAISING}\begin{cases} C_1^2 = \sqrt{2}\, b_1^2 + \sqrt{2}\, b_2^4 + b_3^5 \\ C_1^3 = \sqrt{2}\, b_1^3 + \sqrt{2}\, b_3^6 + b_2^5 \\ C_2^3 = b_2^3 + \sqrt{2}\, b_4^5 + \sqrt{2}\, b_5^6 \end{cases}$$

$$\text{WEIGHT}\begin{cases} C_1^1 = 2 b_1^1 + b_2^2 + b_3^3 \\ C_2^2 = b_2^2 + 2 b_4^4 + b_5^5 \\ C_3^3 = b_3^3 + b_5^5 + 2 b_6^6 \end{cases} \qquad (138)$$

$$\text{LOWERING}\begin{cases} C_2^1 = \sqrt{2}\, b_2^1 + \sqrt{2}\, b_4^2 + b_5^3 \\ C_3^1 = \sqrt{2}\, b_3^1 + \sqrt{2}\, b_6^3 + b_5^2 \\ C_3^2 = b_3^2 + \sqrt{2}\, b_5^4 + \sqrt{2}\, b_6^5 \end{cases}$$

The N-particle interactionless oscillator hamiltonian

$$\sum_{i=1}^{N} H_i^{osc} = \sum_{i=1}^{N} \left\{ \sum_q C_q^q + \frac{3}{2} \right\} \qquad (136)$$

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can likewise be obtained, from (35), in second-quantization language as

$$\mathcal{H}^{osc} = \sum_{\mu\mu'} \langle\mu| \sum_q C_q^q + \frac{3}{2} |\mu'\rangle \, b_\mu^{\mu'}$$

$$= \sum_q C_q^q + \frac{3}{2} \sum_\mu b_\mu^\mu$$

$$= (n_1 + n_{\bar{1}} + n_0 + 3/2) \sum_\mu b_\mu^\mu$$

$$= (\nu + 3/2) \mathcal{N} \qquad (139)$$

$\mathcal{N}$ being the number operator defined in (71). Similarly, the N-particle momentum operator in (118)

$$\sum_{i=1}^{N} L_q(i) = \sum_{i=1}^{N} \sum_{q'q''} (-)^{q'} \epsilon_{q-q'q''} \, C_{q'}^{q''}(i) \qquad (140)$$

now becomes

$$\mathcal{L}_q = \sum_{q'q''} (-)^{q'} \epsilon_{q-q'q''} \left\{ \sum_{\mu\mu'} \langle\mu| C_{q'}^{q''} |\mu'\rangle \, b_\mu^{\mu'} \right\}$$

$$\mathcal{L}_q = \sum_{q'q''} (-)^{q'} \epsilon_{q-q'q''} \, C_{q'}^{q''} \qquad (q = 1, \bar{1}, 0) \qquad (141)$$

so that explicitly, the spherical components

$$\mathcal{L}_1 = -(C_0^{\bar{1}} + C_1^0), \qquad \mathcal{L}_0 = C_1^1 - C_{\bar{1}}^{\bar{1}}, \qquad \mathcal{L}_{\bar{1}} = C_0^1 + C_{\bar{1}}^0 \qquad (142a)$$

are easily seen to obey the commutation relations

$$[\mathcal{L}_0, \mathcal{L}_{\pm 1}] = \pm \mathcal{L}_{\pm 1} \qquad [\mathcal{L}_1, \mathcal{L}_{\bar{1}}] = -\mathcal{L}_0 \qquad (142b)$$

of the group $R_3 \subset U_3$.

In the many-particle case, the group U_3 with generators $\{C_q^{q'}\}$ $(q,q'=1,\bar{1},0 \rightarrow 1,2,3)$ obviously continues to be the symmetry group of the harmonic oscillator as its hamiltonian (136) is a <u>U_3 group invariant</u>, namely,

$$[\mathcal{H}^{osc}, C_q^{q'}] = 0. \qquad (143a)$$

Hence, just as L^2 (in section 4) commuting with the R_3 generators L_q $(q=+,0,-)$ lead to its diagonality (55a) in a base $P_{\ell m}$ transforming irreducibly under R_3, we here have that $\mathcal{H}^{osc}$ will be diagonal in a base irreducible under U_3 (See Appendix). As in (73 a,b) for the group $\mathcal{U}_r$, the maximum weight polynomial belonging to the U_3 basis set is defined by

$$C_q^q \, \mathbb{P}_{(h_{13}h_{23}h_{33})} = h_{q3} \, \mathbb{P}_{(h_{13}h_{23}h_{33})} \qquad \begin{pmatrix} q \rightarrow 1,2,3 \\ q<q' \end{pmatrix} \qquad (144a,b)$$

$$C_q^{q'} \, \mathbb{P}_{(h_{13}h_{23}h_{33})} = 0$$

where C_q^q and $C_q^{q'}$ $(q<q')$ are respectively the weight-giving and raising operators and the second index on the h's refers to U_3 . The raising operators (144b) of U_3 are linear in $\mathcal{U}_r$ raising operators according to (138). If $\mathbb{P}$ is choosen with maximum weight in $\mathcal{U}_r$ as in prescription (79), it will thus also be of maximum weight in U_3 . The <u>full</u> irreducible basis in U_3 could be obtained using lowering <u>operators</u> of U_3 (the $\mathbb{L}_r^k$) as was mentioned in the previous section for $\mathcal{U}_r$. This would give the basis in the $U_3 \supset U_2 \supset U_1$ chain, later to be transformed to one in the physical chain $U_3 \supset R_3 \supset R_2$.

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The second-quantization version of the N-particle hamiltonian (104) will thus be

$$\mathcal{H} = \mathcal{H}^{osc} + \mathcal{H}_{int} \tag{143b}$$

with $\mathcal{H}_{int}$ including both single-and two-body interactions to be dealt with later.

<u>Conclusion</u>: in the $\mathcal{U}_r \supset R_3 \supset R_2$ section of the chain of groups (99) with respect to which our many-particle nuclear states are to transform irreducibly, one can insert the extra (oscillator) symmetry group so that

$$\mathcal{U}_r \supset U_3 \supset R_3 \supset R_2 \tag{145a}$$

$$\uparrow \qquad \uparrow \qquad \uparrow \qquad \uparrow$$

$$|[f]\,\alpha\,(h_{13}h_{23}h_{33})\,\omega\,L\;\; M_L\rangle \tag{145b}$$

will provide the additional classification quantum numbers ($h_{13}\,h_{23}\,h_{33}$) of U_3 needed to further distinguish multiple L-values appearing under a given $[f]$ partition of $\mathcal{U}_r$. Labels α and ω , <u>not</u> proceeding from any particular group, serve respectively to distinguish multiple ($h_{13}\;\; h_{23}\;\; h_{33}$) values under a given $[f]$ and multiple L-values under a given ($h_{13}\;\; h_{23}\;\; h_{33}$). The basis state irreducible under (145) is thus designated as:

$$|[f]\,\alpha\,(h_{13}h_{23}h_{33})\,\omega\,L\;\; M_L\rangle \tag{145c}$$

PERMUTATIONAL SYMMETRY — HARMONIC OSCILLATOR SYMMETRY — SPHERICAL SYMMETRY — AXIAL SYMMETRY

CHAPTER

IV. MODEL RESIDUAL INTERACTIONS AND THEIR GROUP SYMMETRIES.

In the shell model one is dealing with a hamiltonian of the type

$$H = \sum_i \frac{p_i^2}{2m} + \sum_i U(r_i) + \xi \sum_i \vec{l}_i \cdot \vec{s}_i + \sum_{i<j} V(r_{ij}) \left(W + M P_{ij}^r - H P_{ij}^\tau + B P_{ij}^\sigma \right) \tag{147}$$

where 1, P_{ij}^r, P_{ij}^τ and P_{ij}^σ are the Wigner, Majorana, Heisenberg and Bartlett exchange operators[52] defined by

$$P_{ij}^\sigma \equiv \frac{1}{2}\left(1 + \vec{\sigma}_i \cdot \vec{\sigma}_j\right) \qquad P_{ij}^\tau \equiv \frac{1}{2}\left(1 + \vec{\tau} \cdot \vec{\tau}\right)$$

$$P_{ij}^r \equiv -\frac{1}{4}\left(1 + \vec{\sigma}_i \cdot \vec{\sigma}_j\right)\left(1 + \vec{\tau}_i \cdot \vec{\tau}_j\right) \tag{148a,b,c}$$

and W,M,H and B indicate the magnitudes of those components in the exchange mixture.

We shall assume that $U(r_i)$ is a harmonic oscillator common potential

$$U(r_i) = \frac{1}{2} m \omega^2 r_i^2$$

and furthermore restrict ourselves to particles interacting within a single shell of this harmonic oscillator. Under this assumption

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the two-body interaction can be analyzed as a mixture of long- and short-range correlations with definite group theoretical properties.

1. Long Range Central Interaction: The Q^2 force.

Let us take an arbitrary (say, Gaussian) from of the central potential function in (147):

$$V(r_{ij}) = -V_0\, e^{-r_{ij}^2/a^2} \qquad (i,j = 1,2,\cdots,N) \tag{151}$$

having variable well-depth V_0 and range a parameters. In addition to short-ranged forces ($\sim$1 to 2 fermis) within the nucleus there are longer-ranged correlations ($\sim$ nuclear radius) responsible for collective behaviour of the rotational and vibrational types.[6, 7] Assuming a gaussian well, rather flat throughout most of the nuclear volume (large a), we can expand the exponential in (151) retaining only the first few terms, namely

$$V(r_{ij}) = -V_0\left[1 - \frac{r_{ij}^2}{a^2} + \frac{r_{ij}^4}{2a^4} - \cdots\right]$$

where the relative distance $r_{ij} = \sqrt{r_i^2 + r_j^2 - 2\vec{r}_i\cdot\vec{r}_j}$ can be inserted above, and upon convenient rearrangements obtain

$$V(r_{ij}) = -V_0\left\{1 - \frac{r_i^2+r_j^2}{a^2} + \frac{2\vec{r}_i\cdot\vec{r}_j}{a^2} + \frac{r_j^4+r_i^4}{2a^4} + \frac{5r_i^2r_j^2}{3a^4} - \frac{2(r_i^2+r_j^2)\,\vec{r}_i\cdot\vec{r}_j}{a^4} + \frac{2\left[(\vec{r}_i\cdot\vec{r}_j)^2 - \frac{1}{3}r_i^2r_j^2\right]}{a^4} + \cdots\right\}. \tag{152}$$

the energy spectrum is determined by matrix elements of the type $\langle n_i l_i, n_j l_j, LM | V(r_{ij}) | n_i' l_i', n_j' l_j', LM \rangle$ for excitations within a single oscillator shell ν . But, from (105) and (106) one deduces that the 1st. 2nd. and 5th. terms of (152) contribute only as functions of $\nu = 2n + l$. Terms linear in ($\vec{r}_i \cdot \vec{r}_j$) ----- 3rd. and 6th. ----- give zero because states $| n_i' l_i' \rangle$ and $\langle n_i l_i |$ have the same parity while $\vec{r}_i, \vec{r}_j$ have odd parity. The 4th. term is separated in the particle coordinates and thus contributes not as an interaction but as a correction to the common potential. Therefore, the only pertinent term remaining for large a is the 7th:

$$V(r_{ij}) \approx -\frac{2V_0}{a^4}\left[(\vec{r}_i \cdot \vec{r}_j)^2 - \frac{1}{3} r_i^2 r_j^2\right] \tag{153}$$

This term is related to the harmonic oscillator symmetry group U_3 as we shall see.

The creation a_q^+ and annihilation a_q boson operators defined in (113) for harmonic oscillator states (123) transform like contravariant vector components, i.e., like a wave function of orbital angular momentum $l = 1$. Let us vector-couple them together to form the k-rank, q-projection tensors

$$\mathcal{U}_q^{(k)} = \sum_{q'q''} \langle 11 q' q'' | kq \rangle a_{q'}^+ a_{q''}$$

$$= \sum_{q'q''} (-)^{q''} \langle 11 q' -q'' | kq \rangle C_{q'}^{q''} \tag{154}$$

since $C_{q'}^{q''} \equiv a_{q'}^+ a^{q''}$ with $q', q'' = 1, \bar{1}, 0$ are the 9 single-particle generators of U_3 defined in (119) and which obey the Lie algebra

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(120). In linear combinations (154) only k=0,1,2 and q=-k,-k+1,...,k can occur so that there are

$$\sum_{k=0,1,2}(2k+1) = 1+3+5 = 9$$

linearly independent operators $\mathcal{U}_q^{(k)}$ which can equivalently be considered the 9 generators of U_3. (It can easily be seen that the commutators between these 9 operators also form a closed algebra.) In particular,

$$\underline{k=2} \quad \mathcal{U}_q^{(2)} = \sum_{q'q''} (-)^{q''} \langle 11 q' -q'' | 2q \rangle \, C_{q'}^{q''} \tag{155}$$

$$\underline{k=1} \quad \begin{aligned} \mathcal{U}_q^{(1)} &= \sum_{q'q''} (-)^{q''} \langle 11 q' -q'' | 1q \rangle \, C_{q'}^{q''} \\ &= -\frac{1}{\sqrt{2}} \sum_{q'q''} (-)^{q'} \epsilon_{q-q'q''} \, C_{q'}^{q''} = -\frac{1}{\sqrt{2}} L_q \end{aligned} \tag{156}$$

$$\underline{k=0} \quad \begin{aligned} \mathcal{U}_q^{(0)} &= \sum_{q'q''} (-)^{q'} \langle 11 q' -q' | 00 \rangle \, C_{q'}^{q'} \\ &= -\frac{1}{\sqrt{3}} \sum_{q'} C_{q'}^{q'} = -\frac{1}{\sqrt{3}} H_0 \end{aligned} \tag{157}$$

where (156) follows from definitions (118), and (157) save a scaling constant is the single-particle oscillator hamiltonian of (117). Considering the 5 operators for k=2 we have from (113) giving the $a_q^+ a^q$ in terms of coordinates and momenta,

$$\mathcal{U}_q^{(2)} = \sum_{q'q''} \langle 11 q' q'' | 2q \rangle \, a_{q'}^+ a_{q''}$$

$$= \sum_{q'q''} \langle 11q'q''|2q\rangle \left[x_{q'} x_{q''} + p_{q'} p_{q''} \right]$$

$$= \frac{1}{2} \left\{ \sqrt{\frac{8\pi}{15}}\, r^2 Y_{2q}(\theta_r, \varphi_r) + \sqrt{\frac{8\pi}{15}}\, p^2 Y_{2q}(\theta_p, \varphi_p) \right\}$$

given as solid spherical harmonics in coordinate and memontum spaces. The matrix element, between oscillator states of a given shell, of a momentum function is identical with that of the same function of coordinates ----- thus

$$\mathcal{U}_q^{(2)} \sim \sqrt{\frac{8\pi}{15}}\, r^2 Y_{2q}(\theta, \varphi) \tag{158}$$

The mass quadrupole moment induced by a single particle is conventionally[53] taken as the expectation value of the operator

$$Q_0 \equiv 3z^2 - r^2 = \sqrt{\frac{16\pi}{5}}\, r^2 Y_{20}(\theta, \varphi) \tag{159}$$

so that the 5 U_3 operators

$$Q_q = \sqrt{6}\, \mathcal{U}_q^{(2)} \tag{160}$$

can be thought of as representing generators of infinitesimal quadrupole distortions. A <u>quadrupole-quadrupole interaction</u> between the $i^{\underline{th}}$ and $j^{\underline{th}}$ particles is simply the scalar product

$$Q_{ij}^2 \equiv \sum_q (-)^q\, Q_q(i)\, Q_{-q}(j) \tag{161}$$

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From (158) taken separately for particles i and j, and the addition theorem for spherical harmonics,[54)]

$$Q_{ij}^2 = \frac{16\pi}{5} r_i^2 r_j^2 \sum_q (-)^q Y_{2q}(i) Y_{2-q}(j)$$

$$= 4 r_i^2 r_j^2 P_2(\cos \omega_{ij})$$

$$= 4 r_i^2 r_j^2 P_2\left(\frac{\vec{r}_i \cdot \vec{r}_j}{r_i r_j}\right)$$

$$= 2 r_i^2 r_j^2 \left[3 \frac{(\vec{r}_i \cdot \vec{r}_j)^2}{(r_i r_j)^2} - 1 \right]$$

$$Q_{ij}^2 = 6\left[(\vec{r}_i \cdot \vec{r}_j)^2 - \frac{1}{3} r_i^2 r_j^2\right] \qquad (162)$$

which is identical in form to (153) ----- the only significant portion of a long-ranged central potential.

The expressibility of (162) in terms of U_3 generators is the crux of this whole section. The Casimir operator of R_3 defined in (56) as

$$L^2 = \sum_q (-)^q L_q L_{-q} = \sum_q L_q L^q \qquad (163)$$

was particularly useful in solving the asymmetric top problem discussed in the Introduction: its eigenvalue was simply $\ell(\ell+1)$,

dependent only on the irreducible representation label and irregardless of its row. A contraction analogous to (163) yields

$$G_{ij} \equiv \sum_{q'q''} C_{q'}^{q''}(i)\, C_{q''}^{q'}(j) \qquad (i,j = 1,2,\cdots,N) \qquad (164a)$$

Then,

$$\sum_{i<j} G_{ij} \qquad (164b)$$

is the (second degree) Casimir operator of U_3 which should and does commute with all the many-particle U_3 operators $\sum_i C_q^{q'}(i)$. Employing the orthonormality relations between Clebsch-Gordan coefficients[35] one can reverse expansions (154) so that

$$C_{q'}^{q''} = (-)^{q''} \sum_{kq} \langle 11 q' -q'' | kq \rangle \mathcal{U}_q^{(k)} \qquad (165)$$

and which is valid separately for i and j. Using this expression, the Casimir operator (164b) will contain

$$G_{ij} = \sum_k \Big\{ \sum_q (-)^q \mathcal{U}_q^{(k)}(i)\, \mathcal{U}_{-q}^{(k)}(j) \Big\}$$

$$= \sum_q (-)^q \Big\{ \mathcal{U}_q^{(0)}(i)\, \mathcal{U}_{-q}^{(0)}(j) + \mathcal{U}_q^{(1)}(i)\, \mathcal{U}_{-q}^{(1)}(j) + \mathcal{U}_q^{(2)}(i)\, \mathcal{U}_{-q}^{(2)}(j) \Big\}$$

$$= \frac{1}{3} H_0^2(i) + \frac{1}{2} \sum_q (-)^q L_q(i)\, L_{-q}(j) + \sum_q (-)^q Q_q(i)\, Q_{-q}(j)$$

having recalled (156), (157) and (160). Finally

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$$G_{ij} = Q_{ij}^2 + \frac{1}{2} L_{ij}^2 + \frac{1}{3} H_{oi}^2$$

$$Q_{ij}^2 = G_{ij} - \frac{1}{2} L_{ij}^2 - \frac{1}{3} H_{oi}^2 . \qquad (166a,b)$$

Clearly, since the interaction operator $\sum_{i<j} Q_{ij}^2$ is given solely in terms of the Casimir operators of U_3 and R_3, and in terms of $\sum_i H_{oi}$, <u>it will be diagonal in a basis for irreducible representations of the groups $U_3 > R_3$</u>.

To cast the N-particle mass quadrupole moment operator $\sum_{i=1}^{N} Q_q(i)$ into second-quantization formulism we again use formula (35) to give the linear combination

$$\mathcal{Q}_q = \sum_{\mu\mu'} \langle \mu | Q_q | \mu' \rangle \, b_\mu^{\mu'}$$

$$= \sum_{\mu\mu'} \langle \mu | \sqrt{\frac{16\pi}{5}} \, r^2 Y_{2q}(\theta,\varphi) | \mu' \rangle \, b_\mu^{\mu'} \qquad (167)$$

whose expectation value, for q=o, between appropriate states is the quadrupole moment induced by N particles. (More detail concerning quadrupole moments and transition rates are given in Appendix J).

Result (166b) in the new many-particle formulism will be

$$\mathcal{Q}^2 = \mathcal{G} - \frac{1}{2} \mathcal{L}^2 - \frac{1}{3} \mathcal{H}_o^2 \qquad (168)$$

where the many-body Casimir operator $\mathcal{G}$ is constructed as in (164) but now in terms of the many-body U_3 generators defined in (131):

$$\mathcal{G} = \sum_{qq'} C_q^{q'} C_{q'}^{q} \quad , \qquad [\mathcal{G}, C_q^{q'}] = 0 . \tag{169}$$

Moreover, we shall have

$$\mathcal{Q}^2 = \sum_q (-)^q \mathcal{Q}_q \mathcal{Q}_{-q} \tag{170}$$

$$\mathcal{L}^2 = \sum_q (-)^q \mathcal{L}_q \mathcal{L}_{-q} \tag{171}$$

$$\mathcal{H}_0 = \mathcal{H}^{osc} - \tfrac{3}{2} \mathcal{N} = \nu \mathcal{N} \tag{172}$$

in accordance with results (139), (141) and (167).

Our model long-range interaction (168) is diagonal under $U_3 \supset R_3$ as argued above. To find its <u>spectrum</u> is a simple matter precisely because of its group symmetry. The eigenvalues of $\mathcal{L}^2$ and $\mathcal{H}_0$ are simply $L(L+1)$ and νN, respectively, L being the total orbital angular momentum of a given state. Now, $\mathcal{G}$ commutes with <u>all</u> the U_3 generators; the lowering ones can be used successively to generate the full irreducible U_3 basis starting from the function of maximum weight in U_3 defined in (144) as $\mathbb{P}_{(h_{13} h_{23} h_{33})}$. Hence, the eigenvalue g of $\mathcal{G}$ is independent of the <u>row</u> of the irreducible basis and it suffices to find g in the equation

$$\mathcal{G} \mathbb{P}_{(h_{13} h_{23} h_{33})} = g \, \mathbb{P}_{(h_{13} h_{23} h_{33})} .$$

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But, expanding $\mathcal{G}$ one has

$$\mathcal{G} = \sum_{qq'} C_q^{q'} C_{q'}^{q} = \sum_q (C_q^q)^2 + \sum_{q<q'} C_q^{q'} C_{q'}^{q} + \sum_{q>q'} C_q^{q'} C_{q'}^{q}$$

$$= \sum_q (C_q^q)^2 + \sum_{q<q'} [C_q^{q'}, C_{q'}^{q}] + 2\sum_{q'<q} C_q^{q'} C_{q'}^{q}$$

where the effect of the last term, involving a <u>raising</u> operator on the right, vanishes because of (144b) and the second term upon use of relations (135) gives

$$\mathcal{G}\mathbb{P} = \left\{\sum_q (C_q^q)^2 + \sum_{q<q'} (C_q^q - C_{q'}^{q'})\right\} \mathbb{P}$$

$$= \left\{C_1^1 + C_2^2 + C_3^3 + 2(C_1^1 - C_3^3)\right\} \mathbb{P} \qquad (174)$$

$$\therefore\ g = h_{13}^2 + h_{23}^2 + h_{33}^2 + 2(h_{13} - h_{33}) \qquad (175)$$

depending <u>only</u> on the U_3 representation $(h_{13}\ h_{23}\ h_{33})$. This representation can alternatively be characterized by only <u>two</u> numbers $(k_1 k_2) \equiv (h_{13}-h_{33},\ h_{23}-h_{33})$ since (172) gives $h_{13} + h_{23} + h_{33} = \nu N$ = constant for a given problem. (The relation between $(k_1 k_2)$ and the labels $(\lambda\mu)$ used by Elliott in his work[12]) are: $\lambda = k_1 - k_2$ and $\mu = k_2$.) Hence, the eigenvalue of $\mathcal{Q}^2$ is

$$E_L^{(k_1 k_2)} = g - \frac{1}{2} L(L+1) - \frac{1}{3}(\nu N)^2$$

$$E_L^{(k_1 k_2)} = \frac{2}{3}(k_1+k_2)^2 - 2k_1(k_2-1) - \frac{1}{2} L(L+1) \qquad (176)$$

manifesting a rotational band-like structure due to the L(L+1) term. States (145c) are eigenfunctions of $\mathcal{Q}^2$:

$$\mathcal{Q}^2 |[f]\alpha(k_1 k_2)\omega L M_L\rangle = E_L^{(k_1 k_2)} |[f]\alpha(k_1 k_2)\omega L M_L\rangle. \qquad (177)$$

2. <u>Exchange Dependent On Spin and Isopin. The Group U_4.</u>[87]

We shall now include the effect of exchange forces of the Wigner, Mayorana, Bartlett and Heisenberg types for the particularly simple extreme of long range. It will be recalled from (147) that a central two-body interaction with exchange can be taken as

$$\sum_{i<j} I_{ij} V(r_{ij}) \equiv \sum_{i<j} \left(W + M P_{ij}^{r} - H P_{ij}^{\tau} + B P_{ij}^{\sigma}\right) V(r_{ij}). \qquad (178a)$$

The operator of exchange between the ith and j^{th} particles is I_{ij} (for <u>inter</u>change) and is defined as the linear combination within the parenthesis of (178a) of the operators

<u>Bartlett</u>: $P_{ij}^{\sigma} = \frac{1}{2}(1+\vec{\sigma}_i\cdot\vec{\sigma}_j) = \frac{1}{2}(1+4\vec{s}_i\cdot\vec{s}_j)$

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<u>Heisenberg</u>: $P^{\tau}_{ij} = \frac{1}{2}(1+\vec{\tau}_i\cdot\vec{\tau}_j) = \frac{1}{2}(1+4\vec{t}_i\cdot\vec{t}_j)$ (178b,c,d)

<u>Majorana</u>: $P^{r}_{ij} = -\frac{1}{4}(1+4\vec{s}_i\cdot\vec{s}_j)(1+4\vec{t}_i\cdot\vec{t}_j)$

in accordance with (148), $\vec{s}$ and $\vec{t}$ being the spin and isospin operators of a single particle with possible eigenvalues each of 1/2 and -1/2.

Now, by formula (34) our two-body operator (178a) in second-quantization language will become

$$\mathcal{J}\mathcal{V} = \frac{1}{2}\sum_{\substack{\rho_1\rho_2\\ \rho_1'\rho_2'}} \langle \rho_1\rho_2 | I_{12}V_{12} | \rho_1'\rho_2'\rangle \left\{ C^{\rho_1'}_{\rho_1} C^{\rho_2'}_{\rho_2} - \delta^{\rho_1'}_{\rho_2} C^{\rho_2'}_{\rho_1} \right\}$$

where $\rho_i \equiv (\mu_i\sigma_i\tau_i), (i=1,2)$. Since I_{12} depends only on spin-isospin and V_{12} only on the orbital part, then

$$\mathcal{J}\mathcal{V} = \frac{1}{2}\sum_{\substack{\mu_1\sigma_1\tau_1\\ \mu_2\sigma_2\tau_2\\ \mu_1'\sigma_1'\tau_1'\\ \mu_2'\sigma_2'\tau_2'}} \langle \sigma_1\tau_1,\sigma_2\tau_2 | I_{12} | \sigma_1'\tau_1',\sigma_2'\tau_2'\rangle \langle \mu_1\mu_2 | V_{12} | \mu_1'\mu_2'\rangle \times$$
$$\times \left\{ C^{\mu_1'\sigma_1'\tau_1'}_{\mu_1\sigma_1\tau_1} C^{\mu_2'\sigma_2'\tau_2'}_{\mu_2\sigma_2\tau_2} - \delta^{\mu_1'}_{\mu_2}\delta^{\sigma_1'}_{\sigma_2}\delta^{\tau_1'}_{\tau_2} C^{\mu_2'\sigma_2'\tau_2'}_{\mu_1\sigma_1\tau_1} \right\}$$

Assuming $V(r_{12}) \equiv V_{12}$ equal to say $-V_0 e^{-r_{12}^2/\alpha^2}$ to be of <u>long range</u> then we can approximate

$$V_{12} \simeq -V_0 \quad \text{(constant)}$$

so that $\langle \mu_1\mu_2 | V_{12} | \mu_1'\mu_2'\rangle \simeq -V_0\delta^{\mu_1'}_{\mu_1}\delta^{\mu_2'}_{\mu_2}$ and thus, since in this case $\mathcal{V} = -V_0$,

$$\mathcal{J} = \frac{1}{2} \sum_{\substack{\sigma_1\tau_1\sigma_2\tau_2 \\ \sigma_1'\tau_1'\sigma_2'\tau_2'}} \langle \sigma_1\tau_1, \sigma_2\tau_2 | I_{12} | \sigma_1'\tau_1', \sigma_2'\tau_2' \rangle \times \left\{ C_{\sigma_1\tau_1}^{\sigma_1'\tau_1'} C_{\sigma_2\tau_2}^{\sigma_2'\tau_2'} - \delta_{\sigma_2}^{\sigma_1'} \delta_{\tau_2}^{\tau_1'} C_{\sigma_1\tau_1}^{\sigma_2'\tau_2'} \right\} \tag{179}$$

where from (61) the 4^2 operators $C_{\sigma\tau}^{\sigma'\tau'}$, which obey the U_4 commutation relations (62), are the generators of U_4--- the group of transformations in spin-isospin space.

The problem of the complete classification of states of supermultiplet theory, i.e., according to the U_4 chain given in (99), has been solved recently by Moshinsky and Nagel.[46] To completely characterize the rows of the bases for the irreducible representations of U_n, Racah in his Lecture Notes[55] proves that $\frac{1}{2}n(n-1)$ commuting operators that are functions of the group generators are required. This, for U_4 six such operators are needed. Explicit reduction of U_4 representations according to (99) provides four of the six needed: S^2, T^2, S_z, T_z with eigenvalues S(S+1), T(T+1) M_S and M_T. (The other two, whose eigenvalues constitute the additional label β in the chain (99), are derived by Moshinsky & Nagel in the paper referred to above.) The cited authors redefine the U_4 group generators as the following more physical operators, totaling 4^2 in number,

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$$\mathcal{W} \equiv \sum_{\sigma\tau\sigma'\tau'} (M_0)^{\sigma}_{\sigma'} (N_0)^{\tau}_{\tau'} C^{\sigma'\tau'}_{\sigma\tau} = \sum_{\sigma\tau} C^{\sigma\tau}_{\sigma\tau}$$

$$S_k \equiv \frac{1}{2} \sum_{\sigma\tau\sigma'\tau'} (M_k)^{\sigma}_{\sigma'} (N_0)^{\tau}_{\tau'} C^{\sigma'\tau'}_{\sigma\tau}$$

$$(180\ a, b, c, d)$$

$$T_\ell \equiv \frac{1}{2} \sum_{\sigma\tau\sigma'\tau'} (M_0)^{\sigma}_{\sigma'} (N_\ell)^{\tau}_{\tau'} C^{\sigma'\tau'}_{\sigma\tau}$$

$$R_{k\ell} \equiv \frac{1}{4} \sum_{\sigma\tau\sigma'\tau'} (M_k)^{\sigma}_{\sigma'} (N_\ell)^{\tau}_{\tau'} C^{\sigma'\tau'}_{\sigma\tau}$$

where k,ℓ =1,2,3 refer to cartesian components and

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the unit matrix and the well-known Pauli spin-matrices whose rows and columns are given by σ', σ =1/2,-1/2. Similarly, N_0, N_ℓ (ℓ=1,2,3) refer to the unit and Pauli matrices in isospin with rows and columns given by τ', τ =1/2,-1/2. Among the commutation relations between the operators (180), the relations

$$[S_i, S_j] = i \sum_k \epsilon_{ijk} S_k \ ; \ [T_i, T_j] = i \sum_k \epsilon_{ijk} T_k \ ; \ [S_i, T_j] = 0$$

hold ----- thus S_k (k=1,2,3) and T_ℓ (ℓ=1,2,3) are the generators of SU_2(spin) and SU_2(isospin) which in turn are homomorphic to

R_3 (spin) and R_3 (isopin), respectively. Now, a tensor operator $\mathcal{T}_{qr}$ (similar to (154) for U_3) can here be defined as

$$\mathcal{T}_{qr} \equiv \sum_{\sigma\tau\sigma'\tau'} (M_q)^{\sigma}_{\sigma'} (N_r)^{\tau}_{\tau'} \, C^{\sigma'\tau'}_{\sigma\tau} \qquad \begin{cases} q = 0, k = 0,1,2,3 \\ r = 0, \ell = 0,1,2,3 \end{cases} \tag{181}$$

such that

$$\mathcal{T}_{00} = \mathcal{N}, \quad \mathcal{T}_{k0} = 2S_k, \quad \mathcal{T}_{0\ell} = 2T_\ell, \quad \mathcal{T}_{k\ell} = 4R_{k\ell},$$

and whose scalar product

$$\sum_{qr} \mathcal{T}_{qr}\mathcal{T}_{qr} = \mathcal{T}_{00}^2 + \sum_k \mathcal{T}_{k0}^2 + \sum_\ell \mathcal{T}_{0\ell}^2 + \sum_{k\ell} \mathcal{T}_{k\ell}^2$$

$$= \mathcal{N}^2 + 4S^2 + 4T^2 + 16R^2. \tag{182}$$

On the other hand, from definition (181)

$$\sum_{qr} \mathcal{T}_{qr}\mathcal{T}_{qr} = \sum_{\substack{\sigma\tau\sigma'\tau' \\ \bar\sigma\bar\tau\bar\sigma'\bar\tau'}} \sum_q (M_q)^{\sigma}_{\sigma'} (M_q)^{\bar\sigma}_{\bar\sigma'} \sum_r (N_r)^{\tau'}_{\tau} (N_r)^{\bar\tau'}_{\bar\tau} \, C^{\sigma'\tau'}_{\sigma\tau} C^{\bar\sigma'\bar\tau'}_{\bar\sigma\bar\tau}$$

$$= \sum_{\substack{\sigma\tau\sigma'\tau' \\ \bar\sigma\bar\tau\bar\sigma'\bar\tau'}} 2\delta^{\bar\sigma}_{\sigma'}\delta^{\sigma}_{\bar\sigma'} \cdot 2\delta^{\bar\tau'}_{\tau}\delta^{\tau'}_{\bar\tau} \, C^{\sigma'\tau'}_{\sigma\tau} C^{\bar\sigma'\bar\tau'}_{\bar\sigma\bar\tau}$$

$$= 4 \sum_{\substack{\sigma\tau \\ \sigma'\tau'}} C^{\sigma'\tau'}_{\sigma\tau} C^{\sigma\tau}_{\sigma'\tau'}$$

$$= 4\, G_2(U_4) \tag{183}$$

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where $G_2(U_4)$ is the (second degree) Casimir operator of U_4. Therefore,

$$G_2(U_4) = \frac{1}{4}\mathcal{N}^2 + S^2 + T^2 + 4R^2. \qquad (184)$$

So much for the formal properties of the group U_4. Turning our attention to the exchange operator $\mathcal{J}$ of (179), and in view of definition (178a) for I_{ij}, one concludes that

$$\mathcal{J} = W + M\mathcal{P}^r + B\mathcal{P}^\sigma - H\mathcal{P}^\tau \qquad (185)$$

W,M,B and H being arbitrary constants as before; the script $\mathcal{P}$'s referring to the second-quantization formulation of the various exchange operators. For example, the isopin independent Bartlett operator is now

$$\mathcal{P}^\sigma = \frac{1}{2}\sum_{\substack{\sigma_1\sigma_2\\ \sigma_1'\sigma_2'}} \langle\sigma_1\sigma_2|\, P_{12}^\sigma \,|\sigma_1'\sigma_2'\rangle \sum_{\tau_1\tau_2}\left\{ C_{\sigma_1\tau_1}^{\sigma_1'\tau_1} C_{\sigma_2\tau_2}^{\sigma_2'\tau_2} - \delta_{\sigma_2}^{\sigma_1'}\delta_{\tau_2}^{\tau_1} C_{\sigma_1\tau_1}^{\sigma_2'\tau_2}\right\}.$$

But from (178b), the coefficient

$$\frac{1}{2}\langle\sigma_1\sigma_2|\, P_{12}^\sigma \,|\sigma_1'\sigma_2'\rangle = \frac{1}{4}\delta_{\sigma_1}^{\sigma_1'}\delta_{\sigma_2}^{\sigma_2'} + \sum_{k=1}^{3}\langle\sigma_1|\Delta_{k1}|\sigma_1'\rangle\langle\sigma_2|\Delta_{k2}|\sigma_2'\rangle$$

so that from (180a,b) the operator $\mathcal{P}^\sigma$ becomes simply

$$\mathcal{P}^\sigma = \frac{1}{4}\mathcal{N}^2 - \mathcal{N} + S^2$$

if one recalls that, for example $\langle\sigma|\Delta_k|\sigma'\rangle$ is one-half the

(σ, σ')-element of the kth component Pauli spin matrix. In exactly the same manner, the spin-independent Heisenberg operator reduces to

$$\mathcal{P}^{\tau} = \frac{1}{2}\mathcal{W}^2 - \mathcal{W} + T^2.$$

The Majorana operator (178d) is more general than these; it can be written as

$$P_{12}^{r} = -\frac{1}{4} - \vec{s}_1\cdot\vec{s}_2 - \vec{t}_1\cdot\vec{t}_2 - 4(\vec{s}_1\cdot\vec{s}_2)(\vec{t}_1\cdot\vec{t}_2)$$

$$= -\frac{1}{2}\left(P_{12}^{\sigma} + P_{12}^{\tau}\right) + \frac{1}{4} - 4\sum_{k,l=1}^{3} s_{k1}t_{l1}s_{k2}t_{l2}\ .$$

Thus, using formula (179) one has

$$\mathcal{P}^{r} = -\frac{1}{2}(\mathcal{P}^{\sigma} + \mathcal{P}^{\tau}) + \frac{1}{2}\Big\{\frac{1}{4}(\mathcal{W}^2 - \mathcal{W})$$

$$-4\sum_{kl}\sum_{\substack{\sigma_1\tau_1\\ \sigma_1'\tau_1'}}\langle\sigma_1|s_{k1}|\sigma_1'\rangle\langle\tau_1|t_{l1}|\tau_1'\rangle C_{\sigma_1\tau_1}^{\sigma_1'\tau_1'}\times$$

$$\times\sum_{\substack{\sigma_2\tau_2\\ \sigma_2'\tau_2'}}\langle\sigma_2|s_{k2}|\sigma_2'\rangle\langle\tau_2|t_{l2}|\tau_2'\rangle C_{\sigma_2\tau_2}^{\sigma_2'\tau_2'}$$

$$+4\sum_{kl}\sum_{\substack{\sigma_1\tau_1\sigma_2\tau_2\\ \sigma_1'\tau_1'\sigma_2'\tau_2'}}\langle\sigma_1|s_{k1}|\sigma_1'\rangle\langle\sigma_2|s_{k2}|\sigma_2'\rangle\langle\tau_1|t_{l1}|\tau_1'\rangle\langle\tau_2|t_{l2}|\tau_2'\rangle\times$$

$$\times C_{\sigma_1\tau_1}^{\sigma_2'\tau_2'}\delta_{\sigma_2}^{\sigma_1'}\delta_{\tau_2}^{\tau_1'}\Big\}$$

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$$= -\frac{1}{2}(\mathcal{P}^{\sigma}+\mathcal{P}^{\tau}) + \frac{1}{2}\left\{ \frac{1}{4}(\mathcal{W}^2-\mathcal{W}) - 4R^2 + \frac{9}{4}\mathcal{W} \right\}$$

where the scalar product of (180d) with itself is used in evaluating the expression before the last and the last expression is simply $4\cdot\frac{3}{4}\cdot\frac{3}{4}\,\mathcal{W}$ so that

$$\mathcal{P}^{r} = -\frac{1}{2}(\mathcal{P}^{\sigma}+\mathcal{P}^{\tau}) + \frac{1}{2}\left\{ \frac{1}{4}(\mathcal{W}^2-\mathcal{W}) - 4R^2 + \frac{9}{4}\mathcal{W} \right\}$$

$$= 2\mathcal{W} - \frac{1}{2}\left[\frac{1}{4}\mathcal{W}^2 + 4R^2 + S^2 + T^2\right]$$

$$\mathcal{P}^{r} = 2\mathcal{W} - \frac{1}{2}G_2(\mathcal{U}_4)$$

having used here the previous $\mathcal{U}_4$ result (184). To deal with $G_2(\mathcal{U}_r)$ instead of $G_2(\mathcal{U}_4)$ ----- future developments will make this more convenient ----- one can derive a relation between these two Casimir operators which is very simply due to the fact that the $\mathcal{U}_4$ representations must be conjugate to those of $\mathcal{U}_r$ (see example in 80). From the last two steps in (183) and from (61) and (31) one can write

$$G_2(\mathcal{U}_4) = \sum_{ss'} C_s^{s'} C_{s'}^{s} = \sum_{ss'}\sum_{\mu\bar{\mu}} b^{+}_{\mu s}\, b^{\mu s'}\, b^{+}_{\bar{\mu} s'}\, b^{\bar{\mu} s}$$

$$(s, s' = \sigma\tau, \sigma'\tau') \qquad (\mu, \mu' = 1, 2, \cdots, r)$$

which by using anticommutation relations (12) becomes

$$G_2(U_4) = (4+r)W - \sum_{\mu\bar{\mu}} b_\mu^{\bar{\mu}\bullet} b_{\bar{\mu}}^{\mu}$$

$$G_2(U_4) = (4+r)W - G_2(\mathcal{U}_r)$$

the latter being the $\mathcal{U}_r$ Casimir operator. Thus

$$P^r = \frac{1}{2}\left[G_2(\mathcal{U}_r) - r\,W\right].$$

Combining the results just obtained for P, P^r, P^σ and P^τ into (185) we arrive finally at the <u>long-range exchange operator</u>

$$\mathcal{J} = W + \frac{1}{4}(B-H)W(W-4) + BS^2 - HT^2 + \frac{1}{2}M\left[G_2(\mathcal{U}_r) - rW\right]$$

which is an eigenoperator of a basis set transforming irreducibly according to the $U_4 \supset SU_2 \times SU_2$ segment of the chain in (99), that is, when acting on the kets

$$|\{v_1 v_2 v_3 v_4\}\,\beta\, S M_S T M_T\rangle \tag{185}$$

where $\{v_1 v_2 v_3 v_4\} = [\widetilde{h_1 h_2 \cdots h_r}] = [\tilde{f}]$ of (97). The Casimir operator $G_2(\mathcal{U}_r)$ is the operator Γ to be used in the section on pairing and whose eigenvalue is there derived to give (226).

