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# The Analysis of Variable Reluctance Transducers: The Energy Method

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# The Analysis of Variable Reluctance Transducers: The Energy Method

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The operating equations of a variable reluctance transducer are derived in this report through the application of the conservation of energy principle. These equations are formulated in terms of the energy stored in the magnetic field. The basic equations are shown to be nonlinear, and a set of approximate linear equations is derived for a special class of inputs. The method is applied to a double field, double mass transducer as an illustrative example.

This report and NRL Report 6088 are companion reports dealing with the derivation of the equations of variable reluctance transducers.

## INTRODUCTION

The standard analysis technique used for variable reluctance transducers is the application of classical vectorial electromagnetic theory. A detailed discussion of this method has been developed and presented in a companion report,* and it was found that several disadvantages resulted from the vectorial nature of this technique. In this report an energy method will be presented which alleviates some of the problems associated with the classical analysis.

The central problem in transducer analysis is, of course, the development of the electromechanical terms to be inserted in the basic electrical and mechanical laws (*i.e.*, Kirchoff's loop and node laws, Newton's law, the kinematic law). The electrical laws are

$$\sum_N i_n = 0 \qquad \sum_M \Delta v_m = 0 \tag{1}$$

where

$i_n$ = $n$th current entering a node $N$

$\Delta v_m$ = $m$th voltage difference around a loop $M$

and the mechanical laws are

$$\sum_N f_n = 0 \qquad \sum_M \Delta u_m = 0 \tag{2}$$

where

$f_n$ = $n$th force acting on a massless connection point (mechanical node) $N$

$\Delta u_m$ = $m$th velocity difference around a closed path (mechanical loop) $M$.

In the energy method to be developed here, the objective is to derive the electromechanical effects in terms of the total energy stored in the magnetic field. As a prelude to this derivation, it will be useful to outline the development of the energy method. This outline will emphasize the proper perspective for relating each portion of this rather detailed derivation to the whole.

The first step in the analysis is to separate the purely electrical and purely mechanical portions of the transducer from the magnetic field, or fields, in which the energy conversion occurs. This is of course a purely conceptual division of the device and it is made so that the magnetic field can be isolated and studied in detail.

In this detailed examination, interest is centered on the interaction of the energy stored in the field with the external currents and forces. The analysis is based on the conservation of energy principle, and it is assumed that the energy conversion process is completely lossless. That is, it is assumed that all variations in the magnetic

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*R.M. Moore, "The Analysis of Variable Reluctance Transducers: The Classical Method," NRL Report 6088, March, 1964.

field occur in a conservative and reversible fashion, and that the field is quasi-static in the sense that the possibility of electromagnetic radiation loss is completely ignored.

From this study of the ideal magnetic field, a set of relations defining the energy conversion currents and forces in terms of the energy stored in the magnetic field is obtained. These electromechanical effects can then be inserted in Eqs. (1) and (2) to obtain the operating equations of the transducer. Unfortunately, the operating equations contain nonlinear functions of the electrical and mechanical variables associated with the field, and it thus becomes desirable to obtain linear energy exchange relations which will be valid for as wide a range of physical situations as possible.

The technique of obtaining these linearized equations is to expand the exact energy conversion relations in a Taylor series about a given set of values and to neglect the terms of higher than first order in the series. This leads to a set of linear energy exchange equations which are valid for small variations of the system about a fixed state.

The last step in the analysis is to manipulate the linear energy exchange relations into convenient forms for use in deriving the overall transducer equations and to insert them into the appropriate versions of Eqs. (1) and (2). The final linearized operating equations are then written in terms of the electrical and mechanical variables which are observable directly at the electrical and mechanical terminals of the transducer.

In the remainder of this report, attention will be focused on the energy conversion in variable reluctance transducers. The general analysis will be presented first for a simple system, and then an illustrative example will be given for a more complex device.

## A SIMPLE TRANSDUCER

A quantitative description of the energy conversion process in a magnetic field transducer requires a development of the functional dependence of the energy stored in the magnetic field. In particular, it is necessary to study the variation of this stored energy when it is subjected to the action of external electrical and mechanical sources.

In order to develop this subject in the simplest possible context, the first transducer considered will utilize a magnetic field having only one mechanical degree of freedom and one electrical degree of freedom. A schematic diagram of such a transducer is represented in Fig. 1, where the magnetic mass is constrained to translation. The variables shown in the schematic are defined in the following fashion:

$v_0'$ = electrical voltage difference applied at the external terminals

$i_0'$ = electric current entering the positive external terminal

$\lambda'$ = flux linkage associated with the coil

$x_0'$ = mechanical displacement of the magnetic mass with reference to the inertial frame

$f_0'$ = external mechanical force applied to the magnetic mass in the direction of positive $x_0'$.

