NRL Report 6169

First Order Oblateness Perturbations and The U. S. Naval Space Surveillance System

T. B. MURTAGH

Space Surveillance Branch
Applications Research Division

November 5, 1964

U.S. NAVAL RESEARCH LABORATORY
Washington, D.C.

FIRST ORDER OBLATENESS PERTURBATIONS

AND

THE U. S. NAVAL SPACE SURVEILLANCE SYSTEM

by
Thomas Blackhall Murtagh

Thesis submitted to the Faculty of the Graduate School
of the University of Maryland in partial fulfillment
of the requirements for the degree of
Master of Science
1964

ABSTRACT

Title of Thesis: First Order Oblateness Perturbations and the U. S. Naval Space Surveillance System

Thomas B. Murtagh, Master of Science, 1964

Thesis directed by: Dr. Gart Westerhout

An error analysis is presented involving a comparison of position errors in the orbital path of an artificial earth satellite produced either by the inclusion or omission of first order oblateness effects on all of the orbital elements or by the introduction of theoretical errors into the system with which the measurements are made.

The reasons for making the above comparison are twofold. First, if the improved model of the earth's gravitational field produces an increased accuracy in the orbital elements commensurate with the accuracy of the measuring system, then this more accurate model should be used in the derivation of any epoch set of orbital elements from raw data as well as in any scheme for predicting satellite position as a function of time from epoch. Secondly, if the improved model accuracy is indeed not comparable to the measuring system accuracy, then the inclusion of these first order effects is not necessary and their omission will reduce expenditures for the use of the high speed digital computer necessary for orbital element computation and updating.

The system referred to herein is the U. S. Naval Space Surveillance System which is a detection system comprised of three transmitting and four receiving sites located on a great circle across the southern portion of the United States.

The general development of the problem can be divided into three parts.

The first part involves the computation of the position of the satellite at two different times from the observed direction cosines at those times and the subsequent derivation of the orbital elements of the satellite using an approximate model of the earth's gravitational field.

The second involves the differential correction of the computed elements at the two times using a more accurate model of the earth's gravitational field where the force function allows for first order corrections to all of the elements.

Finally, position errors of the satellite in what we shall define as the error plane are produced by perturbations of system direction cosine measurements or by the inclusion or omission of first order perturbation corrections to the elements themselves. A comparison of these errors will then indicate at what point the accuracy of the model used and the accuracy of the measuring system are comparable.

PROBLEM STATUS

This is an interim report on a continuing problem.

AUTHORIZATION

NRL Problem R02-35
BuWeps Project RT 8801-001/652-1/3434-00-01

Manuscript submitted September 2, 1964.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

ACKNOWLEDGMENTS

I would like to thank the Naval Research Laboratory for permission to use as thesis material work done at the Laboratory. Particularly, I am grateful to my immediate supervisor, Mr. D. Lynch, for his encouragement and helpful criticism of the manuscript, and to Dr. Peter Musen of the Goddard Space Flight Center and Dr. Gart Westerhout of the University of Maryland for their guidance in the preparation of the thesis. Thanks are also due to the personnel of the Research Computation Center for their assistance in the preparation and running of the computer programs needed for the thesis and to Miss Faith Batchelder for her capable typing of the manuscript and final copy.

CHAPTER I

THE SPACE SURVEILLANCE SYSTEM

Historical Background. The Naval Space Surveillance System (Fig. 1) is basically an adaptation of the radio interferometer techniques developed by the U. S. Naval Research Laboratory for tracking Project Vanguard satellites, known as Minitrack.[1] This surveillance system was developed by NRL primarily to detect, track, and predict orbital positions of non-radiating artificial earth satellites passing over the continental United States. Illumination of the satellite is produced by one or more of the three transmitting (T) sites and the reflected energy is received by one or more of the four receiving (R) sites, all of which are located on a great circle across the southern portion of the United States. The three transmitters produce a continuous wave of electromagnetic energy in a pattern narrow in the north-south direction and wide in the east-west direction. The receiving station antenna patterns are similar to the transmitter patterns and are co-planar with them.

Direction Cosine Measurement. The position of a satellite in space, at a given instant of time, is obtained by coincident observations from two or more receiving stations. These observations produce a measurement of the direction cosines of an imaginary line at each of the receiving sites, from which the satellite position can be computed in an earth-centered coordinate set.

Computation of satellite position. The earth centered set used in the computation of satellite position has the X axis directed along 0°

longitude (Greenwich), the Y axis 90° east of Greenwich, and the Z axis normal to the plane formed by the X and Y axes, forming a right handed coordinate set. Latitude is measured upwards from the equatorial plane and longitude is measured west from Greenwich. Each receiving station has a set of unit vectors associated with it, $\bar{A}$, $\bar{B}$, and $\bar{C}$. The unit vector $\bar{A}$ is oriented parallel to the east-west antenna field and is the unit vector from which the east-west direction cosine is measured. The unit vector $\bar{B}$ is oriented parallel to the north-south antenna field and is the unit vector from which the north-south direction cosine is measured. The antenna fields are constructed so that $\bar{A}$ and $\bar{B}$ are normal to each other; therefore the vector set $\bar{B}$ $\bar{A}$ $\bar{C}$ form a right-handed coordinate set. Computation of the satellite coordinates can be effected by knowing the coordinates of the receiving stations making the observation, the components of the unit vectors $\bar{A},\bar{B},\bar{C}$, at each receiving site, and the direction cosines measured by the system at the respective receiving stations (Fig. 2).

From figure 2 we can write

$$\bar{R} = \bar{R}_1 + \bar{R}_3 = \bar{R}_2 + \bar{R}_4$$

$$X\bar{i} + Y\bar{j} + Z\bar{k} = (X_1\bar{i} + Y_1\bar{j} + Z_1\bar{k}) + (R_{3A}\,\bar{A}_1 + R_{3B}\,\bar{B}_1 + R_{3C}\,\bar{C}_1) \quad (1)$$

where

$$\bar{A}_1 \cdot \bar{R}_3 = A_1\,R_3 \cos\theta_{A1} = R_3 \cos\theta_{A1} = R_{3A}$$

$$\bar{B}_1 \cdot \bar{R}_3 = B_1\,R_3 \cos\theta_{B1} = R_3 \cos\theta_{B1} = R_{3B}$$

$$\bar{C}_1 \cdot \bar{R}_3 = C_1\,R_3 \cos\theta_{C1} = R_3 \cos\theta_{C1} = R_{3C}$$

and the unit vectors of the first receiving station are defined as

$$\bar{A}_1 = a_{1x}\,\bar{i} + a_{1y}\,\bar{j} + a_{1z}\,\bar{k}$$

$$\bar{B}_1 = b_{1x}\,\bar{i} + b_{1y}\,\bar{j} + b_{1z}\,\bar{k}$$

$$\overline{C_1} = c_{1x}\,\overline{i} + c_{1y}\,\overline{j} + c_{1z}\,\overline{k}\,.$$

Making the above substitutions in equation (1) and equating coefficients of the unit vectors $\overline{i}$, $\overline{j}$, $\overline{k}$ we obtain

$$X = X_1 + R_3\,(a_{1X}\cos\theta_{A1} + b_{1X}\cos\theta_{B1} + C_{1X}\cos\theta_{C1})$$

$$Y = Y_1 + R_3\,(a_{1Y}\cos\theta_{A1} + b_{1Y}\cos\theta_{B1} + C_{1Y}\cos\theta_{C1})$$

$$Z = Z_1 + R_3\,(a_{1Z}\cos\theta_{A1} + b_{1Z}\cos\theta_{B1} + C_{1Z}\cos\theta_{C1}).$$

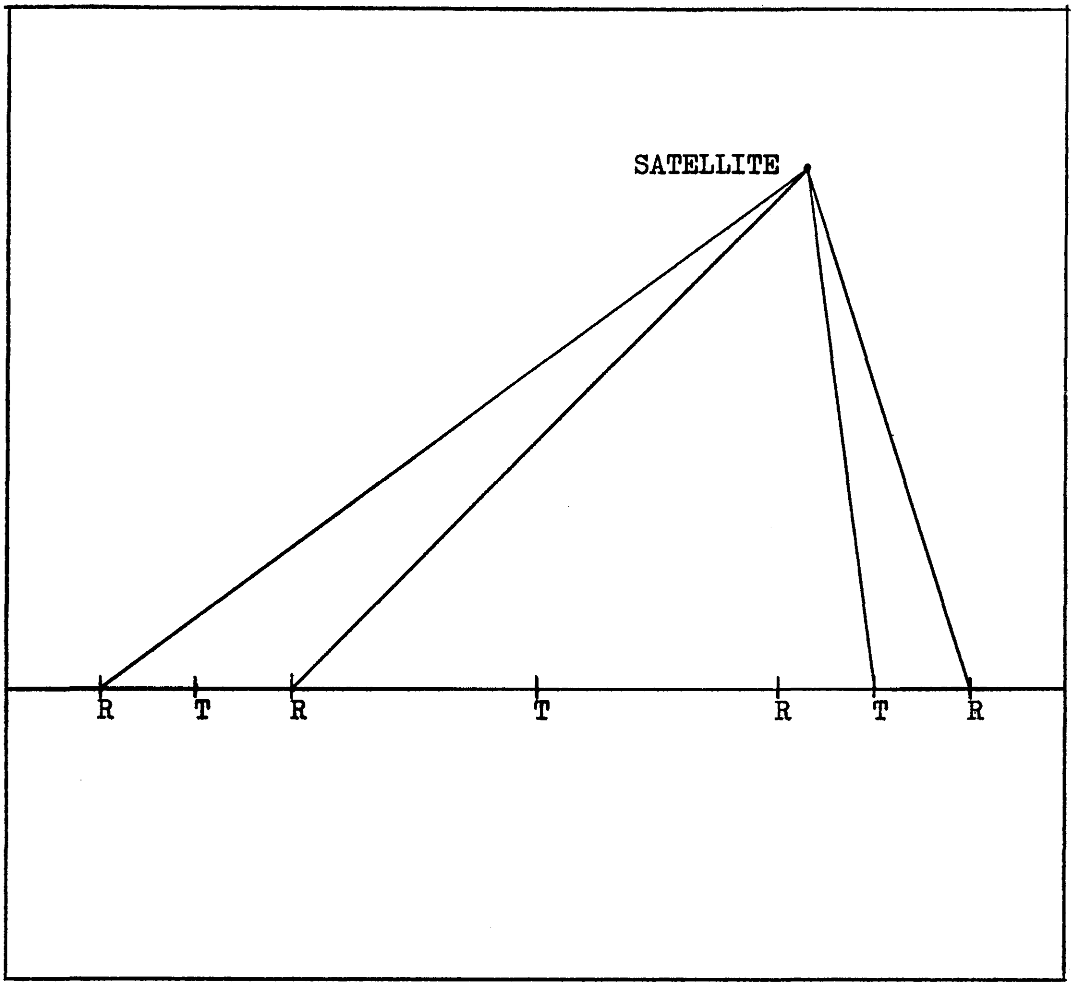

FIGURE 1 - SPACE SURVEILLANCE SYSTEM

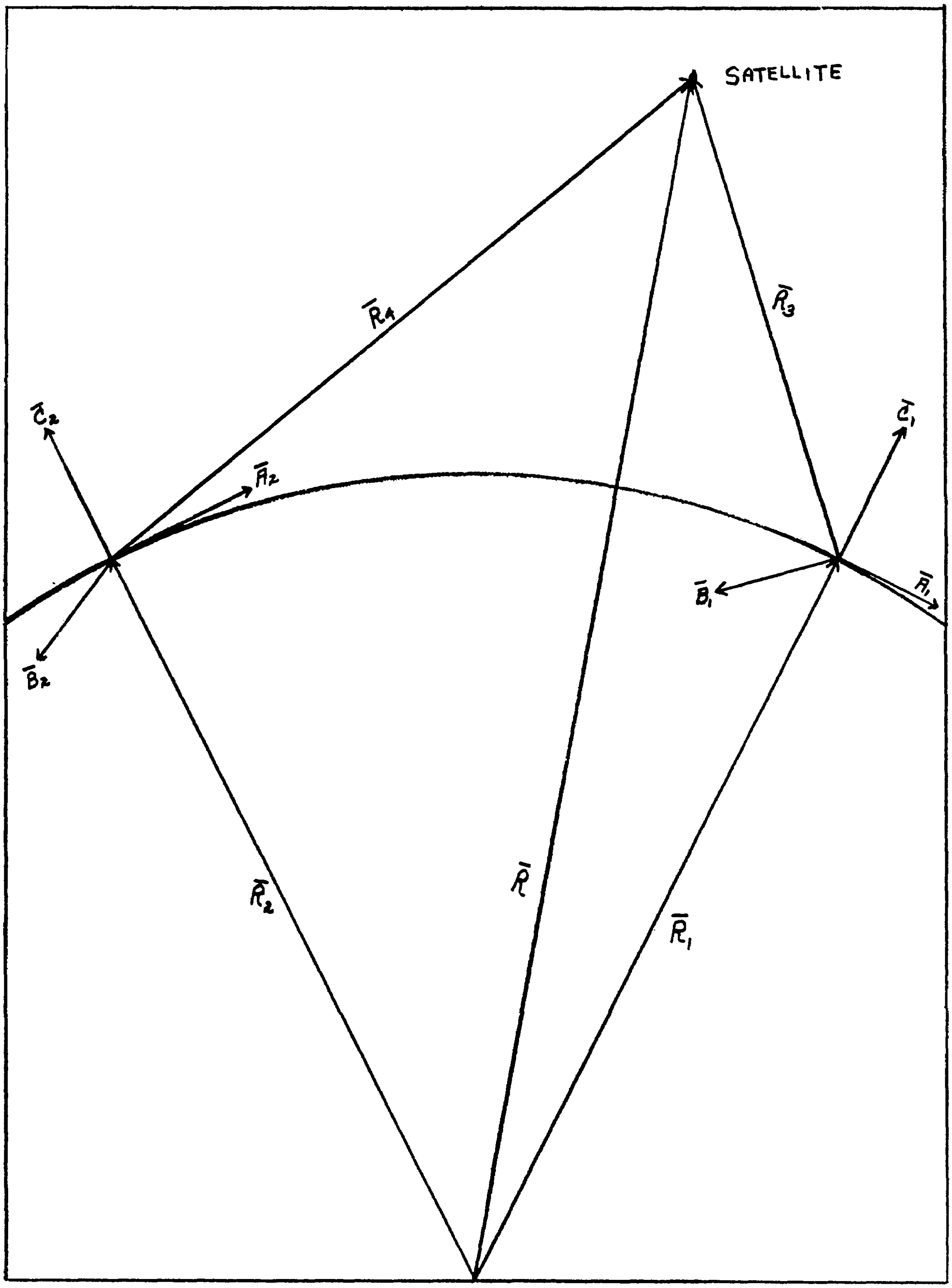

FIGURE 2 - COINCIDENT OBSERVATION

Similarly using

$$X\bar{i} + Y\bar{j} + Z\bar{k} = (X_2\bar{i} + Y_2\bar{j} + Z_2\bar{k}) + (R_{4A}\,\bar{A}_2 + R_{4B}\,\bar{B}_2 + R_{4C}\,\bar{C}_2) \quad (2)$$

we obtain

$$X = X_2 + R_4\,(a_{2x}\cos\theta_{A2} + b_{2x}\cos\theta_{B2} + c_{2x}\cos\theta_{C2})$$

$$Y = Y_2 + R_4\,(a_{2y}\cos\theta_{A2} + b_{2y}\cos\theta_{B2} + c_{2y}\cos\theta_{C2})$$

$$Z = Z_2 + R_4\,(a_{2z}\cos\theta_{A2} + b_{2z}\cos\theta_{B2} + c_{2z}\cos\theta_{C2}).$$

Let

$$\alpha_1 = a_{1x}\cos\theta_{A1} + b_{1x}\cos\theta_{B1} + C_{1x}\cos\theta_{C1}$$

$$\alpha_2 = a_{2x}\cos\theta_{A2} + b_{2x}\cos\theta_{B2} + C_{2x}\cos\theta_{C2}$$

$$\beta_1 = a_{1y}\cos\theta_{A1} + b_{1y}\cos\theta_{B1} + C_{1y}\cos\theta_{C1}$$

$$\beta_2 = a_{2y}\cos\theta_{A2} + b_{2y}\cos\theta_{B2} + C_{2y}\cos\theta_{C2}$$

$$\gamma_1 = a_{1z}\cos\theta_{A1} + b_{1z}\cos\theta_{B1} + C_{1z}\cos\theta_{C1}$$

$$\gamma_2 = a_{2z}\cos\theta_{A2} + b_{2z}\cos\theta_{B2} + C_{2z}\cos\theta_{C2}.$$

Therefore we have

$$X = X_1 + \alpha_1 R_3 \quad (3) \qquad\qquad X = X_2 + \alpha_2 R_4 \quad (6)$$

$$Y = Y_1 + \beta_1 R_3 \quad (4) \qquad\qquad Y = Y_2 + \beta_2 R_4 \quad (7)$$

$$Z = Z_1 + \gamma_1 R_3 \quad (5) \qquad\qquad Z = Z_2 + \gamma_2 R_4. \quad (8)$$