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Finally, one has the algebraic eigenvalue equation

$$\mathcal{J}\,|[\tilde{f}]\beta S M_S T M_T\rangle = I\,|[\tilde{f}]\beta S M_S T M_T\rangle \tag{186}$$

$$I \equiv W + \tfrac{1}{4}(B-H)N(N-4) + BS(S+1) - HT(T+1) + \tfrac{1}{2}M\sum_{\mu=1}^{r} h_\mu(h_\mu - 2\mu + 1)$$

where I is the contribution to the energy at long-range due to exchange.

<u>Conclusion</u>: the simple eigenvalue I given above depends only on S and T for a given $[f]$ partition of N particles. One must multiply the diagonal $\mathcal{Q}^2$ matrix elements of (177) in a base extended to include total spin S and isospin T quantum numbers. Thus

$$\langle [f]\alpha(k_1k_2)\omega L M_L;\, [\tilde{f}]\beta S M_S T M_T\,|\,\mathcal{J}\mathcal{Q}^2\,|\,[f]\alpha(k_1k_2)\omega L M_L;\, [\tilde{f}]\beta S M_S T M_T\rangle$$

$$= \langle [f]\alpha(k_1k_2)\omega L M_L\,|\,\mathcal{Q}^2\,|\,[f]\alpha(k_1k_2)\omega L M_L\rangle \times$$
$$\times \langle [\tilde{f}]\beta S M_S T M_T\,|\,\mathcal{J}\,|\,[\tilde{f}]\beta S M_S T M_T\rangle \tag{187}$$

is diagonal and accounts for the long-range part of a central interaction with exchange.

One final remark regarding different exchange mixtures is in order; for this end we give several of the more popular ones in Table IV.2.1.

MIXTURE NAME	W	M	H	B
ROSENFELD	-.13	.93	-.26	.46
SERBER	.50	.50	0	0
KURATH	0	.80	0	.20
MESHKOV	.40	.40	0	.20
SOPER	.40	.30	.10	.20
SLIV[56]	.70	.15	.15	0

TABLE IV.2.1

Note: $W+M+H+B \equiv 1$

3. The Pairing Force. Short-Range Character.

The general interaction operator $\sum_{i<j} V(r_{ij})$ is equivalent under second-quantization formulation to the operator (25):

$$\mathcal{V} = \frac{1}{2} \sum_{\substack{\rho_1 \rho_2 \\ \rho_1' \rho_2'}} \langle \rho_1 \rho_2 | V_{12} | \rho_1' \rho_2' \rangle \, b^+_{\rho_2} \, b^+_{\rho_1} \, b^{\rho_1'} b^{\rho_2'}, \qquad (188)$$

$$\rho_i \equiv (\nu l_i m_i ; \sigma_i \tau_i) \equiv (\nu l_i m_i ; s_i) \qquad (i = 1, 2)$$

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being the single-particle state labels referring to orbital quantum numbers $\nu l_i m_i$ arising within a single given shell ν and spin-isospin quantum numbers s_i. As before, if $V_{12} \equiv V(r_{12})$ is spin-isospin independent then

$$\langle \rho_1 \rho_2 | V_{12} | \rho_1' \rho_2' \rangle \equiv \langle \nu l_1 m_1 s_1, \nu l_2 m_2 s_2 | V_{12} | \nu l_1' m_1' s_1', \nu l_2' m_2' s_2' \rangle \tag{189}$$

$$= \langle \nu l_1 m_1, \nu l_2 m_2 | V_{12} | \nu l_1' m_1', \nu l_2' m_2' \rangle \delta_{s_1 s_1'} \delta_{s_2 s_2'}$$

holds. Furthermore, letting Λ be the resultant orbital angular momentum eigenvalue and M its projection, one has the coupled or paired state

$$|\nu l_1, \nu l_2, \Lambda M\rangle = \sum_{m_1 m_2} \langle l_1 l_2 m_1 m_2 | \Lambda M \rangle | \nu l_1 m_1, \nu l_2 m_2 \rangle, \tag{190a}$$

the Clebsch-Gordan coefficients obeying the usual orthonormality relations

$$\sum_{m_1 m_2} \langle l_1 l_2 m_1 m_2 | \Lambda M \rangle \langle l_1 l_2 m_1 m_2 | \Lambda' M' \rangle = \delta_{\Lambda\Lambda'} \delta_{MM'}. \tag{190b}$$

Let us rewrite (188) such that particles 1 and 2 are paired to give net angular momentum Λ. Obviously

$$\mathcal{V} = \frac{1}{2} \sum_{\substack{l_1 l_2 l_1' l_2' \\ m_1 m_2 m_1' m_2' \\ \bar{m}_1 \bar{m}_2 \bar{m}_1' \bar{m}_2'}} \langle \nu l_1 \bar{m}_1, \nu l_2 \bar{m}_2 | V_{12} | \nu l_1' \bar{m}_1', \nu l_2' \bar{m}_2' \rangle \delta_{m_1}^{\bar{m}_1} \delta_{m_2}^{\bar{m}_2} \delta_{m_1'}^{\bar{m}_1'} \delta_{m_2'}^{\bar{m}_2'}$$

$$\times \sum_{s_1 s_2} b^{+}_{\nu l_2 m_2 s_2} b^{+}_{\nu l_1 m_1 s_1} b^{\nu l_1' m_1' s_1} b^{\nu l_2' m_2' s_2}$$

is equivalent to (188) with restriction (189). Using (190) and (191) there results

$$\mathcal{V} = \frac{1}{2} \sum_{\substack{\Lambda M \Lambda' M' \\ \ell_1 \ell_2 \ell_1' \ell_2'}} \langle \nu \ell_1, \nu \ell_2, \Lambda M | V_{12} | \nu \ell_1', \nu \ell_2', \Lambda' M' \rangle \delta_{\Lambda\Lambda'} \delta_{MM'}$$

$$\times \sum_{s_1 s_2} \left\{ \sum_{m_1 m_2} \langle \ell_1 \ell_2 m_1 m_2 | \Lambda M \rangle b^{+}_{\nu \ell_2 m_2 s_2} b^{+}_{\nu \ell_1 m_1 s_1} \right\} \left\{ \sum_{m_1' m_2'} \langle \ell_1' \ell_2' m_1' m_2' | \Lambda' M' \rangle b^{\nu \ell_1' m_1' s_1} b^{\nu \ell_2' m_2' s_2} \right\}$$

the $\delta_{\Lambda\Lambda'} \delta_{MM'}$ appearing because of spherical (and of course, also axial) symmetry, i.e., L^2 and L_z commute with V_{12}. The coefficient matrix element, moreover, is independent of the row M= $\Lambda, \Lambda - 1, \ldots, -\Lambda$ (see Appendix F) so that suppressing the redundant label M in that coefficient, one has that

$$\mathcal{V} = \frac{1}{2} \sum \langle \nu \ell_1, \nu \ell_2, \Lambda | V_{12} | \nu \ell_1', \nu \ell_2', \Lambda \rangle \mathcal{P}(\ell_1' \ell_2', \ell_1 \ell_2, \Lambda) \quad (191a)$$

where

$$\mathcal{P}(\ell_1' \ell_2', \ell_1 \ell_2, \Lambda) \equiv \sum_{s_1 s_2 M} \left\{ \sum_{m_1 m_2} \langle \ell_1 \ell_2 m_1 m_2 | \Lambda M \rangle b^{+}_{\nu \ell_2 m_2 s_2} b^{+}_{\nu \ell_1 m_1 s_1} \right\} \quad (191b)$$

$$\times \left\{ \sum_{m_1' m_2'} \langle \ell_1' \ell_2' m_1' m_2' | \Lambda M \rangle b^{\nu \ell_1' m_1' s_1} b^{\nu \ell_2' m_2' s_2} \right\}$$

can be considered a "generalized pairing force"[57] ----- it acts on a many-particle state of the type (21) destroying a particle-pair in the orbitals $\ell_1' \ell_2'$ and of total angular momentum Λ and <u>then</u> creates another pair in the orbitals $\ell_1 \ell_2$ having the same resultant Λ.

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A pairing force with $\Lambda = 0$ considered in the case of superconductivity[58,59] directly predicts the experimental energy gap between Fermi surface and conduction band. A similar interaction between nucleons was considered by Bohr, Mottelson & Pines[14] which among other things accounts for the well-defined large energy spacing between the ground and excited levels in even-even nuclei[14, 15]. As early as 1943 Racah[60] calculated interaction matrix elements of a pairing force of this type in some complex atoms using the expression

$$\langle \ell^2, \Lambda | V_{12} | \ell^2, \Lambda \rangle = (2\ell+1)\delta_{\Lambda 0}$$

for a pair of "equivalent" electrons in an orbital state ℓ. For non-equivalent ones (mixed orbitals $\ell\ell'$) this can be generalized to

$$\langle \ell\ell, \Lambda | V_{12} | \ell'\ell', \Lambda \rangle = \sqrt{(2\ell+1)(2\ell'+1)}\,\delta_{\Lambda 0}.$$

Inserting this value for the coefficients in (191) one obtains Moshinsky's form of the pairing operator

$$\mathcal{P} \equiv \sum_{\ell\ell'} \sqrt{(2\ell+1)(2\ell'+1)}\; \mathcal{P}(\ell'\ell', \ell\ell, 0)$$

$$= \frac{1}{2} \sum_{\substack{\ell m \\ \ell' m'}} \sum_{s_1 s_2} (-)^{m+m'}\, b^{+}_{\nu\ell -m s_2}\, b^{+}_{\nu\ell m s_1}\, b^{\nu\ell' m' s_1}\, b^{\nu\ell' -m' s_2}$$

where use is made of the well-known result that

$$\langle \ell\ell m -m | 00 \rangle = \frac{(-)^{\ell - m}}{\sqrt{2\ell+1}}.$$

The operator $\mathcal{P}$ clearly displays the required pairing property. A

more convenient form in terms of $\mathcal{U}_r$ group generators follows from anticommutation relations (12) and definitions (31) and (37):

$$\mathcal{P} = \frac{1}{2} \sum_{\substack{\ell m \\ \ell' m'}} (-)^{m+m'} b_{\nu \ell m}^{\nu \ell' m'} b_{\nu \ell -m}^{\nu \ell' -m'} - \frac{1}{2} \mathcal{N} \qquad (192)$$

$$\mathcal{N} \equiv \sum_{\ell m} b_{\nu \ell m}^{\nu \ell m}$$

$$(\ell = \nu, \nu-2, \ldots, 1 \text{ or } 0) \qquad (m = \ell, \ell-1, \ldots, -\ell)$$

where $\mathcal{N}$ is the particle-number operator defined previously.

3a. <u>Single-Orbital Pairing and the Group $R_{2\ell+1}$</u>

The special case of pairing between nucleons in a single orbital state ℓ is given by the operator

$$\mathcal{P} = \frac{1}{2} \sum_{mm'} (-)^{m+m'} b_m^{m'} b_{-m}^{-m'} - \frac{1}{2} \mathcal{N}$$

$$\mathcal{N} = \sum_{m=\ell}^{-\ell} b_m^m \qquad (193)$$

which follows immediately from (192) if $\ell \equiv \ell'$. The set of $(2\ell+1)^2$ operators

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$$\mathcal{C}_m^{\ m'} = \sum_s b^+_{\ell m s}\, b^{\ell m' s} \tag{194}$$

are the generators of $\mathcal{U}_{2\ell+1}$. The $(2\ell+1)$-dimensional space is spanned by the $(2\ell+1)$ vector components $b^+_{\ell m s}$ which under rotations in this space transform like the spherical harmonic components $Y_{\ell m}(\theta,\varphi)$. The scalar product

$$\sum_{mm'} g_{mm'}\, b^+_{ms}\, b^{+\,m's'} = \sum_m (-)^m\, b^+_{ms}\, b^+_{-ms'}$$

must remain invariant under $(2\ell+1)$-dimensional rotations. The <u>space metric</u> $g_{mm'}$ is thus seen to be

$$g_{mm'} = (-)^m\, \delta_{m,-m'} \tag{195}$$

which also gives the rule for raising and lowering indices since, for instance,

$$\mathcal{C}_{mm'} = \sum_{m''} g_{m'm''}\, \mathcal{C}_m^{\ m''} = (-)^{m'}\, \mathcal{C}_m^{\ -m'} \tag{196}$$

The $\mathcal{U}_{2\ell+1}$ generators, in doubly-covariant form, thus obey the Lie algebra

$$[\mathcal{C}_{mm'}, \mathcal{C}_{m''m'''}] = g_{m'm''}\, \mathcal{C}_{mm'''} - g_{m'''m}\, \mathcal{C}_{m''m'} .$$

Defining the antisymmetric operator set[33)]

$$\Lambda_{mm'} \equiv \frac{1}{2}\left(\ell_{mm'} - \ell_{m'm}\right), \qquad \Lambda_{mm'} = -\Lambda_{m'm} \qquad (197a,b)$$

one can see that these obey the commutation relations

$$[\Lambda_{mm'}, \Lambda_{m''m'''}] = \qquad (198)$$

$$\frac{1}{2}\left\{g_{m''m'}\Lambda_{mm'''} + g_{m'''m}\Lambda_{m'm''} + g_{mm''}\Lambda_{m'''m'} + g_{m'm'''}\Lambda_{m''m}\right\}$$

which from Appendix E are seen to be the generators for infinitesimal rotations in $(2\ell+1)$-dimensions, i.e., of the group $R_{2\ell+1}$. From (197b) it follows that there are $\ell(2\ell+1)$ linearly independent generators of this kind. The operator set $\{\Lambda_{mm'}\}$ is contained in the set $\{\ell_m^{\ m'}\}$ of $\mathcal{U}_{2\ell+1}$, so that $\mathcal{U}_{2\ell+1} \supset R_{2\ell+1}$ as a subgroup. Using (196) we can express the set (197) more conveniently as

$$\Lambda_m^{\ m'} = \frac{1}{2}\left(\ell_m^{\ m'} + (-)^{m+m'+1}\ell_{-m'}^{\ -m}\right), \qquad \Lambda_m^{\ m'} = (-)^{m+m'+1}\Lambda_{-m'}^{\ -m} \qquad (199)$$

the latter symmetry rule again confirming that one has $\ell(2\ell+1)$ independent generators. From (38) can be derived the relations

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$$[\Lambda_m^{m'}, \Lambda_{m''}^{m'''}] = \tag{200}$$

$$\frac{1}{2}\left\{\Lambda_m^{m'''}\delta_{m''}^{m'} - \Lambda_{m''}^{m'}\delta_m^{m'''} + (-)^{m+m'+1}\Lambda_{-m'}^{m'''}\delta_{m''}^{-m} + (-)^{m+m'}\Lambda_{m''}^{-m}\delta_{-m'}^{m'''}\right\}$$

equivalent to the algebra (198) but now in a form more convenient for us as will be seen presently.

The contraction of $\mathcal{U}_{2\ell+1}$ generators

$$\Gamma \equiv \sum_{mm'} \ell_m^{m'} \ell_{m'}^{m} \tag{201}$$

is seen to be a $\mathcal{U}_{2\ell+1}$ invariant since

$$[\Gamma, \ell_m^{m'}] = 0 \tag{202}$$

The equivalent contraction over $R_{2\ell+1}$ generators

$$\Phi \equiv \sum_{mm'} \Lambda_m^{m'} \Lambda_{m'}^{m} \tag{203}$$

is an $R_{2\ell+1}$ invariant as

$$[\Phi, \Lambda_m^{m'}] = 0 \tag{204}$$

Thus, Γ and Φ satisfy the first two Racah conditions[61] for Casimir operators (given on page 30 in relation to the group R_3). Introducing a numbering convention $\mu = \ell - m + 1$ (integer) such that there is a one-to-one correspondence between $m = \ell, \ell-1, \ldots, -\ell$ and $\mu = 1, 2, \ldots, 2\ell+1$ we have, first of all,

$$\Gamma = \sum_{\mu\mu'=1}^{2l+1} b_\mu^{\mu'} b_{\mu'}^{\mu} = \sum_{\mu} (b_\mu^{\mu})^2 + \sum_{\mu<\mu'} [b_\mu^{\mu'}, b_{\mu'}^{\mu}] + 2\sum_{\mu<\mu'} b_{\mu'}^{\mu} b_\mu^{\mu'}$$

where, from (73b), the effect of the last term when acting on a maximum weight state $\mathbb{P}$ is zero. The equation

$$\Gamma \mathbb{P} = \gamma \mathbb{P}$$

will have an eigenvalue γ (valid for <u>any</u> row of the $\mathcal{U}_{2l+1}$ irreducible basis, since Γ is a group invariant) that is evaluated identically as in (173-175) to give the result:

$$\gamma = \sum_{\mu=1}^{r} [h_\mu (h_\mu - 2\mu + r + 1)] . \qquad (r = 2l+1). \qquad (205)$$

Regarding the group R_{2l+1}, the l linearly independent (hermitean) operators, which from (200) mutually commute, can be used to define weight in R_{2l+1} by

$$\Lambda_m^m \bar{P} = \bar{w}_\mu \bar{P} \qquad (\mu = l - m + 1) \qquad (206)$$

$\bar{P}$ being a polynomial of weight $(\bar{w}_\mu)$. The set of operators $\{\Lambda_m^{m'}\}$ is divisible as before into three clases

$$1)\ \Lambda_m^{m'} (m > m' > -m) \qquad 2)\ \Lambda_m^m (m = l, l-1, \ldots, -l) \qquad 3)\ \Lambda_{m'}^m (m > m' > -m) \qquad (207)$$

of which class 2 are the <u>weight</u> operators. Let two new polynomials $\bar{P}'$ and $\bar{P}''$ be defined as

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$$\left.\begin{aligned}\bar{P}' &\equiv \Lambda_{m''}^{m'''}\,\bar{P}\\ \bar{P}'' &\equiv \Lambda_{m'''}^{m''}\,\bar{P}\end{aligned}\right\}\ (m''>m'''>-m'') \qquad (208)$$

Then,

$$\Lambda_m^m\,\bar{P}' = [\Lambda_m^m, \Lambda_{m''}^{m'''}]\,\bar{P} + \Lambda_{m''}^{m'''}\Lambda_m^m\,\bar{P}$$

$$= \tfrac{1}{2}\left(\Lambda_m^{m'''}\delta_{m''}^{m} - \Lambda_{m''}^{m}\delta_m^{m'''} - \Lambda_{-m}^{m'''}\delta_{m''}^{-m} + \Lambda_{m''}^{-m}\delta_m^{m'''}\right)\bar{P} + \Lambda_{m''}^{m'''}\Lambda_m^m\,\bar{P}$$

$$= \left[\tfrac{1}{2}\delta_{m''}^{m} - \tfrac{1}{2}\delta_m^{m'''} - \tfrac{1}{2}\delta_{m''}^{-m} + \tfrac{1}{2}\delta_{-m}^{m'''} + \bar{w}_\mu\right]\bar{P}'$$

shows that class 1 consists of <u>raising</u> generators and a similar analysis of $\bar{P}''$ proves class 3 to be the <u>lowering</u> generators. Hence, maximum weight polynomials $\bar{\mathbb{P}}$ in $R_{2\ell+1}$ will satisfy

$$\Lambda_m^m\,\bar{\mathbb{P}} = \lambda_\mu\,\bar{\mathbb{P}} \qquad \Lambda_m^{m'}\,\bar{\mathbb{P}} = 0\ (m>m'>-m) \qquad (209a,b)$$

with maximum weight being given by the set of K numbers

$$(\lambda_1\lambda_2\cdots\lambda_K)\quad \begin{cases} K = \tfrac{1}{2}(r-1)\\ r = 2\ell+1\end{cases}\ . \qquad (210)$$

In a way identical with (75) one can deduce that

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_K \geqslant 0\,. \qquad (211)$$

Finally, since

$$\Phi \bar{\mathbb{P}} = \varphi \bar{\mathbb{P}}$$

where φ is the Casimir eigenvalue of $R_{2\ell+1}$ which, after a short deriviation along the lines arriving at (205), results explicitly in[33)]

$$\varphi = \frac{1}{2} \sum_{\mu=1}^{K} \lambda_\mu (\lambda_\mu + r - 2\mu) \quad \begin{cases} K = \frac{1}{2}(r-1) \\ r = 2\ell+1 \end{cases} . \qquad (212)$$

A special case of (201) was used to great advantage in the oscillator symmetry group U_3 to determine the $\mathcal{Q}^2$ interaction eigenvalues. The single-orbital pairing force (193) can also be expressed as a linear combination of Casimir operators (second degree ones-and all further reference to Casimir operators will entail only this type) since, an expanding (203) using definition (199) we have, putting $\lambda \equiv m + m' + 1$,

$$\Phi = \sum_{mm'} \Lambda_m^{\ m'} \Lambda_{m'}^{\ m} = \frac{1}{4} \sum_{mm'} \left(b_m^{\ m'} + (-)^\lambda b_{-m'}^{\ -m} \right) \left(b_{m'}^{\ m} + (-)^\lambda b_{-m}^{\ -m'} \right)$$

$$= \frac{1}{4} \sum_{mm'} b_m^{\ m'} b_{m'}^{\ m} + \frac{1}{4} \sum_{mm'} (-)^\lambda b_m^{\ m'} b_{-m}^{\ -m'}$$

$$+ \frac{1}{4} \sum_{mm'} (-)^\lambda b_{-m'}^{\ -m} b_{m'}^{\ m} + \frac{1}{4} \sum_{mm'} b_{-m'}^{\ -m} b_{-m}^{\ -m'}$$

which upon change of some dummy indices and signs becomes

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$$\Phi = \frac{1}{2}\sum_{mm'} b_m^{\ m'} b_{m'}^{\ m} + \frac{1}{2}\sum_{mm'} (-)^{\lambda} b_m^{\ m'} b_{-m}^{\ -m'}$$

$$R_{2\ell+1}: \qquad \Phi = \frac{1}{2}\Gamma - \mathcal{P} - \frac{1}{2}\mathcal{N} \tag{213}$$

from (201) and (193). Therefore,

$$\mathcal{P} = \frac{1}{2}\Gamma - \Phi - \frac{1}{2}\mathcal{N} \tag{214}$$

will be diagonal in a base transforming irreducibly under $\mathcal{U}_{2\ell+1} \supset R_{2\ell+1}$.

The resulting states can be classified, in addition, by total orbital angular momentum L, i.e., with respect to R_3 whose generators $\mathcal{L}_q$ are expressible as the linear combinations

$$\mathcal{L}_q = \sum_{mm'} \sqrt{\ell(\ell+1)} \langle \ell\, 1 m' q \,|\, \ell m \rangle \Lambda_m^{\ m'} \quad (q = 1, \bar{1}, 0) \tag{215}$$

which, through (200) can be seen to satisfy the Lie algebra

$$[\mathcal{L}_q, \mathcal{L}_{q'}] = \sum_{q''} (-)^{q''} \epsilon_{q-q''q'} \mathcal{L}_q \qquad (q, q' = 1, \bar{1}, 0) \tag{216}$$

coinciding with (142b). Thus, <u>single-orbital pairing is diagonal in a scheme irreducible according to the group chain</u>

$$\mathcal{U}_{2\ell+1} \supset R_{2\ell+1} \supset R_3 \supset R_2 \tag{217}$$

where by R_3 is meant the $\mathcal{D}^{(L)}(R_3)$ representation of dimension $(2L+1)$.

The corresponding ket state is denoted by

$$| [h_1 h_2 \cdots h_{2\ell+1}] \, \alpha' (\lambda_1 \lambda_2 \cdots \lambda_\ell) \, \omega' L M_L \rangle \tag{218}$$

where, again, α' and ω' are additional labels that may be needed to distinguish multiplicities, if they occur.

3b. <u>Mixed-orbital pairing and the $R_{2\ell+1}$ group</u>.

If the particles paired to total angular momentum $\Lambda = 0$ have available more than one orbital then $\ell \neq \ell'$ in (192). In the chapter on harmonic oscillator symmetry we decided to classify our one-particle oscillator states by $\mu \equiv (n_1 n_{\bar{1}} n_0)$ instead of $(\nu \ell m)$, where the former are "partitions" along spherical "modes" (+,-,o) of oscillator quanta $\nu = n_1 + n_{\bar{1}} + n_0$. The single-particle orbital state (creation) operator $b^+_{\nu\ell m}$ transforms like the spherical harmonics $Y_{\ell m}(\theta, \varphi)$ which in turn have the phase convention

$$Y^*_{\ell m}(\theta, \varphi) = (-)^m \, Y_{\ell -m}(\theta, \varphi)$$

of Condon & Shortley.[62] Thus, following this convention

$$(b^+_{\nu\ell m})^* = (-)^m \, b^+_{\nu\ell -m} .$$

On the other hand, the state $b^+_{n_1 n_{\bar{1}} n_0} |\bar{0}\rangle$, where $|0\rangle$ is a <u>vacumn</u> (no-particle) state, should transform exactly like the oscillator state $|n_1 n_{\bar{1}} n_0\rangle$ of (123) given in terms of <u>boson</u> creations operators acting on a <u>ground</u> state $|\bar{0}\rangle$. Now, since $(a^+_{\pm 1})^* = -a_{\mp 1}$ according to (116), then the aforementioned state

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$|n_1 n_{\bar{1}} n_0\rangle$ has the property

$$|n_1 n_{\bar{1}} n_0\rangle^* = (-)^{n_1+n_{\bar{1}}} |n_{\bar{1}} n_1 n_0\rangle$$

and since $b^+_{n_1 n_{\bar{1}} n_0}$ transform like $|n_1 n_{\bar{1}} n_0\rangle$ we have

$$\left(b^+_{n_1 n_{\bar{1}} n_0}\right)^* = (-)^{n_1+n_{\bar{1}}} b^+_{n_{\bar{1}} n_1 n_0} .$$

But in (192) are involved the operators

$$b^+_{\nu l m} ; \qquad (-)^m b^+_{\nu l -m} ,$$

and their respective annihilation counterparts. To the complete set $\{b^+_{\nu l m}\}$ there corresponds the complete set $\{b^+_{n_1 n_{\bar{1}} n_0}\}$ ----- there being in each case $r = \frac{1}{2}(\nu+1)(\nu+2)$ components.* Likewise, to the set $\{(-)^m b^+_{\nu l -m}\}$ there corresponds the set $\{(-)^{n_1+n_{\bar{1}}} b^+_{n_{\bar{1}} n_1 n_0}\}$. The same holds for annihilation operators. Thus with $n_1 n_{\bar{1}} n_0$ indices (192) becomes our (working) <u>pairing operator for mixed-orbitals</u>:

$$P = \frac{1}{2} \sum_{\substack{n_1 n_{\bar{1}} n_0 \\ n_1' n_{\bar{1}}' n_0'}} (-)^{n_1+n_{\bar{1}}+n_1'+n_{\bar{1}}'} b^{n_1' n_{\bar{1}}' n_0'}_{n_1 n_{\bar{1}} n_0} \; b^{n_{\bar{1}}' n_1' n_0'}_{n_{\bar{1}} n_1 n_0} - \frac{1}{2} N . \qquad (219)$$

The generators defined in (199) now become

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*General formulas and tables for the 2s-1d shell of the transformation coefficients from one scheme to the other have been published by Chacon and the author.[63)]

$$\Lambda^{n_1' n_{\bar{1}}' n_0'}_{n_1 n_{\bar{1}} n_0} = \frac{1}{2}\left[\mathcal{C}^{n_1' n_{\bar{1}}' n_0'}_{n_1 n_{\bar{1}} n_0} + (-)^{n_1+n_{\bar{1}}+n_1'+n_{\bar{1}}'+1} \mathcal{C}^{n_{\bar{1}} n_1 n_0}_{n_{\bar{1}}' n_1' n_0'} \right] \tag{220a, b}$$

$$\Lambda^{n_1' n_{\bar{1}}' n_0'}_{n_1 n_{\bar{1}} n_0} = (-)^{n_1+n_{\bar{1}}+n_1'+n_{\bar{1}}'+1} \Lambda^{n_{\bar{1}} n_1 n_0}_{n_{\bar{1}}' n_1' n_0'}$$

where the former (symmetry) relation shows that one has $\frac{1}{2} r(r-1)$ linearly independent such operators.

Let us adopt the shorthand notation

$$\begin{aligned} (n_1 n_{\bar{1}} n_0) &\equiv \mu & (n_1' n_{\bar{1}}' n_0') &\equiv \mu' \\ (n_{\bar{1}} n_1 n_0) &\equiv \mu_- & (n_{\bar{1}}' n_1' n_0') &\equiv \mu_-' \end{aligned} \tag{121}$$

there being $r = \frac{1}{2}(\nu+1)(\nu+2)$ possibilities for each μ. It then becomes simpler to prove that

$$[\Lambda^{\mu'}_{\mu}, \Lambda^{\mu'''}_{\mu''}] = \tag{222}$$

$$\frac{1}{2}\left\{ \delta^{\mu'}_{\mu''} \Lambda^{\mu'''}_{\mu} - \delta^{\mu'''}_{\mu} \Lambda^{\mu'}_{\mu''} + (-)^{\eta+1} \delta^{\mu_-''}_{\mu} \Lambda^{\mu'''}_{\mu_-'} + (-)^{\eta} \delta^{\mu'''}_{\mu_-'} \Lambda^{\mu_-}_{\mu''} \right\}$$

having put $\eta \equiv n_1+n_{\bar{1}}+n_1'+n_{\bar{1}}'$. From the results of Appendix one deduces that, indeed, the $\frac{1}{2} r(r-1)$ independent operators $\Lambda^{\mu'}_{\mu}$ are the generators of r- dimensional rotations namely, of the group R_r.

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The Casimir operator of R_r is defined similarly to (203) for $R_{2\ell+1}$ as

$$R_r: \qquad \Phi = \sum_{\mu\mu'} \Lambda_\mu^{\mu'} \Lambda_{\mu'}^{\mu} \qquad (\mu, \mu' = 1, 2, \ldots, r) \qquad (223)$$

where

$$[\Phi, \Lambda_\mu^{\mu'}] = 0$$

and keeping in mind that μ stands for $(n_1 n_1 n_0)$. Expanding this in a manner exactly analogous with the steps leading to (213) one arrives at

$$R_r: \qquad \Phi = \frac{1}{2}\Gamma - \mathcal{P} - \frac{1}{2}\mathcal{N} \qquad (224)$$

where

$$\Gamma = \sum_{\mu\mu'} b_\mu^{\mu'} b_{\mu'}^{\mu}, \qquad \mathcal{N} = \sum_{\mu} b_\mu^{\mu}$$

refer, respectively, to the Casimir operator of R_r and the (invariant) particle operator; $\mathcal{P}$ is now the mixed-orbital pairing operator (219) which, being the linear combination of invariants

$$\mathcal{P} = \frac{1}{2}\Gamma - \Phi - \frac{1}{2}\mathcal{N}, \qquad (225)$$

is itself an invariant in the chain $\mathcal{U}_r \supset R_r$.

The eigenvalue γ of the operator Γ in a $\mathcal{U}_r$ irreducible basis is, as in (205),

$$\gamma = \sum_{\mu=1}^{r} [h_\mu (h_\mu - 2\mu + r + 1)] \qquad r = \tfrac{1}{2}(\nu+1)(\nu+2) \qquad (226)$$

r now referring to the ν-shell orbital degeneracy. The eigenvalue φ of Φ in an R_r irreducible basis will be (212), i.e.,

$$\varphi = \frac{1}{2}\sum_{\mu=1}^{K} \lambda_\mu (\lambda_\mu + r - 2\mu) \tag{227}$$

$$K = \tfrac{1}{2}(r-1) \text{ for } \underline{r \text{ odd}}\ ; \quad K = \tfrac{1}{2} r \text{ for } \underline{r \text{ even}}. \tag{228}$$

The various shell orbital-degeneracies r are given in Table .

ν	SHELL	r
0	1s	1
1	1p	3
2	2s-1d	6
3	2p-1f	10
4	3s-2d-1g	15
5	3p-2f-1h	21
6	4s-3d-2g-1i	28

TABLE IV.3b.1

For any shell, the mixed-orbital pairing force has eigenvalue $\left(\frac{1}{2}\gamma - \varphi - \frac{1}{2}N\right)$; hence

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$$E^{[f]}_{(\lambda_1\lambda_2\cdots\lambda_K)} = \frac{1}{2}\left[\sum_{\mu=1}^{r} h_\mu(h_\mu-2\mu+r) - \sum_{\mu=1}^{K}\lambda_\mu(\lambda_\mu-2\mu+r)\right] \tag{229a}$$

$$[f] \equiv [h_1 h_2 \cdots h_r] \quad \text{(orbital permutational symmetry)}$$

taking care to follow the correct K values involved in (228). Finally, as each R_r irreducible representation $(\lambda_1\lambda_2\cdots\lambda_K)$ will contain one or more R_3 irreducible representations, namely L-values, one can label the spectrum (229) by total orbital angular momentum values which however will be degenerate. The basis is irreducible under transformations of the chain

$$\mathcal{U}_r \supset R_r \supset R_3 \supset R_2 \tag{229b}$$

and is denoted by the set of kets

$$|\,[f]\,\alpha'(\lambda_1\lambda_2\cdots\lambda_K)\,\omega' L M_L\rangle \tag{229c}$$

where $[f] \equiv [h_1 h_2 \cdots h_r]$ and (α', ω') are distinguishing labels as in (218).

The effect of an attractive pairing force between two particles in a single orbital ℓ will be to <u>lower</u> the level $\Lambda = 0$ from the degenerate group of levels $\Lambda = 0,1,2,\ldots,2\ell$, thus producing a "gap", as can be seen from the Racah matrix element on page 98 which we have used in our second-quantization formulation of the operator $\mathcal{P}$. But more importantly, its resemblance with a short-range central interaction as well as its group invariance properties warrants its use as a <u>model interaction</u>. The latter

characteristics we have just seen. Its short-range character is well displayed in some calculations of wave function overlaps, energy levels and quadrupole moments carried out parallelly between a gaussian central interaction and a "model" central interaction consisting of a variable mixture of $\mathcal{P}$ and $\mathcal{Q}^2$ forces. The cases covered by Mell[64] were for 2,3 and 4 particles in the $2s-1d$ nuclear shell using $\mathcal{U}_6$ partitions $[f] = [2], [11], [3], [21], [111]$ and $[4]$. Energy level and quadrupole moment diagrams for both gaussian and model interactions show marked resemblances at intermediate ranges. In particular, overlaps calculated between wave functions of pure pairing against those of zero-range gaussian (which with proper normalization reduces to a δ-force) are extremely good for the low-lying levels arising from the partitions $[f]$ mentioned above. Good overlaps are also found in the comparison of pure $\mathcal{Q}^2$ and long range gaussian.

One undesirable though not very serious feature of the pairing force is its <u>inability</u> to break degeneracies as well as an ordinary δ-force. One can see this, eg., in the comparison for $N=2$ given by Biedenharn.[65]

4a. <u>A Word About Exchange at Short-Ranges</u>.

At short range, ideally a δ-force, the Majorana exchange operator (178d) becomes the Wigner operator ($\equiv 1$) and consequently the Bartlett and Heisenberg operators (178b,c) coincide.

Treatment of the Bartlett operator has proven difficult for

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short ranges, we shall thus make the not unreasonable assumption that at short ranges the Wigner part predominates over the Bartlett such as to justify neglecting the latter. Consequently, the effect of exchange at this extreme reduces to a constant factor to be ignored, as only energy differences arising from the pairing force matter.

5. Model Central Interaction Composed of $\mathcal{P}$ and $\mathcal{Q}^2$

Combining the interaction operators discussed in the previous sections one could therefore adopt the model central interaction (apart from an overall intensity factor V_0)

$$x\mathcal{P} + (1-x)\,\mathcal{J}\,\mathcal{Q}^2 \qquad (0 \leqslant x \leqslant 1) \tag{230}$$

with the variable parameter x determining the percentage of $\mathcal{P}$ involved in the mixture.

To re-stress the group theoretic simplicity of the model interaction constituents $\mathcal{P}$ and $\mathcal{Q}^2$ we recall from (168) and (225) that, apart from the invariant number operator $\mathcal{N}$,

$\mathcal{Q}^2 \longrightarrow$ Casimir operators of U_3 and R_3

$\mathcal{P} \longrightarrow$ Casimir operators of $\mathcal{U}_r$ and R_r.

Consequently,

$\mathcal{Q}^2$ is diagonal in the scheme $U_3 \supset R_3$

$\mathcal{P}$ is diagonal in the scheme $\mathcal{U}_r \supset R_r$.

The first scheme can be extended to $\mathcal{U}_r \supset U_3 \supset R_3 \supset R_2$ to include permutational and axial symmetries; the second can be enlarged to $\mathcal{U}_r \supset R_r \supset R_3 \supset R_2$ to include spherical and axial symmetries. (These "enlargements" of the group chains will allow us to reduce

the number of states to be dealt with in the calculation of matrix elements.) If one decides upon one of the two schemes, the operator corresponding to it will be diagonal with eigenvalues for which simple algebraic formulas hold, i.e., (176) or (229). The remaining operator is not diagonal of course, and its matrix elements in the chosen basis must be evaluated. For reasons already stated in the Introduction we will here choose the U_3 -scheme.

All interactions dealt with thus far are spin-independent. In the following section we introduce spin-orbit coupling and shall consider the evaluation of its matrix elements in the U_3 scheme.

6. <u>The Spin-Orbit Force</u>.

We are interested in the matrix elements of the single-body spin-orbit operator for N-particles

$$W_{s.o.} = \sum_{i=1}^{N} \vec{\ell}_i \cdot \vec{s}_i \tag{231}$$

calculated in the U_3 scheme. In second-quantized language this operator becomes, by (24),

$$W_{s.o.} \longleftrightarrow \mathcal{W}_{s.o.} = \sum_{\rho\rho'} \langle \rho | \vec{\ell}_1 \cdot \vec{s}_1 | \rho' \rangle C_\rho^{\rho'} \tag{232}$$

recalling that $\rho = (\mu\sigma\tau)$ refer to orbital-spin-isopin single-particle quantum numbers and $C_\rho^{\rho'}$ are the U_{4r} group generators.

The transformation properties of $\mathcal{W}_{s.o.}$ under U_3 will be specified and fully exploited. Accordingly, instead of using a basis set irreducible under transformations according to the chain (99) with U_3 inserted between $\mathcal{U}_r$ and R_3 as in (145), we shall

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use the segment

$$\mathcal{U}_r \supset U_3 \supset U_2 \supset U_1 \qquad \text{(canonical)} \qquad (233)$$

replacing the segment

$$\mathcal{U}_r \supset U_3 \supset R_3 \supset R_2 \qquad \text{(physical)} \qquad (234)$$

in chain (99). The irreducible representations of $U_3 \supset U_2 \supset U_1$ are characterized by $(h_{13}\ h_{23}\ h_{33},\ h_{12}\ h_{22},\ h_{11})$. On the other hand, the weight $(w_1\ w_2\ w_3)$ in U_3 is given by

$$C_q^{\prime q} \left| \begin{matrix} h_{13}\ h_{23}\ h_{33} \\ h_{12}\ h_{22} \\ h_{11} \end{matrix} \right\rangle = w_q \left| \begin{matrix} h_{13}\ h_{23}\ h_{33} \\ h_{12}\ h_{22} \\ h_{11} \end{matrix} \right\rangle \qquad (235)$$

$$(q = 1, \bar{1}, 0 \rightarrow 1, 2, 3)$$

in accordance with definition (67) and numbering convention in q of Section 6. By (94a) one has explicitly the eigenvalues of $C_1^{\prime 1}$, $C_2^{\prime 2}$ and $C_3^{\prime 3}$ respectively as

$$\begin{aligned} w_1 &= h_{11} \\ w_2 &= (h_{12} + h_{22}) - h_{11} \\ w_3 &= (h_{13} + h_{23} + h_{33}) - (h_{12} + h_{22}). \end{aligned} \qquad (236)$$

Now, from the many-particle angular momentum operators $\mathcal{L}_q$ $(q = 1, \bar{1}, 0)$ expressed in terms of U_3 generators $C_q^{\prime q'}$ ----- relations (142a)

----- we see that $\mathcal{L}_z = \mathcal{L}_0 = C_1^1 - C_{\bar{1}}^{\bar{1}}$ will therefore have an eigenvalue

$$M_L = w_1 - w_2 = 2h_{11} - (h_{12} + h_{22})$$

so that classification by $U_2 \supset U_1$ with $(h_{12}\, h_{22}, h_{11})$ is <u>equivalent</u> to classification by $U_2 \supset R_2$ with $(h_{12}\, h_{22}, M_L)$, R_2 being the group of rotations about the z-axis. For convenience, let us call

$$\begin{aligned} (h_{13}\, h_{23}\, h_{33}) &\equiv (h_1\, h_2\, h_3) \\ (h_{12}\;\; h_{22}) &\equiv (q_1 q_2) \\ (h_{11}) &\equiv (r_1) = \tfrac{1}{2}(M_L + q_1 + q_2) \end{aligned} \tag{237}$$

so that our general state would be given by the set of kets

$$|\, [f]\alpha(h_1 h_2 h_3) q_1 q_2 M_L ;\, \beta\, S\, M_S\, T\, M_T \rangle \tag{238}$$

where $\{v\}$ has been suppressed as $\{v\} = [\tilde{f}]$ makes it redundant.

Writing the operator (232) in more detail

$$\mathcal{W}_{S.O.} = \sum_{\substack{\mu\sigma\tau \\ \mu'\sigma'\tau'}} \langle \mu\sigma\tau |\, \vec{l}_1 \cdot \vec{s}_1 \,| \mu'\sigma'\tau' \rangle\, C_{\mu\sigma\tau}^{\mu'\sigma'\tau'}$$

$$= \sum_{q''} (-)^{q''} \sum_{\mu\mu'} \sum_{\substack{\sigma\sigma' \\ \tau}} \langle \mu |\, l_{-q''} \,| \mu' \rangle \langle \sigma |\, s_{q''} \,| \sigma' \rangle\, C_{\mu\sigma\tau}^{\mu'\sigma'\tau}$$

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where q'' designate the spherical component indices $(1,\bar{1},0)$ and the unnecessary particle index is suppressed. Introducing expansion (118) for $\ell_{-q''}$ one gets

$$\mathcal{W}_{S.O.} = \sum_{qq'q''} (-)^{q+q''} \epsilon_{-q''-q\,q'} \sum_{\substack{\sigma\sigma' \\ \tau}} \langle\sigma|\Delta_{q''}|\sigma'\rangle \sum_{\mu\mu'} \langle\mu|C_q^{q'}|\mu'\rangle \mathbb{C}_{\mu\sigma\tau}^{\mu'\sigma'\tau}$$

$$= \sum_{qq'q''} (-)^{q+q''} \epsilon_{-q''-q\,q'} \sum_{\sigma\sigma'\tau} \langle\sigma|\Delta_{q''}|\sigma'\rangle \mathbb{C}_{q\sigma\tau}^{q'\sigma'\tau} \qquad (239)$$

where the set of operators $\mathbb{C}_{q\sigma\tau}^{q'\sigma'\tau'}$ can be interpreted as the generators of a subgroup $U_3 \times U_2(\text{spin}) \times U_2(\text{isospin})$ of $\mathbb{U}_{4r}$ and U_3 being the subgroup of $\mathcal{U}_r$ discussed in Chapter III,6. The set $\{\mathbb{C}_{q\sigma\tau}^{q'\sigma'\tau'}\}$, from the above equations, is hereby expressed as linear combinations of the $\mathbb{U}_{4r}$ generators $\mathbb{C}_{\mu\sigma\tau}^{\mu'\sigma'\tau'}$, namely

$$\mathbb{C}_{q\sigma\tau}^{q'\sigma'\tau'} = \sum_{\mu\mu'} \langle\mu|C_q^{q'}|\mu'\rangle\, \mathbb{C}_{\mu\sigma\tau}^{\mu'\sigma'\tau'}. \qquad (240)$$

If one now defines a <u>tensor</u> (traceless with respect to indices $q, q' = 1, \bar{1}, 0$)

$$\mathbb{T}\left((21)_q^{q'}; 1_{q''}\right) \equiv \sum_{\substack{\sigma\sigma' \\ \tau}} \langle\sigma|\Delta_{q''}|\sigma'\rangle \left[\mathbb{C}_{q\sigma\tau}^{q'\sigma'\tau} - \frac{1}{3}\left(\sum_{\bar{q}} \mathbb{C}_{\bar{q}\sigma\tau}^{\bar{q}\sigma'\tau}\right)\delta_q^{q'}\right], \qquad (241)$$

then from (239) and the fact that $\sum_{qq'q''} (-)^{q+q''} \epsilon_{-q''-q\,q'}\, \delta_q^{q'}$ vanishes we have simply that

$$\mathcal{W}_{S.O.} = \sum_{qq'q''} (-)^{q+q''} \epsilon_{-q''-q\,q'}\, \mathbb{T}\left((21)_q^{q'}; 1_{q''}\right). \qquad (242)$$

Labelling $(21)_q^{q'}; 1_{q''}$ on tensor $\mathbb{T}$ refers to its transformation properties under SU_3 and U_2(spin), respectively, and will now be explained. This can be seen from the commutators between the U_3 generators $C_q^{q'}$ and the operators (240). From (131), (240), (37) (32) and (120) they are found to be

$$[C_{\bar{q}}^{\bar{q}'}, \mathbb{C}_{q\sigma\tau}^{q'\sigma'\tau'}] = [\sum_{\bar{\mu}\bar{\mu}'} \langle\bar{\mu}| C_{\bar{q}}^{\bar{q}'} |\bar{\mu}'\rangle \mathbb{C}_{\bar{\mu}}^{\bar{\mu}'}, \sum_{\mu\mu'} \langle\mu| C_q^{q'} |\mu'\rangle \mathbb{C}_{\mu s}^{\mu' s'}]$$

$$= \sum_{\substack{\mu\mu' \\ \bar{\mu}\bar{\mu}'}} \langle\bar{\mu}| C_{\bar{q}}^{\bar{q}'} |\bar{\mu}'\rangle \langle\mu| C_q^{q'} |\mu'\rangle \sum_{\bar{s}\bar{s}'} \delta_{\bar{s}}^{\bar{s}'} \{\mathbb{C}_{\bar{\mu}\bar{s}}^{\mu' s'} \delta_{\mu}^{\bar{\mu}'} \delta_{s}^{\bar{s}'} - \mathbb{C}_{\mu s}^{\bar{\mu}'\bar{s}'} \delta_{\bar{\mu}}^{\mu'} \delta_{\bar{s}}^{s'}\}$$

$$[C_{\bar{q}}^{\bar{q}'}, \mathbb{C}_{q\sigma\tau}^{q'\sigma'\tau'}] = \mathbb{C}_{\bar{q}\sigma\tau}^{q'\sigma'\tau'} \delta_q^{\bar{q}'} - \mathbb{C}_{q\sigma\tau}^{\bar{q}'\sigma'\tau'} \delta_{\bar{q}}^{q'} \qquad (243)$$

since $s \equiv (\sigma\tau)$. Since (243) is similar to relations (135) comprising the Lie algebra of U_3, operators $\mathbb{C}_{q\sigma\tau}^{q'\sigma'\tau'}$ transform under U_3 like the generators $C_q^{q'}$. But these in turn transform exactly like the single-particle generators $C_q^{q'} \equiv a_q^+ a^q$ or, as

$$\square \otimes \boxminus = \boxplus + \boxminus = (20) + \qquad (111),$$

where $\square$ represents a quanta from a_q^+ and $\boxminus$ a <u>hole</u> from $a^{q'}$, the "$\otimes$" representing the outer product of two representations which is

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calculated by Littlewood's rules for unitary groups (see Hammermesh, ref. 34, p. 249). Hence $\left[C^{q'\sigma'\tau'}_{q\sigma\tau} - \frac{1}{3}\left(\sum_{\bar{q}} C^{\bar{q}\sigma'\tau'}_{\bar{q}\sigma\tau}\right)\delta^{q'}_{q}\right]$ transforms under U_3 like $(h_1 h_2 h_3)=(210)$ since the trace $\left(\sum_{\bar{q}} C^{\bar{q}\sigma'\tau'}_{\bar{q}\sigma\tau}\right)$ transforms like $(h_1 h_2 h_3)=(000)$. Therefore, $\mathbb{T}\left((21)^{q'}_{q};1_{q''}\right)$ transforms <u>irreducibly</u> under SU_3 like $(k_1 k_2)=(21)$ with indices q, q' which determine the row of that representation. The coefficients in (241) can be written, by the Wigner-Eckhart theorem, as

$$\langle\sigma|\, s^{(1)}_{q''}\,|\sigma'\rangle \equiv \langle \tfrac{1}{2}\sigma|\, s^{(1)}_{q''}\,|\tfrac{1}{2}\sigma'\rangle$$

$$= \langle \tfrac{1}{2}\, 1\ \sigma' q''|\, \tfrac{1}{2}\,\sigma\rangle \langle \tfrac{1}{2}\| s^{(1)} \| \tfrac{1}{2}\rangle \qquad (244)$$

the superscript 1 indicating $s^{(1)}_{q''}$ to be a tensor of rank 1 (a vector) and of projection $q'' = 1, \bar{1}, 0$ so that, under U_2(spin), (241) transforms irreducibly like a vector with projection index q''.

Before proceeding to the evaluation of $\mathcal{W}_{s.o.}$ in matrix from between states (238), we require an important result to be sketched below.

<u>Theorem</u>. Recall first of all that any <u>vector operator</u> X_q $(q=1,\bar{1},0)$ obeys with R_3 generators $\mathcal{L}_q$ $(q = 1, \bar{1}, 0)$ commutation relations $[\mathcal{L}_0, X_{\pm 1}] = \pm X_{\pm 1}$ and $[\mathcal{L}_1, X_{\bar{1}}] = -X_0$ similar to relations (142b) between $\mathcal{L}_q$ themselves. Consequently, the X_q transform irreducibly under R_3 like the $\mathcal{L}_q$. The Wigner-Eckhart theorem thus tells us that $\langle LM'|X_q|LM\rangle = A_L \langle LM'|\mathcal{L}_q|LM\rangle$ where the proportionality constant A_L is

independent of the <u>row</u> of R_3, namely M and M'. This constant is furthermore easy to evaluate as for $M = M' = L$ and $q = 0$ we have $A_L = \frac{1}{L}\langle LL|X_0|LL\rangle$. Vectorially, we have added $\vec{L} + \vec{1}$ to give $\vec{L}$ which appears <u>only once</u>.

For the case of U_3, consider the operator set $\{Z_q^{q'}\}$ which obeys the commutation relations

$$[C_q^{q'}, Z_{q''}^{q'''}] = Z_q^{q'''}\delta_{q''}^{q'} - Z_{q''}^{q'}\delta_q^{q'''} \qquad (245)$$

so that $Z_q^{q'}$ transforms like $C_q^{q'}$, specifically like $(h_1 h_2 h_3) = (2\,1\,0)$ or $(k_1 k_2) = (2\,1)$ under SU_3. It is easily seen that since

$$\mathrm{Tr}\, Z = \sum_q Z_q^q \quad \text{and} \quad [C_q^{q'}, \mathrm{Tr}\, Z] = 0$$

then the traceless operators

$$\bar{Z}_q^{q'} \equiv Z_q^{q'} - \frac{1}{3}(\mathrm{Tr}\, Z)\delta_q^{q'} \qquad (246)$$

satisfy the same commutation relations (245) and transform like (21) of SU_3. If one is interested in the matrix elements of $\bar{Z}_q^{q'}$, in the chain $U_3 \supset U_2 \supset U_1$, between the <u>same</u> U_3 representation, that is,

$$\left\langle \begin{matrix} h_{13} & h_{23} & h_{33} \\ & h'_{12} & h'_{22} \\ & & h'_{11} \end{matrix} \middle| \bar{Z}_q^{q'} \middle| \begin{matrix} h_{13} & h_{23} & h_{33} \\ & h_{12} & h_{22} \\ & & h_{11} \end{matrix} \right\rangle, \qquad (247)$$

one must consider the fact that, by Littlewood's rules, the outer product

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$$(h_{13}\, h_{23}\, h_{33}) \otimes (210) = \quad \cdots + (h_{13}\, h_{23}\, h_{33})^2 + \cdots$$

involved in (247) gives, among other representations, the $(h_{13}\, h_{23}\, h_{33})$ <u>twice</u> except when $h_{33} = h_{23}$ or $h_{23} = h_{13}$. Thus, (247) is expandable as the matrix elements of <u>two</u> independent operator sets which must transform under U_3 like (210). We saw that the traceless generators

$$\bar{C}_q^{\,q'} \equiv C_q^{\,q'} - \frac{1}{3}\,(\mathrm{Tr}\,C)\,\delta_q^{\,q'} \tag{248}$$

form one such set. The traceless operator set independent from the above, namely,

$$\bar{\bar{C}}_q^{\,q'} \equiv \sum_{q''} C_q^{\,q''} C_{q''}^{\,q'} - \frac{1}{3}\Big(\sum_{q''q'''} C_{q''}^{\,q'''} C_{q'''}^{\,q''}\Big)\,\delta_q^{\,q'} \tag{249}$$

<u>also</u> transforms like (21) under SU_3 as it is easily shown that

$$\begin{aligned}[\bar{C}_q^{\,q'}, \bar{\bar{C}}_{\bar{q}}^{\,\bar{q}'}] &= [C_q^{\,q'}, \sum_{q''} C_{\bar{q}}^{\,q''} C_{q''}^{\,\bar{q}'}] \\ &= \bar{\bar{C}}_q^{\,\bar{q}'}\,\delta_{\bar{q}}^{\,q'} - \bar{\bar{C}}_{\bar{q}}^{\,q'}\,\delta_q^{\,\bar{q}'}.\end{aligned} \tag{250}$$

Hence, one can write (247) as

$$\left\langle \begin{matrix} h_{13}\, h_{23}\, h_{33} \\ h'_{12}\, h'_{22} \\ h'_{11} \end{matrix} \middle| \; \bar{Z}_q^{\,q'} \; \middle| \begin{matrix} h_{13}\, h_{23}\, h_{33} \\ h_{12}\, h_{22} \\ h_{11} \end{matrix} \right\rangle =$$

$$A_{h_{13}h_{23}h_{33}} \left\langle \begin{matrix} h_{13}\, h_{23}\, h_{33} \\ h'_{12}\, h'_{22} \\ h'_{11} \end{matrix} \middle| \bar{C}_q^{q'} \middle| \begin{matrix} h_{13}\, h_{23}\, h_{33} \\ h_{12}\, h_{22} \\ h_{11} \end{matrix} \right\rangle + B_{h_{13}h_{23}h_{33}} \left\langle \begin{matrix} h_{13}\, h_{23}\, h_{33} \\ h'_{12}\, h'_{22} \\ h'_{11} \end{matrix} \middle| \bar{G}_q^{q'} \middle| \begin{matrix} h_{13}\, h_{23}\, h_{33} \\ h_{12}\, h_{22} \\ h_{11} \end{matrix} \right\rangle \tag{251}$$

the A and B coefficients being <u>independent</u> of the row of U_3. These are evaluated by taking $q = q'$ and maximum weight in both bras and kets, i.e., for

$$h_{12} = h_{11} = h'_{12} = h'_{11} = h_{13}$$

$$h_{22} = h'_{22} = h_{23},$$

from which considerations three inhomogeneous linear equations result, one of which is dependent as

$$\mathrm{Tr}(\bar{Z}) = \mathrm{Tr}(\bar{C}) = \mathrm{Tr}(\bar{G}) \equiv 0.$$

It is simpler, therefore, to evaluate A and B via the 2 independent equations involving matrix elements of

$$\bar{Z}_1^1 - \bar{Z}_2^2 = Z_1^1 - Z_2^2 \quad \text{and} \quad \bar{Z}_2^2 - \bar{Z}_3^3 = Z_2^2 - Z_3^3 .$$

These results are generalizable to the $\mathcal{U}_r$ group, should they be required for a given problem.

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Returning to our specific problem, the matrix elements of (241) between states (238) with the <u>same</u> $[f]\alpha(h_1h_2h_3)$ in bra and ket will be, in accordance with result (251),

$$\langle [f]\alpha(h_1h_2h_3)q_1'q_2'M_L';\beta' S'M_S'T|\mathbb{T}((21)_q^{q'};1_{q''})|[f]\alpha(h_1h_2h_3)q_1q_2M_L;\beta SM_ST\rangle$$

$$=\Big\{A_{S'S}\langle h_1h_2h_3,q_1'q_2',M_L'|\bar{C}_q^{q'}|h_1h_2h_3,q_1q_2,M_L\rangle$$

$$+B_{S'S}\langle h_1h_2h_3,q_1'q_2',M_L'|\bar{\mathfrak{S}}_q^{q'}|h_1h_2h_3,q_1q_2,M_L\rangle\Big\}\langle S1M_Sq''|S'M_S'\rangle \tag{252}$$

where the Wigner - Eckhart theorem has been used in the spin part to give on the right hand side

$$\langle S1M_Sq''|S'M_S'\rangle\langle \tfrac{1}{2}\|\mathcal{S}^{(1)}\|\tfrac{1}{2}\rangle=\langle S1M_Sq''|S'M_S'\rangle\sqrt{\tfrac{3}{4}},$$

the factor $\langle \tfrac{1}{2}\|\mathcal{S}^{(1)}\|\tfrac{1}{2}\rangle=\sqrt{\mathcal{S}(\mathcal{S}+1)}=\sqrt{3/4}$ being incorporated into the coefficients $A_{S'S}$ and $B_{S'S}$ which depend <u>neither</u> on $q_1q_2M_L$ (the row of U_3) nor on M_S (the row of U_2), but <u>do</u> depend on $[f]\alpha(h_1h_2h_3)$ of $\mathcal{U}_r\supset U_3$.

Coefficients $A_{S'S}$ and $B_{S'S}$ can now be evaluated by constructing the two linearly independent inhomogeneous equations corresponding to the matrix elements of the two differences

$$\mathbb{T}((21)_1^1;1_{q''})-\mathbb{T}((21)_2^2;1_{q''})\quad\text{and}\quad\mathbb{T}((21)_2^2;1_{q''})-\mathbb{T}((21)_3^3;1_{q''}) \tag{253 a,b}$$

(as before: $q, q' = 1, \bar{1}, 0 \rightarrow 1, 2, 3$) between states of the <u>same</u> U_3 representation $(h_1\ h_2\ h_3)$ on both bra and ket, of max. wt. in $U_3 \supset U_2 \supset U_1$ and max. proj. of spin, namely for:

$$(h_1' h_2' h_3') = (h_1 h_2 h_3)$$

$$h_1 = q_1 = q_1' = r_1 = r_1'$$

$$h_2 = q_2 = q_2' \qquad (254)$$

$$M_S' = S'$$

$$M_S = S$$

$$\therefore\ q'' = M_S' - M_S = S' - S.$$

The calculation is simple and will be given in Chapter VIII for the case $[f](h_1 h_2 h_3) = [31](710)$ corresponding to the lowest levels of F^{20} in the U_3 scheme.

From (242) and (252) the spin-orbit operator $\mathcal{W}_{S.O.}$ for N particles will thus have matrix elements between states (238) of the same $[f]\alpha\ (h_1 h_2 h_3)$ given by

$$\langle [f]\alpha(h_1h_2h_3)\, q_1' q_2' M_L; \beta' S' M_S' T \,|\, \mathcal{W}_{S.O.} \,|\, [f]\alpha(h_1h_2h_3)\, q_1 q_2 M_L; \beta S M_S T \rangle$$

$$= \sum_{q''} (-)^{q''} \Big\{ A_{S'S} \langle h_1h_2h_3, q_1'q_2', M_L' \,|\, \sum_{qq'} (-)^{q} \epsilon_{-q''-qq'}\, \bar{C}_q^{q'} \,|\, h_1h_2h_3, q_1q_2, M_L \rangle$$

$$+ B_{S'S} \langle h_1h_2h_3, q_1'q_2', M_L' \,|\, \sum_{qq'} (-)^{q} \epsilon_{-q''-qq'}\, \bar{C}_q^{q'} \,|\, h_1h_2h_3, q_1q_2, M_L \rangle \Big\}$$

$$\times \langle S\, 1\, M_S\, q'' | S' M_S' \rangle \qquad (255)$$

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But, from (141) we have that the operator

$$\sum_{qq'} (-)^q \epsilon_{-q''-q\,q'} \; \bar{C}_q^{q'} = \mathcal{L}^{(1)}_{-q''} \; ,$$

the trace term of $\bar{C}_q^{q'} \equiv C_q^{q'} - Tr(C)\,\delta_q^{q'}$ contributing with zero as before, where of course $\mathcal{L}^{(1)}_{q''}$ transforms like a vector (superscript 1). Calling the second operator on the right - hand side of (255)

$$\mathcal{K}_{-q''} \equiv \sum_{qq'} (-)^q \epsilon_{-q''-q\,q'} \; \bar{C}_q^{q'} \qquad (256)$$

it is easily seen that <u>it too transforms like a vector</u> since the commutator

$$[\mathcal{L}_q, \mathcal{K}_{q'}] = \sum_{q''} (-)^{q''} \epsilon_{q-q''q'} \; \mathcal{K}_{q''}$$

is precisely the condition that $\mathcal{K}_{q''}$ be an irreducible Racah tensor under R_3 (compare with Rose, <u>Angular Momentum</u> p.84, ref. (35)). One is thus left with the problem of evaluating matrix elements

$$\langle h_1 h_2 h_3, q_1' q_2', M_L' \,|\, \mathcal{L}^{(1)}_{-q''} \,|\, h_1 h_2 h_3, q_1 q_2, M_L \rangle$$

$$\langle h_1 h_2 h_3, q_1' q_2', M_L' \,|\, \mathcal{K}^{(1)}_{-q''} \,|\, h_1 h_2 h_3, q_1 q_2, M_L \rangle$$

in the basis irreducible under $U_3 \supset U_2 \supset U_1$. The similarity transformation passing from this basis to the basis irreducible under $U_3 \supset R_3 \supset R_2$ with respective quantum labels (145 b)

and given by the kets

$$|(h_1h_2h_3)\omega LM_L\rangle \equiv |(k_1k_2)\omega LM_L\rangle \begin{cases} k_1 \equiv h_1 - h_3 \\ k_2 \equiv h_2 - h_3 \end{cases}$$

(as in eqns. 176 and 177) can be applied to both sides of (255). This transformation will be discussed in detail later. For the moment, if one imagined it applied to (255) the result would evidently be

$$\begin{aligned} &\langle [f]\alpha(k_1k_2)\omega' L' M_L'; \beta' S' M_S' T | \mathcal{W}_{S.O.} | [f]\alpha(k_1k_2)\omega L M_L; \beta S M_S T\rangle \\ &= \sum_{q''} (-)^{q''} \Big\{ A_{S'S} \langle (k_1k_2)\omega' L' M_L' | \mathcal{L}^{(1)}_{-q''} | (k_1k_2)\omega L M_L\rangle \\ &\quad + B_{S'S} \langle (k_1k_2)\omega' L' M_L' | \mathcal{K}^{(1)}_{-q''} | (k_1k_2)\omega L M_L\rangle \Big\} \langle S 1 M_S q'' | S' M_S'\rangle \end{aligned} \tag{257}$$

ω, ω' having the same significance as in chain (145). The matrix elements of $\mathcal{L}^{(1)}_{-q''}$ and $\mathcal{K}^{(1)}_{-q''}$ can now be reduced by the Wigner-Eckhart theorem in R_3 thusly:

$$\langle (k_1k_2)\omega' L' M_L' | \mathcal{L}^{(1)}_{-q''} | (k_1k_2)\omega L M_L\rangle =$$
$$\langle L 1 M_L - q'' | L' M_L'\rangle \langle (k_1k_2)\omega' L' \| \mathcal{L}^{(1)} \| (k_1k_2)\omega L\rangle$$

$$\langle (k_1k_2)\omega' L' M_L' | \mathcal{K}^{(1)}_{-q''} | (k_1k_2)\omega L M_L\rangle =$$
$$\langle L 1 M_L - q'' | L' M_L'\rangle \langle (k_1k_2)\omega' L' \| \mathcal{K}^{(1)} \| (k_1k_2)\omega L\rangle$$

with the same Clebsch-Gordon coefficient as proportionality constant.

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Furthermore, the reduced matrix elements are:

$$\langle (k_1k_2)\omega' L' \| \mathcal{L}^{(1)} \| (k_1k_2)\omega L \rangle = \frac{\langle (k_1k_2)\omega' L' L' | \mathcal{L}^{(1)}_{L'-L} | (k_1k_2)\omega L L \rangle}{\langle L 1 L\, L'-L \,|\, L' L' \rangle}$$

$$= \frac{\langle (k_1k_2)\,\omega' L' L \,|\, \mathcal{L}^{(1)}_0 \,|\, (k_1k_2)\,\omega L L \rangle}{\langle L 1 L 0 \,|\, L' L \rangle}$$

$$= \sqrt{L(L+1)}\,\delta_{\omega'\omega}\,\delta_{L'L}$$

and, for $\mathcal{K}^{(1)}$, arbitrarily choosing $M_L' = M_L = 1$ one has

$$\langle (k_1k_2)\omega' L' \| \mathcal{K}^{(1)} \| (k_1k_2)\,\omega L \rangle = \frac{\langle (k_1k_2) L' 1 \,|\, \mathcal{K}^{(1)}_0 \,|\, (k_1k_2) L 1 \rangle}{\langle L 1 1 0 \,|\, L' 1 \rangle}$$

$$= \frac{1}{\langle L 1 1 0 \,|\, L' 1 \rangle} \sum_{\substack{q_1 q_2 \\ q_1' q_2'}} \langle \omega' L' (k_1 k_2 M_L = 1)\, q_1' q_2' \rangle \langle q_1' q_2' 1 \,|\, \mathcal{K}^{(1)}_0 \,|\, q_1 q_2 1 \rangle \times \langle q_1 q_2 (k_1 k_2 M_L = 1)\, \omega L \rangle \tag{258}$$

where the bracket symbols[66] $\langle \omega' L' (k_1 k_2 M_L)\, q_1' q_2' \rangle$ stand for the elements of the similarity transformation matrix that passes a basis from $U_3 \supset U_2 \supset U_1$ to $U_3 \supset R_3 \supset R_2$, i.e., from the "canonical" to the "physical" chain. One therefore needs first to evaluate (for $M_L = M_L' = 1$) the matrix with elements

$$\langle (k_1k_2)q_1'q_2'M_L' \,|\, \mathcal{K}_0^{(1)} \,|\, (k_1k_2)\, q_1 q_2\, M_L \rangle \tag{259}$$

where by (256) $\mathcal{K}_0^{(1)} = \bar{\mathcal{G}}_1^1 - \bar{\mathcal{G}}_2^2 = \mathcal{G}_1^1 - \mathcal{G}_2^2$ and by definition (249)

$$\mathcal{K}_0^{(1)} = C_1^1 C_1^1 + C_1^2 C_2^1 + C_1^3 C_3^1 - C_2^1 C_1^2 - C_2^2 C_2^2 - C_2^3 C_3^2 \tag{260}$$

so that matrix elements (259) can be found if one has the matrix elements of U_3 generators $C_q^{q'}$ $(q,q' = 1,\bar{1},0 \rightarrow 1,2,3)$ in the chain $U_3 \supset U_2 \supset U_1$ since, for instance, suppressing labels (k_1k_2),

$$\langle (k_1k_2)\, q_1'q_2'M_L' \,|\, C_1^2 C_2^1 \,|\, (k_1k_2)\, q_1 q_2\, M_L \rangle = \tag{261}$$

$$\sum_{\bar{q}_1\bar{q}_2\bar{M}_L} \langle q_1'q_2'M_L' \,|\, C_1^2 \,|\, \bar{q}_1\bar{q}_2\bar{M}_L \rangle \langle \bar{q}_1\bar{q}_2\bar{M}_L \,|\, C_2^1 \,|\, q_1 q_2 M_L \rangle$$

is a typical term. The elements $\langle q_1'q_2'M_L' \,|\, C_q^{q'} \,|\, q_1 q_2\, M_L \rangle$ (for $q,q' = 1,\bar{1},0 \rightarrow 1,2,3$) are listed in Appendix H. Using these results, after a brief calculation (259) becomes: (see next page)

$$\langle (k_1 k_2) q_1' q_2' M_L' \,|\, \mathcal{K}_0^{(1)} \,|\, (k_1 k_2) q_1 q_2 M_L \rangle =$$

$$\delta_{q_1' q_2} \delta_{q_2' q_2} \delta_{M_L' M_L} M_L \left[(q_1+q_2+1) + \frac{(k_1-q_1+1)(q_1-k_2)(q_1+1)}{(q_1-q_2)(q_1-q_2+1)} - \frac{q_2(k_2-q_2+1)(k_1-q_2+2)}{(q_1-q_2+1)(q_1-q_2+2)} \right]$$

$$- \delta_{q_1' q_1+1} \delta_{q_2' q_2-1} \delta_{M_L' M_L} \sqrt{\frac{(k_1-q_1)(q_1-k_2+1)(q_1+2)\, q_2(k_2-q_2+1)(k_1-q_2+2)(q_1-q_2+M_L+2)(q_1-q_2-M_L+2)}{(q_1-q_2+1)(q_1-q_2+2)^2(q_1-q_2+3)}}$$

(262)

$$- \delta_{q_1' q_1-1} \delta_{q_2' q_2+1} \delta_{M_L' M_L} \sqrt{\frac{(q_2+1)(k_2-q_2)(k_1-q_2+1)(k_1-q_1+1)(q_1-k_2)(q_1+1)(q_1-q_2+M_L)(q_1-q_2-M_L)}{(q_1-q_2-1)(q_1-q_2)^2(q_1-q_2+1)}}$$

$$k_1 \geq q_1 \geq k_2 \geq q_2 \geq 0, \qquad q_1 \geq r_1 \geq q_2, \qquad r_1 \equiv q_1 + q_2 + M_L/2$$

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which despite its formidable aspect is quite simple to evaluate for particular values of $(k_1 k_2)$. The inequality conditions listed below are the equivalent of (92) for $U_3 > U_2 > U_1$.

Combining these results---except for the last one regarding the vector $\mathcal{K}^{(1)}$ which will be illustrated in application--- and coupling $\vec{L} + \vec{S}$ to give $\vec{J}$ in the bra and ket of $\mathcal{W}_{s.o.}$ by Clebsch - Gordon coefficients, one obtains

$$\langle [f]\alpha(k_1k_2)\omega' L'; \beta' S', J M_J, T \,|\, \mathcal{W}_{s.o.} \,|\, [f]\alpha(k_1k_2)\omega L, \beta S, J M_J, T \rangle$$

$$= \sum_{\substack{q'' M_L M_S \\ M_S' M_L'}} (-)^{q''} \langle L S M_L M_S | J M_J \rangle \langle L' S' M_L' M_S' | J M_J \rangle \langle L\, 1\, M_L - q'' | L' M_L' \rangle$$

$$\times \langle S\, 1\, M_S\, q'' | S' M_S' \rangle \Big\{ A_{S'S} \sqrt{L(L+1)}\, \delta_{\omega'\omega}\, \delta_{L'L}$$

$$+ B_{S'S} \langle (k_1k_2)\omega' L' \| \mathcal{K} \| (k_1k_2)\omega L \rangle \Big\}$$

$$= (-)^{L'+S-J} \sqrt{(2L'+1)(2S'+1)}\; W(L'S', LS; J1) \times \tag{263}$$

$$\times \Big\{ A_{S'S} \sqrt{L(L+1)}\, \delta_{\omega'\omega}\, \delta_{L'L} + B_{S'S} \langle (k_1k_2)\omega' L' \| \mathcal{K} \| (k_1k_2)\omega L \rangle \Big\}$$

after a slight rearrangement of $\langle S\, 1\, M_S\, q'' | S' M_S' \rangle$ with the aid of Clebsch - Gordan coefficient symmetry rules. $W(L'S'LS; J1)$ is the well-known Racah coefficient which can be defined as a sum over four Clebsch -Gordan coefficients (c.f. Rose, page 110). Matrix element formula (263) can be considered a generalization of Racah's formula[60] for the same operator acting for N particles

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all in the same orbital level 1.