From the above we can write

$$R_3 = \frac{X-X_1}{\alpha_1} \qquad\qquad R_4 = \frac{X-X_2}{\alpha_2}\ .$$

Equating equations (4) and (7) and substituting for R_3 and R_4 the above definitions we can solve for X in terms of known quantities:

$$Y_1 + \beta_1\left(\frac{X-X_1}{\alpha_1}\right) = Y_2 + \beta_2\left(\frac{X-X_2}{\alpha_2}\right)$$

$$X\left(\frac{\beta_1}{\alpha_1} - \frac{\beta_2}{\alpha_2}\right) = Y_2 - Y_1 + \frac{\beta_1}{\alpha_1}X_1 - \frac{\beta_2}{\alpha_2}X_2$$

$$X = \frac{(Y_2-Y_1) + \frac{\beta_1}{\alpha_1} X_1 - \frac{\beta_2}{\alpha_2} X_2}{\left(\frac{\beta_1}{\alpha_1} - \frac{\beta_2}{\alpha_2}\right)} \,. \tag{9}$$

Equating equations (3) and (6) and substituting the values of R_3 and R_4 obtained by solving equations (4) and (7) respectively we can solve for Y in terms of known quantities:

$$X_1 + \alpha_1 \left(\frac{Y-Y_1}{\beta_1}\right) = X_2 + \alpha_2 \left(\frac{Y-Y_2}{\beta_2}\right)$$

$$Y \left(\frac{\alpha_1}{\beta_1} - \frac{\alpha_2}{\beta_2}\right) = X_2 - X_1 + \frac{\alpha_1}{\beta_1} Y_1 - \frac{\alpha_2}{\beta_2} Y_2$$

$$Y = \frac{(X_2-X_1) + \frac{\alpha_1}{\beta_1} Y_1 - \frac{\alpha_2}{\beta_2} Y_2}{\left(\frac{\alpha_1}{\beta_1} - \frac{\alpha_2}{\beta_2}\right)} \,. \tag{10}$$

Similarly an equation for Z in terms of known quantities may be obtained:

$$Y_1 + \beta_1 \left(\frac{Z-Z_1}{\gamma_1}\right) = Y_2 + \beta_2 \left(\frac{Z-Z_2}{\gamma_2}\right)$$

$$Z = \frac{(Y_2-Y_1) + \frac{\beta_1}{\gamma_1} Z_1 - \frac{\beta_2}{\gamma_2}}{\left(\frac{\beta_1}{\gamma_1} - \frac{\beta_2}{\gamma_2}\right)} \,. \tag{11}$$

From equations (9), (10), and (11) the satellite coordinates X, Y, and Z can be computed, using the known values of the station coordinates X_1, Y_1, Z_1, X_2, Y_2, Z_2, the station vector components of $\bar{A}$, $\bar{B}$, and $\bar{C}$, and the measured values of the direction cosines at the respective stations. Since equations (3) through (8) represent six equations in five unknowns, they are considered redundant. Consequently equations (9), (10), and (11) can be modified to compute statistically optimum values of the coordinates, using the X, Y, and Z from equations (9), (10), and (11) as a first guess. Appendix A presents the procedure for computing these statistically optimum values and is based on the second section of Chapter II in a University of Maryland Master's Thesis by H. G. deVezin of the Space Surveillance Branch ("The Computation of Optimum Orbital Elements Derived From a Single Coincident Observation by Two Receiving Stations of the Naval Space Surveillance System," reproduced as NRL Report 6172).

Computation of Standard Elliptic Elements. The standard elliptic elements for a given epoch which will be referred to in this thesis are listed in Table 1 and their definitions may be found in the literature.[2] Of the nine variables in the list, only six are independent, since the semi-major axis and the period are related through Kepler's Third Law, the mean and eccentric anomaly are related through Kepler's equation, $M = E - e \sin E$, and equations like $\cos v = \frac{\cos E - e}{1 - e \cos E}$ relate the eccentric and true anomaly. The computation of these elements from Space Surveillance angle data requires two observations of the satellite on its orbital path, the time between these two passes, the south-north (SN) or north-south (NS) direction of the two passes, the east-west (EW) or west-east (WE) direction of the satellite, the Greenwich Hour Angle of Aries (GHA) at the time of the first pass (epoch), and an approximate value of the anomalistic period of the satellite. An optional third observation of the satellite may be used, in which case an approximate value of the period of the satellite would not be needed. In place of this, the time between this third pass and one of the first two passes having the same NS or SN direction is

TABLE 1 - STANDARD ELLIPTIC ELEMENTS

ELEMENT NAME	SYMBOL	UNITS
SEMI-MAJOR AXIS	a	STATUTE MILES
ECCENTRICITY	e	-------
INCLINATION	i	DEGREES
ARGUMENT OF PERIGEE	ω	DEGREES
RIGHT ASCENSION OF THE ASCENDING NODE	Ω	DEGREES
MEAN ANOMALY	M	DEGREES
TRUE ANOMALY	v	DEGREES
ECCENTRIC ANOMALY	E	DEGREES
ANOMALISTIC PERIOD	T	MINUTES

used, together with the number of revolutions of the satellite during thi time. (The division of this time, in minutes, by the number of revolutio will yield a first guess at the period of the satellite.)

The first step in the derivation of the elliptic elements from angle data is to calculate the position vectors for the two passes, from which the latitude and longitude of these two points can be computed. Similarly, if a third observation is given, the position vector is first computed and then the latitude and longitude of this third point can be calculated. The longitude of the second point is then corrected for the earth's rotation during the elapsed time between the first two passes. (It should be remarked here that the longitude defined herein is east longitude.) From this longitude and latitude information, the central angle between the two position vectors (Fig. 3) can be computed, using the following equation derived from $\bar{R}_1 \cdot \bar{R}_2$,

$$\alpha = \cos^{-1} [\cos L_1 \cos L_2 \cos (\lambda_2 - \lambda_1) + \sin L_1 \sin L_2] \qquad (12)$$

where L = latitude, λ = longitude, and α = central angle. The above equation yields the principle value of the central angle ($0 \leq \alpha \leq 180°$); using this value of central angle, and assuming a non-retrograde orbit, the inclination can be computed using the following equation derived from $\frac{\bar{R}_1 \times \bar{R}_2}{|\bar{R}_1 \times \bar{R}_2|} \cdot \bar{k}$,

$$i = \cos^{-1} \left[\frac{\cos L_1 \cos L_2 \sin (\lambda_2 - \lambda_1)}{\sin \alpha} \right] \quad . \qquad (13)$$

This value of inclination will be used as a temporary value; if the absolute value of this inclination is greater than or equal to 45° we will use SN or NS direction information to determine the true value of inclination, which will either be the above value or 180° minus the above value. If the absolute value of this inclination is less than 45° we use EW or WE direction information to determine the correct value of inclination. Essentially in the above we are attempting to resolve the ambiguity in the direction of the satellite between the two observed points, which implies that we are

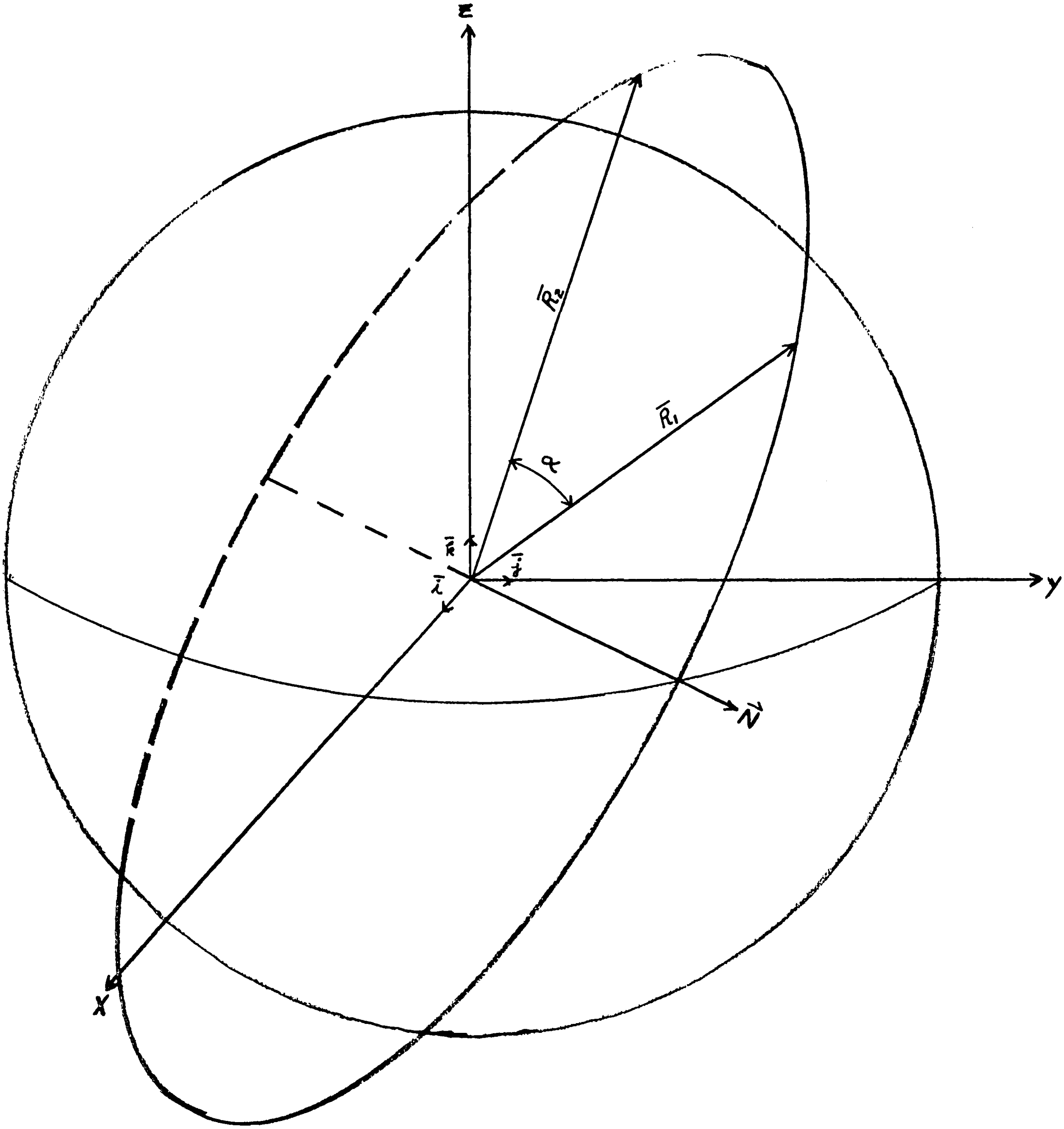

FIGURE 3 - SATELLITE POSITION IN SPACE

attempting to determine whether it passed from the first point to the second through the large or the small central angle.

The next element to be computed is the semi-major axis. This is done in one of two ways, depending on the information given:

1. If the approximate anomalistic period is given, the semi-major axis can be computed immediately from the equation of Kepler's Third Law.

2. If the time between the third pass and one of the other two passes in a like SN or NS direction is divided by the number of revolutions between these passes, an approximate nodal period, uncorrected for latitude difference, is obtained.

In either case, a guess at period is obtained, since anomalistic and nodal period are approximately the same, and the semi-major axis can be computed from the equation

$$a = 205.835706 \quad T^{2/3} \tag{14}$$

where $\underline{T}$ is in minutes and $\underline{a}$ is in statute miles.

The regression of the ascending node during the elapsed time between the two passes can then be computed, assuming a circular orbit, using the equation

$$\frac{d\Omega}{dt} = - 9.9596 \left(\frac{r_e}{a}\right)^{7/2} \frac{\cos i}{(1-e^2)^2} \frac{\text{deg}}{\text{day}} \tag{15}$$

where r_e is the equatorial radius of the earth in statute miles. The longitude of the second point can then be corrected for regression, and subsequently the central angle and the inclination can be corrected for regression. Similarly the rotation of perigee during the elapsed time can be computed using the equation

$$\frac{d\omega}{dt} = 4.9798 \left(\frac{r_e}{a}\right)^{7/2} \frac{(5 \cos^2 i - 1)}{(1-e^2)^2} \frac{\text{deg}}{\text{day}} \quad . \tag{16}$$

If three passes are being used as input information, the approximate nodal period previously computed is then corrected for latitude difference and the rotation of perigee during the elapsed time and the resulting period will be the approximate anomalistic period.

The next pair of elements to be computed are the eccentricity and the true anomaly of the first pass, calculated using the equation

$$R = \frac{a(1-e^2)}{1+e \cos v} \tag{17}$$

which is the defining equation of a conic section. Applying the above equation at t_1 and t_2 (times of the first and second pass respectively) and using the fact that $v_2 - v_1 = \alpha$, a fourth degree equation in eccentricity is derived from which only two solutions are possible. (The derivation is reproduced for convenience in Appendix B.) The existence of two possible solutions implies that two possible ellipses fit the two observed points and the choice of the correct set of values of eccentricity and true anomaly is made by substituting $(v_1 + \alpha)$ and $(2\pi - v_1 + \alpha)$ into equation (17) at t_2 and seeing which of these substitutions yields a value of R closest to R_2. Using this correct value of e and v_1, and computing v_2 with $v_2 = v_1 + \alpha$ we can calculate a time difference (less than a period) between the two passes from the equation

$$t_2 - t_1 = \frac{a^{3/2}}{\sqrt{Gm}} \left[\frac{e\sqrt{1-e^2} \sin v}{1 + e \cos v} + \sin^{-1} \left(\frac{\cos v + e}{1+e \cos v} \right) \right]_{v_2}^{v_1} \tag{18}$$

which is nothing more than the integration of Kepler's equation M = E - e sin E, using the following relationship between E and v,

$$\cos E = \frac{\cos v + e}{1 + e \cos v}$$

and the fact that $M = \frac{\sqrt{GM}}{a^{3/2}} \; (t_2 - t_1)$.

This quantity is evaluated so that an iteration process can be performed in order to obtain an accurate value of anomalistic period.

The first iteration is carried out by computing a second value of period from

$$T_1 = \left(\frac{\Delta T_{OBS}}{\Delta T_{GUESS}}\right) T_{GUESS} \tag{19}$$

where ΔT_{OBS} = observed time difference reduced to a value less than one period

ΔT_{GUESS} = calculated time difference from equation (18)

T_{GUESS} = value of anomalistic period computed previously.

The second value of anomalistic period T_1 is then used to repeat the procedure previously indicated, beginning on the bottom of page 11. After completing the process a second time, a straight line fit is made on the successive periods using

$$T_{n+1} = T_n - \frac{\Delta T_n - \Delta T_{OBS}}{\Delta T_n - \Delta T_{n-1}} \quad (T_n - T_{n-1}) \tag{20}$$

until the successive periods are within a pre-determined tolerance.

Once an accurate period has been obtained, a vector which points along the line of nodes, in the direction of the ascending node, is computed (see Fig.3). The sum of the anomaly and the argument of perigee (argument of the latitude) can then be computed using

$$(v + \omega) = \cos^{-1} \left[\frac{\overline{N}}{/\overline{N}/} \cdot \frac{\overline{R}}{/\overline{R}/} \right] \tag{21}$$

where $\overline{N}$ = the vector along the line of nodes in the direction of the ascending node. Subtraction of the true anomaly from the above quantity will yield the argument of perigee. The longitude of the ascending node, λ_N, can be computed using

$$\lambda_N = \cos^{-1}\left[\frac{N_X}{/\overline{N}/}\right] \tag{22}$$

where N_X is the component of the node vector along the X axis. The right ascension of the ascending node, Ω, can then be calculated using

$$\Omega = GHA - \lambda_N \text{ for } \lambda_n < GHA \tag{23a}$$

or

$$\Omega = 2\pi - (\lambda_N - GHA) \text{ for } \lambda_N > GHA \tag{23b}$$

where GHA = angle measured west from Greenwich to Aries

λ_N = angle measured west from Greenwich to node

Ω = angle measured from first point of Aries east to the node.

The computed values of v_1 and v_2 can then be converted to the mean anomaly at t_1 and t_2 from

$$M_v = \cos^{-1}\left[\frac{\cos v + e}{1 + e \cos v}\right] - \left|\frac{e\sqrt{1-e^2}\sin v}{1 + e \cos v}\right| \tag{24}$$

remembering that if $v < \pi$, $M = M_v$ and that if $v \geq \pi$, $M = 2\pi - M_v$.

The above procedure was adapted by the author to the Naval Research Electronic Computer (NAREC) in order to facilitate the computation of orbital elements from many sets of observed angle data.

CHAPTER II

PERTURBATIONS

<u>Introduction</u>. The main causes of perturbations of a satellite orbit are asphericity of the primary, atmospheric drag, lunar and solar perturbations, perturbations by additional satellites or planets, electric and magnetic fields, relativistic effects, and interplanetary dust. The first two perturbations are of particular concern in the theory of close earth satellites, whereas the third and fourth perturbations involve deep space probes. The other perturbations are negligible by comparison with the first four. The term asphericity includes both oblateness and deviations from the oblate spheroid of revolution.

This work is primarily concerned with the first perturbation in the above list, namely asphericity of the primary, and in particular is concerned with the oblateness perturbations involved therein. The oblateness of the primary, which in this instance is the earth, produces five perturbations on a close earth satellite: regression, rotation of perigee, variation of radial distance, variation of the period of revolution, and periodic variations of the orbital elements.[3] More specifically, this work is concerned only with first order oblateness effects assuming that the only forces acting on the satellite are those due to a gravitational field with axial symmetry.

A large number of articles have appeared in the literature since the advent of artificial earth satellites which discuss the motion of a close earth satellite in the gravitational field of an oblate planet.

Because of this large number, it would be impossible to list them all in the brief confines of this work, but a few of the more pertinent articles are those written by Brouwer[4], Garfinkel[5], King-Hele[6], Kozai[7], Merson[8], Musen[9], and O'Keefe, Eckels, and Squires[10].