<u>In conclusion</u>: evaluation of the spin - orbit interaction matrices in the SU_3 scheme is reduced to:

1) evaluation of coefficients $A_{S'S}$ and $B_{S'S}$.
2) evaluation of the quantities $\langle (k_1 k_2) L' \| \mathcal{K} \| (k_1 k_2) L \rangle$ forming a matrix of rows and columns given by (L', L) with only the principal diagonal and the diagonals immediately above and below it differing from zero from the selection rule $L' = L \pm 1,\ L$ as $\mathcal{K}$ is a vector.
3) Use of tabulated Racah coefficients $W(L'S', LS; J1)$.

Points (1) and (2) will be seen in detail in application (Chap. VIII).

For particular SU_3 representations $(k_1 k_2)$ in which $k_2 = 0$ or $k_1 = k_2$, the second term within the brackets of (263) <u>vanishes</u> thus leaving the problem of evaluating spin-orbit effects an extremely simple one (e.g., F^{18} with $N=2$ in the $2s-1d$ shell, $(k_1 k_2) = (40)$, is such a case[31].

Finally, the case of different values of $(k_1 k_2)$ between bra and ket of $\mathcal{W}_{S.O.}$ --- which is outside our present scope ---- is a problem involving actual use of SU_3 Wigner coupling coefficients and is being studied.

7. <u>The Complete Model Interaction.</u>

The full hamiltonian in second-quantization form (143b) for N particles in an unfilled shell is now explicitly

$$\mathcal{H} = \mathcal{H}^{osc} + \mathcal{H}_{int}$$

$$\mathcal{H}^{osc} \rightarrow \sum_{i=1}^{N} \left\{ \frac{p_i^2}{2m} + \frac{1}{2} m\omega^2 r_i^2 \right\} \qquad (264\ a,\ b,\ c)$$

$$\mathcal{H}_{int} = -\left(x\,\mathcal{P} + y\,\mathcal{Q}^2\mathcal{J} + z\,\mathcal{W}_{s.o.} \right)$$

and since $\mathcal{H}^{osc}$ does not affect the relative spacing between levels it is thus neglected. Only $\mathcal{H}_{int}$ is pertinent. Factoring out $(x+y+z)$ one has

$$\mathcal{H}_{int} = -(x+y+z)\left[\frac{x}{x+y+z}\,\mathcal{P} + \frac{y}{x+y+z}\,\mathcal{Q}^2\mathcal{J} + \frac{z}{x+y+z}\,\mathcal{W}_{s.o.} \right]$$

where the sum of the three coefficients is obviously unity. Letting

$$(x+y+z) \equiv \mathcal{V}_0, \qquad \frac{x}{x+y+z} \equiv \alpha, \qquad \frac{y}{x+y+z} \equiv \beta$$

then

$$\frac{z}{x+y+z} = 1 - \alpha - \beta$$

and since α, β and $(1-\alpha-\beta)$ must be positive or zero for x, y and z positive or zero one obtains

$$\mathcal{H}_{int} = -\mathcal{V}_0 \left[\alpha\,\mathcal{P} + \beta\,\mathcal{Q}^2\mathcal{J} + (1-\alpha-\beta)\,\mathcal{W}_{s.o.} \right]$$

$$\alpha + \beta \leq 1 \qquad (265\ a,b)$$

Coefficient parameters α and β will be varied in compliance with (265 b) and parameter $\mathcal{V}_0$ determined by a least-mean-squares fit to

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the empirical spectrum of the eigenvalues from the matrix

$$\| \langle [f]\alpha(k_1k_2)\omega' L'; \beta S'; JT \,|\, \mathcal{H}_{int}(\mathcal{V}_0, \alpha, \beta) \,|\, [f]\alpha(k_1k_2)\omega L; \beta S; JT \rangle \| \qquad (266)$$

NOTE: distinguish parameters α, β from quantum nos. α, β.

having taken $M_J = J$ and $M_T = T$ because of invariance under rotations in the spin-orbital space and neglect of Coulomb effects.

Each predicted level is labelled by J, T and of course π, the parity, which is positive for excitations within the $2s-1d$ shell. It is expected that for F^{20} the lowest levels are of $T=1$, the lowest $T=2$ levels being above 6 Mev.

We know that the $\mathcal{Q}^2$ term in $\mathcal{H}_{int}$ of (264 c) is diagonal in the SU_3 scheme states of (266). The matrix elements of $\mathcal{P}$ and $\mathcal{W}_{s.o.}$ must then be computed in this scheme, should it be chosen. For the spin-orbit matrix elements one only requires the maximum weight component of the $U_3 \supset U_2 \supset U_1$ base. The <u>whole</u> base is needed to evaluate $\|\mathcal{P}\|$ but fortunately one only needs the <u>diagonal</u> elements for the case of F^{20}. These details will be discussed in application.

Alternatively, one could in principal choose the "pairing scheme" associated with the group R_r under which $\|\mathcal{P}\|$ diagonal but not $\|\mathcal{Q}^2\|$ and $\|\mathcal{W}_{s.o.}\|$. Thirdly, a "j-j N - particle coupling scheme" which diagonalizes $\|\mathcal{W}_{s.o.}\|$ could be decided upon and under which the matrix elements of $\|\mathcal{P}\|$ and $\|\mathcal{Q}^2\|$ would have to be calculated.

All three coupling schemes will be discussed in the following chapter even though only the SU_3 scheme will be adopted for our present calculations and these are relegated to Chapter IX.

V. POSSIBLE CLASSIFICATION SCHEMES FOR MANY - PARTICLE STATES IN AN OSCILLATOR SHELL, IN PARTICULAR THE $2s - 1d$ SHELL.

1. The $\mathcal{Q}^2$ (or SU_3 group) Scheme.

The problem of finding the irreducible representations of U_3 contained under a given one of U_r is similar to the chain calculation illustrated in Appendix A for the reduction $\mathcal{U}_r \supset R_3$ $(r=6)$. Further details, as well as an extensive table for the $2s - 1d$ shell, are found in Elliott.[12] Finding the L-values contained in a given $(k_1\ k_2)$ representation of U_3, i.e., the reduction $U_3 \supset R_3$, are easily gotten by simple inequalities derived rigorously by Bargmann & Moshinsky[67] and tabulated elsewhere [68] by the same authors for some particular cases. The inequalities involved are simply

$$\underline{(k_1-L)\ \text{even}}: \quad 0 \leqslant 2q \leqslant k_2 \qquad k_2 - L \leqslant 2q \leqslant k_1 - L$$

$$\underline{(k_1-L)\ \text{odd}}: \quad 0 \leqslant 2q \leqslant k_2 - 1 \qquad k_2 - L \leqslant 2q \leqslant k_1 - L - 1 \tag{267}$$

where q is a non-negative integer.* Using Elliott's and Bargmann - Moshinsky's tables and inequalities (267) we give

---

* The L-structure of a given $(k_1 k_2)$ representation further breaks down into classes of levels, each class labelled by a number $K \geqslant 0$ and which for quanta $(k_1 \geqslant 2k_2)$ is simply $K = k_2 - 2q$ while for quanta - holes $(k_1 < 2k_2)$ it is $K = k_1 - k_2 - 2q$. Elliot[12] identifies this K with the rotational - band quantum number K of the collective model (Figure 2).

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Table V.1.1 covering $N=1$ to 12 particles for the more symmetrical orbital partitions $[f]$ and the lower (energywise) SU_3 representations $(k_1 k_2)$. For $N \leq 5$, all SU_3 representations present are given.

Some observations regarding these results are in order: 1) The maximum number of allowed nucleons in the $2s-1d$ shell being $4r = 24$, only $N \leq 12$ are included in Table since any number above that can be treated as $N \leq 12$ "holes". Interaction matrix elements for N holes differ from those for N particles by simple phases.[69] 2) Always $L \leq k_1$, which is obvious from (267). 3) Multiple L-values for a given $(k_1 k_2)$ occur only when $k_2 \geq 2$ in which case the additional label ω in (177) must be used to distinguish them. Bargmann & Moshinsky found[67] the operator corresponding to this label. 4) For $N \leq 4$, $(k_1 k_2)$-values appear only once under a given $[f]$. To distinguish multiplicities occurring elsewhere, eg., $N[f](k_1 k_2) = 5[32](40)^2$, additional label α of (177) must be made available. 5) For further convenience, the eigenvalues

$$\mathcal{E}^{(k_1 k_2)} \equiv E_L^{(k_1 k_2)} + \tfrac{1}{2} L(L+1) = \tfrac{2}{3}(k_1+k_2)^2 - 2k_1(k_2-1) \qquad (268)$$

of the $\mathcal{Q}^2$ interaction, save for the term $\frac{1}{2} L(L+1)$, are given in the last column. The SU_3 representations are listed in order of increasing energy recalling that larger $\mathcal{E}^{(k_1 k_2)}$ lie lower in energy because $\mathcal{Q}^2$ is attractive. The base truncation suggested by Elliot consists in taking that $(k_1 k_2)$ lying lowest in energy and limiting calculation to this representation. 6) Finally, some $2s-1d$ shell nuclides are written in parenthesis under that $[f]$ partition which will presumably explain its lowest states. More details are given later on this point.

The SU_3 representation falling lowest in energy (having

maximum Casimir eigenvalue $\mathcal{E}^{(k_1 k_2)}$)seem <u>always to be given by</u>

$$k_1 \equiv h_{13} - h_{33} = 2(h_1 - h_6) + (h_2 - h_5)$$
$$k_2 \equiv h_{23} - h_{33} = 2(h_4 - h_6) + (h_2 - h_3) \qquad (269)$$

where $[h_1 h_2 h_3 \cdots h_r] \equiv [f]$ refer to the Young partition. These relations follow directly from the U_3 weight operators C_1^1, C_2^2 and C_3^3 (with eigenvalues h_{13}, h_{23} and h_{33} listed in (138). The construction of polynomials transforming irreducibly according to <u>this</u> SU_3 representation will be a simple matter if one uses the prescription of which (79) served as an example and will be illustrated in detail for F^{20} and other nuclei.

TABLE V.1.1 (for 2s-1d Shell)

N	U_6 [f]	SU_3 $(k_1 k_2)$	R_3 L	$\mathcal{E}^{(k_1 k_2)}$
1	[1] (O^{17}, F^{17})	(20)	0,2	6⅔
2	[2] (O^{18}, F^{18})	(40)	0,2,4	18⅔
		(22)	0,2	6⅔
	[11]	(31)	1,2,3	10⅔
3	[3] (F^{19}, Ne^{19})	(60)	0,2,4,6	36
		(42)	$0,(2)^2,3,4$	16
		(00)	0	0
	[21] (O^{19})	(51)	1,2,3,4,5	24
		(42)	$0,(2)^2,3,4$	16
		(21)	1,2	6

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4	[4] (Ne^{20})	(80)	0,2,4,6,8	58 2/3
		(62)	$0,(2)^2, 3,(4)^2, 5,6$	30 2/3
		(44)	0,2,4	18 2/3
		(20)	0,2	6 2/3
	[31] (F^{20}, Na^{20})	(71)	1,2,3,4,5,6,7	42 2/3
		(62)	$0,(2)^2, 3,(4)^2, 5,6$	30 2/3
		(53)	$1,2,(3)^2, 4,5$	22 2/3
		(41)	1,2,3,4	16 2/3
		(32)	1,2,3	10 2/3
		(20)	0,2	6 2/3
	[22] (O^{20})	(62)	$0,(2)^2, 3,(4)^2, 5,6$	30 2/3
		(44)	0,2,4	18 2/3
		(41)	1,2,3,4	16 2/3
		(20)	0,2	6 2/3
5	[41] (Ne^{21}, Na^{21})	(91)	1,2,3,4,5,6,7,8,9	66 2/3
		(82)	$0,(2)^2, 3,(4)^2, 5,(6)^2, 7,8$	50 2/3
		(73)	$1,2,(3)^2,(4)^2,(5)^2, 6,7$	38 2/3
		(61)	1,2,3,4,5,6	32 2/3
		(64)	$0,(2)^2, 3,(4)^2, 5,6$	30 2/3
		(52)	$1,2,(3)^2, 4,5$	22 2/3
		(40)	0,2,4	18 2/3
		(43)	1,2,3,4	16 2/3
		(31)	1,2,3	10 2/3
		(22)	0,2	6 2/3
		(82)	$0,(2)^2, 3,(4)^2, 5,(6)^2, 7,8$	50 2/3
		(73)	$1,2,(3)^2,(4)^2,(5)^2, 6,7$	38 2/3
		(61)	1,2,3,4,5,6	32 2/3
		(64)	$0,(2)^2, 3,(4)^2, 5,6$	30 2/3
		(52)	$1,2,(3)^2, 4,5$	22 2/3
		$(40)^2$	0,2,4 for each (40).	18 2/3
		(43)	1,2,3,4	16 2/3
		(31)	1,2,3	10 2/3
		(22)	0,2	6 2/3

6	[42]	(10,2)	$0,(2)^2,3,(4)^2,5,(6)^2,7,(8)^2,9,10$	76
	(Ne^{22}, Na^{22})	(93)	$1,2,(3)^2,(4)^2,(5)^2,(6)^2,(7)^2,8,9$	60
		$(84)^2$	$0,(2)^2,3,(4)^3,(5)^2,(6)^2,7,8$ for each (84)	48
		⋮		
	[41]	(93)	$1,2,(3)^2,(4)^2,(5)^2,(6)^2,(7)^2,8,9$	60
		(75)	$1,2,(3)^2,4,(5)^2,6,7$	40
		(72)	$1,2,(3)^2,4,(5)^2,6,7$	40
		⋮		
7	[43]	(11,3)	$1,2,(3)^2,(4)^2,(5)^2,(6)^2,(7)^2,(8)^2,(9)^2,10,11$	$86\frac{2}{3}$
	(Na^{23}, Mg^{23})	(10,4)	$0,(2)^2,3,(4)^3,(5)^2,(6)^3,(7)^2,(8)^2,9,10$	$70\frac{2}{3}$
		(9,5)	$1,2,(3)^2,(4)^2,(5)^3,(6)^2,(7)^2,8,9$	$58\frac{2}{3}$
		⋮		
	[421]	(10,1)	$1,2,3,4,5,6,7,8,9,10$	$80\frac{2}{3}$
	(Ne^{23})	(10,4)	$0,(2)^2,3,(4)^3,(5)^2,(6)^3,(7)^2,(8)^2,9,10$	$70\frac{2}{3}$
		$(92)^2$	$1,2,(3)^2,4,(5)^2,6,(7)^2,8,9$ for each (92)	$62\frac{2}{3}$
		(95)	$1,2,(3)^2,(4)^2,(5)^3,(6)^2,(7)^2,8,9$	$58\frac{2}{3}$
		⋮		
8	[44]	(12,4)	$0,2^2,3,4^3,5^2,6^3,7^2,8^3,9^2,10^2,11,12$	$98\frac{2}{3}$
	(Mg^{24})	(10,3)	$1,2,3^2,4^2,5^2,6^2,7^2,8^2,9,10$	$72\frac{2}{3}$
		(10,6)	$0,2^2,3,4^3,5^2,6^3,7^2,8^2,9,10$	$70\frac{2}{3}$
		⋮		
	[431]	(11,2)	$1,2,3^2,4,5^2,6,7^2,8,9^2,10,11$	$90\frac{2}{3}$
	(Na^{24}, Al^{24})	(11,5)	$1,2,3^2,4^2,5^3,6^3,7^3,8^2,9^2,10,11$	$82\frac{2}{3}$
		$(10,3)^2$	$1,2,3^2,4^2,5^2,6^2,7^2,8^2,9,10$	$72\frac{2}{3}$
		(10,6)	$0,2^2,3,4^3,5^2,6^3,7^2,8^2,9,10$	$70\frac{2}{3}$
		⋮		

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9	[441]	(12,8)	$1, 2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2, 9^2, 10^2, 11, 12$	102
	(Mg^{25}, Al^{25})	(12,6)	$0, 2^2, 4^3, 6^4, 8^3, 10^2, 12$	96
		(11,4)	$1, 2, 3^2, 4^2, 5^3, 6^2, 7^3, 8^2, 9^2, 10, 11$	84
		⋮		
	[432]	(11,1)	$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$	96
	(Na^{25})	(11,7)	$1, 2, 3^2, 4^2, 5^3, 6^2, 7^3, 8^2, 9^2, 10, 11$	84
		(11,4)	$1, 2, 3^2, 4^2, 5^3, 6^2, 7^3, 8^2, 9^2, 10, 11$	84
		⋮		
10	[442]	(12,2)	$0, 2^2, 3, 4^2, 5, 6^2, 7, 8^2, 9, 10^2, 11, 12$	$106\frac{2}{3}$
	(Mg^{26}, Al^{26})	(12,8)	$0, 2^2, 3, 4^3, 5^2, 6^3, 7^2, 8^3, 9^2, 10^2, 11, 12$	$98\frac{2}{3}$
		(12,5)	$1, 2, 3^2, 4^2, 5^3, 6^3, 7^3, 8^3, 9^2, 10^2, 11, 12$	$96\frac{2}{3}$
		⋮		
	[4411]	(12,5)	$1, 2, 3^2, 4^2, 5^3, 6^3, 8^3, 9^2, 10^2, 11, 12$	$96\frac{2}{3}$
		(11,3)	$1, 2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2, 9^2, 10, 11$	$86\frac{2}{3}$
		(11,6)	$1, 2, 3^2, 4^2, 5^3, 6^3, 7^3, 8^2, 9^2, 10, 11$	$82\frac{2}{3}$
		⋮		
11	[443]	(12,1)	$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$	$112\frac{2}{3}$
	(Mg^{27})	(12,10)	$0, 2^2, 3, 4^2, 5, 6^2, 7, 8^2, 9, 10^2, 11, 12$	$106\frac{2}{3}$
		(12,4)	$0, 2^2, 3, 4^3, 5^2, 6^3, 7^2, 8^3, 9^2, 10^2, 11, 12$	$98\frac{2}{3}$
		(12,7)	$2, 3, 4^2, 5^2, 6^3, 7^2, 8^3, 9^2, 10^2, 11, 12$	$96\frac{2}{3}$
		⋮		
	[4421]	(12,4)	$0, 2^2, 3, 4^3, 5^2, 6^3, 7^2, 8^3, 9^2, 10^2, 11, 12$	$98\frac{2}{3}$
	(Al^{27}, Si^{27})	(12,7)	$2, 3, 4^2, 5^2, 6^3, 7^2, 8^3, 9^2, 10^2, 11, 12$	$96\frac{2}{3}$
		(11,2)	$1, 2, 3^2, 4, 5^2, 6, 7^2, 8, 9^2, 10, 11$	$90\frac{2}{3}$
		⋮		

12	[444]	(12,0)	$0, 2, 4, 6, 8, 10, 12$	120
	(Si^{28})	(12,3)	$1, 2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2, 9^2, 10^2, 11, 12$	102
		(12,6)	$0, 2^2, 4^3, 6^4, 8^3, 10^2, 12$	96
		⋮		
	[4431]	(12,3)	$1, 2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2, 9^2, 10^2, 11, 12$	102
	(Al^{28})	(12,9)	$1, 2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2, 9^2, 10^2, 11, 12$	102
		(12,6)	$0, 2^2, 4^3, 6^4, 8^3, 10^2, 12$	96
		⋮		

2. <u>The Pairing (R_r group) Scheme</u>.

It was shown in (IV, 3. b.) that the mixed-orbital pairing operator $\mathcal{P}$ given in (219) and (225) ---- which approximates the short-range character of our central two - body interaction ---- is invariant in the chain $\mathcal{U}_r \supset R_r$, where as usual r is the ν- shell orbital degeneracy $r = \frac{1}{2}(\nu+1)(\nu+2)$. The spectrum of $\mathcal{P}$ in a base transforming irreducibly according to $\mathcal{U}_r \supset R_r$ is given by (229 a) and is a function only of irreducible representations $[f] \equiv [h_1 h_2 \cdots h_r]$ and $(\lambda_1 \lambda_2 \cdots \lambda_K)$ of $\mathcal{U}_r$ and R_r respectively. The number K of labels required to specify an irreducible representation of R_r is given in terms of r by (228). One can further label the states by orbital angular momentum L and projection M_L by reducing R_r explicitly according to $R_3 \supset R_2$ as in chain (229 b) corresponding to kets (229c.) Additional symmetries (Pauli exclusion, spin and isopin) can be incorporated into the base which transforms irreducibly

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according to the chain

$$\begin{array}{ccccccccccccc} \mathcal{U}_{4r} & \supset & \mathcal{U}_r \times \mathcal{U}_4 & \supset & R_r \times \mathcal{U}_4 & \supset & R_3 \times \mathcal{U}_4 & \supset & R_3 \times \mathcal{U}_2 \times \mathcal{U}_2 & \supset & R_2(0) \times R_2(s) \times R_2(i) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \quad \uparrow \quad \uparrow & & \uparrow \quad \uparrow \quad \uparrow \\ [1^N 0^{4r-N}] & & [f] & & \alpha'(\lambda_1\lambda_2\cdots\lambda_K) & & \beta\omega' L & & S \quad T & & M_L \quad M_S \quad M_T \end{array} \tag{270}$$

$K = \frac{1}{2}(r-1)$ <u>r odd</u> ; $K = \frac{1}{2}r$ <u>r even</u>

where α', β and ω' respectively designate eigenvalues of operators required to distinguish multiplicities in the reductions. $\mathcal{U}_r \supset R_r$, $\mathcal{U}_4 \supset \mathcal{U}_2 \times \mathcal{U}_2$ and $R_r \supset R_3$, when and if they are needed. (This chain of groups under which $\mathcal{P}$ is diagonal should be compared with chains (99) and (145a) under which $\mathcal{Q}^2$ is diagonal.

Calculations in this scheme would thus require the construction of the set of totally anti-symmetric states

$$|[f]\alpha'(\lambda_1\lambda_2\cdots\lambda_K)\beta\omega' LST, M_L M_S M_T\rangle. \tag{271}$$

The irreducible representations $(\lambda_1 \lambda_2 \cdots \lambda_K)$ of R_r contained in a given one $[f] \equiv [h_1 h_2 \cdots h_r]$ of $\mathcal{U}_r$ can be derived by a technique given by Jahn[41] (1950). The L-structure of each R_r representation is in turn found by chain calculations of the type illustrated before. Elliot[12] provides a table for the more symmetric Young partitions in the $2s-1d$ shell $(r=6)$ for $N \leq 4$ particles and we here reproduced that table with the pairing force eigenvalues (229 a).

TABLE V.2.1 (for 2s-1d Shell)

N	U_6 [f]	R_6 $(\lambda_1\lambda_2\lambda_3)$	R_3 L	$E_{(\lambda)}^{[f]}$
1	[1]	(100)	0,2	0
2	[2]	(000)	0	6
		(200)	$0, 2^2, 4$	0
	[11]	(110)	1,2,3	0
3	[3]	(100)	0,2	8
		(300)	$0^2, 2^2, 3, 4^2, 6$	0
	[21]	(100)	0,2	4
		(210)	$1^2, 2^3, 3^2, 4^2, 5$	0
4	[4]	(000)	0	16
		(200)	$0, 2^2, 4$	10
		(400)	$0^2, 2^3, 3, 4^3, 5, 6^2, 8$	0
	[31]	(110)	1,2,3	8
		(200)	$0, 2^2, 4$	6
		(310)	$0, 1^3, 2^4, 3^5, 6^2, 7$ $4^4, 5^3$	0

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More pronounced degeneracy in the L-structure is present here than in the case $\mathcal{U}_6 \supset U_3 \supset R_3$.
Typical example: The 3 particle partition $[21]$ contains $L = 0, 1^2, 2^4, 3^2, 4^2, 5$ which in the $\mathcal{Q}^2$ scheme is broken into three levels (see Table V.1.1) while only two levels result under the $\mathcal{P}$ scheme (Table V. 2.1.).

Multiple $(\lambda_1 \lambda_2 \lambda_3)$ of a given $[f]$ occur only for $N \geqslant 6$ --- the label α' is then needed.

The problem of constructing pairing scheme states (271) is being studied by Chacón at the University of Mexico.

3. <u>The j-j Many - Particle Coupling Scheme Using Unitary Groups</u>.

If the single-particle quantum numbers ρ were to be given as $(\nu l j m_j, \tau)$ of (15) instead of $(\nu l m, \sigma \tau)$ of (14) or, with more explicit reference to the oscillator, by $(n_1 n_T n_0, \sigma \tau)$ as we have done in our work, then our N - particle functions of the type (79) would be <u>eigenfunctions</u> of the single - body spin - orbit operator for N particles

$$\sum_{i=1}^{N} \vec{l}_i \cdot \vec{s}_i = \frac{1}{2} \sum_i (\vec{j}_i^{\,2} - \vec{l}_i^{\,2} - \vec{s}_i^{\,2})$$

with eigenvalues given by

$$\frac{1}{2} \sum_{i=1}^{N} \left[j_i(j_i+1) - l_i(l_i+1) - 3/4 \right]. \qquad (272)$$

What would be the chain of groups under which the corresponding set of eigenfunctions transform irreducibly?

The largest group of transformations is of course U_{4r}, r being the orbital degeneracy involved and 4 denoting the spin - isopin degeneracy. If the major oscillator shell contains but <u>one</u> sub-shell, i.e., a single j value, then one may certainly consider the whole space as broken up into spin-orbital and isospin subspaces (of dimensions $2j+1$ and 2, respectively) leading to the chain

$$U_{4r} \equiv U_{2(2j+1)} \supset \mathcal{U}_{2j+1} \times U_2 \supset R_3 \text{(spin-orbital)} \times U_2 \text{(isospin)}.$$

This chain is easily generalized to the case of K sub-shells within our given mayor shell by considering the "direct sum" of unitary groups $\mathcal{U}_{2j_i+1}$ where $i = 1, 2, \ldots, K$:

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$$\mathbb{U}_{4r} \supset \mathcal{U}_{2r} \times U_2 \supset \begin{pmatrix} \mathcal{U}_{2j_1+1} & & & 0 \\ & \mathcal{U}_{2j_2+1} & & \\ & & \ddots & \\ 0 & & & \mathcal{U}_{2j_k+1} \end{pmatrix} \times U_2 \supset R_3 \times U_2 \qquad (273a)$$

where our notation means the direct product

$$\begin{pmatrix} \mathcal{U}_{2j_1+1} & & & 0 \\ & \mathcal{U}_{2j_2+1} & & \\ & & \ddots & \\ 0 & & & \mathcal{U}_{2j_k+1} \end{pmatrix} \equiv \begin{pmatrix} \mathcal{U}_{2j_1+1} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \times \begin{pmatrix} 1 & & & \\ & \mathcal{U}_{2j_2+1} & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \times \cdots \times \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & \mathcal{U}_{2j_k+1} \end{pmatrix} \qquad (273b)$$

and R_3 refers to rotations in the full spin - orbital sub-space, i.e., in representation language, to

$$R_3 \Rightarrow \begin{pmatrix} \mathcal{D}^{(j_1)} & & & 0 \\ & \mathcal{D}^{(j_2)} & & \\ & & \ddots & \\ 0 & & & \mathcal{D}^{(j_k)} \end{pmatrix} \qquad (273c)$$

while U_2 refers to isospinorial space. The told spin - orbital degeneracy, is simply $2r$ where

$$2r = \sum_{i=1}^{k} (2j_i+1).$$

In the $2s-1d$ shell there are 3 sub-shells, namely, $d_{5/2}$, $s_{1/2}$ and $d_{3/2}$ (see Figure V.III.6.1) and the chain involved is thus:

$$\underset{\substack{\uparrow \\ [1^N 0^{24-N}]}}{\mathbb{U}_{24}} \supset \underset{\substack{\uparrow \\ [g]}}{\mathcal{U}_{12}} \times \underset{\substack{\uparrow \\ T}}{U_2} \supset \begin{pmatrix} \mathcal{U}_6 & & 0 \\ & \mathcal{U}_2 & \\ 0 & & \mathcal{U}_4 \end{pmatrix} \times \underset{\substack{\uparrow \\ T}}{U_2} \supset \underset{\substack{\uparrow \\ J}}{R_3} \times \underset{\substack{\uparrow \\ T}}{U_2} \qquad (274)$$

$$[g'] \quad [g''] \quad [g''']$$

in accordance with (273a, b, c,) the irreducible representation labels (quantum numbers) of the various groups being given below them. Notice that, as desired, our totally antisymmetric N - particle states would be of definite J and T.

Consider a single particle in the $2s - 1d$ shell. The $\mathcal{U}_{12}$ Young diagram $[g]$ is simply $[1]$ and its dual representation $[\tilde{1}] = [1]$ is the corresponding diagram for U_2 (isospin) required by the exclusion principle----thus, $T = 1/2$. The particle can be in either of the three sub-shells $d_{5/2}$, $s_{1/2}$ or $d_{3/2}$ so that there are <u>three</u> irreducible representations, each given by $[g'][g''][g'''] \equiv \{g', g'', g'''\}$, of the subgroup

$$\begin{pmatrix} \mathcal{U}_6 & & 0 \\ & \mathcal{U}_2 & \\ 0 & & \mathcal{U}_4 \end{pmatrix} \qquad (275)$$

contained in the $[g] = [1]$ representation of $\mathcal{U}_{12}$, namely,

$$\begin{aligned} \{1,0,0\} &> J = j_1 = 5/2 \\ \{0,1,0\} &> J = j_2 = 1/2 \\ \{0,0,1\} &> J = j_3 = 3/2 \end{aligned} \qquad (276)$$

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The first, second and third numbers in the brackets refer to the Young diagrams, respectively, of $\mathcal{U}_6$, $\mathcal{U}_2$ and $\mathcal{U}_4$. The irreducible representations of R_3 (J-structure) are shown to the right of each bracket.

Which are the irreducible representations of subgroup (275) contained in a representation $[g] = [1^N\, 0^{12-N}]$, that is, totally antisymmetric, of $\mathcal{U}_{12}$? The 12 single particle states $(\nu\, l_j\, m_j)$ referring to spin - orbital coordinates for the $2s - 1d$ shell can be labelled with μ by, say,

μ	1	2	3	4	5	6	7	8	9	10	11	12
l	2	2	2	2	2	2	0	0	2	2	2	2
j	5/2	5/2	5/2	5/2	5/2	5/2	1/2	1/2	3/2	3/2	3/2	3/2
m_j	5/2	3/2	1/2	-1/2	-3/2	-5/2	1/2	-1/2	3/2	1/2	-1/2	-3/2
	$d_{5/2}$						$s_{1/2}$		$d_{3/2}$			

so that the <u>generators</u> of $\mathcal{U}_{12}$ are $b_\mu^{\mu'} \equiv b_{l_j m_j}^{l_j m_j}$ $(\mu, \mu' = 1, 2, \cdots, 12)$ and obey commutation relations like (35). An N - particle basis irreducible under $\mathcal{U}_{12}$ and also explicitly under subgroup (275) is the general polynomial

$$P^{[g]}_{\{g', g'', g'''\}} |0\rangle \equiv |[g]\{g', g'', g'''\}\rangle =$$

$$\sum_{\mu_1 \mu_2 \cdots \mu_N} A_{\mu_1 \mu_2 \cdots \mu_N}\, b^+_{\mu_1}\, b^+_{\mu_2} \cdots b^+_{\mu_N} |0\rangle$$

of degree N. In accordance with (73), to find the $\{g', g'', g'''\}$'s contained in $[g] = [1^N\, 0^{12-N}]$ one must require the conditions on P that

$$1)\quad b_\mu^{\mu} P = g_\mu P \qquad \begin{cases} [g'] \equiv [g_1 g_2 \cdots g_6] \\ [g''] \equiv [g_7 g_8] \\ [g'''] \equiv [g_9 g_{10} g_{11} g_{12}] \end{cases}$$

$$g_\mu = 0 \text{ or } 1$$

$$g_1 \geqslant g_2 \geqslant \cdots \geqslant g_6 ; \qquad g_7 \geqslant g_8 ; \qquad g_9 \geqslant g_{10} \geqslant g_{11} \geqslant g_{12}$$

$$2)\quad b_\mu^{\mu'} P = 0 \qquad \begin{cases} 1 \leqslant \mu < \mu' \leqslant 6 \\ 7 \leqslant \mu < \mu' \leqslant 8 \\ 9 \leqslant \mu < \mu' \leqslant 12 \end{cases}$$

so that P is of maximum weight in $\mathcal{U}_{12}$ <u>and</u> in subgroup (275). A polynomial of arbitrary weight for, say N = 5, would be labelled by say

$$\{110100, 10, 0100\} \qquad N = 5$$

but to be of <u>maximum</u> weight it would have to be designated by

$$\{111000, 10, 1000\}$$

in accordance with (1) above. Our polynomial would then have to contain the factors b_1^+, b_2^+, b_3^+, b_7^+ and b_9^+ ; in fact the polynomial is simply

$$b_1^+ b_2^+ b_3^+ b_7^+ b_9^+$$

up to a multiplicative constant for normalization. A similar analysis would give us the other $\{g', g'', g'''\}$ for $[g] = [11111]$ apart from the $\{111, 1, 1\}$ already obtained.

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For $N=2$ consider the $[11]$ partition of $\mathcal{U}_{12}$ which has the dual partition $[\tilde{11}]=[2]$ of U_2 implying that $T=1$. The possible representations of subgroups (275) involved here are then:

$$[11] \supset \{11,0,0\}\{1,1,0\}\{1,0,1\}\{0,11,0\}\{0,1,1\} \text{ and } \{0,0,11\} \qquad (278)$$

each possessing a definite J - structure. To find the $\{g', g'', g'''\}$ contained in the $[g]=[2], [\tilde{g}]=[11] \Longrightarrow T=0$ case, we take the external product of $[1]$ with $[1]$ in $\mathcal{U}_{12}$ so that

$$[1]\otimes[1] \equiv \square\otimes\square = \square\square + \begin{smallmatrix}\square\\\square\end{smallmatrix}$$

according to Littlewood's rules. Taking all the possible external products of (276) among themselves we find that

$$\mathcal{U}_{12} \supset \begin{pmatrix} \mathcal{U}_6 & & 0 \\ & \mathcal{U}_2 & \\ 0 & & \mathcal{U}_4 \end{pmatrix}$$

$$\square\otimes\square = \square\square + \begin{smallmatrix}\square\\\square\end{smallmatrix} \supset \{2,0,0\}\{11,0,0\}\{1,1,0\}\{1,0,1\}\{1,1,0\}\{0,2,0\}$$
$$\{0,11,0\}\{0,1,1\}\{1,0,1\}\{0,1,1\}\{0,0,2\}\{0,0,11\},$$

and subtracting (278) from this, one is left with:

$$\square\square \supset \{2,0,0\}\{1,1,0\}\{0,2,0\}\{1,0,1\}\{0,1,1\}\{0,0,2\}.$$

Now, $\{2,0,0\}$ means 2 particles arranged symmetrically in the $d_{5/2}$ sub-shell, the possible J - values are thus $J = 1,3,5$. Representation $\{1,1,0\}$ means a configuration $(d_{5/2})^1\,(s_{1/2})^1$ so that, J being $j_1 + j_2$ vectorially, we have $J = 2, 3$. Then $\{0,2,0\}$ contains simply $J = 1$, and so forth. Tabulating these typical results for $[g] = [2]$ we have

TABLE V.3.1.

$\mathcal{U}_{12}$	U_2	$\begin{pmatrix} \mathcal{U}_6 & \mathcal{U}_2 & 0 \\ & 0 & \mathcal{U}_4 \end{pmatrix}$	J	D
[2]	T=0	$\{2,0,0\}$	1,3,5	21
		$\{1,1,0\}$	2,3	12
		$\{0,2,0\}$	1	3
		$\{1,0,1\}$	1,2,3,4	24
		$\{0,1,1\}$	1,2	8
		$\{0,0,2\}$	1,3	10

$\Sigma D = 78$

where the numbers in the last column refer to the <u>dimensionality</u> (c.f., Hammermesh, p. 387, ref. 34) of the representations involved. The D's are given in each case in terms of the Young partition numbers of the unitary group involved (Weyl dimensionality formula) and are the analogne of $(2J+1)$ for an R_3 representation given by J. Their use provides a good check on the $\{g', g'', g'''\}$'s obtained from a given $[g]$. For example, in the above Table: for $\{2,0,0\} \supset J = 1, 3, 5$ and for which D = 21, one must have (recalling definition (273b))

$$D(\mathcal{U}_6)\,D(\mathcal{U}_2)\,D(\mathcal{U}_4) = \sum_{J=1,3,5} (2J+1)$$

$$21 \times 1 \times 1 = 3+7+11$$

$$21 = 21.$$

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Again, the grand total of 78 must result from

$$D(\mathcal{U}_{12}) = \sum D(\mathcal{U}_6)\, D(\mathcal{U}_2)\, D(\mathcal{U}_4) = \sum_{\text{all } J} (2J+1) = 78$$

for the $[g]=[2]$ representation of $\mathcal{U}_{12}$.

To continue our chain calculation, the $\mathcal{U}_{12}$ representation $[111]$, with conjugate $\mathcal{U}_2$ representation $\widetilde{[111]} = [3] \Rightarrow T = 3/2$, contains the following $\{g', g'', g'''\}$'s of subgroup(275):

$\{111,0,0\}$ $\{11,1,0\}$ $\{11,0,1\}$ $\{1,11,0\}$ $\{1,1,1\}$ $\{1,0,11\}$

$\{0,11,1\}$ $\{0,1,11\}$ and $\{0,0,11\}$.

By a process identical to the one for deriving the structure of $[g] = [2]$ in the previous example we can find the structure of $[g] = [21]$ with T = 1/2 from the structure of $\square \otimes \square = \square + \square$ and of $[111]$ above.