<u>The Undisturbed Two-Body Problem</u>[11]. In the undisturbed two-body problem the equation of motion can be written as

$$\frac{d^2\bar{r}}{dt^2} = -\frac{G\,(M+m)}{r^3}\,\bar{r} \tag{25}$$

where M = mass of central body

m = mass of orbiting body (or satellite)

G = gravitational constant

$\bar{r}$ = position vector between central body and satellite.

Let us now define the force function

$$U_o = \frac{G\,(M+m)}{r}$$

so that the above equation becomes

$$\frac{d^2\bar{r}}{dt^2} = \bar{\nabla}\,U_o\,. \tag{25a}$$

Since the force function in undisturbed motion is time independent the integral of energy does exist and we can write

$$T = U_o + h$$

where T = kinetic energy per unit mass

$$\frac{v^2}{2} = \frac{G\,(M+m)}{r} - \frac{G\,(M+m)}{2a} .$$

Now define $\mu = G\,(M + m)$ and the above equation becomes

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) . \tag{26}$$

Now consider

$$\frac{d}{dt}\left(\bar{r} \times \frac{d\bar{r}}{dt}\right) = \frac{d\bar{r}}{dt} \times \frac{d\bar{r}}{dt} + \bar{r} \times \frac{d^2\bar{r}}{dt^2} .$$

The first term on the right hand side of the above equation is obviously zero and the second term will be found to be zero also by crossing the vector $\bar{r}$ with equation (25). Therefore we have

$$\bar{r} \times \bar{v} = \bar{c} \tag{27}$$

where $\bar{c}$ is a vectorial constant; this equation represents the area or momentum integral. Next we write equation (25) as

$$\frac{d\bar{v}}{dt} = -\frac{\mu}{r^3}\,\bar{r}$$

and cross this equation with the vector $\bar{c}$. The result is

$$\bar{c} \times \frac{d\bar{v}}{dt} + \frac{\mu}{r^3}\,\bar{r} \times \left(\frac{d\bar{r}}{dt} \times \bar{r}\right) = 0$$

$$\bar{c} \times \frac{d\bar{v}}{dt} + \frac{\mu}{r^3}\left[\frac{d\bar{r}}{dt}\,(\bar{r} \cdot \bar{r}) - \bar{r}\,\left(\bar{r} \cdot \frac{d\bar{r}}{dt}\right)\right] = 0$$

$$\bar{c} \times \frac{d\bar{v}}{dt} + \mu\left[\frac{r\,\frac{d\bar{r}}{dt} - \bar{r}\,\frac{dr}{dt}}{r^2}\right] = 0 .$$

But since $\bar{c}$ is a constant vector we can write

$$\frac{d}{dt}(\bar{c} \times \bar{v}) + \mu \frac{d}{dt}\left(\frac{\bar{r}}{r}\right) = 0 .$$

Therefore

$$\bar{c} \times \bar{v} + \frac{\mu \bar{r}}{r} + \mu \bar{e} = 0 \tag{28}$$

where $\bar{e}$ = vector directed towards perigee along the X axis. Equation (28) is known as the Laplacian integral. Still another integral of the motion may be derived, the Hamiltonian integral, which is essentially an algebraic combination of the area integral and the Laplacian integral:

$$\bar{c} \times (\bar{c} \times \bar{v}) + \mu\, \bar{c} \times \left(\frac{\bar{r}}{r}\right) + \mu\, \bar{c} \times \bar{e} = 0$$

$$\bar{c}\,(\bar{c} \cdot \bar{v}) - \bar{v}\,(\bar{c} \cdot \bar{c}) = -\mu\, \bar{c} \times \left(\frac{\bar{r}}{r} + \bar{e}\right)$$

$$\bar{v} = \frac{\mu}{c^2}\, \bar{c} \times \left(\frac{\bar{r}}{r} + \bar{e}\right) \tag{29}$$

Because the Laplacian and area integrals are not entirely independent, only five of the six independent scalar integrals needed to obtain a complete solution to the problem are obtained from them. The sixth independent scalar integral is obtained by using Kepler's equation in conjunction with one of the equations for the position vector in terms of the elliptic elements.

The Disturbed Two-Body Problem. In the disturbed two-body problem the equation of motion becomes

$$\frac{d^2\bar{r}}{dt^2} = -\frac{G\,(M + m)}{r^3}\,\bar{r} + \bar{F} \tag{30}$$

where $\bar{F}$ = the disturbing force per unit mass. An immediately obvious consequence of the addition of the disturbing force is the vanishing of the area integral, since now we have

$$\frac{d\bar{c}}{dt} = \bar{r} \times \bar{F}$$

which implies that $\bar{c}$ is no longer a vectorial constant. In order to obtain a clearer physical picture of what happens to the elliptic elements when a disturbing force is present, it is useful to decompose the disturbing force both along the vector set in the orbital plane and along the radius vector to the satellite. As a result of this decomposition, we come to the obvious conclusion that the elements a, e, ω, and M are primarily affected by components of forces in the orbital plane and the elements i and Ω are primarily affected by components of forces outside of the orbital plane. The equations for the variations of the standard elliptic elements can then be derived in terms of the components of $\bar{F}$ along the above-mentioned vector sets by applying Brown's operator, δ/dt, to equations (26), (27), (28), and (29).[12]

The Disturbing Function. If we write equation (30) as

$$\frac{d^2\bar{r}}{dt^2} = \bar{\nabla} U_o + \bar{F} \tag{30a}$$

and write $\bar{F} = \bar{\nabla} U_n$ for $n > 1$, letting n indicate the order of the corrective term to the force function, and note that for n = 0 we have the force function in the undisturbed two-body problem, we can rewrite equation (30a) as

$$\frac{d^2\bar{r}}{dt^2} = \bar{\nabla} U_n \; . \tag{31}$$

In considering oblateness effects due to an axially symmetric gravitational field (which means that only latitude dependent or zonal harmonics appear in the force function expansion, and longitude dependent or tesseral harmonics are absent) we can write the force function which describes the earth's gravitational field in terms of a series of spherical harmonics

$$U = \sum_{m=0}^{\infty} Am\ r^{-m-1}\ Ym\ (\theta, \lambda)$$

where $Ym\ (\theta, \lambda)$ is the spherical harmonic of order m, θ is the co-latitude, λ is the longitude, and the A_m's are constant coefficients in the expansion. Since we are omitting tesseral harmonics in this work, the force function can be written more simply in terms of Legendre polynomials of order m,

$$U = \frac{GM}{r} \left[1 - \sum_{m=2}^{\infty} ({}^{r}E/r)^m\ J_m\ P_m\ (\cos\theta) \right] \tag{32}$$

where G = the gravitational constant, M = the mass of the earth, r_E = equatorial radius of the earth, and the J_m's are constant coefficients in the expansion. The term for m=1 in the above expansion is omitted because $J_1 = 0$ if the origin of coordinates is chosen to be at the center of the earth, which is the case for all practical purposes. The disturbing function $R = U - \frac{Gm}{r}$ can be derived immediately from equation (32),

$$R = - \frac{GM}{r_E} \sum_{m=2}^{\infty} ({}^{r}E/r)^{m+1}\ J_m\ P_m\ (\cos\theta) \tag{33}$$

and the Legendre polynomials $P_m(\cos\theta)$ are conventionally defined as

$$P_m\ (\cos\theta) = \frac{1}{2^m.m!} \frac{d^m}{d\ (\cos\theta)^m} (\cos^2\theta - 1)^m\ .$$

The explicit forms of the first five Legendre polymonials are given in Table 2. Since it is the purpose of this author to discuss first order periodic perturbations on all of the elements and first and second order secular effects on the argument of perigee and right ascension, only the terms for m = 2, 3, and 4 need be considered in equation (33). Therefore the expression for the disturbing function can be written more explicitly, noting that $\cos \Theta = \sin L$, where L = latitude, as

TABLE 2 - LEGENDRE POLYNOMIALS (m = 2 to m = 6)

$$P_2 = \tfrac{1}{2} (3 \cos^2 \Theta - 1)$$

$$P_3 = \tfrac{1}{2} (5 \cos^3 \Theta - 3 \cos \Theta)$$

$$P_4 = 1/8\ (35 \cos^4 \Theta - 30 \cos^2 \Theta + 3)$$

$$P_5 = 1/8\ (63 \cos^5 \Theta - 70 \cos^3 \Theta + 15 \cos \Theta)$$

$$P_6 = 1/16\ (231 \cos^6 \Theta - 315 \cos^4 \Theta + 105 \cos^2 \Theta - 5)$$

$$\begin{aligned} R = - \frac{GM}{r_E} \Big[&({r_E}/r)^3 \left(\tfrac{1}{2} J_2\right) (3 \sin^2 L - 1) \\ &+ ({r_E}/r)^4 \left(\tfrac{1}{2} J_3\right) (5 \sin^3 L - 3 \sin L) \\ &+ ({r_E}/r)^5 \left(1/8\ J_4\right) (35 \sin^4 L - 30 \sin^2 L + 3) \Big]. \end{aligned} \quad (34)$$

Using the relation $\sin L = \sin i \sin (v + \omega)$, and reducing second, third, and fourth degree terms to ones involving multiple angles, the above equation for the disturbing function becomes

$$R = GM \Big[(\tfrac{3}{2} J_2) \frac{r_E^2}{a^3} (a/r)^3 \left\{ \tfrac{1}{3} - \tfrac{1}{2} \sin^2 i + \tfrac{1}{2} \sin^2 i \cos 2 (v + \omega) \right\}$$

$$- J_3 (\frac{r_E^3}{a^4}) (a/r)^4 \left\{ (\tfrac{15}{8} \sin^3 i - \tfrac{3}{2} \sin i) \sin (v + \omega) \right.$$

$$\left. - \tfrac{5}{8} \sin^3 i \sin 3 (v + \omega) \right\}$$

$$- (\tfrac{35}{8} J_4) (\frac{r_E^4}{a^5}) (a/r)^5 \left\{ \tfrac{3}{35} - \tfrac{3}{7} \sin^2 i + \tfrac{3}{8} \sin^4 i \right.$$

$$\left. + (\tfrac{3}{7} \sin^2 i - \tfrac{1}{2} \sin^4 i) \cos 2 (v + \omega) + \tfrac{1}{8} \sin^4 i \cos 4 (v + \omega) \right\} \Big]. \tag{35}$$

The constant J_2 is of first order, the constants J_3 and J_4 are of second order,[13, 14] and the values used for them in this work are $J_2 = 1.08219 \times 10^{-3}$, $J_3 = -2.29 \times 10^{-6}$, and $J_4 = -2.12 \times 10^{-6}$. It would now be convenient to separate the disturbing function into a first order secular part, a second order secular part, a first order long periodic part, and a first order short periodic part. Terms in R not depending on M or ω are considered secular, terms depending on ω but not M are considered long periodic, and terms depending on M but not on ω are considered short periodic.[15] We shall designate the first order secular part of R by R_1, the second order secular part by R_2, the first order long periodic part by R_3 and the first order short periodic part by R_4. In order to smooth out the rapid variations due to changes in the mean anomaly we first integrate the disturbing function with respect to the mean anomaly as

$$[R] = \frac{1}{2\pi} \int_0^{2\pi} R \, dM \tag{36}$$

and then we can write down the expressions for R_1, R_2, R_3, and R_4 according to the above criterion. In performing the above operation it is useful to recall that

$$\frac{1}{2\pi} \int_0^{2\pi} \left(\frac{r}{a}\right)^p \sin q\, v\, dM = 0 \tag{36a}$$

and that

$$\frac{1}{2\pi} \int_0^{2\pi} \left(\frac{r}{a}\right)^p \cos q\, v\, dM = X_o^{p,q} \tag{36b}$$

where q and p are positive or negative integers, and $X_o^{p,q}$ is a Hansen's coefficient and is defined as

$$X_o^{p,q} = (-1)^q \left(\frac{e}{2}\right)^q \begin{pmatrix} p+q+1 \\ q \end{pmatrix} F\left(\frac{q-p-1}{2}, \frac{q-p}{2}, 1+q, e^2\right).$$

In the above $\begin{pmatrix} p+q+1 \\ q \end{pmatrix}$ is the notation for the binomial coefficients and $F(\frac{q-p-1}{2}, \frac{q-p}{2}, 1+q, e^2)$ is the conventional hypergeometric function form. The evaluation of the integrals of the form of equations (36a) and (36b) may be found in the literature.[16] The resulting expressions for R_1, R_2, R_3, and R_4 are

$$R_1 = GM \left(\frac{3}{2} J_2\right) \frac{r_E^2}{a^3} \left(\frac{1}{3} - \frac{1}{2} \sin^2 i\right) (1 - e^2)^{-3/2} \tag{37}$$

$$R_2 = - GM \left(\frac{35}{8} J_4\right) \frac{r_E^4}{a^5} \left(\frac{3}{35} - \frac{3}{7} \sin^2 i + \frac{3}{8} \sin^4 i\right) (1-e^2)^{-7/2}\left(1+\frac{3}{2} e^2\right) \tag{38}$$

$$\begin{aligned} R_3 = - GM \Bigg[& \frac{3}{2} J_3 \left(\frac{r_E^3}{a^4}\right) \sin i \left(\frac{5}{4} \sin^2 i - 1\right) e \,(1-e^2)^{-5/2} \sin \omega \\ & +\left(\frac{35}{8} J_4\right) \frac{r_E^4}{a^5} \sin^2 i \left(\frac{9}{28} - \frac{3}{8} \sin^2 i\right) e^2 (1-e^2)^{-7/2} \cos 2\omega \Bigg] \end{aligned} \tag{39}$$

$$\begin{aligned} R_4 = GM \left(\frac{3}{2} J_2\right) \left(\frac{r_E^2}{a^3}\right) \Bigg[& \left(\frac{1}{3} - \frac{1}{2}\sin^2 i\right) \left\{(a/r)^3 - (1 - e^2)^{-3/2}\right\} \\ & + \tfrac{1}{2} (a/r)^3 \sin^2 i \cos 2 (v +\omega)\Bigg] \end{aligned} \tag{40}$$

where

$$(a/r)^3 = (1-e^2)^{-3} \left[(1 + \frac{3}{2} e^2) + (3e + \frac{3}{4} e^3) \cos v + \frac{3}{2} e^2 \cos 2v + \frac{1}{4} e^3 \cos 3v \right].$$

The above expressions for R_1, R_2, R_3, and R_4 will then enable us to write down the secular, long periodic, and short periodic variations of the standard elliptic elements.

CHAPTER III

VARIATION OF ARBITRARY CONSTANTS

Basic Principles of the Method.[17, 18] In the undisturbed two-body problem, if the position and velocity of a satellite at a given instant of time is known, or if two positions of the satellite for two instants of time are known, the six orbital elements of the satellite can be determined, and these six elements do not change with time. In the disturbed two-body problem, however, the six orbital elements determined from position and velocity measurements at a given time, or from two position measurements at two times, do vary with time, and the set of elements determined at an epoch are exactly the elements of the ellipse that the satellite would follow if all perturbations on the satellite would cease from that moment on. This ellipse is called an "osculating" ellipse and the elements are called the "osculating" elements. Our procedure is to first obtain the elements as functions of time and then to substitute these elements into the equations relating the coordinates and the elements, and thus obtain the coordinates as functions of time. This is the basic principle of the method of the variation of arbitrary constants or variation of parameters[19, 20], and in celestial mechanics it is applied to a system of sixth order differential equations.[21] The equations which relate the coordinates and the standard elliptic elements are

$$X = \frac{a(1 - e^2)}{1 + \cos v} \sqrt{1 - \sin^2 (v + \omega) \sin^2 i} \cos (2\pi - \lambda_n + \beta) \qquad (41)$$

$$Y = \frac{a(1 - e^2)}{1 + e \cos v} \sqrt{1 - \sin^2 (v + \omega) \sin^2 i} \sin (2\pi - \lambda_n + \beta) \qquad (42)$$

$$Z = \frac{a(1 - e^2)}{1 + e \cos v} \quad \sin (v + \omega) \sin i \tag{43}$$

where λ_n = longitude of the ascending node and

$$\beta = \sin^{-1} \left[\frac{\sin (v + \omega) \cos i}{\sqrt{1 - \sin^2 (v + \omega) \sin^2 i}} \right]$$

where the value of β is taken to be the principle value of the arcsine. In order to resolve the ambiguity in the determination of β, check the sign of the quantity $\cos (v + \omega)$. If $\cos (v + \omega) > 0$, then the principle value obtained is correct; if $\cos (v + \omega) < 0$, then π minus the principle value is the correct value of β.

Equations for $\frac{da}{dt}$, $\frac{de}{dt}$, etc. The equations for the variation of the elements will be derived using two methods. The first method used is referred to as the classical method, and can be found in Brouwer and Clemence's book but will be reproduced in part here for convenience. The second method, using the Pfaffian expression $\delta \phi_d - d\phi_\delta = 0$[22], was developed by Dr. Peter Musen in his lecture notes from an Advanced Celestial Mechanics course taught by him at the University of Maryland, part of which will be found in Appendix C.