The process is repeated for any desired number N of particles and results like Table V.3.2 tabulated for any wanted $[g]$ representation. In particular, we shall give the complete table for the case of F^{20} (N = 4) whose lowest levels are given by T = 1, where one has $[g] = [211]$ and $[\tilde{g}] = [31]$ which implies $T_4 = 1$. We included the triplet of numbers $(n_1 n_2 n_3)$ giving the distribution of N particles among the $2s-1d$ shell according to the configuration $(d_{5/2})^{n_1} (s_{1/2})^{n_2} (d_{3/2})^{n_3}$. Also, the eigenvalues (272) corresponding to each such configuration and <u>normalized</u> with respect to the single - particle oxygen - 17 spectrum (c.f., Preston, p. 184, ref. 53) are given in the last column.

TABLE V.3.2. $\mathcal{U}_{12}$: $[211]$ (T=1) F^{20}

$\mathcal{U}_6$	$\mathcal{U}_2$	$\mathcal{U}_4$	$(n_1 n_2 n_3)$	J	D	NORMALIZED EIGENVALUES OF EQ. (272)
211	0	0	400	$1^2, 2^2, 3^3, 4^2, 5^2, 6, 7$	105	0 (Mev)
111	1	0	310	$1, 2^2, 3, 4, 5$	40	.871
21	1	0	310	$0, 1^2, 2^3, 3^4, 4^3, 5^2, 6^2, 7$	140	.871
11	2	0	220	$1^2, 2, 3^2, 4, 5$	45	1.742
111	0	1	301	$0, 1^2, 2^2, 3^3, 4^2, 5, 6$	80	5.08
21	0	1	301	$0, 1^4, 2^6, 3^6, 4^6, 5^5, 6^3, 7^2, 8$	280	5.08
11	1	1	211	$0, 1^3, 2^4, 3^4, 4^3, 5^2, 6$	120	5.951
2	1	1	211	$0, 1^3, 2^4, 3^4, 4^4, 5^3, 6^2, 7$	168	5.951
1	11	1	121	$1, 2, 3, 4$	24	6.822
1	2	1	121	$0, 1^2, 2^3, 3^3, 4^2, 5$	72	6.822
1	1	11	112	$0, 1^2, 2^3, 3^3, 4^2, 5$	72	11.031
1	1	2	112	$0, 1^3, 2^4, 3^4, 4^3, 5^2, 6$	120	11.031
11	0	11	202	$0^2, 1, 2^4, 3^2, 4^3, 5, 6$	90	10.16

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11	0	2	202	$1^4, 2^3, 3^5, 4^3, 5^3, 6, 7$	150	10.16
1	21	0	130	2,3	12	2.613
11	11	0	220	0,2,4	15	1.742
2	11	0	220	1,3,5	21	1.742
11	1	1	211	$0, 1^3, 2^4, 3^4, 4^3, 5^2, 6$	120	5.951
2	0	11	202	$1^3, 2^2, 3^4, 4^2, 5^3, 6, 7$	126	10.16
1	0	21	103	$0, 1^3, 2^4, 3^4, 4^3, 5^2, 6$	120	15.24
1	11	1	121	1,2,3,4	24	6.822
0	21	1	031	1,2	8	7.693
0	11	2	022	1,3	10	11.902
1	1	11	112	$0, 1^2, 2^3, 3^3, 4^2, 5$	72	etc.
0	11	11	022	0,2	6	
0	2	11	022	$1^2, 2, 3$	18	
0	1	21	013	$0, 1^2, 2^2, 3^2, 4$	40	
1	0	111	103	1,2,3,4	24	
0	1	111	013	1,2	8	
0	0	211	004	1,2,3	15	
				$\Sigma D =$	2145	

Conclusion: One would naturally only be interested in the first few lowest configurations (see Table V.3.2) to carry out calculations in the " j-j N - particle coupling scheme" described here. Notice that even these lowest states are very degenerate in J, but it is presumably removed by the $\mathcal{P}$ and $\mathcal{Q}^2$ interactions. Construction of the lowest states of the basis irreducible under the chain of groups(274) has not been attempted as we have decided upon the SU_3 classification scheme for which the construction of states we turn to the following chapter.

VI. CONSTRUCTION OF MANY - PARTICLE WAVE FUNCTIONS IN THE SU_3 SCHEME.

From elementary angular momentum theory we recall that there are two basic methods involved in the construction of an N - particle basis set of definite total angular momentum L out of single - particle functions of definite ℓ. One is a lowering - (or step -) operator technique and the other utilizes Clebsch - Gordon vector - coupling coefficients. Both will be illustrated briefly as a prelude to their generalization for SU_3.

The lowering operator technique is essentially contained in (53), from which one can easily see that

$$\mathcal{Y}_{LM-1} = \frac{1}{\sqrt{(L+M)(L-M+1)}} L_- \mathcal{Y}_{LM} \equiv C_{LM} L_- \mathcal{Y}_{LM}. \quad (279)$$

The operator L_- lowers the value of index M in steps of one and is given from (279) by

$$L_- = \sum_{i=1}^{N} L_-(i) = \sum_{i=1}^{N} \left[x_0^{(i)} \frac{\partial}{\partial x_+^{(i)}} + x_-^{(i)} \frac{\partial}{\partial x_0^{(i)}} \right]$$

so that <u>if one possessed</u> all the $\mathcal{Y}_{LL}$ functions of maximum weight for a given N - particle configuration $(\ell_1, \ell_2, \cdots, \ell_N)$ one could by (279) <u>generate</u> the full $(2L+1)$-dimensional bases according to the row index M where $-L \leq M \leq L$. In particular, for maximum L, namely, $L = \ell_1 + \ell_2 + \cdots + \ell_N$ one has simply $\mathcal{Y}_{LL} = \mathcal{Y}_{\ell_1\ell_1} \mathcal{Y}_{\ell_2\ell_2} \cdots \mathcal{Y}_{\ell_N\ell_N}$ where from (51) and (52) $\mathcal{Y}_{\ell_i\ell_i} = A_{0i} (x_+^{(i)})^{\ell_i}$ $(i = 1, 2, \ldots, N)$. For non - maximum L-values the $\mathcal{Y}_{LL}$'s could be obtained, e.g., by the following procedure for 2 particles:

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The weight operation

$$L_0 \mathcal{Y}_{l_1 m_1} \mathcal{Y}_{l_2 m_2} = M \mathcal{Y}_{l_1 m_1} \mathcal{Y}_{l_2 m_2}$$

gives that $M = m_1 + m_2$ as $L_0 = L_0^{(1)} + L_0^{(2)}$. A maximum weight (i.e., M = L) two-particle function transforming irreducibly as $L = l_1+l_2,\ l_1+l_2-1,\ \ldots,\ l_1-l_2$ is then the linear combination

$$\mathcal{Y}_{LL} = \sum_m A_m \mathcal{Y}_{l_1 m} \mathcal{Y}_{l_2 L-m}$$

where $m \equiv m_1$, $m_2 = L - m$. The coefficients A_m are easily found through the fact that

$$L_+ \mathcal{Y}_{LL} = 0$$

and $L_+^{(i)} \mathcal{Y}_{l_i m_i} = \sqrt{(l_i - m_i)(l_i + m_i + 1)}\ \mathcal{Y}_{l_i\ m_i+1}$.
One gets then the result

$$\frac{A_{m+1}}{A_m} = -\sqrt{\frac{(l_1-m)(l_1+m+1)}{(l_2-L+m+1)(l_2-L-m)}}$$

allowing us to determine A_m up to a constant A_0 , that is:

$$\mathcal{Y}_{LL} = A_0 \sum_m (-)^m \sqrt{\frac{(l_1+m)!(l_2-L-m)!}{(l_1-m)!(l_2-L+m)!}}\ \mathcal{Y}_{l_1 m} \mathcal{Y}_{l_2 L-m}.$$

By (279) one could then generate all the $\mathcal{Y}_{LM}$ functions for $L = l_1+l_2, l_1+l_2-1, \ldots, l_1-l_2$ and $-L \leq M \leq L$, in other words: the full basis for irreducible representations L of the group R_3.

Alternatively, to build $\{\mathcal{Y}_{LM}\}$ for N- particles one could vector couple the first two particles to definite $l'm'$ by

$$\mathcal{Y}_{l'm'} = \sum_{m_1 m_2} \langle l_1 l_2\, m_1 m_2 | l'm' \rangle\, \mathcal{Y}_{l_1 m_1}\, \mathcal{Y}_{l_2 m_2}$$

and couple this to particle 3 to give

$$\mathcal{Y}_{l''m''} = \sum_{m_3 m'} \langle l_3\, l'm_3\, m' | l''m'' \rangle\, \mathcal{Y}_{l_3 m_3}\, \mathcal{Y}_{l'm'},$$

and so on, up to the $N^{\underline{th}}$ particle ---- thus obtaining the complete set $\mathcal{Y}_{LM}$ for each R_3 irreducible representation L. (Note: we disregard the problem of antisymmetrization in our example.)

Both techniques ---- of lowering operators and Clebsch - Gordan coefficients ---- are relatively simple. Coefficients $\langle l_1 l_2 m_1 m_2 | lm \rangle$ which refer to the coupling of two R_3 irreducible representations l_1 and l_2 to give a third one l, have been tabulated extensively. Using the three lowering generators of U_3 and the Wigner coefficients for U_3 which have also been tabulated to some extent, one can generalize for U_3 the above well - known methods.

1. Lowering Operator Functions.

From the introductory discussion of (IV,d) we recall that the irreducible representations of $U_3 \supset U_2 \supset U_1$ are characterized by $(k_1 k_2, q_1 q_2, r_1)$ through the general bra

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$$\underline{R_3 \supset R_2 \text{ ANALOGUE:}} \quad ; \quad \left| \begin{matrix} k_1 k_2 \, 0 \\ q_1 q_2 \\ r_1 \end{matrix} \right\rangle \equiv P^{(k_1 k_2)}_{q_1 q_2, r_1}(b^+_{\mu s})|0\rangle \qquad (280)$$

$$|\ell m\rangle \equiv \mathcal{Y}^\ell_m$$

the superscript on the polynomial P referring to the irreducible representation of SU_3, the subscript denoting the <u>row</u> indices of that representation. From (235) and (236) moreover, the <u>weight</u> of (280) is given by the three eigenvalues of C_1^1, C_2^2 and C_3^3

$$\underline{R_3 \supset R_2 \text{ ANALOGUE:}} \quad ; \quad \begin{aligned} w_1 &= r_1 \\ w_2 &= (q_1+q_2) - r_1 \\ w_3 &= (k_1+k_2) - (q_1+q_2), \end{aligned} \qquad (281)$$

$$m = \ell, \ell-1, \ldots, -\ell$$

and the polynomial of maximum weight in SU_3 is designated by

$$\underline{R_3 \supset R_2 \text{ ANALOGUE:}} \quad ; \quad \left| \begin{matrix} k_1 k_2 \, 0 \\ k_1 k_2 \\ k_1 \end{matrix} \right\rangle = \mathbb{P}^{(k_1 k_2)}_{k_1 k_2, k_1}(b^+_{\mu s})|0\rangle \qquad (282)$$

$$|\ell\ell\rangle \equiv \mathcal{Y}^\ell_\ell$$

and satisfying (144 a,b), remembering that $k_1 = h_{13} - h_{33}$ and $k_2 = h_{23} - h_{33}$.

The operation in (53) yields an arbitrary $|\ell m\rangle \equiv \mathcal{Y}^\ell{}_m$ from the maximum weight $|\ell\ell\rangle = \mathcal{Y}^\ell{}_\ell(\vec{r})$ in R_3. An arbitrary state (280) in U_3 can be generated [39] from (282) by

$$P^{(k_1 k_2)}_{q_1 q_2, r_1} = \mathcal{R}(C^{\sigma'}_{\sigma})\, \mathbb{P}^{(k_1 k_2)}_{k_1 k_2, k_1} \qquad (283)$$

where $\mathcal{R}(C_\sigma^{\sigma'})$ must be polynomial in the 3 <u>lowering</u> generators of U_3, namely,

$$\mathcal{R}(C_\sigma^{\sigma'}) = \sum_{\alpha\beta\gamma} A_{\alpha\beta\gamma} \, (C_2^1)^\alpha (C_3^2)^\beta (C_3^1)^\gamma . \qquad (284)$$

Analogously with (279), it is desirable to have lowering operators which reduce the representation label of a given subgroup in <u>steps of one</u>, keeping the representation in maximum weight with regard to that subgroup. The rows of $(k_1 k_2)$ being characterized by three indices $(q_1 q_2, r_1)$, there will be <u>three</u> such operators obtainable directly from (284) by imposing on it the appropriate restrictions: The <u>first</u> such operator $\mathcal{R}_I$ is to lower by one the first index q_1 of U_2 maintaining maximum weight in U_2, i.e.,

$$\mathcal{R}_I \, P_{q_1 q_2, q_1} \equiv P_{q_1 - 1, q_2, q_1 - 1}$$

having suppressed the unneeded $(k_1 k_2)$ superscripts. Now, from (144) and (281) one can write

$$C_1^1 \, \mathcal{R}_I P = (q_1 - 1) \, \mathcal{R}_I P$$

$$C_2^2 \, \mathcal{R}_I P = q_2 \, \mathcal{R}_I P$$

$$C_1^2 \, \mathcal{R}_I P = 0$$

where $P \equiv P_{q_1 q_2, q_1}$, that is, of maximum weight in U_2. These three equations can be rewritten as

$$[C_1^1, \mathcal{R}_I] P + q_1 \mathcal{R}_I P = (q_1 - 1) \mathcal{R}_I P \qquad (a)$$

$$[C_2^2, \mathcal{R}_I] P + q_2 \mathcal{R}_I P = q_2 \mathcal{R}_I P \qquad (b)$$

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$$[C_1^2, \mathcal{R}_I]\, P = 0. \qquad \text{(c)}$$

Carrying out these simple commutations using (284) and the fact that from (135), e.g.,

$$[C_1^1, (C_3^1)^\gamma] = -\gamma\,(C_3^1)^\gamma$$

then equation (a) gives condition $\gamma = 1-\alpha$ and (b) gives $\beta = \alpha$ so that

$$\mathcal{R}_I = \sum_\alpha A_\alpha\,(C_2^1)^\alpha (C_3^2)^\alpha (C_3^1)^{1-\alpha}$$

with the requirement that it stay polynomial $(\alpha = 0,1)$ becomes

$$\mathcal{R}_I = A_0\, C_3^1 + A_1\, C_2^1\, C_3^2 .$$

Equation (c) then eliminates 1 of the constants so that

$$\mathcal{R}_I = A_1 [C_3^1 (C_1^1 - C_2^2 + 1) + C_2^1\, C_3^2] \qquad (285)$$

having used commutation relations (135). Our second operator must lower the second index q_2,

$$\mathcal{R}_{II}\, P_{q_1 q_2, q_1} \equiv P_{q_1\, q_2 - 1, q_1} .$$

Exactly the same reasoning as before leads to conditions $\alpha = \gamma = 0$ and $\beta = 1$ on (284) leaving

$$\mathcal{R}_{II} = A_1' C_3^2. \tag{286}$$

The <u>third</u> operator $\mathcal{R}_{III}$ is to lower the <u>third</u> index r_1 by one

$$\mathcal{R}_{III} P_{q_1 q_2, r_1} = P_{q_1 q_2, r_1 - 1}$$

and one similarly obtains that $\alpha = \beta + 1, \ \beta = \gamma = 0$ so (284) here becomes

$$\mathcal{R}_{III} = A_1'' C_2^1.$$

The operators $\mathcal{R}_I$, $\mathcal{R}_{II}$ and $\mathcal{R}_{III}$ are precisely the $U_3 \supset U_2 \supset U_1$ equivalent of the normalized operators $\mathbb{L}_r^k$ mentioned on page 43 in relation to the canonical chain $\mathcal{U}_r \supset \mathcal{U}_{r-1} \supset \cdots \supset \mathcal{U}_1$. They have been discussed thoroughly by Nägel & Moshinsky[45], who also obtained the normalization constants for the general case of $\mathcal{U}_r$ by an elegant method relying <u>only</u> on the group and subgroup generator commutation relations. The normalization constants here are A_1, A_1' and A_1'' and are given in the above reference allowing us to write

$$U_3 \supset U_2 \begin{cases} \mathcal{R}_I \equiv \mathbb{L}_3^1 = \dfrac{\left[(q_1 - q_2 + 1) C_3^1 + C_2^1 C_3^2\right]}{\sqrt{(q_1 - q_2 + 1)(k_1 - q_1 + 1)(q_1 - k_2)(q_1 + 1)}} \\ \mathcal{R}_{II} \equiv \mathbb{L}_3^2 = \sqrt{\dfrac{(q_1 - q_2 + 2)}{(k_2 - q_2 + 1)(k_1 - q_2 + 2) q_2}}\, C_3^2 \end{cases} \tag{287a, b, c}$$

$$U_2 \supset U_1 \left\{ \mathcal{R}_{III} \equiv \mathbb{L}_2^1 = \frac{C_2^1}{\sqrt{(r_1 - q_2)(q_1 - r_1 + 1)}} \right.$$

in normalized form. As before, the upper index on $\mathbb{L}_r^k$

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refers to the representation label of the subgroup; the lower index designates the unitary group order immediately before that subgroup.

In conclusion, the full base of the lowest - energy U_3 irreducible representation $(k_1 k_2)$ within a given Young pattern $[f]$ can be generated by operators $\mathbb{L}^1_3$, $\mathbb{L}^2_3$ and $\mathbb{L}^1_2$ of (287) in the chain $U_3 \supset U_2 \supset U_1$. The desired $[f]$ symmetry function of maximum weight is constructed by prescription (79). The $(k_1 k_2)$ representation corresponding to it is given by (269) which is the one lying lowest in energy and of maximum weight in U_3. One then applies (287) to build the full base but with the U_3 lowering generators $C^{\sigma'}_{\sigma}$ $(\sigma > \sigma')$ replaced by linear combinations (138) of $\mathcal{U}_r$ lowering generators $b^{\mu'}_{\mu}$ $(\mu > \mu')$ which acting on a $b^{+}_{\mu s}$ convert it into a $b^{+}_{\mu s}$. Thus, making that replacement one has the normalized lowering operators

$$\mathbb{L}^1_3 = \frac{1}{\sqrt{(q_1-q_2+1)(k_1-q_1+1)(q_1-k_2)(q_1+1)}} \times$$

$$\times \left[(q_1-q_2+1)\left(\sqrt{2}\, b^1_3 + \sqrt{2}\, b^3_6 + b^2_5\right) + \left(\sqrt{2}\, b^1_2 + \sqrt{2}\, b^2_4 + b^3_5\right)\left(b^2_3 + \sqrt{2}\, b^4_5 + \sqrt{2}\, b^5_6\right)\right]$$

$$\mathbb{L}^2_3 = \sqrt{\frac{(q_1-q_2+2)}{(k_2-q_2+1)(k_1-q_2+2)q_2}}\left(b^2_3 + \sqrt{2}\, b^4_5 + \sqrt{2}\, b^5_6\right) \qquad (289a,b,c)$$

$$\mathbb{L}^1_2 = \frac{1}{\sqrt{(r_1-q_2)(q_1-r_1+1)}}\left(\sqrt{2}\, b^1_2 + \sqrt{2}\, b^2_4 + b^3_5\right)$$

which by definition accomplish the following:

$$\mathbb{L}^1_3 \, P^{(k_1 k_2)}_{q_1 q_2, q_1} \equiv P^{(k_1 k_2)}_{q_1 - 1 \, q_2, q_1 - 1}$$

$$\mathbb{L}^2_3 \, P^{(k_1 k_2)}_{q_1 q_2, q_1} \equiv P^{(k_1 k_2)}_{q_1 \, q_2 - 1, q_1} \qquad (290\,a,b,c)$$

$$\mathbb{L}^1_2 \, \mathbb{P}^{(k_1 k_2)}_{q_1 q_2, r_1} \equiv P^{(k_1 k_2)}_{q_1 q_2, r_1 - 1}$$

the row $(q_1 q_2, r_1)$ of $(k_1 k_2)$ from (92) being subject to the conditions

$$SU_3 \supset U_2 \supset U_1 : \quad \begin{array}{c} k_1 \geqslant q_1 \geqslant k_2 \geqslant q_2 \geqslant 0 \\ q_1 \geqslant r_1 \geqslant q_2 \end{array} \qquad (291)$$

analogous to $-\ell \leqslant m \leqslant \ell$ for $R_3 \supset R_2$. Just as the latter gives a dimensionality of $(2\ell+1)$ for the R_3 irreducible representation ℓ, one can deduce [45] from (291) the dimensionality of a given $(k_1 k_2)$ of SU_3,

$$D(SU_3) = \tfrac{1}{2}(k_1 - k_2 + 1)(k_1 + 2)(k_2 + 1) \qquad (292)$$

this being a special case of the Weyl formula [70] for $\mathcal{U}_r$.

2. <u>Alternative: SU_3 Wigner Coupling Coefficients</u>.

We recall from (237) that classification by the cano-

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nical chain $U_3 \supset U_2 \supset U_1$ with a basis set

$$P^{(k_1 k_2)}_{q_1 q_2, r_1}(b^+_{\mu s})|0\rangle \equiv \left| \begin{matrix} k_1 k_2 \; 0 \\ q_1 q_2 \\ r_1 \end{matrix} \right\rangle$$

was equivalent to classifying by $U_3 \supset U_2 \supset R_2$ with basis set $P^{(k_1 k_2)}_{q_1 q_2, M_L}$ where $M_L = 2r_1 - q_1 - q_2$ is the projection of the total orbital angular momentum along a given z-axis. Taking the generators C^1_1, C^2_2, C^2_1 and C^1_2 of U_2 in the three linear combinations

$$T_{+1} \equiv \frac{1}{\sqrt{2}} C^2_1 \qquad T_{-1} \equiv -\frac{1}{\sqrt{2}} C^1_2$$
$$T_0 \equiv \frac{1}{2}(C^1_1 - C^2_2) \equiv \frac{1}{2}\mathcal{L}_0 \tag{293}$$

one can directly verify them to be the normalized generators of SU_2 (hom-omorphic to R_3) since, upon using relations (135),

$$[T_0, T_{\pm 1}] = \pm T_{\pm 1} \; ; \quad [T_{\pm 1}, T_{\mp 1}] = \mp T_0$$

is a Lie algebra identical to (142b) of R_3. Thus, the irreducible basis of $SU_2 \supset R_2$ can be characterized by the eigenvalues of T^2 and T_0, say $t(t+1)$ and τ, or by the kets $|t\tau\rangle$ in analogy to the $|lm\rangle$ of $R_3 \supset R_2$.

One can immediately find that

$$t = \frac{1}{2}(q_1 - q_2)$$

$$\tau = r - \tfrac{1}{2}(q_1+q_2) = \frac{M_L}{2} \qquad (294)$$

$$\tau = t, t-1, \ldots, -t$$

so that our base can be designated by $P^{(k_1 k_2)}_{q_1 q_2, \tau}$.

The outer product of two irreducible representations $\mathcal{D}^{(l_1)}$ and $\mathcal{D}^{(l_2)}$ of R_3 gives the well known Clebsch - Gordan equation

$$\mathcal{D}^{(l_1)} \otimes \mathcal{D}^{(l_2)} = \mathcal{D}^{(l_1+l_2)} \dotplus \mathcal{D}^{(l_1+l_2-1)} \dotplus \cdots \dotplus \mathcal{D}^{(l_1-l_2)}$$

where $+$ means that the resulting irreducible representations are of the form

$$\mathcal{D}^{(l_1)} \otimes \mathcal{D}^{(l_2)} = \begin{pmatrix} \boxed{\mathcal{D}^{(l_1+l_2)}} & & & 0 \\ & \boxed{\mathcal{D}^{(l_1+l_2-1)}} & & \\ & & \ddots & \\ 0 & & & \boxed{\mathcal{D}^{(l_1-l_2)}} \end{pmatrix}$$

The basis of a given irreducible representation $l = l_1+l_2, l_1+l_2-1, \ldots, l_1-l_2$ is then simply

$$|lm\rangle = \sum_{m_1 m_2} \langle l_1 l_2 m_1 m_2 | lm\rangle \, |l_1 m_1\rangle |l_2 m_2\rangle . \qquad (295)$$

For unitary groups, the outer product of two irreducible representations labelled by $[f']$ and $[f'']$ is

$$\mathcal{D}^{[f']} \otimes \mathcal{D}^{[f'']} = \mathcal{D}^{[f_1]} \dotplus \mathcal{D}^{[f_2]} \dotplus \cdots$$

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where the resulting $[f] = [f_1], [f_2], \ldots$ are determined by Littlewood's rules (See Appendix A) and the basis transforming irreducibly according to a given resulting $[f]$ is in principle obtainable by generalized (Clebsch - Gordan, or) <u>Wigner coefficients for the unitary group in question</u>.

In particular, for SU_3 one has $[f'] \equiv (k_1' k_2')$ and $[f''] \equiv (k_1'' k_2'')$ whose rows are respectively given by $q_1' q_2' \tau'$ and $q_1'' q_2'' \tau''$ so that the full base corresponding to a given $(k_1 k_2)$ resulting from $(k_1' k_2') \otimes (k_1'' k_2'')$ shall be [66]

$$P^{(k_1 k_2)}_{q_1 q_2 \tau} = \tag{296}$$

$$\sum_{\substack{q_1' q_2' \tau' \\ q_1'' q_2'' \tau''}} \langle (k_1' k_2') q_1' q_2' \tau' ; (k_1'' k_2'') q_1'' q_2'' \tau'' | (k_1 k_2) q_1 q_2 \tau \rangle P^{(k_1' k_2')}_{q_1' q_2' \tau'} P^{(k_1'' k_2'')}_{q_1'' q_2'' \tau''}$$

in analogy to (295) for R_3. The Wigner SU_3 coupling coefficient factorizes[66] into parts referring to $SU_3 \supset SU_2$ and $SU_2 \supset R_2$:

$$\langle (k_1' k_2') q_1' q_2' \tau' ; (k_1'' k_2'') q_1'' q_2'' \tau'' | (k_1 k_2) q_1 q_2 \tau \rangle = \tag{297}$$

$$\langle (k_1' k_2') q_1' q_2' ; (k_1'' k_2'') q_1'' q_2'') (k_1 k_2) q_1 q_2 \rangle \langle t' \tau' t'' \tau'' | t \tau \rangle$$

$$t' \equiv \tfrac{1}{2}(q_1' - q_2'), \qquad t'' \equiv \tfrac{1}{2}(q_1'' - q_2''), \qquad t = \tfrac{1}{2}(q_1 - q_2)$$

the latter factor being the ordinary Clebsch Gordan coefficient widely tabulated. The first factor in (297), called the <u>reduced Wigner SU_3 coefficient</u>, was obtained by Moshinsky in closed algebraic form [66] and extensive tables for them are being prepared by T. A. Brody [71] at the University of Mexico.

To construct a given N - particle basis transforming irreducibly under $U_3 \supset U_2 \supset U_1$ one could use (296) and (297)

to couple two particles, then a third, and so on up to N particles. The one - particle Young partition $[1]$ of $\mathcal{U}_r$ contains one and only one SU_3 irreducible representation $(k_1 k_2) = (\nu 0)$, where $r = \frac{1}{2}(\nu+1)(\nu+2)$. (The dimensionality of this SU_3 representation is, by (292), simply r. The dimensionality of $[1]$ of $\mathcal{U}_r$ is, by the Weyl formula for $\mathcal{U}_r$, also r.) For the $2s-1d$ shell $r=6$ states, those given in (137) single-particle quantum numbers $\mu \rightarrow n_1 n_{\bar{1}} n_0$. Conditions (291) tell us that, since

$$2 \geqslant q_1 \geqslant 0 \geqslant q_2 \geqslant 0$$

$$q_1 \geqslant r_1 \geqslant q_2$$

then $q_2 \equiv 0$ and

$$\begin{array}{lccc} q_1 = & 2 & 1 & 0 \\ r_1 = & 2 & 1 & 0 \\ & 1 & 0 & \\ & 0 & & \end{array}$$

will define the 6 states $\mu \rightarrow q_1 q_2 r_1$ or $\mu \rightarrow q_1 q_2 \tau$ recalling that $\tau = r_1 - \frac{1}{2}(q_1+q_2)$. The correspondence between set $(n_1\ n_{\bar{1}}\ n_0)$ and set $(q_1 q_2 \tau)$ is one - to - one and is given in Table VI.2.1.

TABLE VI.2.1

μ	1	2	3	4	5	6
$n_1 n_{\bar{1}} n_0$	200	110	101	020	011	002
$q_1 q_2 \tau$	201	200	$10\frac{1}{2}$	$20\bar{1}$	$10-\frac{1}{2}$	000
$P^{(20)}_{q_1 q_2 \tau} =$	Δ'_1	Δ'_2	Δ'_3	Δ'_4	Δ'_5	Δ'_6

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In the last row of our table are given the single-particle functions: Using prescriptions (79) for $[1]$ of $\mathcal{U}_6$ one gets $\mathbb{P} = b^+_{11}$, which by (138) and (144) is of maximum weight in U_3 so that

$$\mathbb{P} \equiv P^{(20)}_{201}(b^+_{\mu s}) \equiv b^+_{11} \equiv \Delta^1_1 \; (\mu=1, s=1) \qquad (298)$$

the remaining states labeled from 2 to 6 in order of decreasing weight $(n_1 n_{\bar{1}} n_0)$ will thus correspond respectively to $(\mu s) = (2,1)(3,1)(4,1)(5,1)(6,1)$. (These could also be obtained by successive application of lowering operators (289) on maximum weight state (298) .) As in Appendix A, we shall use the simplified notation

$$\Delta^{s_1 s_2 \cdots s_r}_{\mu_1 \mu_2 \cdots \mu_r} \equiv \sum_{P_s} (-)^{P_s} P_s \; b^+_{\mu_1 s_1} b^+_{\mu_2 s_2} \cdots b^+_{\mu_r s_r} \qquad (299)$$

where P_s is a permutation of the indices $s_1 s_2 \cdots s_r$ so that $\Delta^{s_1 s_2 \cdots s_r}_{\mu_1 \mu_2 \cdots \mu_r}$ is <u>antisymmetric</u> with regards to spin - isospin indices s and, since the $b^+_{\mu s}$ anti-commute, <u>symmetric</u> with respect to permutation of orbital indices μ. In short, a form $\Delta^{s_1 s_2 s_3}_{\mu_1 \mu_2 \mu_3}$, say, transforms under $\mathcal{U}_r$ like $[f] = [3]$ and under U_4 like $[\tilde{f}] = [111]$. Thus of course, $\Delta^s_\mu \equiv b^+_{\mu s}$.

To couple two $2s-1d$ shell particles one has, by Littlewood's rules, in SU_3:

$$(20)\otimes(20) = (40) + (22) + (31)$$

of which (40) and (22) are contained in $[2]$ and $(31) \subset [11]$ (see Table V.1.1). Representation (40) is the lowest - energy SU_3 representation of both O^{18} and F^{18}. Should we then construct the base in $(k_1 k_2) = (40)$ by the Wigner coefficient method (296), our states $P^{(40)}_{q_1 q_2 \tau}$ would be linear combinations of $\Delta^1_{\mu'} \Delta^2_{\mu''}$ which terms would have to be <u>regrouped</u> to give terms of the type $\Delta^{12}_{\mu'\mu''}$, that is, of symmetry $[f] = [2]$ under $\mathcal{U}_6$. If the permutation symmetry is <u>maintained</u> throughout the whole construction however, great labor will be saved. Thus, to construct $P^{(40)}_{q_1 q_2 \tau}$ it is preferable to use the lowering operator method beginning with

$$P^{(40)}_{40,2} = b^+_{11} b^+_{12} \equiv \frac{1}{2} \Delta^{12}_{11} ,$$

which is of maximum weight in $U_3 \supset U_2 \supset U_1$. The symmetry $[f] = [2]$ shall then be maintained throughout as operations (289) affect <u>only</u> the lower indices of Δ^{12}_{11} without altering the <u>form</u> $\Delta^{12}_{\mu'\mu''}$.

Efficient construction of an N - particle $U_3 \supset U_2 \supset U_1$ base for a definite $(k_1 k_2)$ contained in a given $[f]$ partition requires an adequate <u>combination of both methods discussed above</u>. By expressing N - particle states in terms of determinants of the type (299) one <u>avoids</u> destroying the permutation symmetry of the desired partition $[f]$. Consider the $N = 8$ particle partition $[f] = [431]$.

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According to formula (269), the lowest - energy SU_3 representation contained here is $(k_1 k_2) = (11, 2)$. To construct our 8 particle base $P^{(11,2)}_{q_1 q_2 \tau}$ we proceed as follows:

1) Construct by prescription (79) the maximum weight polynomial for $(k_1 k_2)$ = (80), this being the lowest-energy SU_3 representation contained in $[f] = [4]$. Applying lowering operators (289), proceed to generate the full base $P^{(80)}_{q_1 q_2 \tau}$. This is simple since here $k_2 = 0$. Each of the $D(U_3) = 36$ polynomials (see formula (292)) of the full base will be simple linear combinations of terms $\Delta^{1\,2\,3\,4}_{\mu_1 \mu_2 \mu_3 \mu_4}$ with symmetry $[f] = [4]$.

2) Repeat the process for $(k_1 k_2) = (60)$ which is the lowest - energy SU_3 representation contained in $[f] = [3]$, i.e., construct the $D(U_3) =$ 28 polynomials $P^{(60)}_{q_1 q_2 \tau}$.

3) The lowest $(k_1 k_2)$ of $[f] = [43]$ is by (269) equal to (11,3) so that we then <u>couple</u> $P^{(80)}_{q_1' q_2' \tau'}$ and $P^{(60)}_{q_1'' q_2'' \tau''}$ obtained in (1) and (2) by (296) to give us $P^{(11,3)}_{q_1 q_2 \tau}$. These will result as linear combinations of $\Delta^{1\,2\,3\,4}_{\mu_1' \mu_2' \mu_3' \mu_4'}$ $\Delta^{1\,2\,3}_{\mu_1'' \mu_2'' \mu_3''}$ with obvious symmetry $[f] = [43]$.

4) Finally, couple $P^{(11,3)}_{q_1' q_2' \tau'}$ with $P^{(20)}_{q_1'' 0 \tau''}$ of Table to give $P^{(11,2)}_{q_1 q_2 \tau}$ by method (296). These are linear combinations of $\Delta^{1\,2\,3\,4}_{\mu_1' \mu_2' \mu_3' \mu_4'}$ $\Delta^{1\,2\,3}_{\mu_1'' \mu_2'' \mu_3''}$ $\Delta^{1}_{\mu_1'''}$ and of symmetry $[f] = [431]$ as desired.

Notice that permutational symmetry has been maintained in each step. There will be $D(U_3) = 195$ polynomials for $(k_1 k_2)$ = (11,2) but one need only construct those $P^{(11,2)}_{q_1 q_2 \tau}$ with τ non-negative as negative values of $M_L = 2\tau$ are unnecessary to calculate matrix elements of a central interation like pairing.

3. Explicit Construction of the F^{20} Base for Lowest $(k_1 k_2)$ in $SU_3 \supset U_2 \supset R_2$.

In F^{20} one has 4 nucleons (1 proton and 3 neutrons) in the $2s-1d$ shell. Lowest levels are thus given by $T=1$. The most symmetric Young partition of N = 4 with conjugate representation containing $T=1$ is, according to Appendix A, $[f] = [31]$. The lowest energy $(k_1 k_2)$ value contained in $[31]$ from Table V.1.1 is (71). Contributions to the wave function proceeding from higher-energy $(k_1 k_2)$ values of will be neglected (Elliott hypothesis) in our calculations.

The polynomial basis $P^{(71)}_{q_1 q_2 \tau}(b^+_{\mu s})$ is composed of $D(U_3:(71))$ = 63 polynomial components of which only 35 possess τ non - negative. This set of 35 polynomials was constructed separately by both lowering - operator and the combined technique explained in the last section. Identical results were obtained but the combined technique involved much less labor and time. Since the chain of representations $[31] \supset (71)$ is desired, one starts by using lowering operators (289) on the maximum weight $(k_1 k_2)$ = (60) function of symmetry $[3]$:

$$\mathbb{P} \equiv P^{(60)}_{60,3} = b^+_{11} b^+_{12} b^+_{13} \equiv \frac{1}{6} \Delta^{123}_{111}$$

in accordance with (79) and definition (299). A set of $D(U_3:(60))$ = 28 normalized polynomials is generated, each of which is a linear combination of terms $\Delta^{123}_{\mu_1\mu_2\mu_3}$. The longest polynomial of this set was, for example,

$$P^{(60)}_{40,0} = \frac{1}{3\sqrt{10}} \left(\Delta^{123}_{622} + 2\sqrt{2}\, \Delta^{123}_{532} + \Delta^{123}_{334} + \Delta^{123}_{641} + \Delta^{123}_{551} \right).$$

(Incidentally, this $(k_1 k_2)$ = (60) base corresponds to the lowest energy SU_3 representation of F^{19}).