In the classical method, one of the basic assumptions is that each of the components of the disturbing force can be written as a derivative of some disturbing function R. In addition we should remember that if the disturbing force should cease to exist (an instantaneous ellipse) the solutions for the position and velocity components would be of the form

$$X = f_1 (e_1, e_2, \ldots e_6 ; t) \qquad \dot{X} = g_1 (e_1, e_2, \ldots e_6 ; t)$$

$$Y = f_2 (e_1, e_2, \ldots e_6 ; t) \qquad \dot{Y} = g_2 (e_1, e_2, \ldots e_6 ; t) \tag{44}$$

$$Z = f_3\,(e_1, e_2, \dots e_6\,;\,t) \qquad \dot{Z} = g_3\,(e_1, e_2, \dots e_6\,;\,t)$$

where $e_1, e_2, \dots e_6$ represent the constants of integration, which in this case are the standard elliptic elements. The problem now is to satisfy equation (30) by the equations (44) which apply to elliptic motion, which means that we must first derive the equations for the variable elements $e_1, e_2, \dots e_6$ in perturbed motion. Now in perturbed motion it is true that

$$\frac{dx}{dt} = \frac{\partial f_1}{\partial t} + \sum_{j=1}^{6} \frac{\partial f_1}{\partial e_j}\,\frac{de_j}{dt}$$

$$\frac{dy}{dt} = \frac{\partial f_2}{\partial t} + \sum_{j=1}^{6} \frac{\partial f_2}{\partial e_j}\,\frac{de_j}{dt} \qquad (45)$$

$$\frac{dz}{dt} = \frac{\partial f_3}{\partial t} + \sum_{j=1}^{6} \frac{\partial f_3}{\partial e_j}\,\frac{de_j}{dt}$$

and when these equations are differentiated again and substituted into

$$\frac{d^2x}{dt^2} + \frac{\mu}{r^3}\,x = \frac{\partial R}{\partial x}$$

$$\frac{d^2y}{dt^2} + \frac{\mu}{r^3}\,y = \frac{\partial R}{\partial y} \qquad (46)$$

$$\frac{d^2z}{dt^2} + \frac{\mu}{r^3}\,z = \frac{\partial R}{\partial z}$$

three equations result for the six variables $e_1, e_2, \dots e_6$, which equations could be satisfied in an infinite number of ways. Therefore, we introduce an additional set of conditions to make the problem well defined. The choice of these conditions is dictated by the fact that, in unperturbed motion,

$$\frac{dx}{dt} = \frac{\partial f_1}{\partial t}, \quad \frac{dy}{dt} = \frac{\partial f_2}{\partial t}, \quad \frac{dz}{dt} = \frac{\partial f_3}{\partial t} \tag{47}$$

and that, in perturbed motion, we would like to be able to express the coordinates and velocity in the form of equations (44), where e_1, e_2, ... e_6 would again be constants of integration. The additional conditions are then

$$\sum_{j=1}^{6} \frac{\partial f_1}{\partial e_j} \frac{de_j}{dt} = 0$$

$$\sum_{j=1}^{6} \frac{\partial f_2}{\partial e_j} \frac{de_j}{dt} = 0 \tag{48}$$

$$\sum_{j=1}^{6} \frac{\partial f_3}{\partial e_j} \frac{de_j}{dt} = 0.$$

Differentiating equations (47) once again with respect to the time and substituting the result into equations (46) yields

$$\frac{\partial^2 f_1}{\partial t^2} + \sum_{j=1}^{6} \frac{\partial g_1}{\partial e_j} \frac{de_j}{dt} + \frac{\mu}{r^3} f_1 = \frac{\partial R}{\partial x}$$

$$\frac{\partial^2 f_2}{\partial t^2} + \sum_{j=1}^{6} \frac{\partial g_2}{\partial e_j} \frac{de_j}{dt} + \frac{\mu}{r^3} f_2 = \frac{\partial R}{\partial y} \tag{49}$$

$$\frac{\partial^2 f_3}{\partial t^2} + \sum_{j=1}^{6} \frac{\partial g_3}{\partial e_j} \frac{de_j}{dt} + \frac{\mu}{r^3} f_3 = \frac{\partial R}{\partial z} .$$

But since f_1, f_2, f_3 were defined such as to satisfy the equations of elliptic motion the first and third terms in each of the equations (49) cancel each other, and we have

$$\sum_{j=1}^{6} \frac{\partial g_1}{\partial e_j} \frac{de_j}{dt} = \frac{\partial R}{\partial x}$$

$$\sum_{j=1}^{6} \frac{\partial g_2}{\partial e_j} \frac{de_j}{dt} = \frac{\partial R}{\partial y} \qquad (50)$$

$$\sum_{j=1}^{6} \frac{\partial g_3}{\partial e_j} \frac{de_j}{dt} = \frac{\partial R}{\partial z} .$$

The equations (48) and (50), which represent six first order equations, are exactly equivalent to equations (46), which represent three second order equations. If we then multiply equations (48) and (50) successively by $-\partial\dot{x}/\partial e_i$, $-\partial\dot{y}/\partial e_i$, $-\partial\dot{z}/\partial e_i$, $+\partial x/\partial e_i$, $+\partial y/\partial e_i$, $+\partial z/\partial e_i$ and replace f_1, f_2, f_3 by x,y,z, and g_1, g_2, g_3 by $\dot{x}$, $\dot{y}$, $\dot{z}$ we obtain

$$-\sum_{i=1}^{6} \sum_{j=1}^{6} \frac{\partial x}{\partial e_j} \frac{\partial \dot{x}}{\partial e_i} \frac{de_j}{dt} = 0$$

$$-\sum_{i=1}^{6} \sum_{j=1}^{6} \frac{\partial y}{\partial e_j} \frac{\partial \dot{y}}{\partial e_i} \frac{de_j}{dt} = 0 \qquad (48a)$$

$$-\sum_{i=1}^{6} \sum_{j=1}^{6} \frac{\partial z}{\partial e_j} \frac{\partial \dot{z}}{\partial e_i} \frac{de_j}{dt} = 0$$

$$\sum_{i=1}^{6} \sum_{j=1}^{6} \frac{\partial \dot{x}}{\partial e_j} \frac{\partial x}{\partial e_i} \frac{de_j}{dt} = \sum_{i=1}^{6} \frac{\partial R}{\partial x} \frac{\partial x}{\partial e_i}$$

$$\sum_{i=1}^{6} \sum_{j=1}^{6} \frac{\partial \dot{y}}{\partial e_j} \frac{\partial y}{\partial e_i} \frac{de_j}{dt} = \sum_{i=1}^{6} \frac{\partial R}{\partial y} \frac{\partial y}{\partial e_i} \qquad (50a)$$

$$\sum_{i=1}^{6} \sum_{j=1}^{6} \frac{\partial \dot{z}}{\partial e_j} \frac{\partial z}{\partial e_i} \frac{de_j}{dt} = \sum_{i=1}^{6} \frac{\partial R}{\partial z} \frac{\partial z}{\partial e_i} .$$

Adding equations (48a) and (50a), introducing the Lagrange brackets defined as

$$[e_i, e_j] = \frac{\partial x}{\partial e_i} \frac{\partial \dot{x}}{\partial e_j} - \frac{\partial x}{\partial e_j} \frac{\partial \dot{x}}{\partial e_i} + \frac{\partial y}{\partial e_i} \frac{\partial \dot{y}}{\partial e_j} - \frac{\partial y}{\partial e_j} \frac{\partial \dot{y}}{\partial e_i} + \frac{\partial z}{\partial e_i} \frac{\partial \dot{z}}{\partial e_j} - \frac{\partial z}{\partial e_j} \frac{\partial \dot{z}}{\partial e_i}$$

and noting that the right-hand side of the new equations becomes $\partial R/\partial e_i$, we obtain

$$\sum_{i=1}^{6} \sum_{j=1}^{6} [e_i, e_j] \frac{de_j}{dt} = \sum_{i=1}^{6} \frac{\partial R}{\partial e_i} . \qquad (51)$$

The quantities to be evaluated in the above six equations are the thirty-six Lagrange brackets $[e_i, e_j]$. By the very definition of these Lagrange brackets we can see that $[e_i, e_i] = 0$ and that $[e_i, e_j] = -[e_j, e_i]$. Therefore the determinant formed by the Lagrange brackets in equation (51) is antisymmetric and the diagonal elements are zero.

(A discussion of the time independence of the Lagrange brackets and Whittaker's method for evaluating them are contained in Brouwer and Clemence's text in some detail and hence will not be reproduced here.)

The equation which is used to compute the necessary Lagrange brackets is[23]

$$[e_i, e_j] = \frac{\partial(\sigma, L)}{\partial(e_i, e_j)} + \frac{\partial(\omega, G)}{\partial(e_i, e_j)} + \frac{\partial(\Omega, H)}{\partial(e_i, e_j)} \tag{52}$$

where n is defined in Kepler's Third Law $\mu = n^2a^3$, $\sigma = M - nt$, $L = \sqrt{\mu a}$, $G = \sqrt{\mu a(1-e^2)}$, and $H = \sqrt{\mu a(1-e^2)}\ \cos i$. In addition we shall define $a = e_1$, $e = e_2$, $i = e_3$, $\omega = e_4$, $\Omega = e_5$, and $\sigma = e_6$. The only non-zero brackets then are

$$[e_1, e_4] = [a, \omega] = -\tfrac{1}{2}\, na\sqrt{1-e^2} = -[\omega, a] \tag{52a}$$

$$[e_1, e_5] = [a, \Omega] = -\tfrac{1}{2}\, na\sqrt{1-e^2}\cos i = -[\Omega, a] \tag{52b}$$

$$[e_1, e_6] = [a, \sigma] = -\tfrac{1}{2}\, na = -[\sigma, a] \tag{52c}$$

$$[e_2, e_4] = [e, \omega] = \frac{na^2e}{\sqrt{1-e^2}} = -[\omega, e] \tag{52d}$$

$$[e_2, e_5] = [e, \Omega] = \frac{na^2e\cos i}{\sqrt{1-e^2}} = -[\Omega, e] \tag{52e}$$

$$[e_3, e_5] = [i, \Omega] = na^2\sin i\ \sqrt{1-e^2} = -[\Omega, i]. \tag{52f}$$

Substituting these into the six equations represented by equation (51) gives the following set of equations

$$-\tfrac{1}{2}\, na\sqrt{1-e^2}\ \frac{d\omega}{dt} - \tfrac{1}{2}\, na\sqrt{1-e^2}\ \cos i\ \frac{d\Omega}{dt} - \tfrac{1}{2}\, na\ \frac{d\sigma}{dt} = \frac{\partial R}{\partial a} \tag{53a}$$

$$\frac{na^2e}{\sqrt{1-e^2}}\ \frac{d\omega}{dt} + \frac{na^2e\cos i}{\sqrt{1-e^2}}\ \frac{d\Omega}{dt} = \frac{\partial R}{\partial e} \tag{53b}$$

$$na^2 \sin i \sqrt{1-e^2} \frac{d\Omega}{dt} = \frac{\partial R}{\partial i} \tag{53c}$$

$$\tfrac{1}{2} na \sqrt{1-e^2} \frac{da}{dt} - \frac{na^2 e}{\sqrt{1-e^2}} \frac{de}{dt} = \frac{\partial R}{\partial \omega} \tag{53d}$$

$$\tfrac{1}{2} na \sqrt{1-e^2} \cos i \frac{da}{dt} - \frac{na^2 e \cos i}{\sqrt{1-e^2}} \frac{de}{dt} - na^2 \sin i \sqrt{1-e^2} \frac{di}{dt} = \frac{\partial R}{\partial \Omega} \tag{53e}$$

$$\tfrac{1}{2} na \frac{da}{dt} = \frac{\partial R}{\partial \sigma} \ . \tag{53f}$$

The above equations can then be solved for the time derivatives of the elliptic elements. Equation (53f) yields immediately the time derivative of the semi-major axis as

$$\frac{da}{dt} = \frac{2}{na} \frac{\partial R}{\partial M} \tag{54}$$

remembering that $\frac{\partial R}{\partial \sigma}$ can be replaced by $\frac{\partial R}{\partial M}$ for a given epoch. Likewise equation (53c) yields the time derivative of the right ascension of the ascending node as

$$\frac{d\Omega}{dt} = \frac{1}{na^2 \sqrt{1-e^2} \sin i} \frac{\partial R}{\partial i} \ . \tag{55}$$

The other equations for the time derivatives of the rest of the elliptic elements follow from appropriate substitutions in the other equations and are

$$\frac{d\omega}{dt} = \frac{\sqrt{1-e^2}}{na^2 e} \frac{\partial R}{\partial e} - \frac{\cos i}{na^2 \sqrt{1-e^2} \sin i} \frac{\partial R}{\partial i} \tag{56}$$

$$\frac{d\sigma}{dt} = -\frac{2}{na} \frac{\partial R}{\partial a} - \frac{(1-e^2)}{na^2 e} \frac{\partial R}{\partial e} \tag{57}$$

$$\frac{de}{dt} = \frac{(1-e^2)}{na^2 e} \frac{\partial R}{\partial M} - \frac{1-e^2}{na^2 e} \frac{\partial R}{\partial \omega} \tag{58}$$

$$\frac{di}{dt} = \frac{\cos i}{na^2 \sin i \sqrt{1-e^2}} \frac{\partial R}{\partial \omega} - \frac{1}{na^2 \sin i \sqrt{1-e^2}} \frac{\partial R}{\partial \Omega} . \tag{59}$$

If we remember that $\frac{d\sigma}{dt} = \frac{dM}{dt} - n$, equation (57) can be re-written as

$$\frac{dM}{dt} = n - \frac{2}{na} \frac{\partial R}{\partial a} - \frac{(1-e^2)}{na^2 e} \frac{\partial R}{\partial e} \tag{57a}$$

where it should be noted that n appearing as the first term in the above expression is itself a function of time as can be deduced from equation (54) and the fact that

$$\frac{dn}{dt} = - \frac{3}{2} \frac{n}{a} \frac{da}{dt} .$$

Substitution of the disturbing functions R_1, R_2, R_3, or R_4 into the above equations will produce first order secular, second order secular, first order long periodic, or first order short periodic variations of all of the standard elliptic elements.

Elliptic Elements as Functions of Time. In order to simplify the long periodic part of the disturbing function, and because we are primarily interested in short periodic effects on the elements, we will ignore the cos 2ω term which appears in equation (39) in this work, and rewrite the equation as

$$R_3 = - GM \left[\frac{3}{2} J_3 (r_E^3/a^4) \sin i \left(\frac{5}{4} \sin^2 i - 1\right) e (1-e^2)^{-5/2} \sin\omega \right] . \tag{39a}$$

It should also be noted that $\frac{\partial R}{\partial \Omega} = 0$ for R equals R_1, R_2, R_3, or R_4 since tesseral harmonics were not included in the force function expansion. Upon examination of equations (54), (55), (56), (57a), (58), and (59), in conjunction with the expressions for R_1, R_2, R_3, and R_4 it is found that the semi-major axis exhibits only short periodic effects, eccentricity and inclination exhibit both long and short periodic effects, mean anomaly exhibits secular and short periodic effects, and argument of perigee and right ascension exhibit secular, long periodic, and short periodic effects due to the axially symmetric model of the gravitational field of the earth. An example of the calculation of the elliptic elements as functions of time will be given using the equation for the time derivative of the inclination

$$\frac{di}{dt} = \frac{\cos i}{na^2 \sin i \sqrt{1-e^2}} \frac{\partial R}{\partial \omega} . \tag{59a}$$

Since R_1 and R_2 are by definition functions of neither v or ω, $\frac{\partial R_1}{\partial \omega} = 0$,

$\frac{\partial R_2}{\partial \omega} = 0$, and

$$\frac{\partial R_3}{\partial \omega} = - GM \left[\frac{3}{2} J_3 (r_E^3/a^4) \sin i \left(\frac{5}{4} \sin^2 i - 1\right) e (1-e^2)^{-5/2} \cos \omega\right]$$

and

$$\frac{\partial R_4}{\partial \omega} = - GM \left(\frac{3}{2} J_2\right) (r_E^2/a^3) \left[(a/r)^3 \sin^2 i \sin 2 (v + \omega)\right]$$

since $(a/r)^3$ is a function of v only. Therefore

$$\frac{di}{dt} = \frac{\cos i}{na^2 \sin i \sqrt{1-e^2}} \left[\frac{\partial R_3}{\partial \omega} + \frac{\partial R_4}{\partial \omega}\right]$$

and

$$di = \frac{GM \cos i}{na^2 \sin i \sqrt{1-e^2}} \left[- \frac{3}{2} J_3 (r_E^3/a^4) \sin i \left(\frac{5}{4} \sin^2 i - 1\right) e (1-e^2)^{-5/2} \cos \omega - \frac{3}{2} J_2 (r_E^2/a^3) (a/r)^3 \sin^2 i \sin 2 (v + \omega)\right] dt. \tag{60}$$

In order to facilitate the integration of the above equation, it is useful to substitute

$$dt = \frac{a^{3/2}}{\sqrt{GM}} (a/r_E)^2 \frac{(1-e^2)^2}{3 J_2 (1 - 5/4 \sin^2 i)} d\omega$$

when integrating the long periodic part of the inclination equation and to substitute

$$dt = \frac{dM}{n}$$

when integrating the short periodic part of the equation for the inclination. Upon making the above substitutions in equation (60) we obtain

$$di = \tfrac{1}{2}(J_3/J_2) (r_E/a) \frac{e \cos i}{(1-e^2)} \cos\omega \, d\omega$$

$$- \frac{3}{4} J_2 (r_E/a)^2 \frac{\sin 2i}{\sqrt{1-e^2}} (a/r)^3 \sin 2 (v + \omega) \, dM \quad (60a)$$

the first term of which is readily integrated, and the second term of which can be integrated with the aid of equations (36a) and (36b), for $p = -3$ and $q = 2$.

Since in this work we are considering only first order perturbations, the solutions for the orbital elements will be of the form

$$a = a_o + \Delta a_1$$

$$e = e_o + \Delta e_1$$

$$i = i_o + \Delta i_1$$

$$\omega = \omega_{EPOCH} + \dot{\omega} t + \Delta\omega_1$$

$$\Omega = \Omega_{EPOCH} + \dot{\Omega} t + \Delta\Omega_1$$

$$M = M_{EPOCH} + \dot{M} t + \Delta M_1$$

where the Δ's represent the first order periodic corrections and the zero subscripts indicate the constants of integration for the particular elements. The quantities ω_{EPOCH}, Ω_{EPOCH}, and M_{EPOCH} are the values of the argument of perigee, right ascension, and mean anomaly for t = o and from which periodic terms have been subtracted. Consequently, in this approximation, it is permissible to substitute a_o, e_o, and i_o for the elements which appear as coefficients of the partial derivatives in equations (54), (55), (56), (57a), (58), and (59). The substitution of these integration constants for the semi-major axis, eccentricity, and inclination makes the integration of equation (60a) and the other orbital element equations comparatively simple.

Upon integrating equation (60a) we then obtain

$$i = i_o + \tfrac{1}{2} (J_3/J_2) (r_E/p_o) e_o \cos i_o \sin \omega$$

$$+ \frac{3}{8} J_2 (r_E/p_o)^2 \sin 2 i_o \Big[e_o \cos (v + 2\omega) + \cos 2 (v + \omega) + \frac{1}{3} e_o \cos (3v + 2\omega) \Big] \quad (61)$$

where $p_o = a_o (1-e_o^2)$ and is defined as the semi-latus rectum of the instantaneous ellipse. Similar procedures are used to integrate the other equations for the variation of the elements, which may be found in the literature[24]. The results are written below for the other elements.

The equation for the variation of the semi-major axis is

$$a = a_o + \frac{3 J_2 r_E^2}{a_o (1-e_o^2)^3} \Bigg\{ (1 - \frac{3}{2} \sin^2 i_o) \Big[- \frac{1}{3} (1-e_o^2)^{3/2} + \frac{1}{3} (1 + \frac{3}{2} e_o^2)$$

$$+ (e_o + \tfrac{1}{4} e_o^3) \cos v + \tfrac{1}{2} e_o^2 \cos 2v + \frac{1}{12} e_o^3 \cos 3v \Big]$$

$$+ \tfrac{1}{2} \sin^2 i_o \left[\frac{3}{8} e_o^3 \cos (v + 2\omega) + \frac{1}{8} e_o^3 \cos (v - 2\omega) \right.$$

$$+ (1 + \frac{3}{2} e_o^2) \cos 2 (v + \omega) + \frac{3}{8} e_o^3 \cos (3v + 2\omega)$$

$$\left. + \frac{3}{4} e_o^2 \cos (4v + 2\omega) + \frac{1}{8} e_o^3 \cos (5v + 2\omega) + \frac{3}{4} e_o^2 \cos 2\omega \right]$$

$$\left. + \frac{3}{4} e_o \sin^2 i_o \left[\cos (v + 2\omega) + \cos (3v + 2\omega) \right] \right\} . \tag{62}$$

The variation of the eccentricity can be written as

$$e = e_o - \tfrac{1}{2}(J_3/J_2) (r_E/a_o) \sin i_o \sin\omega + e_s \tag{63}$$

where the short periodic part e_s is defined as

$$e_s = \frac{3}{2} J_2(r_E/p_o)^2 \left\{ (1 - \frac{3}{2} \sin^2 i_o) \left[- \frac{1}{3e_o} (1-e_o^2)^{3/2} + \frac{1}{3e_o} (1 + \frac{3}{2} e_o^2) \right. \right.$$

$$\left. + (1 + \tfrac{1}{4} e_o^2) \cos v + \tfrac{1}{2} e_o \cos 2v + \frac{1}{12} e_o^2 \cos 3v \right]$$

$$+ \sin^2 i_o \left[(\frac{3}{4} + \frac{3}{16} e_o^2) \cos (v + 2\omega) + \frac{1}{16} e_o^2 \cos (v - 2\omega) \right.$$

$$+ (\frac{1}{2e_o} + \frac{3}{4} e_o) \cos 2 (v + \omega) + (\frac{3}{4} + \frac{3}{16} e_o^2) \cos (3v + 2\omega)$$

$$\left. \left. + \frac{3}{8} e_o \cos (4v + 2\omega) + \frac{1}{16} e_o^2 \cos (5v + 2\omega) + \frac{3}{8} e_o \cos 2\omega \right] \right\}$$

$$- \frac{3}{2} J_2(r_E/a_o)^2 \frac{\sin^2 i_o}{(1-e_o^2)} \left\{ \tfrac{1}{2} \cos (v + 2\omega) + \frac{1}{2e_o} \cos 2(v + \omega) \right.$$

$$\left. + \frac{1}{6} \cos (3v + 2\omega) \right\} . \tag{63a}$$

The mean anomaly variation is

$$M = M_{EPOCH} + \sqrt{\frac{GM}{a_o^3}} \left[1 + \frac{3}{2} J_2 (r_E/p_o)^2 (1 - \frac{3}{2} \sin^2 i_o) \sqrt{1-e_o^2} \right] t + M_s \tag{64}$$

where t represents the time after epoch and the short periodic part M_s is given by

$$M_s = \frac{3}{2} J_2(r_E/p_o)^2 \frac{\sqrt{1-e_o^2}}{e_o} \Bigg\{ - (1 - \frac{3}{2}\sin^2 i_o) \Big[(1 - \tfrac{1}{4} e_o^2) \sin v$$

$$+ \tfrac{1}{2} e_o \sin 2v + \frac{1}{12} e_o^2 \sin 3v \Big] + \sin^2 i_o \Big[(\tfrac{1}{4} + \frac{5}{16} e_o^2) \sin (v + 2\omega)$$

$$- \frac{1}{16} e_o^2 \sin (v - 2\omega) + (- \frac{7}{12} + \frac{1}{48} e_o^2) \sin (3v + 2\omega)$$

$$- \frac{3}{8} e_o \sin (4v + 2\omega) - \frac{1}{16} e_o^2 \sin (5v + 2\omega) \Big] \Bigg\} . \tag{64a}$$

The variation of the right ascension of the ascending node is given by

$$\Omega = \Omega_{EPOCH} + \dot{\Omega} t - \tfrac{1}{2}(J_3/J_2)(r_E/p_o) \frac{e_o \cos i_o}{\sin i_o} \cos \omega + \Omega_s \tag{65}$$

where t is again the time after epoch, $\dot{\Omega}$ is given by

$$\dot{\Omega} = - 9.9596 \frac{(r_E/a_o)^{7/2} \cos i_o}{(1-e_o^2)^2}$$

$$- 0.048778 \frac{(r_E/a_o)^{11/2} \cos i_o}{(1-e_o^2)^4} (1 - \frac{7}{4} \sin^2 i_o)(1 + \frac{3}{2} e_o^2) \tag{65a}$$

expressed in units of degrees per day, and the short period variation Ω_s is

$$\Omega_s = - \frac{3}{2} J_2 (r_E/p_o)^2 \cos i_o \Big[(v-M) + e_o \sin v - \tfrac{1}{2} e_o \sin (v + 2\omega)$$

$$- \tfrac{1}{2} \sin 2(v + \omega) - \frac{1}{6} e_o \sin (3v + 2\omega) \Big] . \tag{65b}$$

Finally the argument of perigee variation is given by

$$\omega = \omega_{EPOCH} + \dot{\omega} t - \tfrac{1}{2}(J_3/J_2)(r_E/p_o) \frac{\sin^2 i_o - e_o^2 \cos^2 i_o}{e_o \sin i_o} \cos \omega + \omega_s \tag{66}$$

where again t represents time after epoch, $\dot{\omega}$ is given by

$$\dot{\omega} = 4.9798 \frac{(r_E/a_o)^{7/2} (5 \cos^2 i_o - 1)}{(1-e_o^2)^2}$$

$$+ 0.097555 \frac{(r_E/a_o)^{11/2} \cos i_o}{(1-e_o^2)^4} \left[1 - \frac{31}{8} \sin^2 i_o + \frac{49}{16} \sin^4 i_o \right] \qquad (66a)$$

expressed in units of degrees per day, and the short period variation ω_s is

$$\omega_s = \frac{3}{2} J_2 (r_E/p_o)^2 \left\{ (2 - \frac{5}{2} \sin^2 i_o) \left[(v - M) + e_o \sin v \right] \right.$$

$$+ (1 - \frac{3}{2} \sin^2 i_o) \left[(\frac{1}{e_o} - \frac{1}{4} e_o) \sin v + \tfrac{1}{2} \sin 2v + \frac{1}{12} e_o \sin 3v \right]$$

$$- (\tfrac{1}{2} e_o - \frac{15}{16} e_o \sin^2 i_o + \frac{\sin^2 i_o}{4 e_o}) \sin (v + 2\omega)$$

$$+ \frac{1}{16} e_o \sin^2 i_o \sin (v - 2\omega) - \tfrac{1}{2} (1 - \frac{5}{2} \sin^2 i_o) \sin 2 (v + \omega)$$

$$+ (\frac{7 \sin^2 i_o}{12 e_o} - \frac{1}{6} e_o + \frac{19}{48} e_o \sin^2 i_o) \sin (3v + 2\omega)$$

$$\left. + \frac{3}{8} \sin^2 i_o \sin (4v + 2\omega) + \frac{1}{16} e_o \sin^2 i_o \sin (5v + 2\omega) \right\} . \qquad (66b)$$

In equations (61), (62), and (63), a_o, e_o, i_o, represent the constants of integration; in equations (64), (65), and (66), M_{EPOCH}, Ω_{EPOCH}, and ω_{EPOCH} represent the values of mean anomaly, right ascension, and argument of perigee at epoch (t = o) from which periodic perturbations have been subtracted.

CHAPTER IV

MODEL ACCURACY VERSUS SYSTEM ACCURACY

Measurement of Satellite Position Errors. Measurement of satellite position errors along the orbital path involve first the construction of what we shall call an error plane normal to the reference orbit plane. This is accomplished by constructing a unit vector normal to the plane of the orbit and a unit vector in the direction of a line which passes through the center of the earth (see figure 4). The vector cross product of a unit vector along the line of nodes in the direction of the ascending node with the position vector from the center of the earth (which here is taken to be $\bar{R}_1$) produces a vector normal to the orbital plane. Therefore

$$\bar{A} = \frac{\bar{N} \times \bar{R}_1}{|\bar{N} \times \bar{R}_1|} \tag{67}$$

where $\bar{N}$, $\bar{R}_1$, and $\bar{A}$ are unit vectors, and $\bar{A}$ is the vector normal to the orbital plane. A unit vector normal to the error plane is then constructed from

$$\bar{B} = \frac{\bar{A} \times \bar{R}_1}{|\bar{A} \times \bar{R}_1|} \text{ .} \tag{68}$$

Once the error plane is determined, we can measure satellite position errors in this plane between an unperturbed or reference orbit and some other orbit which we shall call the perturbed orbit. In figure 5 orbit 1 represents the unperturbed or reference orbit and P_1 is the point where this satellite path intersects the error plane; orbit 2 represents the perturbed orbit and P_2 is the point where this orbit intersects the error plane.

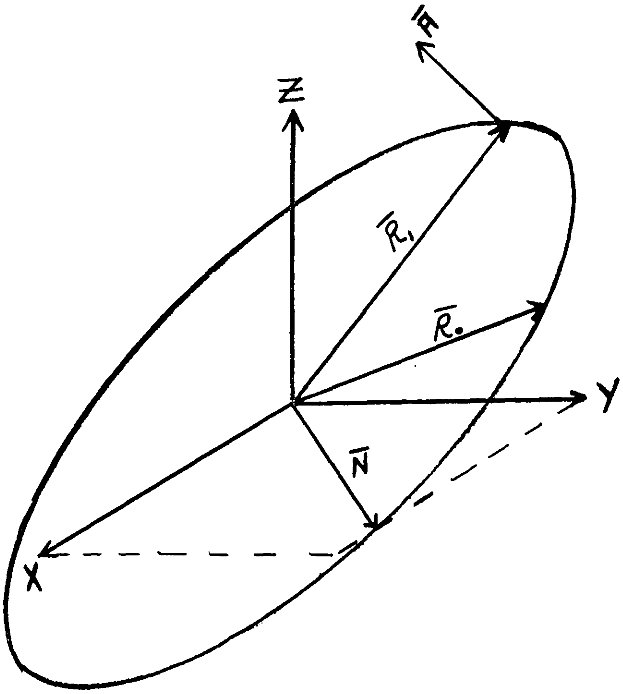

FIGURE 4 - CONSTRUCTION OF ERROR PLANE

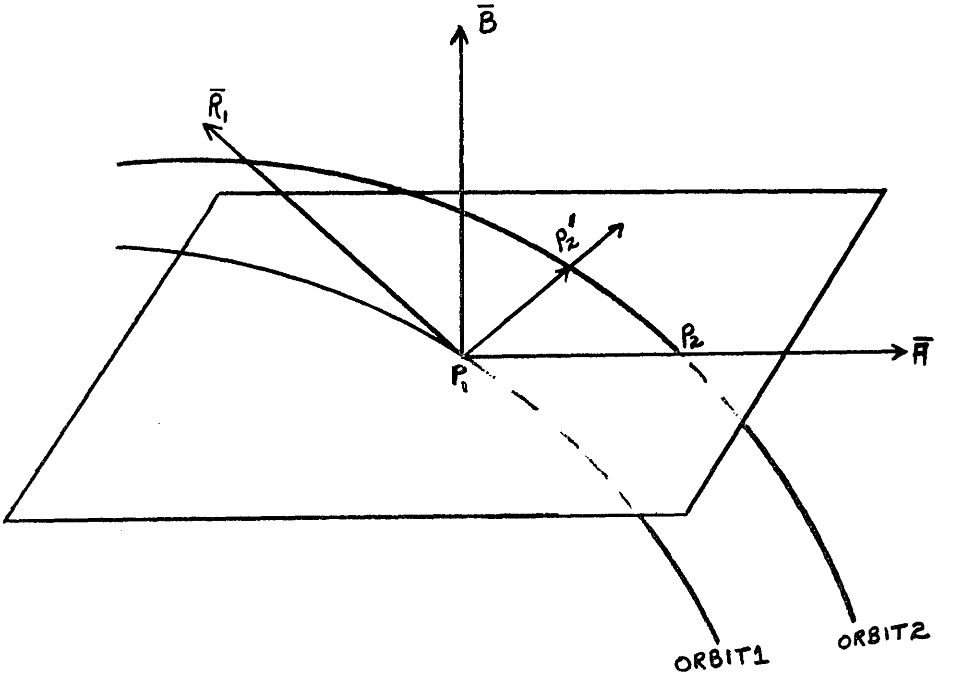

FIGURE 5 - UNPERTURBED AND PERTURBED ORBIT PATHS

In the computational procedure for calculating these position errors, we begin with a set of unperturbed orbital elements at some epoch, represented by the position vector $\overline{R}_o$ in figure 4, and we also have a set of perturbed elements at this same epoch, represented by a slightly different value of $\overline{R}_o$. The unperturbed true anomaly at this epoch is then updated by some arbitrary small increment and a corresponding time difference is computed from equation (18). Using this time difference the argument of perigee can be updated using equation (16), and longitude of the node at this time can be computed from equations (23a) or (23b) once GHA has been updated and regression during the elapsed time has been computed from equation (15). Assuming the semi-major axis, eccentricity, and inclination are essentially constant during this time interval, the coordinates of this point, and hence the position vector $\overline{R}_1$, can be calculated using equations (41), (42), and (43). The vector $\overline{N}$ can be computed from

$$\overline{N} = \cos\lambda n \ \overline{i} - \sin\lambda n \ \overline{j} \tag{69}$$

and then vector $\overline{A}$ can be computed from equation (67). The unit vector normal to the error plane is then calculated from equation (68). This procedure establishes the error plane at this time, normal to the orbital plane, where the point P_1 is given by the components of $\overline{R}_1$, namely X_1, Y_1, and Z_1.

(It should be noted here that the above updating is done using only a very approximate model of the earth's potential.)

The perturbed true anomaly is then updated by the same small increment and the corresponding time difference is again computed from equation (18). All of the orbital elements are then updated according to the first order model of the earth's potential using equations (61), (62), (63), (64), (65), and (66). This updating procedure involves first the computation of the constant set of elements a_o, e_o, i_o, ω_{EPOCH}, Ω_{EPOCH}, and M_{EPOCH} from the given set of perturbed elements at epoch, using this given set as a first guess for the constant set and performing an iteration process until successive constant elements are within a pre-determined tolerance. Having obtained this constant set, the argument of perigee and mean anomaly can be updated approximately assuming only a first order secular variation for them. The mean anomaly can then be converted to true anomaly and a first guess for all of the updated elements for the given time interval can be calculated from equations (61), (62), (63), (64), (65), and (66). This computation produces more accurate values for the argument of perigee and mean anomaly, and having again converted this more accurate value of mean anomaly to true anomaly, we can begin an iteration process by substituting the successively more accurate values of argument of perigee and true anomaly into equations (61), (62), (63), (64), (65), and (66). This iteration process is terminated when successive values of each of the updated orbital elements is within a pre-determined tolerance. This final set of updated perturbed elements can then be substituted into equations (41), (42), and (43) and in general will produce a point P_2' (see figure 5) which does not lie in the error plane. In addition, the time at which the perturbed satellite reaches the error plane will differ from the time of arrival of the unperturbed satellite by some small amount. The point P_2' is then adjusted to the point P_2 in the error plane by adjusting the time difference between the epoch time and the time of arrival of the perturbed satellite in the error plane until the distance of the perturbed satellite out of the plane is less than a tolerance of 0.01 mile. Once this is done, the errors between the two paths in the error plane can be computed as follows:

1. The cross track error, Δx, is the component of the vector $\overline{P_2P_1}$ along the vector $\overline{A}$ and is given by $\overline{P_2P_1} \cdot \overline{A}$.
2. The height error, Δh, is the component of the vector $\overline{P_2P_1}$ along the vector $\overline{R_1}$ and is given by $\overline{P_2P_1} \cdot \overline{R}_1$.
3. The range error, ΔR, is merely the square root of the sum of the squares of the cross track and height errors.
4. The time error, Δt, is the difference between the time of arrival of the unperturbed and the perturbed satellite in the error plane.