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Next, we add the 4th particle to our 3 - particle $P^{(60)}_{q_1 q_2 \tau}$ using SU_3 Wigner coefficients. The fourth particle is $(k_1 k_2)$ = (20) so that

$$(60)\otimes(20) = (80) + (71) + (62)$$

where the first, second and third resulting $(k_1 k_2)$ values correspond to the lowest energy SU_3 representations of Ne^{20}, F^{20} and O^{20}, respectively (see Table V.1.1). We take the (71), and thus require the SU_3 reduced Wigner coefficients

$$\langle (60) q_1' 0 ; (20) q_1'' 0) (71) q_1 q_2 \rangle$$

$$6 \geqslant q_1' \geqslant 0 \qquad 2 \geqslant q_1'' \geqslant 0 \qquad 7 \geqslant q_1 \geqslant 1 \geqslant q_2$$

which are available in Brody's tables.[71] One also requires the ordinary Clebsch - Gordan coefficients

$$\langle t' \tau' t'' \tau'' | t \tau \rangle = \langle \tfrac{1}{2} q_1' \, \tau' \, \tfrac{1}{2} q_1' \, \tau'' | \tfrac{1}{2}(q_1 - q_2) \, \tau \rangle$$

$$\text{for} \quad \tau = \tfrac{1}{2}(q_1 - q_2), \; \tfrac{1}{2}(q_1 - q_2) - 1, \; \cdots, \begin{cases} 0 \\ 1/2 \end{cases}$$

which are also tabulated in various sources. Then by (296) and (297) the F^{20} lowest - energy $U_3 \supset U_2 \supset R_2$ base is gotten by evaluating, for $\tau \geqslant 0$,

$$P^{(71)}_{q_1 q_2 \tau} \left(\Delta^1_{\mu_1} \, \Delta^{1\,2\,3}_{\mu_2 \mu_3 \mu_4} \right) =$$

$$\sum_{q_1' q_1''} \langle (60) q_1' 0 ; (20) q_1'' 0) (71) q_1 q_2 \rangle \sum_{\tau' \tau''} \langle \tfrac{1}{2} q_1' \, \tfrac{1}{2} q_1'' \, \tau' \tau'' | \tfrac{1}{2}(q_1 - q_2) \, \tau \rangle \, X$$

$$\times P^{(60)}_{q_1'0\tau'}\left(\Delta^{123}_{\mu_2\mu_3\mu_4}\right) P^{(20)}_{q_1''0\tau''}\left(\Delta^{1}_{\mu_1}\right) \qquad (300)$$

where single-particle functions $P^{(20)}_{q_1''0\tau''}$ are those of Table VI.2.1. The 35 polynommals $P^{(71)}_{q_1q_2\tau}$ of $\tau \geqslant 0$ are linear combinations of terms $\Delta^{123}_{\mu_1\mu_2\mu_3}\ \Delta^{1}_{\mu_4}$ (which have symmetry [31]). Let us abbreviate

$$\Delta^{1}_{\mu_1}\ \Delta^{1\ 2\ 3}_{\mu_2\mu_3\mu_4} \equiv (\mu_1, \mu_2\mu_3\mu_4) \qquad (301)$$

as the upper (spin - isospin) indices are identical to every term. The polynomial $P^{(71)}_{71,3}$ resulting from (300) is however

$$P^{(71)}_{71,3} = \frac{\sqrt{3}}{12}\left[(2,111)-(1,211)\right] = \frac{2}{\sqrt{3}}\left[\frac{1}{6}(2,111)\right] \qquad (302)$$

since (as is easily shown) $\Delta^{1}_{1}\Delta^{123}_{211} = -\frac{1}{3}\Delta^{1}_{2}\Delta^{123}_{111}$. Prescription (79) allows us, on the other hand, to construct this polynomial immediately as it is of maximum weight in SU_3 namely:

$$P^{(71)}_{71,3} = b^{+}_{11}\ b^{+}_{12}\ b^{+}_{13}\ b^{+}_{21} \equiv \frac{1}{6}(2,111)$$

which is moreover <u>normalized</u>. Thus, in view of (302), we must <u>multiply</u> by $\sqrt{3}/2$ all polynomials resulting from (300) in order to obtain a normalized set. Moreover, the number of terms $\Delta^{1}_{\mu_1}\ \Delta^{1\ 2\ 3}_{\mu_2\mu_3\mu_4}$ resulting in each $P^{(71)}_{q_1q_2\tau}$ from (300) can be reduced by use of the following easily derived identities:

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$$(a, bbb) = -3(b, abb)$$
$$(a, bbc) + 2(b, abc) + (c, bba) = 0 \qquad (303)$$
$$(a, abb) = -(b, aba)$$
$$(a, bcd) + (b, acd) + (c, abd) + (d, abc) = 0.$$

The complete list of $P^{(71)}_{q_1 q_2 \tau}$ for $\tau \geq 0$ is given in Table where $\lambda \equiv (q_1 q_2 \tau) = 1, 2, \ldots, 35.$

T A B L E VI.3.1

$q_1 q_2 \tau$ (2)	$\lvert [31](71) q_1 q_2 \tau ; [211] S=1, M_S=S, T=1, M_T=T\rangle \equiv P^{(71)}_{q_1 q_2 \tau} \lvert 0\rangle$
713 (1)	$\frac{1}{6}(2,111)$
603 (2)	$\frac{1}{6\sqrt{3}}\left[3(3,311) + (6,111)\right]$
712 (3)	$\frac{1}{6\sqrt{3}}\left[3(2,211) + (4,111)\right]$
512 (4)	$\frac{1}{24\sqrt{5}}\left[12(2,331) + 7\sqrt{2}(5,311) + 4(3,321) + 6(2,611) + 2(6,211) + \sqrt{2}(3,511)\right]$
602 (5)	$\frac{1}{6\sqrt{2}}\left[2\sqrt{2}(3,321) + \sqrt{2}(6,211) + (5,311) + (3,511)\right]$
402 (6)	$\frac{1}{2\sqrt{10}}\left[2(3,631) + 2(6,331) + (6,611)\right]$
711 (7)	$\frac{1}{2\sqrt{15}}\left[2(2,221) + 2(4,211) + (2,411)\right]$

511 ⑧	$\frac{1}{24\sqrt{5}}\Big[4\sqrt{2}(2,332) + 16(5,321) + 6\sqrt{2}(2,621) + 2\sqrt{2}(6,221) + 4(3,521) + 6\sqrt{2}(4,331) + 2\sqrt{2}(3,431) + 3\sqrt{2}(4,611) + \sqrt{2}(6,411) + 12(2,531) + 4\sqrt{2}(5,511)\Big]$
311 ⑨	$\frac{1}{2\sqrt{30}}\Big[(2,661) - (6,332) - 3(3,632) + (6,621)\Big] + \frac{1}{12\sqrt{15}}\Big[15(5,631) + 6(6,531) + 3(3,651) + 4(5,333)\Big]$
601 ⑩	$\frac{1}{6\sqrt{5}}\Big[2(3,322) + 2(6,221) + 2\sqrt{2}(5,321) + 2\sqrt{2}(3,521) + 2(3,341) + (6,411) + (5,511)\Big]$
401 ⑪	$\frac{1}{2\sqrt{10}}\Big[\sqrt{2}(3,632) + \sqrt{2}(6,332) + \sqrt{2}(6,621) + (5,631) + (3,651) + 2(6,531)\Big]$
201 ⑫	$\frac{1}{2\sqrt{3}}\Big[(6,633) + (6,661)\Big]$
710 ⑬	$\frac{1}{2\sqrt{10}}\Big[2(4,221) + 2(2,421) + (4,411)\Big]$
510 ⑭	$\frac{1}{4\sqrt{30}}\Big[\sqrt{2}(5,322) - \sqrt{2}(3,522) + 2(4,332) - 2(3,432) + 2(4,621) + 6(5,521) + 3\sqrt{2}(5,431) + \sqrt{2}(3,541) + 4\sqrt{2}(4,531) + 2(2,551) - 2(1,642)\Big]$
310 ⑮	$\frac{1}{2\sqrt{30}}\Big[(4,661) - (6,433) - 3(3,643) + (6,641) + \sqrt{2}(5,632) - \sqrt{2}(3,652) + 3(5,651) + (6,551) + 2(5,533)\Big]$
110 ⑯	$\frac{1}{4\sqrt{2}}\Big[(5,663) - (3,665)\Big]$
600 ⑰	$\frac{1}{6\sqrt{15}}\Big[\sqrt{2}(6,222) + 3(5,322) + 3(3,522) + 3\sqrt{2}(3,432) + 3\sqrt{2}(6,421) + 3\sqrt{2}(5,521) + 3(5,431) + 3(3,541)\Big]$

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400 (18)	$\frac{1}{2\sqrt{15}}\left[(6,622)+\sqrt{2}(5,632)+\sqrt{2}(3,652)+2\sqrt{2}(6,532)+(3,643)+(6,433)+(6,641)+(5,651)+(6,551)\right]$
200 (19)	$\frac{1}{2\sqrt{6}}\left[2(6,653)+\sqrt{2}(6,662)\right]$
$61\frac{5}{2}$ (20)	$\frac{1}{24\sqrt{21}}\left[21\sqrt{2}(2,311)+8(5,111)+3\sqrt{2}(3,211)\right]$
$70\frac{5}{2}$ (21)	$\frac{1}{6\sqrt{7}}\left[3\sqrt{2}(3,211)+(5,111)\right]$
$50\frac{5}{2}$ (22)	$\frac{1}{2\sqrt{15}}\left[2(3,331)+(3,611)+2(6,311)\right]$
$61\frac{3}{2}$ (23)	$\frac{1}{8\sqrt{105}}\left[28(2,321)+9\sqrt{2}(5,211)+4(3,221)+14(4,311)+2(3,411)+7\sqrt{2}(2,511)\right]$
$41\frac{3}{2}$ (24)	$\frac{1}{60}\left[4(2,333)+15(2,631)+6(6,321)+3(3,621)\right]+\frac{1}{20\sqrt{2}}\left[6(5,331)+3(5,611)+2(3,531)+(6,511)\right]$
$70\frac{3}{2}$ (25)	$\frac{1}{2\sqrt{21}}\left[2(3,221)+\sqrt{2}(5,211)+(3,411)\right]$
$50\frac{3}{2}$ (26)	$\frac{1}{10\sqrt{3}}\left[2\sqrt{2}(3,332)+2\sqrt{2}(3,621)+4\sqrt{2}(6,321)+2(5,331)+4(3,531)+2(6,511)+(5,611)\right]$
$30\frac{3}{2}$ (27)	$\frac{1}{6\sqrt{15}}\left[12(6,631)+2(6,333)+3(3,661)\right]$
$61\frac{1}{2}$ (28)	$\frac{1}{12\sqrt{210}}\left[30(5,221)-4\sqrt{2}(3,222)+42\sqrt{2}(4,321)+9\sqrt{2}(3,421)+42(2,521)+21\sqrt{2}(2,431)+15(5,411)+21(4,511)\right]$
$41\frac{1}{2}$ (29)	$\frac{1}{60\sqrt{3}}\left[3\sqrt{2}(2,632)-3\sqrt{2}(3,622)+4\sqrt{2}(4,333)+9\sqrt{2}(4,631)-6\sqrt{2}(1,643)-3\sqrt{2}(3,641)-18(3,532)+3(2,651)-12(1,652)+9(5,621)+6(5,332)+21\sqrt{2}(5,531)+3\sqrt{2}(3,551)\right]$

$21\frac{1}{2}$ (30)	$\frac{1}{6\sqrt{2}}\left[-(6,632)-(3,662)\right]+\frac{1}{12}\left[3(5,633)+2(5,661)+(6,533)+2(6,651)\right]$
$70\frac{1}{2}$ (31)	$\frac{1}{6\sqrt{35}}\left[2\sqrt{2}(3,222)+6(5,221)+6\sqrt{2}(3,421)+3(5,411)\right]$
$50\frac{1}{2}$ (32)	$\frac{1}{5\sqrt{6}}\left[(3,622)+2(6,322)-\sqrt{2}(2,533)+2\sqrt{2}(6,521)+\sqrt{2}(5,621)\right.$ $\left.+(3,433)+(3,641)+2(6,431)-(1,553)\right]$
$30\frac{1}{2}$ (33)	$\frac{1}{6\sqrt{5}}\left[4\sqrt{2}(6,632)+\sqrt{2}(3,662)+4(6,651)+2(6,533)+(5,661)\right]$
$10\frac{1}{2}$ (34)	$\frac{1}{2}\left[(6,663)\right]$
$70\frac{7}{2}$ (35)	$\frac{1}{6}\left[(3,111)\right]$

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The base $P^{(71)}_{q_1 q_2 \tau}$ given in terms of $(\mu_1, \mu_2 \mu_3 \mu_4)$ is convenient as any interaction operator is expressible, from (35) and (36), in terms of operators $\mathcal{b}^{\mu_2}_{\mu_1}$ which replace μ_2 by μ_1, that is

$$\mathcal{b}^{\mu_2}_{\mu_1}(\mu_1, \mu_2 \mu_3 \mu_4) \equiv (\mu_1, \mu_1 \mu_3 \mu_4) \qquad (304)$$
$$\mathcal{b}^{\mu_5}_{\mu_1}(\mu_1, \mu_2 \mu_3 \mu_4) \equiv 0, \quad etc.$$

To construct the base for any number of particles N in 2s-1d shell with orbital symmetry $[f]$ and associated lowest-energy SU_3 representation $(k_1\ k_2)$, one can proceed in the following systematic fashion. If one has the bases for N=1,2,3 and 4 particles with $[f]\,(k_1\ k_2)$ = [1] (20), [2](40), [3](60) and [4](80), respectively, any $[f]\,(k_1\ k_2)$ for any N can then be built by appropriately combining these "elemental bases" with the aid of SU_3 Wigner coupling coefficients in accord with equation (296). The [1](20) base (O^{17} and F^{17}) is simply given in Table VI.2.1. Base [2](40) (O^{18} and F^{18}) could easily be generated from $\mathbb{P}^{(40)} = b^+_{11}\, b^+_{12}$ (of orbital symmetry [2]) by the lowering operator method (290). The [3](60) base (F^{19} and Ne^{19}) can likewise be gotten from $\mathbb{P}^{(60)} = b^+_{11}\, b^+_{12}\, b^+_{13}$, of orbital symmetry [3] (See VI. 3). Finally, the base [4](80) (Ne^{20}) follows $\mathbb{P}^{(80)} = b^+_{11}\, b^+_{12}\, b^+_{13}\, b^+_{14}$, with orbital symmetry [4]. By formula (292) the respective dimensionalities are 6, 15, 28 and 40 but, as mentioned before, one may restrict oneself to components of the base having <u>positive</u> M_L row-index values. A few schematic examples follow:

$$(k_1' k_2') \otimes (k_1'' k_2'') = \cdots + (k_1 k_2) + \cdots \text{ (in formula 296).}$$

$$[3](60) \otimes [1](20) = \cdots + [31](71) + \cdots \qquad (F^{20}, Na^{20})$$

$$[4](80) \otimes [31](71) = \cdots + [431](11,2) + \cdots \qquad (Na^{24}, Al^{24})$$

$$[4](80) \otimes [431](11,2) = \cdots + [4431](12,3) + \cdots \qquad (Al^{28})$$

$$[4](80) \otimes [4](80) = \cdots + [44](12,4) + \cdots \qquad (Mg^{24})$$

$$[1](20) \otimes [44](12,4) = \cdots + [441](12,3) + \cdots \qquad (Al^{25})$$

$$[2](40) \otimes [44](12,4) = \cdots + [442](11,1) + \cdots \qquad (Na^{26})$$

$$[1](20) \otimes [442](11,1) = \cdots + [4421](12,4) + \cdots \qquad (Al^{27}, Si^{27})$$

$$\cdots \text{ etc. } \cdots \text{ etc. } \cdots \text{ etc. } \cdots$$

Global studies using the SU_3 classification scheme can thus be carried out for several nuclei at a time, leaving as much as possible to the computer.

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CHAPTER

VII. PAIRING FORCE MATRICES.

The many - orbital pairing force matrices in the physical (angular momentum) scheme $\mathcal{U}_r \supset U_3 \supset R_3 \supset R_2$ for a single SU_3 representation $(k_1 k_2)$ for N particles in an oscillator shell are designated by

$$\| \langle [f] \alpha (k_1 k_2) \omega' L M_L | \mathcal{P} | [f] \alpha (k_1 k_2) \omega L M_L \rangle \| \quad (305)$$

where $\mathcal{P}$ is the pairing force operator (219). Since $\mathcal{P}$ was assumed a central spin - isospin - independent operator, then $[\mathcal{L}^2, \mathcal{P}] = [\mathcal{L}_0, \mathcal{P}] = 0$ so that (Appendix F) it cannot connect different L- values in (305) and is moreover independent of the row label M_L.

In the canonical scheme $\mathcal{U}_r \supset U_3 \supset U_2 \supset U_1$ our matrices would be denoted by

$$\| \langle [f] \alpha (k_1 k_2) q_1' q_2' M_L | \mathcal{P} | [f] \alpha (k_1 k_2) q_1 q_2 M_L \rangle \| \quad (306)$$

whose elements can be evaluated readily if states $| [f] \alpha (k_1 k_2) q_1 q_2 M_L \rangle$ are constructed as shown in the previous Chapter since the effect of operator (219) on these states, as well as the resulting scalar products, is straightforward. This will be illustrated shortly.

In view of transformation (I.1) of Appendix I, the passage from matrices (306) to (305) would then involve the similarity transformation

$$\| \mathcal{P} \|_{R_3} = \| \mathcal{S} \| \times \| \mathcal{P} \|_{U_2} \times \| \widetilde{\mathcal{S}} \| , \quad (307)$$

the R_3 and U_2 subscripts referring respectively to (305) and (306), and $\|\mathcal{S}\|$ standing for orthogonal transformation matrices of the type $(I.9)$ of Appendix I.

If, however, $\omega'=\omega$ in (305)----which is the case for the (k_1k_2) = (71) of our present interest --- the matrix is diagonal and of dimension equal to as many L - values as are contained in (k_1k_2). Hence, in view of (307), the sum of the eigenvaluesof P, which are labeled by L, is simply

$$\mathrm{Tr}\left\|\langle[f]\alpha(k_1k_2)\,\omega L\,M_L|\,\mathcal{P}\,|\,[f]\alpha(k_1k_2)\,\omega L\,M_L\rangle\right\| = \mathrm{Tr}\left\|\langle[f]\alpha(k_1k_2)\,q_1'q_2'\,M_L|\,\mathcal{P}\,|\,[f]\alpha(k_1k_2)\,q_1q_2\,M_L\rangle\right\| \tag{308}$$

so that <u>only the diagonal elements of (306) need be calculated to obtain the eigenvalues of (305)</u> that will figure in the calculation of energy levels. Matrix (306) is explicitly reduced into submatrices labeled by $M_L = L_{max}, L_{max}-1, \ldots, 1, 0$ (where $L_{max} = k_1$) and of dimensions equal to the multiplicity of each M_L involved. Matrix (305), which is diagonal, can be imagined to be ordered into <u>groups</u> of diagonal elements labeled also by the M_L's. Then, according to (308), the eigenvalues p_L of $\mathcal{P}$ associated with a definite L can readily be found through

$$p_L = \left\{\mathrm{Tr}\left\|\langle[f]\alpha(k_1k_2)q_1'q_2'\,M_L=L\,|\,\mathcal{P}\,|\,[f]\alpha(k_1k_2)q_1q_2\,M_L=L\rangle\right\| - \mathrm{Tr}\left\|\langle[f]\alpha(k_1k_2)q_1'q_2'\,M_L=L+1\,|\,\mathcal{P}\,|\,[f]\alpha(k_1k_2)q_1q_2\,M_L=L+1\rangle\right\|\right\} \tag{309}$$

We now turn to the evaluation of the diagonal elements of (306). Applying the numbering convention of Table $VI.2.1$ for $\mu \equiv (n_1 n_1 n_0)$ to pairing operator (219), which is bilinear in

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$\mathcal{U}_r$ group generators $\mathcal{b}_\mu^{\mu'}$ defined in (37), one obtains

$$\begin{aligned}\mathcal{P} = {} & \mathcal{b}_1^2\mathcal{b}_4^2 + \mathcal{b}_1^6\mathcal{b}_4^6 + \mathcal{b}_2^1\mathcal{b}_2^4 + \mathcal{b}_6^1\mathcal{b}_6^4 + \mathcal{b}_1^4\mathcal{b}_4^1 + \mathcal{b}_3^5\mathcal{b}_5^3 \\ & + \tfrac{1}{2}\mathcal{b}_2^6\mathcal{b}_2^6 + \tfrac{1}{2}\mathcal{b}_6^2\mathcal{b}_6^2 + \mathcal{b}_1^1\mathcal{b}_4^4 + \mathcal{b}_3^3\mathcal{b}_5^5 - \mathcal{b}_1^3\mathcal{b}_4^5 - \mathcal{b}_1^5\mathcal{b}_4^3 \\ & - \mathcal{b}_2^3\mathcal{b}_2^5 - \mathcal{b}_3^1\mathcal{b}_5^4 - \mathcal{b}_3^2\mathcal{b}_5^2 - \mathcal{b}_3^4\mathcal{b}_5^1 - \mathcal{b}_3^6\mathcal{b}_5^6 - \mathcal{b}_6^3\mathcal{b}_6^5 \\ & - \mathcal{b}_1^1 - \mathcal{b}_3^3 + \tfrac{1}{2}\mathcal{b}_2^2(\mathcal{b}_2^2 - 1) + \tfrac{1}{2}\mathcal{b}_6^6(\mathcal{b}_6^6 - 1). \qquad (310)\end{aligned}$$

Recalling (304), the effect of this operator $\mathcal{P}$ on states of the type given in Table VI.3.1 is clear; one must be careful to use the <u>same numbering convention</u> regarding μ for both operator $\mathcal{P}$ and single - particle states $P^{(20)}_{q_1q_2\tau}$ as in Table VI.2.1. For (k_1k_2) = (71), operator (310) must be applied to all 35 polynomials given in Table VI.3.1 to get

$$\mathcal{P}\,|[f]\alpha(k_1k_2)q_1q_2M_L\rangle \equiv \mathcal{P}\,P^{(71)}_{q_1q_2\tau=\frac{M_L}{2}}\left(\Delta^1_{\mu_1}\,\Delta^{123}_{\mu_2\mu_3\mu_4}\right)|0\rangle \qquad (311)$$

for all 35 cases. A typical result of (311) for the polynomial with $q_1q_2\tau = 41\frac{3}{2}$ is, having used simplifying relations (303) and regrouping terms,

$$\begin{aligned}\mathcal{P}P^{(71)}_{41\frac{3}{2}} = \frac{1}{10\sqrt{2}}\Big[& 9(3,531) + 7(5,331) - 7(2,321) - (3,221) - 7(6,631) \\ & -(3,661) + 5(3,411) + 7(1,431)\Big] \qquad (312)\end{aligned}$$

where as in notation (301), e.g.,

$$(3,531) \equiv \Delta^{1}_{3}\, \Delta^{123}_{531}\,,$$

lower indices referring to orbital and upper to spin - isospin quantum numbers. Multiplication of (311) on the left by

$$\langle [f]\alpha(k_1k_2)q_1q_2 M_L | \equiv \langle 0 | \left(P^{(k_1k_2)}_{q_1q_2\tau=M_L/2} \right)^{+}$$

will result in only a few different elemental scalar products of the type $\left(\Delta^{1}_{\mu_1}\, \Delta^{123}_{\mu_2\mu_3\mu_4},\ \Delta^{1}_{\mu_1'}\, \Delta^{123}_{\mu_2'\mu_3'\mu_4'}\right) \equiv (\mu_1\mu_2\mu_3\mu_4 / \mu_1'\mu_2'\mu_3'\mu_4')$ which are simple to evaluate with the aid of anticommutation relations (12) and definition (299). Calculation of diagonal elements of (306) for $(k_1k_2) = (71)$ required <u>only</u> those $(\mu_1\mu_2\mu_3\mu_4 / \mu_1'\mu_2'\mu_3'\mu_4')$ given in the following list:

$$\underline{(a,bcd/a',b'c'd') \equiv \Delta^{1}_{a}\, \Delta^{123}_{bcd},\ \Delta^{1}_{a'}\, \Delta^{123}_{b'c'd'}} \qquad (313)$$

$(a,bcd/a,bcd) = 6$	$(a,bbb/b,abb) = -12$	$(a,abb/a,abb) = 8$
$(a,bbc/a,bbc) = 12$	$(a,bcc/b,acc) = -4$	$(a,abc/a,abc) = 4$
$(a,bbb/a,bbb) = 36$	$(a,bcc/c,abc) = -4$	$(a,aab/a,aab) = 4$
	$(a,bcd/b,acd) = -2$	

If in $(\mu_1\mu_2\mu_3\mu_4 / \mu_1'\mu_2'\mu_3'\mu_4')$ the set of indices $\mu_1\mu_2\mu_3\mu_4$ differs from the set $\mu_1'\mu_2'\mu_3'\mu_4'$, <u>in any order</u>, the elemental scalar product <u>vanishes</u> because of (11). To finish our example for $q_1q_2\tau = 413/2$ with $(k_1k_2) = (71)$ and $[f] = [31]$, (α being unneeded):

$$\langle [31](71)413 | P | [31](71)\,413 \rangle \equiv \langle 0 | \left(P^{(71)}_{413/2}\right)^{+} P\, P^{(71)}_{413/2} | 0 \rangle$$

$$= \frac{1}{20\cdot 10\cdot 2}\Big[6\cdot 9\underbrace{(5,331/3,531)}_{-4} + 6\cdot 7\underbrace{(5,331/5,331)}_{12} + 2\cdot 9\underbrace{(3,531/3,531)}_{4}$$

$$+ 2\cdot 7\underbrace{(3,531/5,331)}_{-4} \Big]$$

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$$= \frac{1}{5\cdot 10\cdot 2}\left[-6\cdot 9 + 6\cdot 7\cdot 3 + 2\cdot 9 - 2\cdot 7\right] = \frac{76}{5\cdot 10\cdot 2} = \frac{19}{25}.$$

Calculating the other 34 diagonal elements of (306) in a similar way and then using (309), one obtains the following eigenvalues $\mathcal{P}_L$ for $L=1,2,\cdots,7$ of (305):

Eigenvalues of $\|\langle [31](71)LM_L|\mathcal{P}|[31](71)LM_L\rangle\|$

L	1	2	3	4	5	6	7
$-\mathcal{P}_L$	$-\frac{21}{5}$	$-\frac{23}{5}$	$-\frac{11}{5}$	$-\frac{11}{3}$	0	0	0

(314)

constituting <u>five</u> different levels the highest of which is degenerate in L = 5, 6, 7.

In cases of certain $(k_1 k_2)$ values where $\omega \neq \omega'$ in (305), the trace technique (309) would not suffice to obtain all the $\mathcal{P}_L$ eigenvalues as then matrix (305) is not entirely diagonal and would contain small submatrices along the diagonal for those L- values which occur multiply under the given $(k_1 k_2)$. These submatrices would be of dimension equal to the number of ω's that occur for the given L and to obtain the corresponding eigenvalues $\mathcal{P}_{\omega L}$ of $\|\mathcal{P}\|_{R_3}$ the similarity transformation (307) would be applied explicitly upon $\|\mathcal{P}\|_{U_2}$ for those M_L-labeled submatrices of $\|\mathcal{P}\|_{R_3}$ containing non-diagonal terms. The resulting $\|\mathcal{P}\|_{R_3}$ are then diagonalized and the $\mathcal{P}_{\omega L}$ obtained. A typical instance of this $\omega \neq \omega'$ situation is found in Table for M_g^{24} with $[f] = [44]$, $(k_1 k_2) = (12, 4)$ which contains up to <u>three</u> - <u>fold</u> multiplicity in the L-structure, that is, $\omega \neq \omega' \neq \omega''$.

Result (314), after coupling each L- value with the associated total spin S - values of each level to give total J, will provide(diagonal) matrices with elements

$$\langle [f](k_1k_2)\omega' L\, S\, J M_J\, T | \mathcal{P} | [f](k_1k_2)\omega L\, S\, J M_J\, T \rangle =$$

$$\sum_{\substack{M_L' M_S' \\ M_L M_S}} \langle L S M_L' M_S' | J M_J \rangle \langle L S M_L M_S | J M_J \rangle \times$$

$$\times \langle [f](k_1k_2)\omega' L M_L' ; [\tilde{f}] S M_S' T | \mathcal{P} | [f](k_1k_2)\omega L M_L ; [\tilde{f}] S M_S, T \rangle$$

where the latter matrix element is identical with those of (305) as $\mathcal{P}$ is spin - isospin independent. This element moves outside the sum since it is independent of row labels M_L', M_S', M_L, M_S so that by Clebsch - Gordan Coefficient orthogonality

$$\langle [f](k_1k_2)\omega' L\, S\, J\, T | \mathcal{P} | [f](k_1k_2)\omega L\, S\, J\, T \rangle = \tag{315}$$

$$\langle [f](k_1k_2)\omega' L ; [\tilde{f}] S\, T | \mathcal{P} | [f](k_1k_2)\omega L ; [\tilde{f}] S\, T \rangle$$

being independent of row label M_J. Matrices (315) with rows labeled by L and S shall form part of the total interaction hamiltonian matrices whose eigenvalues are the energies of definite J and T values.

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CHAPTER

VIII. SPIN-ORBIT FORCE MATRICES FOR F^{20}.

To permit evaluation of the spin-orbit interaction matrix elements (263) in the SU_3 scheme one must evaluate coefficients $A_{s's}$, $B_{s's}$ and the simple matrix of elements $\langle (k_1k_2)\omega' L' \| \mathcal{K} \| (k_1k_2)\omega L \rangle$. This will be illustrated here for F^{20}; extension to other nuclei will then be quite straightforward.

Coefficients $A_{s's}$ and $B_{s's}$ being independent of U_3 row indices $q_1 q_2 r_1$ and of U_2 (spin) row index M_s, conditions (254) may be imposed on the two linearly independent inhomogeneous equations resulting from the application of (252) to both operators (253 a,b). Typically,

$$\langle [f]\alpha(h_1h_2h_3)\, h_1h_2(h_1-h_2);\beta' S'S'T \,|\, \mathbb{T}((21)^1_1;1_{s'-s}) - \mathbb{T}((21)^2_2;1_{s'-s}) \,|\, \times$$
$$\times\, |\, [f]\alpha(h_1h_2h_3)\, h_1h_2(h_1-h_2);\beta SST\rangle \tag{316}$$
$$= \Big\{ A_{s's}\langle h_1h_2,(h_1-h_2)|\, C^1_1 - C^2_2 \,|\, h_1h_2,(h_1-h_2)\rangle$$
$$+ B_{s's}\langle h_1h_2,(h_1-h_2)|\, \sigma^1_1 - \sigma^2_2 \,|\, h_1h_2,(h_1-h_2)\rangle \Big\} \langle S1S\, S'-S|S'S'\rangle,$$

and similarly for (253 b); the $(h_1\ h_2\ h_3)$ labeling of U_3 is here suppressed for brevity on the right hand side. The operator on the left is by definition (241)

$$\mathbb{T}((21)^1_1;1_{s'-s}) - \mathbb{T}((21)^2_2;1_{s'-s}) =$$
$$\sum_{\sigma\sigma'\tau} \langle \sigma | \Delta_{s'-s} | \sigma' \rangle \left[C^{1\sigma'\tau}_{1\sigma\tau} - C^{2\sigma'\tau}_{2\sigma\tau} \right], \tag{317}$$

and similarly for (253 b); and of course

$$\langle\sigma|A_{S'-S}|\sigma'\rangle = \sqrt{\tfrac{3}{4}}\langle\tfrac{1}{2}1\sigma'S'-S|\tfrac{1}{2}\sigma\rangle \tag{318}$$

by the Wigner-Eckhart theorem. Moreover, from expansion (240) we have

$$\mathbb{C}^{q\sigma'\tau}_{q\sigma\tau} = \sum_{\mu\mu'}\langle\mu|C^q_q|\mu'\rangle\,\mathbb{C}^{\mu'\sigma'\tau}_{\mu\sigma\tau} \qquad (q = 1,\bar{1},0 \rightarrow 1,2,3) \tag{319}$$

while

$$\langle\mu|C^q_q|\mu'\rangle \equiv \langle n_1 n_{\bar{1}} n_0|\,C^q_q\,|n_1' n_{\bar{1}}' n_0'\rangle = n_q' \tag{320}$$

from Appendix G. Combining these results carefully one arrives at operators (317) expanded in terms of $\mathbb{U}_{4r}$ generators $\mathbb{C}^{\mu's'}_{\mu s}$ $(s, s'$ $= 1,2,3,4)$ and $(\mu, \mu' = 1,2,\cdots, 6$ for $r=6)$. That is, for $S'=S=1$ say, (317a) shall be $\frac{1}{2}[\mathbb{C}^{31}_{31} + \mathbb{C}^{32}_{32} + \mathbb{C}^{53}_{53} + \mathbb{C}^{54}_{54}] - \frac{1}{2}[\mathbb{C}^{33}_{33} + \mathbb{C}^{34}_{34} + \mathbb{C}^{51}_{51} + \mathbb{C}^{52}_{52}] + [\mathbb{C}^{11}_{11} + \mathbb{C}^{12}_{12} + \mathbb{C}^{43}_{43} + \mathbb{C}^{44}_{44}] - [\mathbb{C}^{13}_{13} + \mathbb{C}^{14}_{14} + \mathbb{C}^{41}_{41} + \mathbb{C}^{42}_{42}]$.

The ket states on the left of (316) being of maximum weight in U_s may be designated briefly as

$$|[f]\alpha(h_1h_2h_3)\,h_1h_2(h_1-h_2);\beta SST\rangle \equiv \mathbb{P}_{SS}$$

the first index referring to total intrinsic spin, the second to projection of same. Now, operators R_{kl} of (180 d) related to cartesian components $(k, l = 1,2,3)$ can be case in spherical component form: $R_{qq'} = \frac{1}{4}\sum_{\sigma\tau\sigma'\tau'}(M_q)^{\sigma}_{\sigma'}(N_{q'})^{\tau}_{\tau'}\,C^{\sigma'\tau'}_{\sigma\tau}$

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where the Pauli matrices now in spherical components are

$$M_1 = -\sqrt{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad M_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad M_{\bar{1}} = \sqrt{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with rows and columns given by $\sigma, \sigma' = 1/2, -1/2$ and similarly for N_q $(q = 1, 0, \bar{1})$ with rows and columns given by $\tau, \tau' = 1/2, -1/2$. The operator $2R_{\bar{1}0}$ is then

$$2R_{\bar{1}0} = \frac{1}{\sqrt{2}}\left(C_3^1 - C_4^2\right)$$

in terms of U_4 lowering generators belonging to the set (61). It is the lowering operator for <u>total</u> intrinsic spin S, i.e.,

$$2R_{\bar{1}0}\,\mathbb{P}_{S,S} = \mathbb{P}_{S-1,S-1}$$

and moreover keeps the result in maximum projection. Likewise, $2R_{0\bar{1}}$ would be the lowering operator for total isospin T. The case of F^{20} involves $(h_1\ h_2\ h_3) = (710)$ and from Table VI.3.1 for S = 1 is simply

$$\mathbb{P}_{11} = \frac{1}{6}(2,111) \equiv b^+_{21}\, b^+_{11}\, b^+_{12}\, b^+_{13}$$

and consequently

$$\mathbb{P}_{00} = 2R_{\bar{1}0}\,\mathbb{P}_{11} = \frac{1}{\sqrt{2}}\left(b^+_{23}\, b^+_{11}\, b^+_{12}\, b^+_{13} - b^+_{21}\, b^+_{11}\, b^+_{14}\, b^+_{13}\right).$$

Evaluation of the left-hand members of equations (316) now follows easily. On the right hand side we have, from (144 a),

$$\langle h_1 h_2, (h_1 - h_2) \,|\, C_1^1 - C_2^2 \,|\, h_1 h_2, (h_1 - h_2)\rangle = h_1 - h_2$$

and similarly for $\mathcal{C}_2^2 - \mathcal{C}_3^3$. Furthermore, by definition (249) and condition (144 b) $\mathcal{C}_1^1 - \mathcal{C}_2^2 = (C_1^1)^2 - (C_2^2)^2 + C_1^1 - C_2^2$

and similarly for $\mathcal{G}_2^2 - \mathcal{G}_3^3$, so that

$$\langle h_1 h_2, (h_1 - h_2) | \mathcal{G}_1^1 - \mathcal{G}_2^2 | h_1 h_2, (h_1 - h_2) \rangle = h_1^2 - h_2^2 + h_1 - h_2$$

and likewise for $\mathcal{G}_2^2 - \mathcal{G}_3^3$. Finally, for $(h_1\ h_2\ h_3) = (710)$ and $S' = S = 1$, say, equations (316) reduce to

$$1 = (6 A_{11} + 60 B_{11}) \frac{1}{\sqrt{2}} \qquad \frac{1}{2} = (A_{11} + 2 B_{11}) \frac{1}{\sqrt{2}}$$

from which follow A_{11} and B_{11} numerically. The cases $(S', S) = (1,0)$ and $(0,1)$ then involve use of operator $2R_{To}$ as indicated before and the complete results are:

TABLE VIII.1

F^{20}

$(k_1 k_2) = (71)$

S'S	11	10	01
$A_{S'S}$	$7\sqrt{2}/12$	$-2/3$	$2\sqrt{3}/3$
$B_{S'S}$	$-\sqrt{2}/24$	$1/12$	$-\sqrt{3}/12$

Notice that the case $S = S' = 0$ does not appear as then matrix element (263) vanishes, since the factor $\langle S 1 M_S q'' | S' M_S' \rangle$ would be zero by violation of triangularity. This reflects the fact that spin-orbit interaction is zero in first-order for singlet states.

To compute reduced matrix elements $\langle (k_1 k_2) \omega' L' \| \mathcal{K} \| (k_1 k_2) \omega L \rangle$ recall the second expression for it in (258) and algebraic formula (262). Transformation coefficients $\langle q_1 q_2 (k_1 k_2 M_L = 1) \omega L \rangle$ form the matrix diagonalizing $\| \langle (k_1 k_2) q_1' q_2' M_L | \mathcal{L}^2 | (k_1 k_2) q_1 q_2 M_L \rangle \|$ for $M_L = 1$, *i.e.*, for $(k_1\ k_2) = (71)$ the former matrix is (I.9). Thus, one begins by constructing $\| \langle (k_1 k_2) q_1' q_2' M_L | \mathcal{K}_0^{(1)} | (k_1 k_2) q_1 q_2 M_L \rangle \|$

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for $(k_1\ k_2) = (71)$ and $M_L = 1$ by formula (262) which shows that our matrix will have many zeros. In accordance with (258), the result is then transformed by similarity with matrix (I.9) to give $\| \langle (71) L'1 | \mathcal{K}_0^{(1)} | (71)\, L1 \rangle \|$ which has non-zero elements only along the principal diagonal and the ones contiguous to this above and below. This results from $\vec{L}' = \vec{L} + \vec{1}$ which implies $L' = L, L \pm 1$. Each element is then divided by the appropriate $\langle L110 | L'1 \rangle$ coefficient and the final result is:

TABLE VIII.2

MATRIX $\| \langle (k_1k_2)\omega' L' \| \mathcal{K}^{(1)} \| (k_1k_2)\omega L \rangle \|$ for $(k_1k_2) = (71)$, $\omega = \omega'$.