The above procedure is then repeated by incrementing the unperturbed and perturbed true anomalies for one or more complete revolutions of the unperturbed and perturbed satellite paths. Since the difference between the epoch anomaly and some other anomaly at a later time, corrected for the rotation of perigee, is defined as the central angle, this enables us to plot height, cross track, range, and time errors as a function of central angle for one or more revolutions.

The procedure outlined above for the construction of the error plane and updating the unperturbed elements was developed and adapted for the Naval Research Electronic Computer by J. A. Buisson and H. G. DeVezin of the Space Surveillance Branch and the updating procedure for the perturbed set of orbital elements was developed and adapted for the NAREC by the author in order to incorporate first order oblateness effects on all of the orbital elements.

<u>System Angle Perturbations</u>. When speaking of system angle accuracy, it has been found experimentally from optical measurements that there is

a constant error in the system measurement of direction cosines when the error in angle is plotted against the cosine of the angle. This error in system measurement is usually spoken of in terms of some nominal number of degrees at zenith, which for example could be 0.01°. This means that in order to compute a theoretical perturbation in the system angle measurement consistent with the nominal system accuracy we take the measured angle, convert it from degrees to radians, subtract the sine of 0.01° (or the cosine of 89.99°) from it, take the arc cosine of the resulting number and this is what we shall call the perturbed angle. This procedure enables us to generate an unperturbed and eight perturbed sets of orbital elements from the orbital element computation procedure outlined in Chapter I. The eight perturbed sets of elements are the result of perturbing one of the eight input angles at a time. By updating these eight perturbed sets and the unperturbed set in the manner cited for the reference orbit in the previous section (that is we assume a, e, i are constant and that ω, Ω, and M exhibit only first order secular variations) we are able to generate errors in satellite position due to system measurement accuracy.

First Order Differential Corrections to the Elements. The functional form of the orbital element variation equations is

$$\begin{aligned}
a &= F_1 \left(a_o, e_o, i_o, v, \omega\right) \\
e &= F_2 \left(a_o, e_o, i_o, v, \omega\right) \\
i &= F_3 \left(a_o, e_o, i_o, v, \omega\right) \\
\Omega &= F_4 \left(a_o, e_o, i_o, \Omega_{EPOCH}, v, \omega, M, t\right) \\
\omega &= F_5 \left(a_o, e_o, i_o, \omega_{EPOCH}, v, \omega, M, t\right) \\
M &= F_6 \left(a_o, e_o, i_o, M_{EPOCH}, v, \omega, t\right)
\end{aligned} \tag{70}$$

where we will assume that the elements a_o, e_o, i_o, Ω_{EPOCH}, ω_{EPOCH}, and M_{EPOCH} have been computed from a measured set of elements for some given epoch.

We would like to have the coordinates of the two observed points on the orbital path in the form

$$
\begin{aligned}
x_1 &= X_1\ (a_o,\ e_o,\ i_o, \omega_{EPOCH}, \Omega_{EPOCH},\ M_{EPOCH},\ t_1) \\
y_1 &= Y_1\ (a_o,\ e_o,\ i_o, \omega_{EPOCH}, \Omega_{EPOCH},\ M_{EPOCH},\ t_1) \\
z_1 &= Z_1\ (a_o,\ e_o,\ i_o, \omega_{EPOCH},\ M_{EPOCH},\ t_1) \\
x_2 &= X_2\ (a_o,\ e_o,\ i_o, \omega_{EPOCH}, \Omega_{EPOCH},\ M_{EPOCH},\ t_2) \\
y_2 &= Y_2\ (a_o,\ e_o,\ i_o, \omega_{EPOCH}, \Omega_{EPOCH},\ M_{EPOCH},\ t_2) \\
z_2 &= Z_2\ (a_o,\ e_o,\ i_o, \omega_{EPOCH},\ M_{EPOCH},\ t_2)\,.
\end{aligned}
\tag{71}
$$

Consequently, if we were able to write our coordinate equations in the form of equations (71), the first order differential correction solution would involve solving the following six equations for the six unknowns da_o, de_o, di_o, $d\omega_E$, $d\Omega_E$, and dM_E, where for simplicity we shall write $d\omega_E$, $d\Omega_E$, and dM_E for $d\omega_{EPOCH}$, $d\Omega_{EPOCH}$, and dM_{EPOCH}:

$$
\begin{aligned}
dx_1 &= \frac{\partial X_1}{\partial a_o}\, da_o + \frac{\partial X_1}{\partial e_o}\, de_o + \frac{\partial X_1}{\partial i_o}\, di_o + \frac{\partial X_1}{\partial \omega_E}\, d\omega_E + \frac{\partial X_1}{\partial \Omega_E}\, d\Omega_E + \frac{\partial X_1}{\partial M_E}\, dM_E \\
dy_1 &= \frac{\partial Y_1}{\partial a_o}\, da_o + \frac{\partial Y_1}{\partial e_o}\, de_o + \frac{\partial Y_1}{\partial i_o}\, di_o + \frac{\partial Y_1}{\partial \omega_E}\, d\omega_E + \frac{\partial Y_1}{\partial \Omega_E}\, d\Omega_E + \frac{\partial Y_1}{\partial M_E}\, dM_E \\
dz_1 &= \frac{\partial Z_1}{\partial a_o}\, da_o + \frac{\partial Z_1}{\partial e_o}\, de_o + \frac{\partial Z_1}{\partial i_o}\, di_o + \frac{\partial Z_1}{\partial \omega_E}\, d\omega_E + \frac{\partial Z_1}{\partial M_E}\, dM_E \\
dx_2 &= \frac{\partial X_2}{\partial a_o}\, da_o + \frac{\partial X_2}{\partial e_o}\, de_o + \frac{\partial X_2}{\partial i_o}\, di_o + \frac{\partial X_2}{\partial \omega_E}\, d\omega_E + \frac{\partial X_2}{\partial \Omega_E}\, d\Omega_E + \frac{\partial X_2}{\partial M_E}\, dM_E
\end{aligned}
\tag{72}
$$

$$dy_2 = \frac{\partial Y_2}{\partial a_o} da_o + \frac{\partial Y_2}{\partial e_o} de_o + \frac{\partial Y_2}{\partial i_o} di_o + \frac{\partial Y_2}{\partial \omega_E} d\omega_E + \frac{\partial Y_2}{\partial \Omega_E} d\Omega_E + \frac{\partial Y_2}{\partial M_E} dM_E$$

$$dz_2 = \frac{\partial Z_2}{\partial a_o} da_o + \frac{\partial Z_2}{\partial e_o} de_o + \frac{\partial Z_2}{\partial i_o} di_o + \frac{\partial Z_2}{\partial \omega_E} d\omega_E + \frac{\partial Z_2}{\partial M_E} dM_E$$

where the partial derivatives of the coordinates with respect to time do not appear since we assume time is measured correctly and consequently $dt_1 = dt_2 = 0$.

Unfortunately, the actual functional form of the coordinate equations is

$$\begin{aligned} x_1 &= X_1' (a_1, e_1, i_1, \omega_1, \Omega_1, v_1) \\ y_1 &= Y_1' (a_1, e_1, i_1, \omega_1, \Omega_1, v_1) \\ z_1 &= Z_1' (a_1, e_1, i_1, \omega_1, v_1) \\ x_2 &= X_2' (a_2, e_2, i_2, \omega_2, \Omega_2, v_2) \\ y_2 &= Y_2' (a_2, e_2, i_2, \omega_2, \Omega_2, v_2) \\ z_2 &= Z_2' (a_2, e_2, i_2, \omega_2, v_2) \end{aligned} \tag{73}$$

so that the first order differential correction equations are of the form

$$\begin{aligned} dx_1 &= \frac{\partial X_1'}{\partial a_1} da_1 + \frac{\partial X_1'}{\partial e_1} de_1 + \frac{\partial X_1'}{\partial i_1} di_1 + \frac{\partial X_1'}{\partial \omega_1} d\omega_1 + \frac{\partial X_1'}{\partial \Omega_1} d\Omega_1 + \frac{\partial X_1'}{\partial v_1} dv_1 \\ dy_1 &= \frac{\partial Y_1'}{\partial a_1} da_1 + \frac{\partial Y_1'}{\partial e_1} de_1 + \frac{\partial Y_1'}{\partial i_1} di_1 + \frac{\partial Y_1'}{\partial \omega_1} d\omega_1 + \frac{\partial Y_1'}{\partial \Omega_1} d\Omega_1 + \frac{\partial Y_1'}{\partial v_1} dv_1 \\ dz_1 &= \frac{\partial Z_1'}{\partial a_1} da_1 + \frac{\partial Z_1'}{\partial e_1} de_1 + \frac{\partial Z_1'}{\partial i_1} di_1 + \frac{\partial Z_1'}{\partial \omega_1} d\omega_1 + \frac{\partial Z_1'}{\partial v_1} dv_1 \\ dx_2 &= \frac{\partial X_2'}{\partial a_2} da_2 + \frac{\partial X_2'}{\partial e_2} de_2 + \frac{\partial X_2'}{\partial i_2} di_2 + \frac{\partial X_2'}{\partial \omega_2} d\omega_2 + \frac{\partial X_2'}{\partial \Omega_2} d\Omega_2 + \frac{\partial X_2'}{\partial v_2} dv_2 \end{aligned} \tag{74}$$

$$dy_2 = \frac{\partial Y_2'}{\partial a_2} da_2 + \frac{\partial Y_2'}{\partial e_2} de_2 + \frac{\partial Y_2'}{\partial i_2} di_2 + \frac{\partial Y_2'}{\partial \omega_2} d\omega_2 + \frac{\partial Y_2'}{\partial \Omega_2} d\Omega_2 + \frac{\partial Y_2'}{\partial v_2} dv_2$$

$$dz_2 = \frac{\partial X_2'}{\partial a_2} da_2 + \frac{\partial X_2'}{\partial e_2} de_2 + \frac{\partial X_2'}{\partial i_2} di_2 + \frac{\partial X_2'}{\partial \omega_2} d\omega_2 + \frac{\partial X_2'}{\partial v_2} dv_2$$

where the differentials da_1, de_1, di_1, $d\omega_1$, $d\Omega_1$, dv_1, da_2, de_2, di_2, $d\omega_2$, $d\Omega_2$, dv_2 can be evaluated by forming the total differentials of equations (70) at t_1 and t_2, and by transforming dv_1 and dv_2 to dM_1 and dM_2 using the relationship

$$dv = \frac{(1 + e \cos v)^2}{(1 - e^2)^{3/2}} dM + \frac{(2 + e \cos v) \sin v}{(1 - e^2)} de \qquad (75)$$

applied at t_1 and t_2, which is the equation relating the true and mean anomaly in perturbed motion. The total differentials of equations (70) to be applied at t_1 and t_2 are

$$da = \frac{\partial F_1}{\partial a_o} da_o + \frac{\partial F_1}{\partial e_o} de_o + \frac{\partial F_1}{\partial i_o} di_o + \frac{\partial F_1}{\partial v} dv + \frac{\partial F_1}{\partial \omega} d\omega$$

$$de = \frac{\partial F_2}{\partial a_o} da_o + \frac{\partial F_2}{\partial e_o} de_o + \frac{\partial F_2}{\partial i_o} di_o + \frac{\partial F_2}{\partial v} dv + \frac{\partial F_2}{\partial \omega} d\omega$$

$$di = \frac{\partial F_3}{\partial a_o} da_o + \frac{\partial F_3}{\partial e_o} de_o + \frac{\partial F_3}{\partial i_o} di_o + \frac{\partial F_3}{\partial v} dv + \frac{\partial F_3}{\partial \omega} d\omega \qquad (76)$$

$$d\Omega = \frac{\partial F_4}{\partial a_o} da_o + \frac{\partial F_4}{\partial e_o} de_o + \frac{\partial F_4}{\partial i_o} di_o + \frac{\partial F_4}{\partial \Omega_E} d\Omega_E + \frac{\partial F_4}{\partial v} dv$$

$$+ \frac{\partial F_4}{\partial \omega} d\omega + \frac{\partial F_4}{\partial M} dM$$

$$d\omega = \frac{\partial F_5}{\partial a_o} da_o + \frac{\partial F_5}{\partial e_o} de_o + \frac{\partial F_5}{\partial i_o} di_o + \frac{\partial F_5}{\partial \omega_E} d\omega_E + \frac{\partial F_5}{\partial v} dv$$

$$+ \frac{\partial F_5}{\partial \omega} d\omega + \frac{\partial F_5}{\partial M} dM$$

$$dM = \frac{\partial F_6}{\partial a_o} da_o + \frac{\partial F_6}{\partial e_o} de_o + \frac{\partial F_6}{\partial i_o} di_o + \frac{\partial F_6}{\partial M_E} dM_E + \frac{\partial F_6}{\partial v} dv + \frac{\partial F_6}{\partial \omega} d\omega$$

where again the partial derivatives of the elements with respect to time do not appear since it is assumed that $dt = 0$. The solution of the problem to obtain first order differential corrections, once having applied equations (76) at t_1 and t_2, is to substitute the resulting equations for da_1, de_1, di_1, $d\Omega_1$, $d\omega_1$, dM_1, da_2, de_2, di_2, $d\Omega_2$, $d\omega_2$, and dM_2 into equations (74) and to replace dv_1 and dv_2 in equations (74) by equation (75). The result of this substitution will be the set of equations (72), which must then be solved for da_o, de_o, di_o, $d\omega_E$, $d\Omega_E$, and dM_E.

This is accomplished by evaluating dx_2, dy_2, and dz_2 and by assuming that dx_1, dy_1, and dz_1 are zero as a first guess. The first guess for values of dx_2, dy_2, and dz_2 is obtained by calculating the difference between the coordinates obtained directly from the observed direction cosines at t_2, and the coordinates obtained by updating the orbital elements at t_1, using an improved model of the earth's potential, and substituting the resulting orbital elements into equations (41), (42), and (43). After solving equations (72) for the differential corrections to the constant set of elements, these results are then substituted into equations (76) for t_1 and t_2 and the resulting differential corrections for the orbital elements at t_1 and t_2 are obtained. These corrections are then added to the elements at t_1 and t_2, and the resulting elements are substituted into equations (41), (42), and (43) in order to evaluate new coordinates for the two times. The difference between these coordithe two times yields new values of dx_1, dy_1, dz_1, dx_2, dy_2, and dz_2. The above process is then repeated until the corrections to the position vectors $d\bar{R}_1$ and $d\bar{R}_2$ are within 0.001 mile.

These corrected elements can then be used as the perturbed set of elements, and with the elements obtained using an approximate model of the earth's potential, as outlined in Chapter I, as the unperturbed or reference set, the comparison of these two sets using the procedure outlined in the first section of this chapter enables us to produce errors in satellite position due to model accuracy.

This procedure was also adapted by the author for the Naval Research Electronic Computer (NAREC) in order to evaluate the corrected set of elements for many observed sets of data.

CHAPTER V

EXPERIMENTAL RESULTS

Experimental Data. In order to minimize the effects of other perturbations and hence to maximize the effect of perturbations due to first order gravitational anomalies, experimental data from the Vanguard I satellite was used in the computation of the experimental error curves for the comparison of the measuring system accuracy and the model accuracy.

In order to generate curves which would display cross-track, height, time, and range errors as a result of the accuracy of the measuring system, the following procedure was used.

First a set of eight observed angles, obtained from two coincident observations of the satellite along its orbital path, were used to derive a set of unperturbed orbital elements, using an approximate model of the earth's gravitational field as outlined in Chapter I. (These eight observed angles consist of an east-west and a north-south angle for each of the two observing stations on the first pass, and a corresponding set for the two observing stations on the second pass of the satellite.) Then eight sets of perturbed orbital elements were derived by perturbing one of the eight input angles at a time, leaving the other seven angles unaffected, and using the seven unperturbed angles and the one perturbed angle in the procedure of Chapter I to derive the elements. (This angle perturbation procedure was discussed in Chapter IV, and for these experimental data runs, nominal system accuracies of 0.01°, 0.05°, and 0.1° at zenith were used.) The unperturbed or reference orbital elements were then updated using an approximate model of the earth's gravitational field and were used to construct the error plane at 10° anomaly intervals along the orbital path; each of the eight perturbed sets of orbital elements were also updated in a similar manner and the errors between these sets, expressed in terms of cross-track, height, time, and range errors in the error plane, were then calculated for the 10° anomaly intervals along the orbital path for one complete revolution of the satellite. The result of this procedure was eight sets of error curves in Δx, Δh, Δt, and ΔR corresponding to each of the eight perturbed sets of angles. In order to evaluate the effect of a system angle error being present in all of the eight input angles at the same time and in order to avoid the effect of canceling errors, we calculated the square root of the sum of the squares of each of the errors at the 10° anomaly intervals along the path for Δx, Δh, Δt, and ΔR. The result of this computation was one set of error curves in Δx, Δh, Δt, and ΔR for each of the nominal system accuracies used.

In generating curves which would display cross-track, height, time, and range errors due to model accuracy, a similar procedure to the above was used.