$L' \backslash L$	1	2	3	4	5	6	7
1	$5\sqrt{2}$	$-6\sqrt{2/3}$					
2	$6\sqrt{2/5}$	$\frac{26\sqrt{6}}{3}$	$\frac{8}{3}\sqrt{\frac{231}{35}}$				
3		$-\frac{8}{3}\sqrt{\frac{33}{7}}$	$\frac{50}{\sqrt{3}}$	$-30\sqrt{\frac{2}{105}}$			
4			$10\sqrt{\frac{2}{15}}$	$\frac{96}{\sqrt{5}}$	$-\frac{\sqrt{55 \times 1307}}{36}$		
5				$\frac{\sqrt{5 \cdot 1307}}{12}$	$131\sqrt{\frac{2}{15}}$	$\frac{1}{75}\sqrt{\frac{25699 \cdot 13}{7}}$	
6					$\frac{-1}{75}\sqrt{\frac{25699 \cdot 11}{[illegible]7[illegible]}}$	$206\sqrt{\frac{2}{21}}$	$\frac{-1}{4}\sqrt{\frac{897 \cdot 35}{31}}$
7						$\frac{-1}{4}\sqrt{\frac{299 \cdot 91}{31}}$	$124\sqrt{\frac{2}{7}}$

In conclusion, formula (263) can now be used to construct the matrix of spin-orbit interaction $\mathcal{W}_{s.o.}$ for a given SU_3 representation, wherein $\omega = \omega'$ automatically, the results for F^{20} with $(k_1\ k_2) = (71)$ are thus:

$J=0$

LS	11
	$-3/4$

$J=1$

LS	10	11	21
	0	$-\sqrt{2}/4$	$-1/\sqrt{6}$
		$-3/8$	$\sqrt{3}/12$
			$-2/3$

$J=8$

LS	71
	1.5

$J=2$

LS	20	11	21	31
	0	$1/\sqrt{10}$	$1/3\sqrt{6}$	$\frac{2}{3}\sqrt{\frac{11}{15}}$
		$3/8$	$3/4\sqrt{15}$	0
			$-2/9$	$-\frac{1}{9}\sqrt{\frac{22}{5}}$
				$-17/18$

$J=3$

LS	30	21	31	41
	0	$-\frac{2}{9}\sqrt{\frac{33}{7}}$	$\frac{1}{6\sqrt{3}}$	$-\sqrt{\frac{5}{42}}$
		$4/9$	$-\frac{2}{9}\sqrt{\frac{11}{7}}$	0
			$-17/72$	$\frac{1}{4}\sqrt{\frac{5}{14}}$
				$-11/12$

$J=4$

LS	40	31	41	51
	0	$\frac{5}{6}\sqrt{\frac{2}{15}}$	$\frac{4\sqrt{5}}{15}$	$-\frac{\sqrt{55\cdot 1307}}{12\cdot 36}$
		$\frac{17}{24}$	$\frac{5}{12\sqrt{6}}$	0
			$-\frac{11}{60}$	$\frac{\sqrt{11\cdot 1307}}{12\cdot 36}$
				$-\frac{79}{60}$

$J=5$

LS	50	41	51	61
	0	$\frac{\sqrt{5\cdot 1307}}{144}$	$\frac{11}{12}\sqrt{\frac{2}{15}}$	$\frac{\sqrt{25699\cdot 13}}{12\cdot 75\sqrt{7}}$
		$11/15$	$\frac{\sqrt{7842}}{288}$	0
			$-79/360$	$-\frac{\sqrt{390\cdot 25699}}{6\cdot 24\cdot 75\sqrt{7}}$
				$-11/9$

$J=6$

LS	60	51	61	71
	0	$-\frac{\sqrt{25699\cdot 11}}{12\cdot 75\sqrt{7}}$	$\frac{19}{9}\sqrt{\frac{3}{14}}$	$\frac{\sqrt{35\cdot 897}}{48\sqrt{31}}$
		$79/72$	$-\frac{\sqrt{66\cdot 25699}}{144\cdot 75}$	0
			$-11/63$	$-\frac{\sqrt{26910}}{96\sqrt{31}}$
				$-12/7$

$J=7$

LS	70	61	71
	0	$-\frac{\sqrt{299\cdot 91}}{48\sqrt{31}}$	$\sqrt{2/7}$
		$\frac{22}{21}$	$-\frac{\sqrt{26\cdot 299}}{48\sqrt{31}}$
			$-3/14$

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IX. PREDICTED LOW-LYING LEVEL STRUCTURE OF F^{20}.

The interaction model to be used was discussed in Chapter IV with special emphasis on their group theoretical symmetries. It should be kept in mind that $\mathcal{P}$ refers to orbital pairing as explained in IV. 3., and $\mathcal{Q}^2$ to orbital quadrupole-quadrupole ---- extension of the former to include spin might provide a more realistic type of pairing but would introduce the symplectic group Sp_6 instead of R_6 and force us to use a jj coupling base. A linear combination of our $\mathcal{P}$ and $\mathcal{Q}^2$ may be used as a model for central two-body residual forces between extra-closed-shell nucleons and evidence in support of this was cited at the end of IV.3. Regarding exchange effects, an exchange operator $\mathcal{A}$ taking a simple form at long ranges as discussed in IV.2 is employed with the $\mathcal{Q}^2$ portion of the model which approximates long range. Exchange at short ranges, i.e., in association with $\mathcal{P}$, is neglected under the assumption of predominance of the Wigner component (into which the Majorana collapses at zero range) over the Bartlett component (into which the Heisenberg collapses) ---- for a Rosenfeld mixture the relative intensities are 80 to 20% respectively. To this is then added the single-body spin-orbit interaction discussed in IV.6. Our total interaction hamiltonian is stated by (264) or, parametrically more convenient, by (265) and is acting amongst the four extra-closed-shell nucleons only. The doubly magic ${}_8O_8^{16}$ core is presupposed spherical and inert though departure from this simple state of affairs is hopefully expected to be simulated at least approximately by the long-ranged $\mathcal{Q}^2$ interaction between extra-closed-shell particles.

The lowest levels of F^{20} will be given by $M_T = T = 1$. From (A.4) the most antisymmetric spin-isospin Young partition which by Wigner[47] is lowest in energy is $[\tilde{f}] = [211]$ to which, for $T = 1$, are associated $S=0$ and 1. The corresponding orbital symmetry is then $[f] = [31]$ whose lowest-energy SU_3 irreducible representation is by Table V.1.1 $(k_1\ k_2) = (71)$. This representation, to which our set of basis functions is restricted (Elliott hypothesis[12]), contains L-values equal to 1, 2, 3, 4, 5, 6 and 7. The resulting J-values are thus 0, 1^3, 2^4, 3^4, 4^4, 5^4, 6^4, 7^3 and 8 so that our largest matrix for $\mathcal{H}_{int}$ is 4 X 4. The base is designated by

$$|[f]\alpha'(k_1k_2)\omega L;\beta' S;JT\rangle \begin{cases} [f](k_1k_2) = [31](71) \\ \alpha' \text{ unneeded for } [f] = [31] \\ \omega' \quad " \quad " \quad (k_1k_2) = (71) \\ \beta' \quad " \quad " \quad [\tilde{f}] = [211] \\ M_J = J \text{ and } M_T = T \end{cases}$$

The (diagonal) matrices of $\mathcal{Q}^2\mathcal{J}$ in this base are given by eigenvalues (176) for $\mathcal{Q}^2$ and (186) for exchange $\mathcal{J}$, in accordance with (187). The matrices for $\mathcal{P}$, which are also diagonal in the orbital angular momentum quantum number L, are composed of pairing eigenvalues (314) in the manner of (315). Finally, the matrices of $\mathcal{W}_{S.O.}$ are given at the end of the previous chapter and their construction was discussed there.

The matrices were fed into an IBM 1620 computer for diagonalization by the Jacobi method, with parameters α and β of (265) varied.

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Scanty and rather ambiguous information on the spins of low-lying F^{20} is known to date. The β-decay[79] from its ground state to the second excited Ne^{20} level $J^{\pi} = 2^{+}$ suggests a $J^{\pi} = 2^{+}$ spin for the F^{20} ground state. Considerations[80] involving the Nilsson rotational model corroborate this assignment. The β-decay from the O^{20} ground state $J^{\pi}=0^{+}$ to the 1.064 Mev level of F^{20} suggests a $J^{\pi}=1^{+}$ spin[81]. Mazari and co-workers[82] have studied the F^{19} (d, pγ)F^{20} stripping reaction measuring gamma-ray energies for 15 excited states as well as angular distributions of proton groups. Distorted-wave theory fits to the data permitted generally reliable neutron-capture angular momenta ℓ_n assignments (see following Figure). The same reaction was studied by Chagnon[83] who measured (p,γ)-directional correlations[85] through four low-lying excited states. These results were interpreted by him via DWBA computations, thus providing him further restrictions on possible spin values for some of these levels. To the 0.66 level he assigns a $J^{\pi}=3^{+}$, but a 2^{+} is not inconsistent, and suggests that separate measurement of the mixing ratio for this transition would suffice to determine the spin uniquely. (Based on the stripping reduced width, Dazai[84] previously obtained a $J^{\pi}=3^{+}$ for this level.) For the 0.989 level the angular distribution correlation is isotropic and this a 0^{+} assignment would not be inconsistent. Chagnon finds a 1^{+} for the 1.064 level admissible, but was unable to discard 2^{+} or 3^{+}. The 1.312 level he finds decaying to the ground state with an intensity ratio of E2 / total >0.97 leaving little doubt of a $J^{\pi}=2^{+}$. These results are summarized on the extreme left of the Figure, all levels have T=1 and T=2 levels are expected to begin appearing in the neighborhood of as high as 6 Mev.

Variation of parameters α and β by tenths ($\alpha+\beta\leq 1$) in the computor diagonalization program results in a set of 66 predicted spectra. The non-degenerate ones fall into three groups of level-order J=2, 3, 1, 0, 2, 1,$\cdots$ and 2,1,3,0,2,1,$\cdots$ and 2,1,0,3,2,1,$\cdots$. The first and second J=2 predicted levels were associated with the empirical ground and 1.312 levels, respectively, and the second J=1 predicted level related to the 1.064 empirical level. The remaining predicted levels, i.e., J=3,0 and first 1 were associated with remaining empirical levels in order of appearance and least-mean-squares (involving above J-levels only) calculated for the three groups. The over-all intensity parameter Vo was found from the least-mean-squares procedure. These results are displayed in the Figure to the right of the empirical spectrum. Parameters x, y, and z given by $x = V_0\alpha$, $y=V_0\beta$ and $z = V_0(1-\alpha-\beta)$ denote the relative intensities, respectively, of pairing (short-range), quadrupole-quadrupole (long-range) and spin-orbit interactions.

In <u>conclusion</u> we may state the following. The spectrum in optimum agreement by least-mean-squares (a in Figure) predicts a J=3 for the first excited level, followed by a J=1 and J=0 levels. The first J=4 level is excessively depressed here. However, adding a slight amount more of pairing (moving from a to b) raises it considerably. Our most encouraging result is perhaps the reproduction of the <u>gross characteristics</u> of the empirical spectrum at low energies, i.e., at some point slightly away from (a) and toward (b) one obtains a "gap" above the ground state followed by a group of 5 levels, followed in turn by another gap above which is another group of levels. For all (α, β) parameter values allowed, the second J=2

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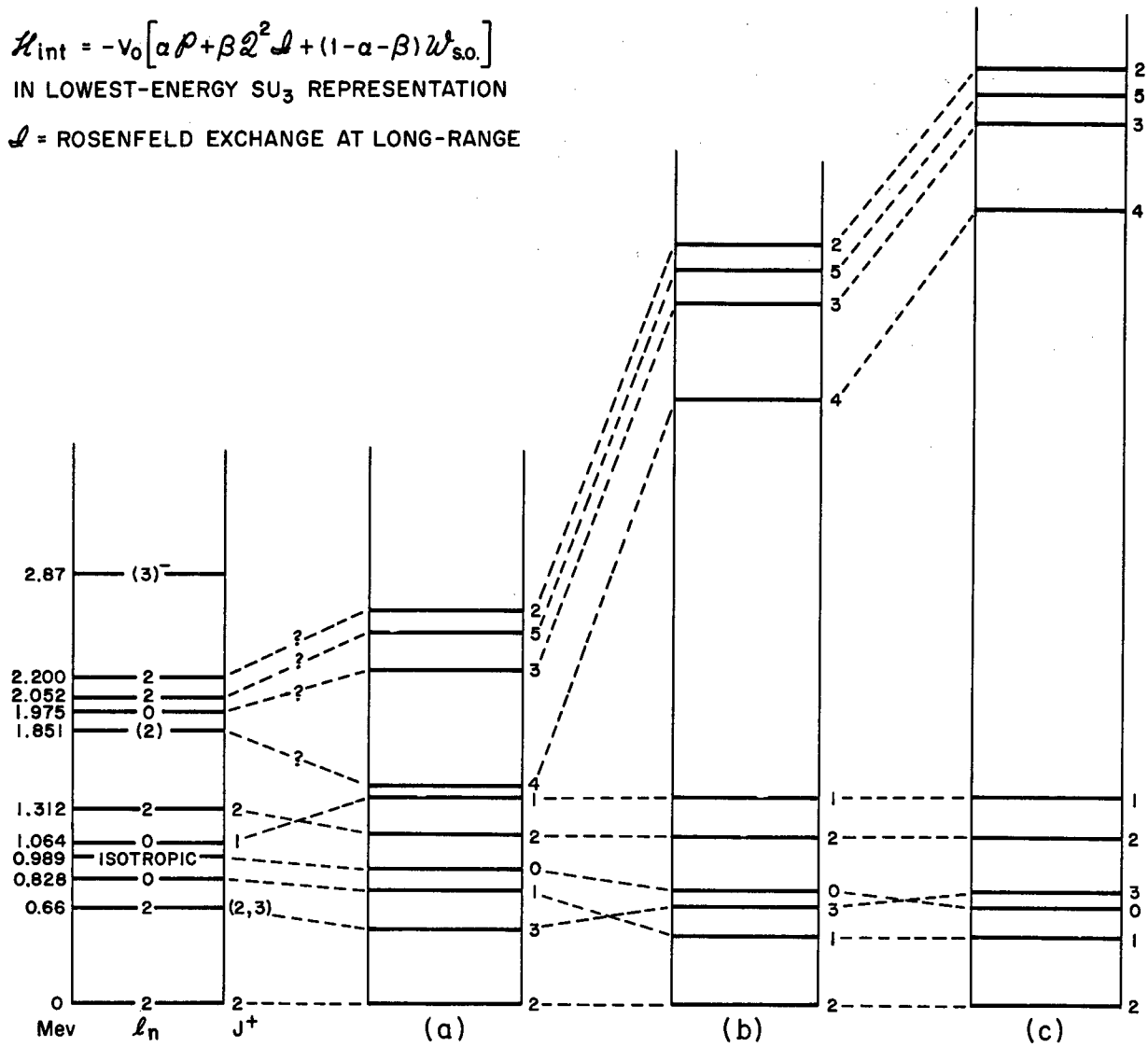
$$\mathcal{H}_{int} = -V_0\left[\alpha\mathcal{P} + \beta\mathcal{Q}^2\mathcal{J} + (1-\alpha-\beta)\mathcal{W}_{s.o.}\right]$$

IN LOWEST-ENERGY SU_3 REPRESENTATION

$\mathcal{J}$ = ROSENFELD EXCHANGE AT LONG-RANGE



EXPERIMENT $F^{19}(d,p\gamma)F^{20}$	THEORY (a)	THEORY (b)	THEORY (c)
	$(\alpha,\beta) = (0.1, 0.1)$	$(\alpha,\beta) = (0.5, 0.1)$	$(\alpha,\beta) = (0.6, 0.1)$
	$V_0 = 0.963$ Mev	$V_0 = 1.659$ Mev	$V_0 = 1.988$ Mev
	x = 0.0963	x = 0.8295	x = 1.1928
	y = 0.0963	y = 0.1659	y = 0.1988
	z = 0.7704	z = 0.6636	z = 0.5964
	m.s.d. = 0.236*	m.s.d. = 0.252*	m.s.d. = 0.267*

F^{20}

*EXCLUDING UPPER FOUR LEVELS

appeared below the second J=1 ---- in seeming discrepancy with experiment. A similar situation arose from this model for the second J=2 level of F^{18} for a wide range of the parameters[86] (with and without the restriction of the base to the lowest-energy SU_3 representation). More definite experimental information on spins is required, in particular for the first three excited states, to render a more complete test of these results. Explicit wave functions for the low-lying states of spectrum (a) will be available in a forthcoming paper.[86]

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X. SUMMARY AND CONCLUSIONS.

Moshinsky's approach by group theoretical techniques to the nuclear shell many-body problem is presented in general form and illustrated with a calculation of the low-lying energy levels of Fluorine-20. On comparison with experiment, encouraging results are found but more definitive conclusions regarding detailed agreement must await further empirical spin assignments for this nucleus.

Every low-energy nuclear level structure calculation is beset by two basic difficulties requiring basic approximations of one sort or another: uncertainties with regard to the nuclear force and those regarding the number of nucleon configurations to be considered.

The customary use of a reasonably shaped two-body residual interaction potential well is justified only "a posteriori" on phenomenological grounds but not on fundamental theory. Therefore, an equally phenomenological model hamiltonian consisting of a linear combination of orbital pairing, quadrupole-quadrupole (with exchange character) and single-body spin-orbit interactions was employed. The first and second interactions respectfully approximate the short- and long-ranged correlations of the central two-nucleon residual interaction. The advantages of this model lie in the fact that its various portions possess group symmetries of considerable convenience in the calculation of matrices.

The enormous number of totally antisymmetric states arising from all possible particle configurations was limited to a smaller, more feasible number by making the following restrictive assumptions. (1) Only configurations arising from a single mayor

oscillator shell (the 2s-1d shell) are considered so that only positive parity states will result. (2) Assuming with Wigner that the residual two-nucleon forces inside the nucleus are attractive and to a large extent independent of the nucleon spins and charges, one may further restrict the number of states remaining in (1) to those corresponding to the most symmetric partition under permutation of the space variables, compatible with the total isospin T of the lowest energy for the given nucleus. (3) Accepting on a tentative basis the proposal of Elliott's that the lowest-energy SU_3 (group of unitary unimodular transformations in 3 dimensions) representation contained in the partition chosen by (2) will to a reasonable degree determine the low-lying states for nuclei in the 2s-1d shell, one further delimits the number of states to be used in calculating the energy matrix. Whereas the first two restrictive assumptions are usual in shell model calculations, the third is more group theoretical in character and is based on Elliott's work showing that SU_3 provides a link between the shell and collective models. For the simpler nuclear p-shell (lying immediately below our 2s-1d shell) this group theoretic classification scheme by SU_3 collapses into the ordinary LS coupling scheme. There is moreover no conclusive evidence that SU_3 cannot serve usefully in dealing with more complex nuclear shells beyond the 2s-1d shell.

The language adopted for operators as well as state-vectors is the second-quantization formalism involving creation and annihilation operators for fermions. Thus, a given problem becomes more transparent to possible group symmetries that may prevail in a given operator or which are to be "built-into" a given state-vector. Consequently, the computation of matrix elements is reduced to simple commutator algebra.

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The construction of N-particle state-vectors possessing the specific quantum numbers demanded by a given problem was thus approached group theoretically. These techniques rest on the studies of S. Lie and E. Cartan in the theory of continuous groups of transformations and are essentially generalizations of angular momentum theory to symmetries higher than rotational. They are discussed in as strict analogy as possible to the better-known concepts of angular momentum and its deduced consequences.

For nuclei with a number of extra-closed-shell nucleons $\gtrsim 4$ the present methods are felt to be an improvement over conventional fractional parentage methods which then become particularly cumbersome. The results obtained thus far in this work with these methods are considered to justify further work along these lines for a larger number of nuclei. This program is being pursued at the University of Mexico where an exhaustive study of 2s-1d shell nuclei, their energies, moments, transition rates and other low-energy properties is in progress.

APPENDIX A

SUPERMULTIPLETS OF A GIVEN SYMMETRY PATTERN IN THE 2s-1d SHELL.

Derivation of the irreducible representation labels of R_3 contained in a given one of U_r has been discussed, among others, by Jahn[41] who considers N particles in the p shell (r=3) and in the d shell (r=5). In the 2s -1d shell r=6. A single particle within this shell has only one possible Young partition, namely, $[1] \equiv \square$ and since s means l=0 and d means l=2, one has

$$\mathcal{U}_6 > R_3$$

$$\underline{N=1}: \quad \square > L=0,2$$

For two particles one can form the <u>outer</u> product of two one-particle $\mathcal{U}_6$ representations and according to Littlewood's rules[72,34] get

$$\square \otimes \square = \square\square + \begin{smallmatrix}\square\\ \square\end{smallmatrix}$$

i.e., the symmetric [2] and the antisymmetric [11] irreducible representations of $\mathcal{U}_6$. Defining $\Delta^{s_1}_{\mu_1} \equiv b^+_{\mu_1 s_1}$ and

$$\Delta^{s_1 s_2}_{\mu_1 \mu_2} \equiv b^+_{\mu_1 s_1} b^+_{\mu_2 s_2} - b^+_{\mu_1 s_2} b^+_{\mu_2 s_1}$$

which is obviously <u>symmetric</u> under interchange of <u>orbital</u> coordinates (the indices μ); two particles in the 2s-1d shell give rise the following orbital configurations with associated angular momenta and

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wave functions in second-quantization form:

$(s)^2$	$(s)^1(d)^1$	$(d)^2$
$L=0$	$L=2^2$	$L=0,1,2,3,4$
$\Delta^{s_1 s_2}_{00,00}$	$\Delta^{s_1 s_2}_{00,2m}$; $\Delta^{s_1}_{00}\Delta^{s_2}_{2m}$	$\sum_{m_1 m_2}\langle 22 m_1 m_2 \mid L M_L\rangle \Delta^{s_1 s_2}_{2m_1,2m_2}$ $=(-)^L\sum_{m_1 m_2}\langle 22 m_1 m_2 \mid L M_L\rangle \Delta^{s_1 s_2}_{2m_1,2m_2}$

(A.1)

where $\mu_i \equiv \nu_i \ell_i m_i$ and as ν_i refers always to the same shell, it is suppressed.

In configuration $(d)^2$, the function is symmetric for L=0,2,4 and antisymmetric for L=1,3. Hence, the total L-structures of the symmetric ($\square\square$) and antisymmetric ($\begin{smallmatrix}\square\\\square\end{smallmatrix}$) partitions are;

$$\square\square > L = 0^2, 2^2, 4$$

$$\begin{smallmatrix}\square\\\square\end{smallmatrix} > L = 1,2,3 \qquad (A.2)$$

Now, the 3-particle antisymmetric partition is $[111] \equiv \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$ with wave function $\Delta^{s_1 s_2 s_3}_{\mu_1\mu_2\mu_3}$, where $\mu_i = \ell_i m_i$ (i=1,2,3) and m_i = 2,1,0,0,½,-2. since ℓ_i =0,2. The total (positive) M values can be formed in the following ways:

$$\left.\begin{matrix}2+1+0\\2+1+0'\end{matrix}\right\}=3 \qquad \left.\begin{matrix}2+1-1\\2+0+0'\end{matrix}\right\}=2 \qquad \left.\begin{matrix}2+1-2\\2+0-1\\2+0'-1\\1+0+0'\end{matrix}\right\}=1 \qquad \left.\begin{matrix}2+0-2\\2+0'-2\\1+0-1\\1+0'-1\end{matrix}\right\}=0$$

Hence,

$$M = \begin{matrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ & 2 & 2 & \\ & 3 & 3 & \end{matrix}$$

so that $L=1^2, 3^2$ and thus, in the 2s-1d shell

$$\begin{smallmatrix}\square\\\square\end{smallmatrix} > L = 1^2, 3^2.$$

Knowing the L-structure of $\mathcal{U}_6$ partitions $[1]$, $[2]$, $[11]$ and $[111]$, one may proceed to deduce the rest by a <u>chain calculation</u>:

$$\begin{smallmatrix}\square\\\square\end{smallmatrix} \otimes \square = \begin{smallmatrix}\square\\\square\end{smallmatrix} + \begin{smallmatrix}\square\square\\\square\end{smallmatrix}$$

will have an L-structure given by the vectorial sum of L's

$$1,2,3 + 0,2 = 0, 1^4, 2^4, 3^4, 4^2, 5$$

from which, if we subtract the $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ L-structure one obtains

$$\begin{smallmatrix}\square\square\\\square\end{smallmatrix} > L = 0, 1^2, 2^4, 3^2, 4^2, 5.$$

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Continuing,

$$\square\square \otimes \square = \begin{smallmatrix}\square\square\\ \square\end{smallmatrix} + \square\square\square$$

has L-structure

$$0^2, 2^2, 4 + 0, 2 = 0^4, 1^2, 2^7, 3^3, 4^4, 5, 6$$

which leaves, after subtraction of the $\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}$ L-structure,

$$\square\square\square \supset L = 0^3, 2^3, 3, 4^2, 6.$$

The partition [1111] has the same L-structure as [11], for $\mathcal{U}_6$. Proceeding with the chain calculation sketched here one further deduces

[f]	L- structure
[211]	$1^5, 2^3, 3^5, 4^2, 5^2$
[22]	$0^3, 1, 2^5, 3^2, 4^4, 5, 6$
[31]	$0^2, 1^4, 2^7, 3^6, 4^5, 5^3, 6^2, 7$
[4]	$0^4, 2^5, 3, 4^4, 5, 6^2, 8$
⋮	

(A.3)

To find the (S,T)-structure (or multiplet structure) of a given Young partition [f] one must effect the reduction

$$U_4 \supset U_2 \times U_2$$

the irreducible representations of U_4 being $[\tilde{f}]$. A chain calculation similar to the previous reduction $\mathcal{U}_6 \supset R_3$ is again useful; one starts with the following primitive cases:

The $[1111]$ representation of U_4 is equivalent to $[0]$ which contains (S,T) = (0,0). Representation $[1]$ of U_4 , explicitly reduced into those of $U_2 \times U_2$ is

$$\square \supset \square \times \square$$

the $\times$ symbol standing for the <u>inner</u> product. Thus

$$\square \text{ of } U_4 \supset (S,T) = \left(\tfrac{1}{2}, \tfrac{1}{2}\right).$$

Representations $[2]$ and $[11]$ of U_4 are formed by

$$\begin{array}{ll} \square\square \supset \square\square \times \square\square & \begin{smallmatrix}\square\\\square\end{smallmatrix} \supset \begin{smallmatrix}\square\square \\ \square\end{smallmatrix} \times \begin{smallmatrix}\square \\ \square\end{smallmatrix} \\ \quad\quad \supset \begin{smallmatrix}\square \\ \square\end{smallmatrix} \times \begin{smallmatrix}\square \\ \square\end{smallmatrix} & \quad\quad\quad\quad \begin{smallmatrix}\square \\ \square\end{smallmatrix} \times \square\square \end{array}$$

that is, the symmetric spin-isospin two-particle function is either spin symmetric times isospin antisymmetric or spin antisymmetric times isospin antisymmetric; the antisymmetric spin-isospin function is either spin symmetric times isospin antisymmetric or spin anti-symmetric times isospin symmetric. The corresponding (S,T) structures are thus

$$\square\square \text{ of } U_4 \supset (S,T) = (1,1), (0,0)$$

$$\begin{smallmatrix}\square \\ \square\end{smallmatrix} \text{ of } U_4 \supset (S,T) = (1,0), (0,1).$$

Tabulating, one has

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U_4	$U_2 \times U_2$	(S, T)
$[1111] = [0]$	$[0]\times[0]$	$(0,0)$
$[1]$	$\square \times \square$	$(1/2, 1/2)$
$[2]$	$\square\square \times \square\square$; $\begin{smallmatrix}\square\\\square\end{smallmatrix} \times \begin{smallmatrix}\square\\\square\end{smallmatrix}$	$(1,1)$ $(0,0)$
$[11]$	$\square\square \times \begin{smallmatrix}\square\\\square\end{smallmatrix}$; $\begin{smallmatrix}\square\\\square\end{smallmatrix} \times \square\square$	$(1,0)$ $(0,1)$

Then, in U_4 one uses Littlewood's rules for external products to form

$$\square\square \otimes \square = \begin{smallmatrix}\square\square\\\square\end{smallmatrix} + \square\square\square$$

whose (S,T)-structure is found by vector addition of S and T separately,

$$\begin{matrix}(1,1)\\(0,0)\end{matrix} + \left(\tfrac{1}{2},\tfrac{1}{2}\right) = \left(\tfrac{1}{2},\tfrac{1}{2}\right)^2 \left(\tfrac{1}{2},\tfrac{3}{2}\right)\left(\tfrac{3}{2},\tfrac{1}{2}\right)\left(\tfrac{3}{2},\tfrac{3}{2}\right).$$

Again, in U_4

$$\begin{smallmatrix}\square\\\square\end{smallmatrix} \otimes \square = \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix} + \begin{smallmatrix}\square\square\\\square\end{smallmatrix} = \square + \begin{smallmatrix}\square\square\\\square\end{smallmatrix},$$

since $[111] \equiv [1]$, has (S,T)-structure

$$\begin{matrix}(1,0)\\(0,1)\end{matrix} + \left(\tfrac{1}{2},\tfrac{1}{2}\right) = \left(\tfrac{1}{2},\tfrac{1}{2}\right)^2 \left(\tfrac{1}{2},\tfrac{3}{2}\right)\left(\tfrac{3}{2},\tfrac{1}{2}\right)$$

which upon subtracting that of $[1]$ leaves

$$\begin{smallmatrix}\square\square\\\square\end{smallmatrix} \text{ of } U_4 \supset (S,T) = \left(\tfrac{1}{2},\tfrac{1}{2}\right)\left(\tfrac{1}{2},\tfrac{3}{2}\right)\left(\tfrac{3}{2},\tfrac{1}{2}\right).$$

Subtracting this from the multiplet structure of $[21] + [3]$ above one gets

$$\square\square\square \text{ of } U_4 \supset (S,T) = \left(\tfrac{1}{2},\tfrac{1}{2}\right)\left(\tfrac{3}{2},\tfrac{3}{2}\right).$$

The calculation can be continued nad extensive tables prepared ----- Jahn and Hammermesh (refs. 34 and 41) have rather complete ones. We extract from those sources the table for N=4 particles as it will concern us directly.

N	U_4: $[\tilde{f}]$	(S,T) - structure
4	[1111]	(0,0)
	[211]	(1,0) (0,1) (1,1)
	[22]	(0,0) (0,2) (2,0) (1,1)
	[31]	(0,1) (1,0) (1,1) (1,2) (2,1)
	[4]	(0,0) (1,1) (2,2)

(A.4)

If multiple (S,T) values occur for a given $[\tilde{f}]$ -----does <u>not</u> happen for $N \leqslant 5$ ----- the additional label β in chain (99) is required.[46)]

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APPENDIX B.

UNIQUENESS OF TOTALLY ANTISYMMETRIC MANY-FERMION POLYNOMIAL OF MAXIMUM WEIGHT.

A general many-particle eigenpolynomial is given by (21), namely

$$P = \sum_{\mu_1 s_1, \mu_2 s_2, \cdots, \mu_N s_N} B_{\mu_1 s_1, \mu_2 s_2, \cdots, \mu_N s_N} \; b^{+}_{\mu_1 s_1} b^{+}_{\mu_2 s_2} \cdots b^{+}_{\mu_N s_N} .$$

For a given set of values $s_1 s_2 \cdots s_N$ the corresponding term can be chosen to transform irreducibly according to the permutation group Π_N as $[h_1 h_2 \cdots h_p]$. Likewise, for a given set $\mu_1 \mu_2 \cdots \mu_N$ one can choose the corresponding result to transform irreducibly as $[v_1 v_2 v_3 v_4]$ of the same Π_N group. Due to Weyl's theorem,[43] a function transforming irreducibly under Π_N as $[h_1 h_2 \cdots h_t]$ will also transform so under $\mathcal{U}_t$ (where t is the dimensionality of the coordinate space), and the irreducible representations of the latter are characterized by the same label $[h_1 h_2 \cdots h_t]$. Therefore, the (maximum weight) representations $[h_1 h_2 \cdots h_r]$ of $\mathcal{U}_r$ and $\{v_1 v_2 v_3 v_4\}$ of U_4, specified by (73) and (74), are <u>irreducible</u>.

The Young pattern of Π_N for permutations in the total space of $4r$ dimensions is $[1^N 0^{4r-N}]$; this is also the irreducible representation of $\mathbb{U}_{4r}$. And since

$$\mathbb{U}_{4r} \supset \mathcal{U}_r \times U_4$$

then $$[1^N\, 0^{4r-N}] \supset [h_1 h_2 \cdots h_r] \times \{v_1 v_2 v_3 v_4\}$$

where, as in example (80), $[h_1 h_2 \cdots h_r] = \overline{[v_1 v_2 v_3 v_4]}$

<u>Simple examples</u> follow:

N= <u>1 partice</u> $[1] \supset [1] \times [1]$ (a)

N= <u>2 particles</u> $[11] \supset [2] \times [11]$ (b)

N= <u>3 particles</u> $[111] \supset [3] \times [111]$ (c)

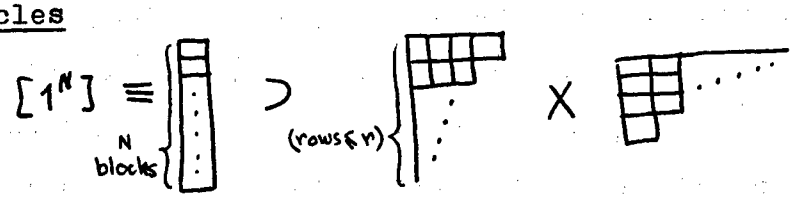
$\supset [21] \times [21]$ (d)

N= <u>4 particles</u> $[1111] \supset [4] \times [1111]$ (e)

$\supset [31] \times [211]$ (f)

$\supset [22] \times [22]$ (g)

<u>N particles</u>



(h)

In each case, the orbital and spin-isospin permutation symmetries must be such that a totally antisymmetric orbit-spin-isopin permutation pattern results. In (a), (b), (c) and (e) we have examples of "pure symmetries", i.e., symmetric x antisymmetric = antisymmetric. In all other examples "mixed symmetries" such as [21] are involved,

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but the net result is allowed by the Pauli principle.

We have shown in (79) how to construct the function which is of maximum weight in the representation of the subgroup $\mathcal{U}_r \times \mathcal{U}_4$ contained in the $[1^N 0^{4r-N}]$ representation of $\mathbb{U}_{4r}$. This polynomial is <u>unique</u> as otherwise it would imply that with a given representation $[h_1 h_2 \cdots h_r]$ we can associate besides the conjugate $\{v_1 v_2 v_3 v_4\}$ <u>other</u> representations of $\mathcal{U}_4$ and still get the totally antisymmetric representation of $\mathbb{U}_{4r}$. However, it is shown by Bayman[73] that this is not possible for representations of the permutation group Π_N which are closely connected with those of the unitary groups.

<u>Conclusion</u>. Polynomial $\mathbb{P}$ as a solution of eqns. (73) and (74) is <u>unique</u>.

APPENDIX C.

ANGULAR MOMENTUM COMPONENTS AS GENERATORS OF THE GROUP R_3.

A real infinitesimal orthogonal transformation of the cartesian components is simply

$$x_i' = \sum_j R_{ij} x_j \quad (i,j=1,2,3) \; ; \; R = I + \epsilon A \; ; \; R\tilde{R} = I \tag{C.1}$$

ϵ being an infinitesimal real number. Now

$$R\tilde{R} = (I + \epsilon A)(I + \epsilon \tilde{A}) = I + \epsilon(A + \tilde{A}) = I$$

$$\therefore \; \tilde{A} = -A \tag{C.2}$$

shows that matrix A must be antisymmetric so that one can write

$$\epsilon A_{ij} = \sum_{k=1}^{3} \varepsilon_{ijk} \, \epsilon_k \tag{C.3}$$

using the completely antisymmetric tensor ε_{ijk} (see page 64). Now, a function F (x_i) $\equiv$ F (x_1, x_2, x_3) will transform to

$$F(x_i') = F\left(\sum_j R_{ij} x_j\right)$$

$$F(x_i') = F\left(x_i + \sum_{jk} \varepsilon_{ijk} \epsilon_k x_j\right)$$

$$= F(x_i) + \sum_{ljk} \varepsilon_{ljk} \epsilon_k x_j \frac{\partial}{\partial x_l} F(x_i) + \cdots \tag{C.4}$$

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using the Taylor expansion in the last step. But the three cartesian angular momentum operators given explicitly in (39) can be written collectively as

$$L_k \equiv \frac{1}{i} \sum_{\ell j} \varepsilon_{kj\ell}\, x_j \frac{\partial}{\partial x_\ell} \tag{C.5}$$

so that (4) now reads

$$F(x_i') = F(x_i) - i \sum_{k=1}^{3} \epsilon_k L_k\, F(x_i) \tag{C.6}$$

and an infinitesimal increment in F(x_i) due to an arbitrary rotation of the axes is

$$\delta F(x_i) \equiv F(x_i') - F(x_i) = -i \sum_{k=1}^{3} \epsilon_k L_k\, F(x_i). \tag{C.7}$$

<u>Conclusion:</u> The angular momentum component operators L_k (k=1,2,3) which obey the cyclic commutation relations $[L_1, L_2] = i L_3$ can be considered as the <u>generators of an infinitesimal transformation</u> of the three-dimensional rotation group R_3. The relations $[L_1, L_2] = i L_3$ form a Lie algebra of the Lie group R_3.

APPENDIX D.

UNITARY GROUP GENERATORS.

In similar fashion, consider an infinitesimal unitary transformation acting in a t-dimensional space vector ($x_1, x_2, \cdots, x_t$). The vector-components will transform as

$$x_\ell' = \sum_j U_{\ell j}\, x_j \qquad (\ell, j = 1, 2, \ldots, t) \tag{D.1}$$

where by definition

$$\sum_j U_{\ell j} U_{kj}^* = \delta_{\ell k} \qquad \text{or} \qquad U U^\dagger = I \tag{D.2}$$

and

$$U = I + i\epsilon S \tag{D.3}$$

ϵ being a real, infinitesimal quantity and $S^\dagger = S$ (hermitean) since

$$U U^\dagger = (I + i\epsilon S)(I - i\epsilon S^\dagger) \simeq I + i\epsilon (S - S^\dagger) = I. \tag{D.4}$$

An arbitrary function in this space $F(x_i) \equiv F(x_1, x_2, \cdots, x_t)$ will transform as

$$F(x_\ell') = F\left(x_\ell + i\epsilon \sum_j S_{\ell j}\, x_j\right) \tag{D.5}$$

and using Taylor's theorem:

$$F(x_\ell') = F(x_\ell) + i\epsilon \sum_{ij} \left(S_{ij}\, x_j \frac{\partial}{\partial x_i} F(x_\ell)\right) + \cdots . \tag{D.6}$$

To first order in ϵ, the change in the function

$$\delta F(x_\ell) \equiv F(x_\ell') - F(x_\ell) = i\epsilon \sum_{ij} \left(S_{ij}\, x_j \frac{\partial}{\partial x_i}\right) F(x_\ell) \tag{D.7}$$

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is accomplished by t^2 linearly independent operators

$$i\epsilon \sum_{ij} S_{ij}\, x_j \frac{\partial}{\partial x_i} \tag{D.8}$$

where since S is an arbitrary hermitean matrix the t^2 operators

$$x_j \frac{\partial}{\partial x_i} \equiv C_{ji} \tag{D.9}$$

could be taken as the <u>generators for infinitesimal unitary transformations</u> in the t-dimensional space. Their commutation relations, following from (D.9), are immediately seen to constitute the commutator algebra

$$[C_{ij}, C_{i'j'}] = C_{ij'}\,\delta_{i'j} - C_{i'j}\,\delta_{ij'} \tag{D.10}$$

identical in structure to (32) since here the metric $g_{ij} = \delta_{ij}$ so that $C_{ij} = C_i^{\,j}$.

APPENDIX E.

ROTATIONAL GROUP GENERATORS.

In the same t-dimensional space spanned by the vector $(x_1, x_2, \cdots, x_t)$ consider the real orthogonal transformations

$$x_i' = \sum_j R_{ij} x_j \qquad (i, j = 1, 2, \ldots, t)$$

where by definition

$$\sum_j R_{ij} R_{i'j} = \delta_{ii'} \qquad \text{or} \qquad R\tilde{R} = I$$

with the added condition that $\det |R_{ij}| = +1$ (rotations only, excluding reflections); moreover

$$R = I + \epsilon A$$

ϵ an infinitesimal real quantity and $\tilde{A} = -A$ since

$$R\tilde{R} = (I + \epsilon A)(I + \epsilon \tilde{A}) = I + \epsilon(A + \tilde{A}) = I.$$

An arbitrary function in this space $G(x_\ell) = G(x_1, x_2, \cdots, x_t)$ will then transform like

$$G(x_\ell') = G\left(x_\ell + \epsilon \sum_j A_{\ell j} x_j\right)$$

which on Taylor expansion becomes

$$G(x_\ell') = G(x_\ell) + \epsilon \sum_{ij} A_{\ell j} x_j \frac{\partial}{\partial x_i} G(x_\ell) + \cdots$$

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$$G(x_\ell') = G(x_\ell) + \epsilon \sum_{ij} A_{\ell j} C_{ji} G(x_\ell) + \cdots$$

C_{ji} being the operators (D.9) of Appendix D. Then

$$G(x_\ell') = G(x_\ell) + 2\epsilon \sum_{i<j} A_{\ell j} \Lambda_{ji} G(x_\ell) + \cdots$$

where we have called the operators

$$\Lambda_{ij} \equiv \frac{1}{2}(C_{ij} - C_{ji})$$

the generators of infinitesimal rotations in t-dimensions, of which there are $\frac{1}{2}t(t-1)$ independent ones as the real antisymmetric matrix A has, for $i<j$, $\frac{1}{2}t(t-1)$ independent elements, In view of relations (10) of Appendix D one derives the relations

$$[\Lambda_{ij}, \Lambda_{i'j'}] = \frac{1}{2}\left(\Lambda_{ij'}\delta_{i'j} + \Lambda_{ji'}\delta_{ij'} + \Lambda_{j'j}\delta_{ii'} + \Lambda_{i'i}\delta_{jj'}\right)$$

which constitute the Lie algebra of the generators of the group R_t. The space metric here is again simply $g_{ij} = \delta_{ij}$.

APPENDIX F.

GROUP INVARIANCE OF AN OPERATOR.

Again, consider the t-dimensional linear vector space spanned by the vector $\vec{x} \equiv (x_i) \equiv (x_1, x_2, \cdots, x_t)$ which transforms unitarily as:

$$x_i = \sum_j U_{ji}^* x_j'. \tag{F.1}$$

An arbitrary function $F(x_i) \equiv F(x_1, x_2, \ldots, x_t)$ of the vector components will transform, let us say, as

$$\mathcal{O}_U F(x_i) = F(x_i') , \qquad \mathcal{O}_U^+ \mathcal{O}_U = I. \tag{F.2}$$

Consider an operator H for which $F(x_i)$ is an eigenfunction, that is,

$$H F(x_i) = E F(x_i). \tag{F.3}$$

Then, providing that

$$[\mathcal{O}_U, H] = 0 \qquad \text{or} \qquad \mathcal{O}_U^+ H \mathcal{O}_U = H \tag{F.4}$$

one has that the transformed function $F(x_i')$ is <u>also</u> an eigenfunction of H, since

$$H F(x_i) = \mathcal{O}_U^{-1} H \mathcal{O}_U F(x_i) = E F(x_i)$$

$$\mathcal{O}_U \mathcal{O}_U^{-1} H \mathcal{O}_U F(x_i) = H \mathcal{O}_U F(x_i) = E \mathcal{O}_U F(x_i)$$

$$\therefore \; H F(x_i') = E F(x_i'), \tag{F.5}$$

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from which follows the <u>Theorem</u>: the group of t-dimensional unitary transformations U_t is the <u>symmetry group of the operator</u> $\underline{H}$ if H commutes with operation $\mathcal{O}_U$. (Common examples from both atomic and nuclear physics for $\mathcal{O}_U$ are the permutation and rotation operations----in which cases the hamiltonian H is an <u>invariant</u>, respectively, of the permutation group Π_N and the rotation group R_3. Alternatively, these are the symmetry groups of H.)

Now, for a continuous group it is sufficient for it to be a symmetry group of H that $\mathcal{O}_U$ is an infinitesimal transformation of that group namely, as in Appendix D,

$$\mathcal{O}_U \equiv \left(I + i\epsilon \sum_{ij} S_{ij}\, \mathcal{b}_{ji}\right) \qquad (F.6)$$

recalling that S_{ij} is an arbitrary hermitean matrix and $\mathcal{b}_{ji}$ the t^2 generators of infinitesimal transformations of the group U_t. Hence if (4) is fullfilled it follows that

$$[H, \mathcal{b}_{ji}] = 0.$$

<u>Conclusion</u>: A necessary and sufficient condition for the invariance of an operator H under a group U_t is that H commute with the group generators.

Consider two different, complete, linearly independent sets of functions of x_i $(i=1,2,\ldots,t)$,

$$\psi_m^{(j)} \equiv \{\psi_1^{(j)}, \psi_2^{(j)}, \ldots, \psi_{n_j}^{(j)}\} \qquad (F.8)$$

$$\phi_m^{(j')} \equiv \{\phi_1^{(j')}, \phi_2^{(j')}, \ldots, \phi_{n_{j'}}^{(j')}\} \qquad (F.9)$$

which are basis for two distinct irreducible representations labelled by j and j', with rows by m and m', respectively, of a group U_t. Since $\mathcal{O}_U$ is unitary, the scalar product

$$\left(\psi_m^{(j)}, \phi_{m'}^{(j')}\right) = \left(\mathcal{O}_U \psi_m^{(j)}, \mathcal{O}_U \phi_{m'}^{(j')}\right). \tag{F.10}$$

But, by definition of an irreducible basis,[74)]

$$\mathcal{O}_U \psi_m^{(j)} = \sum_k \psi_k^{(j)} \mathcal{D}_{km}^{(j)}(u) \tag{F.11}$$

$$\mathcal{O}_U \phi_{m'}^{(j')} = \sum_{k'} \phi_{k'}^{(j')} \mathcal{D}_{k'm'}^{(j')}(u) \tag{F.12}$$

the $\mathcal{D}_{km}^{(j)}(u)$ being the irreducible representation j with rows and columns k,m and u refers to the elements of the group U_t. Likewise for $\mathcal{D}_{k'm'}^{(j')}(u)$. These obey the well-known orthonormality relation[74)]

$$\sum_u \mathcal{D}_{km}^{(j)*}(u)\, \mathcal{D}_{k'm'}^{(j')}(u) = \frac{h}{n_j} \delta_{jj'} \delta_{kk'} \delta_{mm'} \tag{F.13}$$

h being the number of elements in the group (which is irrelevant for our present purposes) and from (8) n_j is the dimensionality of $\mathcal{D}_{km}^{(j)}(u)$. Thus (F.10) becomes

$$\left(\psi_m^{(j)}, \phi_{m'}^{(j')}\right) = \sum_{kk'} \mathcal{D}_{km}^{(j)*}(u)\, \mathcal{D}_{k'm'}^{(j')}(u) \left(\psi_k^{(j)}, \phi_{k'}^{(j')}\right) \tag{F.14}$$

which upon summation over all the elements, h appearing as a factor on both sides, one obtains

$$\left(\psi_m^{(j)}, \phi_{m'}^{(j')}\right) = \frac{1}{n_j} \delta_{jj'} \delta_{mm'} \sum_k \left(\psi_k^{(j)}, \phi_k^{(j')}\right). \tag{F.15}$$

Remembering that, by stipulation, the operator H commutes with $\mathcal{O}_U$ we have

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$$H\,\mathcal{O}_U\,\psi_m^{(j)} = \mathcal{O}_U\left(H\,\psi_m^{(j)}\right) = \sum_k \left(H\,\psi_k^{(j)}\right) \mathcal{D}_{km}^{(j)}(u) \qquad (F.16)$$

wherefrom clearly $H\psi_m^{(j)}$ is an irreducible basis if $\psi_m^{(j)}$ is such a one, and thus the result (15) can be applied to give

$$\left(\phi_{m'}^{(j')}, H\,\psi_m^{(j)}\right) = \frac{1}{n_j}\,\delta_{jj'}\,\delta_{mm'} \sum_k \left(\phi_m^{(j')}, H\,\psi_m^{(j)}\right) \qquad (F.17)$$

which is one of the most useful results of the theory of group representations, it states:

<u>Conclusion</u>: An operator H having U_t as a symmetry group will have non-zero matrix elements <u>only</u> between states of a given irreducible representation j of U_t and furthermore these elements are <u>independent</u> of the row m of that representation.

APPENDIX G.

THE COEFFICIENTS $\langle n_1' n_{\bar{1}}' n_0' | C_q^{q'} | n_1 n_{\bar{1}} n_0 \rangle$.

The coefficients appearing in the U_3 group generator expansion (133) are easily found. The annihilation boson operator $a^{q'}$ becomes a differential operator (124). Then, in view of (119)

$$C_q^{q'} | n_1 n_{\bar{1}} n_0 \rangle = a_q^+ \frac{\partial}{\partial a_{q'}^+} \frac{(a_1^+)^{n_1} (a_{\bar{1}}^+)^{n_{\bar{1}}} (a_0^+)^{n_0}}{\sqrt{n_1!\ n_{\bar{1}}!\ n_0!}} | \bar{0} \rangle$$

$| \bar{0} \rangle$ referring to the ground state in our tri-dimensional harmonic oscillator. Differentiating and multiplying by a_q^+ one gets

$$C_q^{q'} | n_1 n_{\bar{1}} n_0 \rangle = n_{q'} \frac{(a_1^+)^{n_1 - \delta_{q'1} + \delta_{q1}} (a_{\bar{1}}^+)^{n_{\bar{1}} - \delta_{q'\bar{1}} + \delta_{q\bar{1}}} (a_0^+)^{n_0 - \delta_{q'0} + \delta_{q0}} | \bar{0} \rangle}{\sqrt{n_1!\ n_{\bar{1}}!\ n_0!}}$$

Thus

$$\langle n_1' n_{\bar{1}}' n_0' | C_q^{q'} | n_1 n_{\bar{1}} n_0 \rangle =$$

$$= \langle n_1 - \delta_{q'1} + \delta_{q1},\ n_{\bar{1}} - \delta_{q'\bar{1}} + \delta_{q\bar{1}},\ n_0 - \delta_{q'0} + \delta_{q0} | C_q^{q'} | n_1 n_{\bar{1}} n_0 \rangle$$

$$= n_{q'} \sqrt{\frac{(n_1 - \delta_{q'1} + \delta_{q1})!\ (n_{\bar{1}} - \delta_{q'\bar{1}} + \delta_{q\bar{1}})!\ (n_0 - \delta_{q'0} + \delta_{q0})!}{n_1!\ n_{\bar{1}}!\ n_0!}}$$

$$= \sqrt{n_{q'} (n_q - \delta_{qq'} + 1)} \ .$$

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APPENDIX H.

MATRIX ELEMENTS OF U_3 GENERATORS:

Calculation of the matrix elements of the generators of a unitary group $\mathcal{U}_r$ in a base characterized by the canonical chain $\mathcal{U}_r \supset \mathcal{U}_{r-1} \supset \cdots \supset \mathcal{U}_2 \supset \mathcal{U}_1$ has been discussed by Gel'fand & Zetlin,[42] Biedenharn[75] and Moshinsky.[45] We here simply list the results for the 9 U_3 generators $C_q^{q'}$ in the chain $U_3 \supset U_2 \supset U_1$, in a manner useful for calculations (cf. Section on Spin Orbit Force for notation). The labels (k_1, k_2) in the bras and kets, being redundant, are suppressed.

$$\langle q_1 q_2 M | C_1^1 | q_1 q_2 M \rangle = w_1 = \tfrac{1}{2}(M + q_1 + q_2)$$

$$\langle q_1 q_2 M | C_2^2 | q_1 q_2 M \rangle = w_2 = \tfrac{1}{2}(q_1 + q_2 - M) \qquad [q, q' = 1, \bar{1}, 0 \to 1, 2, 3]$$

$$\langle q_1 q_2 M | C_3^3 | q_1 q_2 M \rangle = w_3 = k_1 + k_2 - q_1 - q_2$$

$$\langle q_1' q_2' M' | C_1^2 | q_1 q_2 M \rangle = \tfrac{1}{2}\sqrt{(q_1 - q_2 + M + 2)(q_1 - q_2 - M)}\, \delta_{q_1' q_1} \delta_{q_2' q_2} \delta_{M', M+2}$$

$$\langle q_1' q_2' M' | C_2^1 | q_1 q_2 M \rangle = \tfrac{1}{2}\sqrt{(q_1 - q_2 + M)(q_1 - q_2 - M + 2)}\, \delta_{q_1' q_1} \delta_{q_2' q_2} \delta_{M', M-2}$$

$$\langle q_1' q_2' M' | C_1^3 | q_1 q_2 M \rangle =$$

$$\sqrt{\frac{(k_1-q_1)(q_1-k_2+1)(q_1+2)(q_1-q_2+M+2)}{2(q_1-q_2+1)(q_1-q_2+2)}}\,\delta_{q_1' q_1+1}\,\delta_{q_2' q_2}\,\delta_{M',M+1}$$

$$-\sqrt{\frac{(q_2+1)(k_2-q_2)(k_1-q_2+1)(q_1-q_2-M)}{2(q_1-q_2)(q_1-q_2+1)}}\,\delta_{q_1' q_1}\,\delta_{q_2' q_2+1}\,\delta_{M',M+1}$$

$$\langle q_1' q_2' M' | C_3^1 | q_1 q_2 M \rangle =$$

$$\sqrt{\frac{(k_1-q_1+1)(q_1-k_2)(q_1+1)(q_1-q_2+M)}{2(q_1-q_2)(q_1-q_2+1)}}\,\delta_{q_1' q_1-1}\,\delta_{q_2' q_2}\,\delta_{M',M-1}$$

$$-\sqrt{\frac{(q_2)(k_2-q_2+1)(k_1-q_2+2)(q_1-q_2-M+2)}{2(q_1-q_2+1)(q_1-q_2+2)}}\,\delta_{q_1' q_1}\,\delta_{q_2' q_2-1}\,\delta_{M',M-1}$$

$$\langle q_1' q_2' M' | C_2^3 | q_1 q_2 M \rangle =$$

$$\sqrt{\frac{(k_1-q_1)(q_1-k_2+1)(q_1+2)(q_1-q_2-M+2)}{2(q_1-q_2+2)(q_1-q_2+1)}}\,\delta_{q_1' q_1+1}\,\delta_{q_2' q_2}\,\delta_{M',M-1}$$

$$+\sqrt{\frac{(q_2+1)(k_2-q_2)(k_1-q_2+1)(q_1-q_2+M)}{2(q_1-q_2+1)(q_1-q_2)}}\,\delta_{q_1' q_1}\,\delta_{q_2' q_2+1}\,\delta_{M' M-1}$$

$$\langle q_1' q_2' M' | C_3^2 | q_1 q_2 M \rangle =$$

$$\sqrt{\frac{(k_1-q_1+1)(q_1-k_2)(q_1+1)(q_1-q_2-M)}{2(q_1-q_2+1)(q_1-q_2)}}\,\delta_{q_1' q_1-1}\,\delta_{q_2' q_2}\,\delta_{M',M+1}$$

$$+\sqrt{\frac{q_2(k_2-q_2+1)(k_1-q_2+2)(q_1-q_2+M+2)}{2(q_1-q_2+2)(q_1-q_2+1)}}\,\delta_{q_1' q_1}\,\delta_{q_2' q_2-1}\,\delta_{M',M+1}$$

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APPENDIX I.

TRANSFORMATION BETWEEN $U_3 \supset U_2 \supset U_1$ AND $U_3 \supset R_3 \supset R_2$ BASES.

Both the basis set $|(k_1 k_2) q_1 q_2 M_L\rangle$ transforming irreducibly under $U_3 \supset U_2 \supset U_1$ and the set $|(k_1 k_2) \omega L M_L\rangle$ irreducible under $U_3 \supset R_3 \supset R_2$ constitute complete sets of normalizable wave functions in second-quantization formulism. Thus one may expand

$$|(k_1 k_2) \omega L M_L\rangle = \sum_{q_1 q_2} \langle q_1 q_2 (k_1 k_2 M_L) \omega L\rangle |(k_1 k_2) q_1 q_2 M_L\rangle \qquad (I.1)$$

where $\langle q_1 q_2 (k_1 k_2 M_L) \omega L\rangle$ denote (orthogonal) transformation coefficients.[66] Calculation of matrix elements in the canonical chain $U_3 \supset U_2 \supset U_1$ is simpler than directly in $U_3 \supset R_3 \supset R_2$ as contruction of the base in the former scheme is more direct, as was seen in Chapter VI. However, the latter chain is the physically significant one as it provides the quantum number L. Coefficients (I.1) are thus needed.

We know that

$$\mathcal{L}^2 |(k_1 k_2) \omega L M_L\rangle = L(L+1) |(k_1 k_2) \omega L M_L\rangle \qquad (I.2)$$

but, in general,

$$\mathcal{L}^2 |(k_1 k_2) q_1 q_2 M_L\rangle = \sum_{q_1' q_2'} \langle (k_1 k_2) q_1' q_2' M_L | \mathcal{L}^2 | (k_1 k_2) q_1 q_2 M_L\rangle |(k_1 k_2) q_1' q_2' M_L\rangle \qquad (I.3)$$

Equations (1), (2) and (3) combined give us

$$\sum_{\substack{q_1 q_2 \\ q_1' q_2'}} \langle q_1 q_2 (k_1 k_2 M_L) \omega L \rangle \langle (k_1 k_2) q_1' q_2' M_L | \mathcal{L}^2 | (k_1 k_2) q_1 q_2 M_L \rangle | (k_1 k_2) q_1' q_2' M_L \rangle$$

$$= L(L+1) \sum_{q_1 q_2} \langle q_1 q_2 (k_1 k_2 M_L) \omega L \rangle | (k_1 k_2) q_1 q_2 M_L \rangle . \qquad (I.4)$$

Multiplying both sides by $\langle (k_1 k_2) q_1'' q_2'' M_L |$ and summing over $q_1'' q_2''$, supposing set $| (k_1 k_2) q_1'' q_2'' M_L \rangle$ to be normalized, one gets

$$\sum_{q_1 q_2} \langle q_1 q_2 (k_1 k_2 M_L) \omega L \rangle \langle (k_1 k_2) q_1'' q_2'' M_L | \mathcal{L}^2 | (k_1 k_2) q_1 q_2 M_L \rangle$$

$$= L(L+1) \langle q_1'' q_2'' (k_1 k_2 M_L) \omega L \rangle$$

and multiplying by the (transposed) coefficients $\langle \omega' L' (k_1 k_2 M_L) q_1'' q_2'' \rangle$ and summing again over $q_1'' q_2''$ we obtain the result:

$$\sum_{\substack{q_1 q_2 \\ q_1'' q_2''}} \langle q_1 q_2 (k_1 k_2 M_L) \omega L \rangle \langle (k_1 k_2) q_1'' q_2'' M_L | \mathcal{L}^2 | (k_1 k_2) q_1 q_2 M_L \rangle \langle \omega' L' (k_1 k_2 M_L) q_1'' q_2'' \rangle$$

$$= L(L+1) \sum_{q_1'' q_2''} \langle q_1'' q_2'' (k_1 k_2 M_L) \omega L \rangle \langle \omega' L' (k_1 k_2 M_L) q_1'' q_2'' \rangle$$

$$= L(L+1) \delta_{\omega\omega'} \delta_{LL'} \qquad (I.5)$$

$$\| S \| \times \| \mathcal{L}^2 \| \times \| \tilde{S} \| = L(L+1) I$$

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meaning that the desired coefficients $\langle q_1 q_2 (k_1 k_2 M_L) \omega L \rangle$ are nothing more than the <u>eigenvector components of the matrix representation of operator $\mathcal{L}^2$ in the base irreducible under $U_3 \supset U_2 \supset U_1$</u>.

To obtain them, we can construct the matrix $\| \langle (k_1 k_2) q_1' q_2' M_L | \mathcal{L}^2 | (k_1 k_2) q_1 q_2 M_L \rangle \|$, diagonalize it by computor and obtain at the same time the associated eigenvectors. The construction is simplified since $\mathcal{L}^2$ by (141) is expressible bilinearly in terms of U_3 group generators $C_q^{q'}$ whose matrix elements in the scheme $U_3 \supset U_2 \supset U_1$ are known (Appendix H). We know from (171) that

$$\mathcal{L}^2 = \sum_q (-)^q \mathcal{L}_q \mathcal{L}_{-q} \qquad (q = 1, \bar{1}, 0)$$

$$= -2 \mathcal{L}_{\bar{1}} \mathcal{L}_1 + \mathcal{L}_0 (\mathcal{L}_0 + 1)$$

if use is made of R_3 commutation relations (142b). Suppressing labels $(k_1 k_2)$ we have

$$\langle q_1' q_2' M_L | \mathcal{L}^2 | q_1 q_2 M_L \rangle =$$

$$-2 \langle q_1' q_2' M_L | \mathcal{L}_{\bar{1}} \mathcal{L}_1 | q_1 q_2 M_L \rangle + M_L (M_L + 1) \delta_{q_1' q_1} \delta_{q_2' q_2}$$

since $\mathcal{L}_0$ is diagonal in $U_3 \supset U_2 \supset U_1$ by (142a), (236) and (237). Now, since $\mathcal{L}_{\bar{1}}^{\dagger} = -\mathcal{L}_1$

$$\langle q_1' q_2' M_L | \mathcal{L}^2 | q_1 q_2 M_L \rangle =$$

$$= 2 \sum_{q_1'' q_2''} \langle q_1'' q_2'' M_L | \mathcal{L}_1 | q_1' q_2' M_L \rangle^* \langle q_1'' q_2'' M_L | \mathcal{L}_1 | q_1 q_2 M_L \rangle$$
$$+ M_L (M_L + 1) \delta_{q_1' q_1} \delta_{q_2' q_2}$$

$$\langle q_1' q_2' M_L | \mathcal{L}^2 | q_1 q_2 M_L \rangle =$$

$$2 \sum_{q_1'' q_2''} \langle q_1'' q_2'' M_L | C_1^3 + C_3^2 | q_1' q_2' M_L \rangle \langle q_1'' q_2'' M_L | C_1^3 + C_3^2 | q_1 q_2 M_L \rangle$$
$$+ \; M_L(M_L+1) \delta_{q_1' q_1} \delta_{q_2' q_2} \qquad (I.6)$$

since $\langle \; | \mathcal{L}_1 | \; \rangle$ are real and from (142a), $\mathcal{L}_1 = -(C_1^3 + C_3^2)$. To evaluate (6) in general form one needs essentially the alegebraic expressions of matrix elements

$$\langle q_1' q_2' M_L | C_1^3 | q_1 q_2 M_L \rangle \quad \text{and} \quad \langle q_1' q_2' M_L | C_3^2 | q_1 q_2 M_L \rangle$$

which are included in the list of Appendix H. Carrying out the calculation one obtains the formula in the following page. Only <u>one</u> of the seven terms with double deltas will contribute to a given element of the matrix so that it is relatively simple to use.

The matrix $\| \langle (k_1 k_2) q_1' q_2' M_L | \mathcal{L}^2 | (k_1 k_2) q_1 q_2 M_L \rangle \|$ is formed in blocks or submatrices labeled by M_L and with rows and columns given by $q_1 q_2$ where M_L refers to the <u>positive</u> projections of all the L values contained in the given $(k_1 k_2)$ representation of interest (See Table V.1.1). For $(k_1 k_2) = (71)$, for instance, L = 1,2,3,4,5,6,7 so that

$$M_L = 0^7, 1^7, 2^6, 3^5, 4^4, 5^3, 6^2, 7.$$

Our formula is:

$$\langle (k_1k_2)q_1q_2M_L | \mathcal{L}^2 | (k_1k_2)q_1'q_2'M_L \rangle =$$

$$\delta_{q_1'q_1}\delta_{q_2'q_2}\Bigg\{ M_L(M_L+1) + \frac{(q_1-q_2-M_L)}{(q_1-q_2)(q_1-q_2+1)}\Big[(k_2-q_2)(k_1-q_2+1)(q_2+1)+(k_1-q_1+1)(q_1-k_2)(q_1+1)\Big]$$
$$+\frac{(q_1-q_2+M_L+2)}{(q_1-q_2+2)(q_1-q_2+1)}\Big[(k_1-q_1)(q_1-k_2+1)(q_1+2)+(k_2-q_2+1)(k_1-q_2+2)q_2\Big]\Bigg\}$$

$$+\delta_{q_1'q_1+1}\delta_{q_2'q_2+1}\frac{\sqrt{(k_1-q_1)(q_1-k_2+1)(q_1+2)(k_2-q_2)(k_1-q_2+1)(q_2+1)}}{(q_1-q_2+1)}\left[\frac{(q_1-q_2+M_L+2)}{(q_1-q_2+2)}-\frac{(q_1-q_2-M_L)}{(q_1-q_2)}\right]$$

$$+\delta_{q_1'q_1-1}\delta_{q_2'q_2-1}\frac{\sqrt{(k_2-q_2+1)(k_1-q_2+2)q_2(k_1-q_1+1)(q_1-k_2)(q_1+1)}}{(q_1-q_2+1)}\left[\frac{(q_1-q_2+M_L+2)}{(q_1-q_2+2)}-\frac{(q_1-q_2-M_L)}{(q_1-q_2)}\right]$$

$$+\delta_{q_1'q_1+2}\delta_{q_2'q_2}\sqrt{\frac{(k_1-q_1)(q_1-k_2+1)(q_1+2)(q_1-q_2+M_L+2)(k_1-q_1-1)(q_1-k_2+2)(q_1+3)(q_1-q_2-M_L+2)}{(q_1-q_2+1)(q_1-q_2+2)^2(q_1-q_2+3)}} \qquad (I.7)$$

$$+\delta_{q_1'q_1-2}\delta_{q_2'q_2}\sqrt{\frac{(k_1-q_1+1)(q_1-k_2)(q_1+1)(q_1-q_2-M_L)(k_1-q_1+2)(q_1-k_2-1)q_1(q_1-q_2+M_L)}{(q_1-q_2+1)(q_1-q_2)^2(q_1-q_2-1)}}$$

$$-\delta_{q_1'q_1}\delta_{q_2'q_2+2}\sqrt{\frac{(k_2-q_2)(k_1-q_2+1)(q_2+1)(q_1-q_2-M_L)(k_2-q_2-1)(k_1-q_2)(q_2+2)(q_1-q_2+M_L)}{(q_1-q_2+1)(q_1-q_2)^2(q_1-q_2-1)}}$$

$$-\delta_{q_1'q_1}\delta_{q_2'q_2-2}\sqrt{\frac{(k_2-q_2+1)(k_1-q_2+2)q_2(q_1-q_2+M_L+2)(k_2-q_2+2)(k_1-q_2+3)(q_2-1)(q_1-q_2-M_L+2)}{(q_1-q_2+1)(q_1-q_2+2)^2(q_1-q_2+3)}}$$

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where it is to be noted the the sum of block dimensions equals the dimensionality of $P^{(71)}_{q_1 q_2 M_L}$ for $M_L \geqslant 0$ (page 176), i.e., 35.

Our calculations will however <u>only require one submatrix</u> (see chapter VIII), that for M =1, e.g., which is 7 X 7 and is associated with all seven values L = 1,2,...,7. Using formula on page 229 one gets

$$\| \langle (71)\, q_1 q_2\, M_L = 1 \,|\, \mathcal{L}^2 \,|\, (71)\, q_1' q_2'\, M_L = 1 \rangle \| =$$

$q_1 q_2$ \ $q_1' q_2'$	61	41	21	70	50	30	10
61	22	$12\sqrt{\frac{7}{5}}$	0	0	$\frac{8}{\sqrt{35}}$	0	0
41	$12\sqrt{\frac{7}{5}}$	36	$20\sqrt{\frac{2}{3}}$	0	0	$8\sqrt{\frac{2}{15}}$	0
21	0	$20\sqrt{\frac{2}{3}}$	34	0	0	0	8
70	0	0	0	8	$4\sqrt{\frac{30}{7}}$	0	0
50	$\frac{8}{\sqrt{35}}$	0	0	$4\sqrt{\frac{30}{7}}$	26	$12\sqrt{\frac{6}{5}}$	0
30	0	$8\sqrt{\frac{2}{15}}$	0	0	$12\sqrt{\frac{6}{5}}$	28	$4\sqrt{5}$
10	0	0	8	0	0	$4\sqrt{5}$	14

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which must fulfill the requirement

$$\mathrm{Tr}\,\|\langle (k_1 k_2)\, q_1 q_2\, M_L \,|\, \mathcal{L}^2 \,|\, (k_1 k_2)\, q_1' q_2'\, M_L \rangle\| = \sum_L L(L+1) \qquad (I.8)$$

the sum on the right extending over all L-values contained in $(k_1 k_2)$. The normalized eigenvector components obtained by computor diagonalization of above matrix are:

$$\|\langle q_1 q_2 (71\, M_L{=}1)\, \omega\, L \rangle\| = \qquad (I.9)$$

$q_1 q_2$ \ L	1	2	3	4	5	6	7
61	0.180702	-0.269374	0.645553	-0.163451	-0.596548	-0.064210	0.302220
41	-0.213809	0.318728	-0.462317	-0.032232	-0.321474	-0.151947	0.715184
21	0.218218	-0.325300	0.164122	0.263181	0.627675	-0.124064	0.583945
70	0.590169	0.659829	0.166450	0.400371	-0.057870	0.157280	0.015423
50	-0.427618	-0.159364	0.080403	0.580193	-0.153748	0.645777	0.089398
30	0.390360	-0.145479	-0.256892	-0.500216	0.051037	0.693542	0.163218
10	-0.436436	0.487950	0.492366	-0.394771	0.342368	0.186097	0.145986

The label ω is ignored as no multiple L-values appear under $(k_1 k_2) = (71)$ (See Table V.11).

APPENDIX J.

QUADRUPOLE MOMENT AND TRANSITION RATE OPERATORS IN SECOND-QUANTIZATION FORMULISM.

The mass quadrupole moment of a given N-particle state is defined[76] as the expectation value of the operator

$$Q_0 \equiv \sum_i^N (3z_i^2 - r_i^2) \qquad (J.1)$$

which is of the single-body type (23a). Being spin-isospin independent, (J.1) cast into second-quantization language by (35) becomes

$$Q_0 = \sum_{\mu\mu'} \langle \mu | 3z^2 - r^2 | \mu' \rangle \, b_\mu^{\mu'} \qquad (J.2)$$

where $\mu \equiv (n_1 n_{\bar{1}} n_0)$ denote quantum numbers of a single particle in a harmonic oscillator common potential and $\{b_\mu^{\mu'}\}$ the set of $\mathcal{U}_r$ generators discussed before. Restriction of states to a single harmonic oscillator mayor shell means $\nu = n_1 + n_{\bar{1}} + n_0 = n_1' + n_{\bar{1}}' + n_0'$ and moreover that

$$\langle \mu | 3z^2 - r^2 | \mu' \rangle = \tfrac{1}{2} \langle \mu | 3z^2 - r^2 + 3p_z^2 - p^2 | \mu' \rangle$$

$$= \langle \mu | 3c_0^0 - \sum_q c_q^q - \tfrac{3}{2} | \mu' \rangle$$

because of definitions (113) and (117). From (125) we know that $c_q^q | \mu \rangle \equiv c_q^q | n_1 n_{\bar{1}} n_0 \rangle = n_q | n_1 n_{\bar{1}} n_0 \rangle$ for $q = 1, \bar{1}, 0$ so that the desired matrix elements

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$$\langle \mu | 3z^2 - r^2 | \mu' \rangle = \left[2n_0 - (n_1 + n_{\bar{1}}) - \tfrac{3}{2} \right] \delta_{\mu\mu'}$$

$$\delta_{\mu\mu'} \equiv \delta_{n_1 n_1'} \delta_{n_{\bar{1}} n_{\bar{1}}'} \delta_{n_0 n_0'} .$$

The mass quadrupole operator $\mathcal{Q}_0$ in terms of $\mathcal{U}_{r=6}$ generators is thus very simply

$$\mathcal{Q}_0 = -\frac{7}{2}\left(\ell_1^1 + \ell_2^2 + \ell_4^4\right) - \frac{1}{2}\left(\ell_3^3 + \ell_5^5\right) + \frac{5}{2}\ell_6^6 \qquad (J.3)$$

whose effect on states of the type $\Delta_\mu^1 \Delta_{\mu_1\mu_2\mu_3}^{123}$ is clear from (304).

The <u>transition rate</u> for quadrupole de-excitations will involve[77] the operator $Q_2 = \sum_i^N x_{+i}^2 \equiv \sum_i^N (x+iy)_i^2$ which here becomes

$$\mathcal{Q}_2 = \sum_{\mu\mu'} \langle \mu | x_+^2 | \mu' \rangle \ell_\mu^{\mu'} . \qquad (J.4)$$

Again, restriction to a single oscillator shell ν implies

$$\begin{aligned} \langle \mu | x_+^2 | \mu' \rangle &= \tfrac{1}{2} \langle \mu | x_+^2 + p_+^2 | \mu' \rangle \\ &= \langle \mu | C_1^1 + \tfrac{3}{2} | \mu' \rangle \\ &= \left(n_1 + \tfrac{3}{2} \right) \delta_{\mu\mu'} \end{aligned}$$

and therefore our operator (J.4) is simply

$$\mathcal{Q}_2 = \frac{7}{2}\ell_1^1 + \frac{5}{2}\left(\ell_2^2 + \ell_3^3\right), \qquad (J.5)$$

of the same type as $\mathcal{Q}_0$ of (J.3).

The low-lying states, e.g., of F^{20} are designated by $\left| [31](71)\, J M_J = J, T = 1 \right\rangle$ which will be linear combinations of terms $\Delta_\mu^1 \Delta_{\mu_1\mu_2\mu_3}^{123}$ of $\mathcal{U}_6 \supset U_3$ symmetry $[f](k_1 k_2) = [31](71)$.

Enhancement of calculated quadrupole moments and transition rates due to collective core effects can be estimated by the addition to each nucleon of a certain fraction of the proton charge e ($\sim$ o.5e for O^{17}).[7, 22]

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13. ABSTRACT

Based on the studies of S. Lie and E. Cartan in the theory of continuous groups of transformations, M. Moshinsky has reformulated the nuclear shell many-body problem in second-quantization language. The methods due to J.P. Elliott for simplifying the basis set of state functions by classification according to the group SU(3) are recast into the above-mentioned reformulation. The purpose is to make low-energy nuclear calculations feasible for nuclei with 4 and more particles in the 2s-1d shell and thus render the possibility of probing for SU(3) symmetries in these nuclei. A hamiltonian model consisting of pairing and quadrupole-quadrupole terms is known to approximate respectively the short- and long-ranged correlations between nucleons given by an arbitrary, reasonably shaped two-nucleon central interaction potential. The former model is generalized to include exchange effects at the long range as well as spin-orbit coupling, and is studied in detail from the viewpoint of its various group symmetries. It is then employed to calculate the low-lying levels of Fluorine-20 which show reasonable accord with the empirical level-scheme.

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