In this instance, the unperturbed or reference elements were the same as in the previous procedure and were updated in the same manner in order to construct the error plane at 10° anomaly intervals along the orbital path. However, the perturbed elements were derived by differentially correcting the unperturbed set, using the first order oblateness model of the earth's potential. This perturbed set was then updated using the same first order model, and the two sets were compared for the same 10° anomaly intervals for one complete revolution of the satellite. The square root of the sum of the squares of each of the errors at 10° anomaly

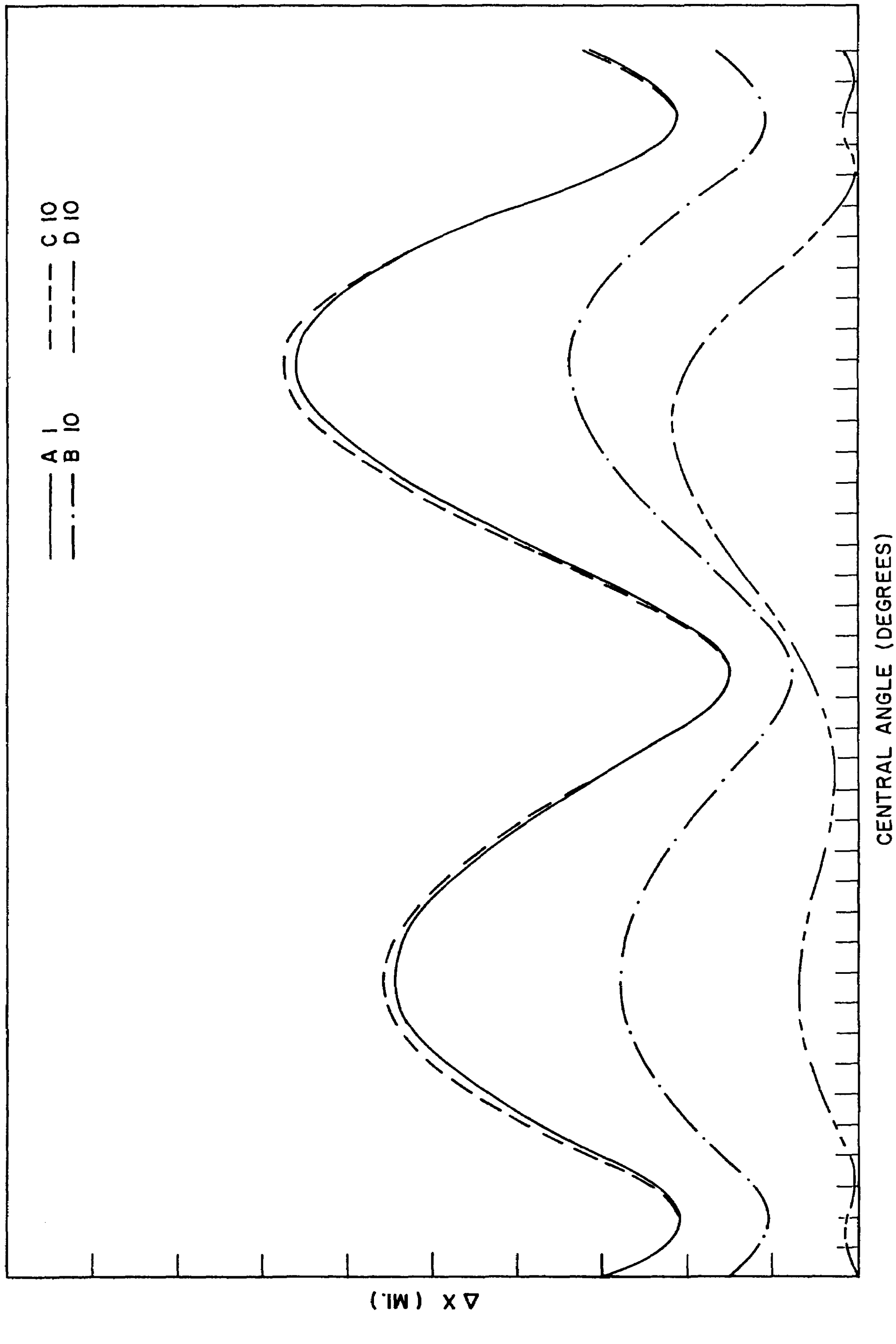

FIGURE 6 - CROSS-TRACK ERRORS

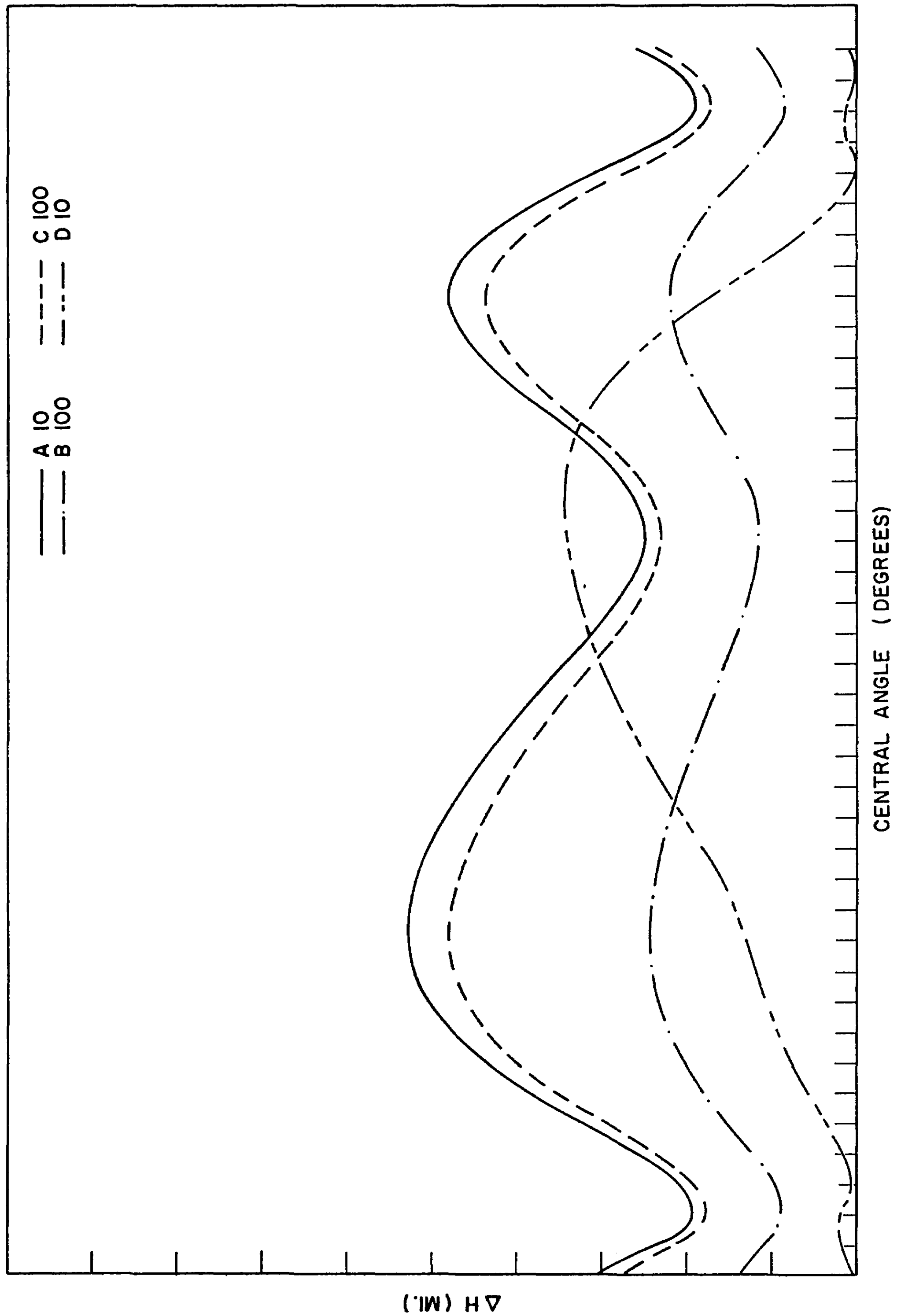

FIGURE 7 - HEIGHT ERRORS

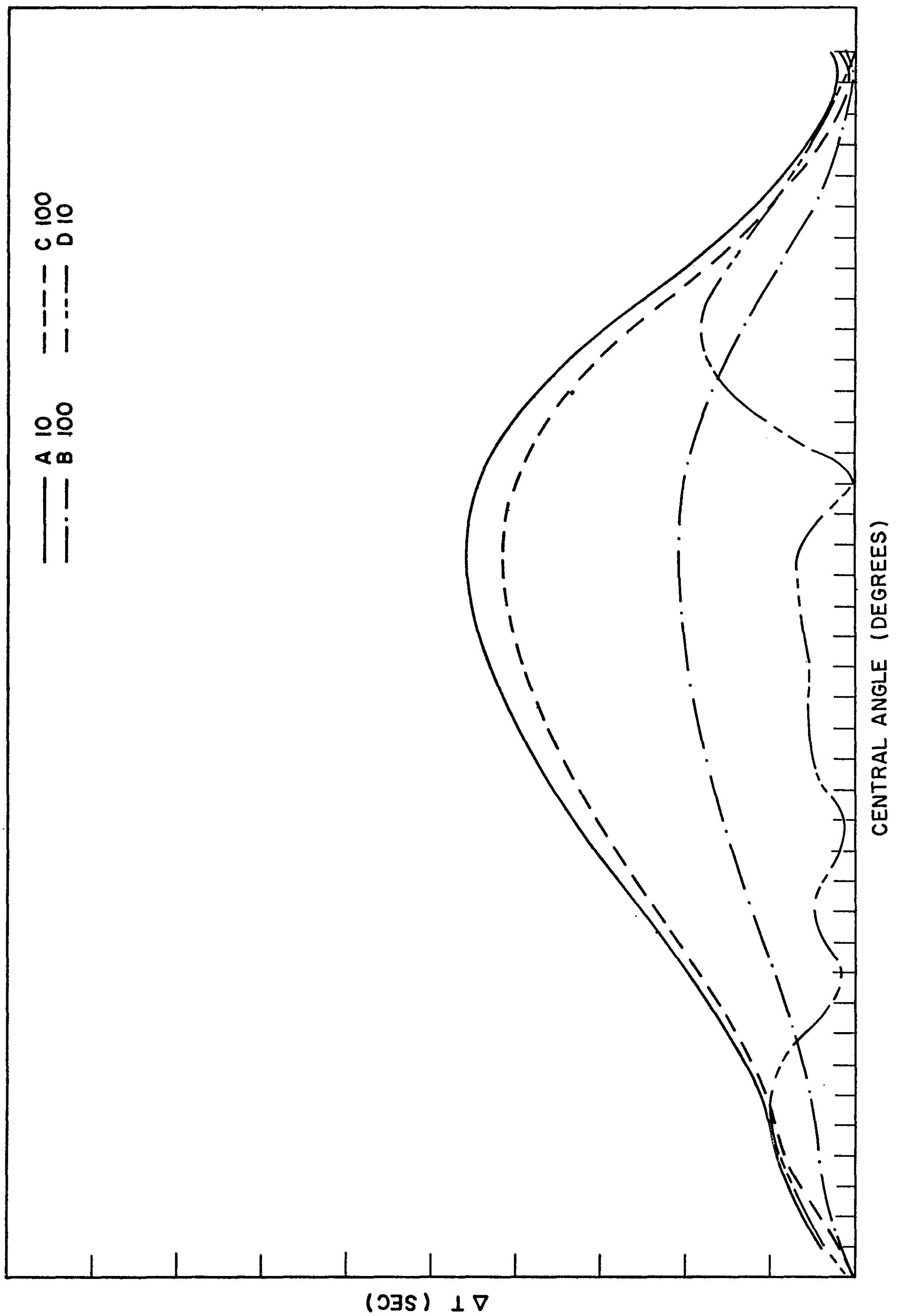

FIGURE 8 - TIME ERRORS

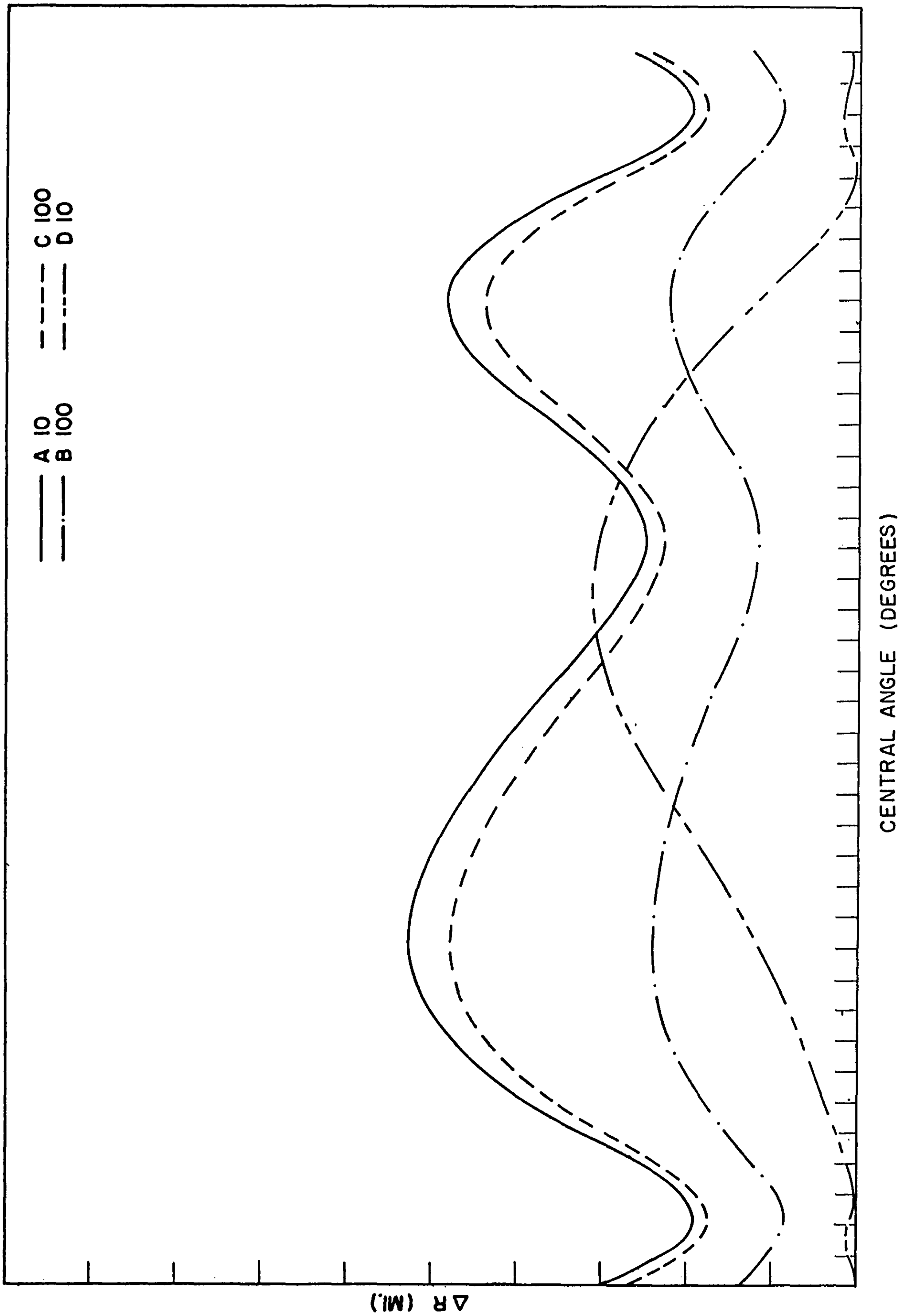

FIGURE 9 - RANGE ERRORS

intervals along the path were calculated here also.

The error curves for a typical set of data, displaying cross-track, height, time, and range errors appear in figures 6, 7, 8, and 9. The curves labelled A in these figures correspond to the 0.01° angle accuracy, those labelled B correspond to the 0.05° angle accuracy, and those labelled C correspond to the 0.1° accuracy. The D curves in these figures represent the errors due to the model accuracy. The ordinate axes in the cross-track, height, and range error curves are expressed in units of normalized statute miles, each interval being one-tenth of a mile; the ordinate axis in the time error curve is in normalized seconds, each interval here being one-tenth of a second. The abscissa axes in all of the curves are expressed in 10° intervals of central angle from the epoch pass of the satellite. Computation of the actual value of the amplitudes of any of the curves is effected by multiplying the normalized ordinate value by the appropriate scale factor of the curve of interest.

Conclusions. The conclusions which were drawn from the typical data displayed in figures 6 through 9 and the other data which was run were that if the measuring system possessed nominal accuracies of either 0.05° or 0.1° at zenith, the errors produced by the system would be approximately ten times the errors produced by the model when plotting these errors for one complete revolution of the satellite. This can be seen by noting the maximum values of the Δt and ΔR error curves for one revolution. The maximum value of the ΔR curve for 0.05° is 23.8 miles and for 0.1° it is 47.5 miles; the maximum value of the ΔR curve for the model is only 3.4 miles. Similarly the maximum value of the Δt curve for 0.05° is 20.6 seconds and for 0.1° it is 41.1 seconds; the maximum value of the Δt curve for the model is only 1.8 seconds. Now if the measuring system possessed an accuracy of 0.01° at zenith, the errors produced by the measuring system would be of the same order of magnitude as the errors produced by the model. Again considering the maximum values of the Δt and ΔR error curves for one revolution of the satellite, it is seen that for 0.01° Δt has a maximum value of 4.5 seconds and ΔR has a maximum value of 5.2 miles; for the model accuracy curves Δt has a maximum value of 1.8 seconds and ΔR has a maximum value of 3.4 miles.

Therefore, if the measuring system is capable of an accuracy of 0.01° or better at zenith, and we wish to derive a set of orbital elements from two observed points on the orbital path of the satellite and predict for small intervals of time (of the order of one revolution of the satellite) using these epoch elements, the model of the earth's gravitational field which we use should at least include the first order oblateness effects discussed in this work. Conversely, if the accuracy of the measuring system is more closely related to the 0.05° or 0.1° nominal accuracies, then the procedure for deriving a set of orbital elements from two observed positions of the satellite, and the subsequent updating of these elements for small intervals of time, could be effected using a procedure similar to that outlined in Chapter I of this work.

APPENDIX A

OPTIMUM VALUES OF XYZ

In the procedure for computing statistically optimum values of the coordinates we begin with the set of equations

$$\begin{aligned} X' &= X + \Delta X \\ Y' &= Y + \Delta Y \\ Z' &= Z + \Delta Z \end{aligned} \qquad \text{(I)}$$

where X, Y, and Z are first order approximations to the coordinates calculated from equations (9), (10), and (11) of Chapter I, X', Y', and Z' represent second order coordinate approximations, and ΔX, ΔY, and ΔZ are small corrective terms. The equations for the cosine of the north-south angle and for the cosine of the east-west angle can be written in the form

$$\begin{aligned} \cos \phi_i &= F_1 \, (X + \Delta X, \; Y + \Delta Y, \; Z + \Delta Z) \\ \cos \theta_i &= F_2 \, (X + \Delta X, \; Y + \Delta Y, \; Z + \Delta Z) \end{aligned} \qquad \text{(II)}$$

where ϕ_i represents the north-south angle, θ_i represents the east-west angle, i = 1 designates the eastern receiving station, and i = 2 designates the western receiving station. If we then make a Maclaurin series expansion about the point $\Delta X = 0$, $\Delta Y = 0$, $\Delta Z = 0$ using equations (II) we obtain equations of the form

$$\begin{aligned} \cos \phi_i &= A_i + B_i \Delta X + C_i \Delta Y + D_i \Delta Z \\ \cos \theta_i &= A_i' + B_i' \Delta X + C_i' \Delta Y + D_i' \Delta Z \end{aligned} \qquad \text{(III)}$$

where higher order terms in ΔX, ΔY, and ΔZ are neglected since the corrective terms are assumed small. We can then form a function

$$
\begin{aligned}
\phi = {} & \frac{1}{2\sigma_1^2}\left[A_1 + B_1\Delta X + C_1\Delta Y + D_1\Delta Z - \cos\phi_{1m}\right]^2 \\
& + \frac{1}{2\sigma_1^2}\left[A_2 + B_2\Delta X + C_2\Delta Y + D_2\Delta Z - \cos\phi_{2m}\right]^2 \\
& + \frac{1}{2\sigma_2^2}\left[A_1' + B_1'\Delta X + C_1'\Delta Y + D_1'\Delta Z - \cos\theta_{1m}\right]^2 \\
& + \frac{1}{2\sigma_2^2}\left[A_2' + B_2'\Delta X + C_2'\Delta Y + D_2'\Delta Z - \cos\theta_{2m}\right]^2 .
\end{aligned}
\qquad \text{(IV)}
$$

which is nothing more than the exponent of the exponential in the expression for the normal probability density function. In equation (IV) $\cos\phi_{1m}$, $\cos\phi_{2m}$, $\cos\theta_{1m}$, and $\cos\theta_{2m}$ are the measured values of the direction cosines from the north-south and east-west antenna fields, the quantities A_i, B_i, C_i, D_i, A_i', B_i', C_i', and D_i' are given, and the sigma values σ_1 and σ_2 are known from optical calibration of the measuring system. In order to maximize the probability that a satellite will be in a given position for a given time using the measured direction cosines at that time, we must minimize the function in equation (IV). This requires that we form the expressions

$$
\frac{\partial\phi}{\partial\Delta X} = 0; \quad \frac{\partial\phi}{\partial\Delta Y} = 0; \quad \frac{\partial\phi}{\partial\Delta Z} = 0 \qquad \text{(V)}
$$

which will produce three linear equations in the three unknowns ΔX, ΔY, and ΔZ. Solving equations (V) these terms are then substituted into equation (I), and the above procedure is repeated using X', Y', and Z' in place of the original X, Y, and Z until the square root of the sum of the squares of these corrective terms is within some arbitrary tolerance.

APPENDIX B

DETERMINATION OF ECCENTRICITY FROM TWO OBSERVED SATELLITE POSITIONS

The measurement of two positions of a satellite, the central angle between these two position vectors, and the semi-major axis of the orbit yield sufficient information to determine the eccentricity of the satellite orbit. Using the defining equation for a conic section at each of the positions of the satellite we can write

$$R_1 = \frac{a\ (1-e^2)}{1+e\ \cos\ v_1} \qquad \text{(I)}$$

$$R_2 = \frac{a\ (1-e^2)}{1+e\ \cos\ v_2} \qquad \text{(II)}$$

where $v_2 - v_1 = \alpha$ is defined as the central angle between the two position vectors. Therefore we can write

$$R_1 = \frac{a\ (1-e^2)}{1+e\ \cos\ (v_2 - \alpha)} \qquad \text{(III)}$$

and from simple trigonometry we know that

$$\cos\ (v_2 - \alpha) = \cos\ v_2\ \cos\alpha + \sin\ v_2\ \sin\alpha \qquad \text{(IV)}$$

and

$$\sin\ v_2 = \sqrt{1 - \cos^2\ v_2}\ . \qquad \text{(V)}$$

From equation (II) we obtain

$$\cos\ v_2 = \frac{a\ (1-e^2) - R_2}{eR_2}\ . \qquad \text{(VI)}$$

Substituting equation (VI) into equation (V) we obtain the result

$$\sin v_2 = \frac{1}{eR_2} \sqrt{(1-e^2) \left[2aR_2 - a^2(1-e^2) - R_2^2\right]} \tag{VII}$$

and now we substitute equation (IV) into equation (III) to obtain

$$R_1 + eR_1 \left[\cos v_2 \cos\alpha + \sin v_2 \sin\alpha\right] = a(1-e^2)\,. \tag{VIII}$$

Then we substitute equations (VI) and (VII) into equation (VIII) and group together the terms with e^4 and e^2 as coefficients. After some algebraic manipulation we obtain an equation of the form

$$Ae^4 + Be^2 + C = 0 \tag{IX}$$

where

$$A = a^2 (\bar{R}_2 - \bar{R}_1)^2$$

$$B = -2a^2 (\bar{R}_2 - \bar{R}_1)^2 + 2a (R_1R_2^2 + R_1^2R_2)(1-\cos\alpha) - R_1^2 R_2^2 \sin^2\alpha$$

$$C = a^2 (\bar{R}_2 - \bar{R}_1)^2 + 2(1-\cos\alpha)\left[R_1^2 R_2^2 - a (R_1R_2^2 + R_1^2R_2)\right].$$

The solution of equation (IX) is

$$e^2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}\,. \tag{X}$$

Equation (X) yields two values for e^2; for each positive value of e^2, take the positive square root, and these are the two eccentricities.

APPENDIX C

DERIVATION OF ORBITAL ELEMENT VARIATION EQUATIONS USING THE PFAFFIAN EXPRESSION

The derivation of the Pfaffian expression $\delta \phi_d - d\phi_\delta = 0$, and a discussion of the application of Pfaff's method to celestial mechanics is found in the article by Bilimovitch; a general discussion of Pfaff's expression in analytical dynamics may be found in the literature.[25] The expressions ϕ_d and ϕ_δ are defined as

$$\phi_d = \sum_{i=1}^{n} p_i \, dq_i - H dt \tag{I}$$

$$\phi_\delta = \sum_{i=1}^{n} p_i \quad q_i - H \ t \tag{II}$$

where p_i = generalized momentum

q_i = generalized coordinate

t = time

H = Hamiltonian of the system.

The $\underline{\delta}$ operator indicates changes in initial conditions and the $\underline{d}$ operator implies changes along a trajectory. Now assume we have picked a transformation for which

$$p = p\ (z_1, z_2, \ldots z_{2n}; t)$$

$$q = q\ (z_1, z_2, \ldots z_{2n}; t).$$

Therefore we have

$$\phi_d = \sum_{i=1}^{2n} Z_i\ (z_1,\ z_2,\ \ldots\ z_{2n};\ t)\ dz_i + Hdt$$

$$\phi_\delta = \sum_{i=1}^{2n} Z_i\ (z_1,\ z_2,\ \ldots\ z_{2n};\ t)\ \delta z_i + H\delta t \qquad \text{(III)}$$

and $$\delta\phi_d - d\phi_\delta = \sum_{i=1}^{2n} \left(\frac{\partial\phi_d}{\delta z_i} - dZ_i\right)\delta z_i + \left(\frac{\partial\phi_d}{\delta t} - dF\right)\delta t = 0\ .$$

Since the above variations are arbitrary the coefficients of δz_i and δt may be set equal to zero, and we obtain

$$\sum_{i=1}^{2n} \left(\frac{\partial\phi_d}{\delta z_i} - dZ_i\right) = 0 \qquad \text{(IV)}$$

$$\frac{\partial\phi_d}{\delta t} - dF = 0\ . \qquad \text{(V)}$$

Equation (IV) is the important equation to be considered in the derivation of the equations for the variation of the orbital elements. In the disturbed two-body problem the Pfaffian expression is written as

$$\phi_d = Ldl + Gdg + Hdh + Fdt \qquad \text{(VI)}$$

where $F = \dfrac{\mu}{2a} + R$

and R = the disturbing function. The quantities L, G, H, l, g, h are the DeLaunay canonical elements and are defined as

$$\begin{aligned} L &= \sqrt{\mu a} & l &= n\ (t - t_o) \\ G &= \sqrt{\mu a(1-e^2)} & g &= \omega \qquad \text{(VII)} \\ H &= \sqrt{\mu a(1-e^2)}\ \cos i & h &= \Omega\ . \end{aligned}$$

Substituting equations (VII) into equation (VI) we have

$$\phi_d = \sqrt{\mu a}\ dl + \sqrt{\mu a(1-e^2)}\ (d\omega + \cos i\ d\Omega) + \left(\frac{\mu}{2a} + R\right)\ dt\ . \qquad \text{(VIII)}$$

Using equation (VIII) the equations for the variation of the orbital elements can be derived immediately. For example, if we let $z_1 = l$, then $Z_1 = L$, as can be deduced by comparing equations (III) and (VI), and therefore equation (IV) is written as

$$\frac{\partial \phi_d}{\partial l} - d\ (\sqrt{\mu a}\ ) = 0\ . \qquad \text{(IX)}$$

From equation (VIII) it is obvious that

$$\frac{\partial \phi_d}{\partial l} = \frac{\partial R}{\partial l}\ dt$$

so that equation (IX) becomes

$$\frac{\partial R}{\partial l}\ dt - d(\sqrt{\mu a}\ ) = 0\ . \qquad \text{(IXa)}$$

From equation (IXa) the equation for the variation of the semi-major axis is immediately derived as

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial R}{\partial l}\ . \qquad \text{(X)}$$

Similarly the equations for the variation of the other orbital elements can be derived by letting $z_2 = i$, $z_3 = \omega$, $z_4 = \Omega$, $z_5 = e$, and $z_6 = a$. The corresponding equations derived for the above sequence are the time derivatives of the right ascension, eccentricity, inclination, argument of perigee, and mean anomaly respectively.

REFERENCES

1. P. J. Klass, Spasur Net Giving Vital Norad Coverage, Aviation Week and Space Technology, 77, Number 22, 52 (Nov. 26, 1962).

2. R. W. Wolverton, Flight Performance Handbook for Orbital Operations (John Wiley and Sons, Inc., New York and London, 1963, Chapter 2.

3. K. A. Ehricke, Space Flight (D. Van Nostrand Company, Inc., Princeton, New Jersey, 1962), Volume II, Page 140.

4. D. Brouwer, Solution of the Problem of Artificial Satellite Theory Without Drag, Astronom. J., 64, 378 (1959).

5. B. Garfinkel, The Orbit of a Satellite of an Oblate Planet, Astronom. J., 64, 353 (1959).

6. D. G. King-Hele, The Earth's Gravitational Potential Deduced from the Orbits of Artificial Satellites, Geophys. J., 4, 3 (1961).

7. Y. Kozai, The Motion of a Close Earth Satellite, Astronom. J., 64, 367 (1959).

8. R. H. Merson, The Motion of a Satellite in an Axi-Symmetric Gravitational Field, Geophys. J., 4, 17 (1961).

9. P. Musen, Application of Hansen's Theory to the Motion of an Artificial Satellite in the Gravitational Field of the Earth, J. Geophys. Res., 64, 2271 (1959).

10. J. A. O'Keefe, A. Eckels and R. K. Squires, The Gravitational Field of the Earth, Astronom. J., 64, 245 (1959).

11. S. W. McCuskey, Introduction to Celestial Mechanics (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1963, Chapter 3.

12. P. Musen, Special Perturbations of the Vectorial Elements, Astronom. J., 59, 262 (1954).

13. Y. Kozai, The Gravitational Field of the Earth Derived from Motions of Three Satellites, Astronom. J., 66, 8 (1961).

14. D. G. King-Hele, op. cit.

15. Y. Kozai, The Motion of a Close Earth Satellite, op. cit.

16. F. Tisserand, Traité de Mécanique Céleste (Paris, Gauthier-Villars et Fils, 1889), Volume I.

17. F. R. Moulton, An Introduction to Celestial Mechanics (The MacMillian Company, New York, 1914), Chapters 9 and 10.

18. D. Brouwer and G. M. Clemence, Methods of Celestial Mechanics (Academic Press, New York and London, 1961), Chapter 11.

19. N. Miller, A First Course in Differential Equations (Oxford University Press, 1935), Page 80.

20. W. Leighton, An Introduction to the Theory of Differential Equations (McGraw-Hill Book Company, Inc., New York, 1952), Page 41.

21. D. Brouwer and G. M. Clemence, op. cit. Page 273.

22. A. Bilimovitch, Über Die Anwendungen der Pfaffschen Methode in der Störungstheorie, Astronomische Nachrischten, 273, 161 (1942).

23. D. Brouwer and G. M. Clemence, op. cit. Page 283.

24. Y. Kozai, The Motion of a Close Earth Satellite, op. cit.

25. E. T. Whittaker, A Treatise on the Analytical Dynamics of Particles and Rigid Bodies (Dover Publications, New York, 1944), Chapter XI.

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* U.S. Naval Research Laboratory Washington, D.C. - 20390	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP

3. REPORT TITLE

FIRST ORDER OBLATENESS PERTURBATIONS AND THE U.S. NAVAL SPACE SURVEILLANCE SYSTEM

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Interim report

5. AUTHOR(S) *(Last name, first name, initial)*

Murtagh, Thomas B.

6. REPORT DATE November 5, 1964	7a. TOTAL NO. OF PAGES 76	7b. NO. OF REFS 25

8a. CONTRACT OR GRANT NO. NRL Problem R02-35 b. PROJECT NO. BuWeps RT 8801-001/652-1/3434-00-01 c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) NRL Report 6169 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*

10. AVAILABILITY/LIMITATION NOTICES

Unlimited availability

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Navy Department (Bureau of Naval Weapons)

13. ABSTRACT

An error analysis is presented involving a comparison of position errors in the orbital path of an artificial earth satellite produced either by the inclusion or omission of first order oblateness effects on all of the orbital elements or by the introduction of theoretical errors into the system with which the measurements are made.

The reasons for making the above comparison are twofold. First, if the improved model of the earth's gravitational field produces an increased accuracy in the orbital elements commensurate with the accuracy of the measuring system, then this more accurate model should be used in the derivation of any epoch set of orbital elements from raw data as well as in any scheme for predicting satellite position as a function of time from epoch. Secondly, if the improved model accuracy is indeed not comparable to the measuring system accuracy, then the inclusion of these first order effects is not necessary and their omission will reduce expenditures for the use of the high speed digital computer necessary for orbital element computation and updating.

The system referred to herein is the U.S. Naval Space Surveillance System which is a detection system comprised of three transmitting and four receiving sites located on a great circle across the southern portion of the United States.

The general development of the problem can be divided into three parts.

(Abstract cont'd.)

DD FORM 1 JAN 64 1473

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Theory						
U.S. Naval Space Surveillance System						
Satellite						
Error Analysis						
Position errors						
Orbital elements						
First order oblateness perturbations						
Model of Earth's gravitational field						
Model versus system accuracy						
Experimental error curves						

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2a. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7a. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8a. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through
________________________________."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through
________________________________."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through
________________________________."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS)*, *(S)*, *(C)*, or *(U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, roles, and weights is optional.

NRL Report 6169 Unclassified
13. Abstract (cont'd.)

The first part involves the computation of the position of the satellite at two different times from the observed direction cosines at those times and the subsequent derivation of the orbital elements of the satellite using an approximate model of the earth's gravitational field.

The second involves the differential correction of the computed elements at the two times using a more accurate model of the earth's gravitational field where the force function allows for first order corrections to all of the elements.

Finally, position errors of the satellite in what we shall define as the error plane are produced by perturbations of system direction cosine measurements or by the inclusion or omission of first order perturbation corrections to the elements themselves. A comparison of these errors will then indicate at what point the accuracy of the model used and the accuracy of the measuring system are comparable.