Following the general method of attack outlined in the introductory section, this transducer is first conceptually divided into three portions as represented in Fig. 2, where the new variables that are to be associated with the lossless energy conversion are defined in terms of the magnetic field as

$v'$ = electrical voltage difference associated with the magnetic field

$i'$ = electric current associated with the magnetic field

$x'$ = mechanical displacement associated with the magnetic field

$f'$ = mechanical force exerted on the magnetic mass in the direction of positive $x'$.

To emphasize the distinction between these field variables and the previously defined external variables, the relationships between the external and field variables, for the transducer of Fig. 1, are

$$v_0' = i_0' R_w + v'$$

$$i_0' = i'$$

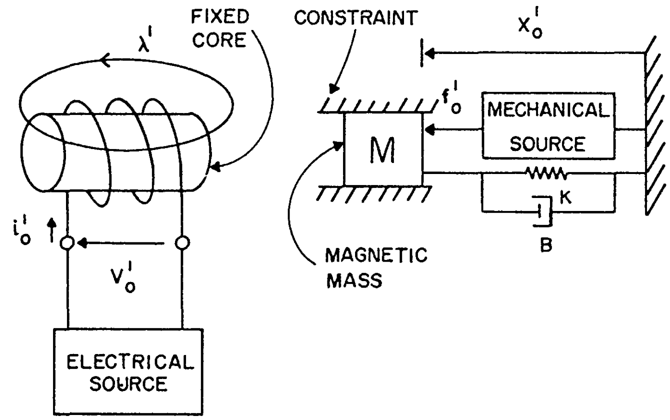


Fig. 1 - Transducer utilizing a magnetic field having one mechanical degree of freedom and one electrical degree of freedom

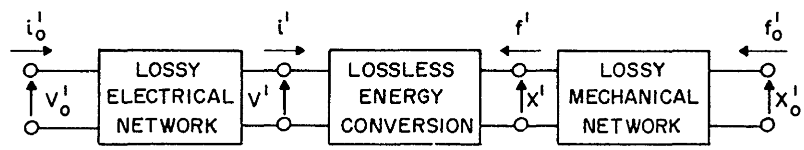


Fig. 2 - Conceptual division of the transducer of Fig. 1 into three portions

where

$$v' = \frac{d\lambda'}{dt}$$

$R_w$ = resistance of the wire in the coil

and

$$x_0' = x'$$

$$f_0' = M\ddot{x}_0' + B\ddot{x}_0' + K(x_0' - D) + f'$$

where

$M$ = mass of the movable magnetic mass

$B$ = damping coefficient associated with the spring and constraint

$K$ = stiffness or spring constant of the spring

$D$ = value of $x_0'$ for which the spring is unstressed.

These equations are, of course, the results obtained from applying Eqs. (1) and (2) to Fig. 1.

The relationships governing the purely electrical and purely mechanical portions of any system of the type represented in Fig. 2 can be obtained by standard techniques. Therefore the following treatment is devoted to developing the appropriate relations for the field variables associated with the lossless energy conversion.

Consider the energy stored in the magnetic field of the transducer represented in Fig. 1. This stored energy can be determined by a consideration of the total input energy, both electrical and mechanical, which contributes to the field.

Introducing the following symbols to represent the input power and energy functions

$p_e$ = instantaneous electrical input power to the field

$dw_e$ = incremental electrical input energy to the field

$p_m$ = instantaneous mechanical input power to the field

$dw_m$ = incremental mechanical input energy to the field

$dw$ = incremental increase in energy in the field

$w$ = total energy stored in the field

the electrical power and incremental energy can be related to the previously defined field variables by the equations

$$p_e = i' v' = i' \frac{d\lambda'}{dt}$$

$$dw_e = p_e\, dt = i' \frac{d\lambda'}{dt}\, dt = i'\, d\lambda'$$

and, since the mechanical portion of the device is constrained to translation, the mechanical power and incremental energy can be expressed in terms of the scalars $f'$ and $x'$ as

$$p_m = f' \frac{dx'}{dt}$$

$$dw_m = p_m\, dt = f' \frac{dx'}{dt}\, dt = f'\, dx'.$$

Thus the incremental change in the stored energy will be

$$dw = dw_e + dw_m$$

or

$$dw = i'\, d\lambda' + f'\, dx'. \tag{3}$$

Equation (3) simply states that the incremental increase in the total energy stored in the magnetic field is the sum of the incremental input energies.

Notice that in Eq. (3) the flux linkage $\lambda'$ and the displacement $x'$ appear as independent variables and that the current $i'$ and force $f'$ appear as dependent quantities. That is, the incremental change in the stored energy $dw$ results from incremental changes in $\lambda'$ and $x'$. This choice of independent and dependent variables is different from that implicitly used in the classical analysis, as given in NRL Report 6088 where the basic laws of the electromagnetic field are applied to this type of transducer. The classical approach implies that the current is the independent variable upon which the flux and flux linkage depend. Thus the current and displacement are implicitly used as independent variables, and the flux linkage and force appear in the guise of dependent quantities. However, there is no fundamental physical requirement that this viewpoint be adopted; it is used in the classical approach only because it is the most convenient when the electromagnetic laws are to be utilized.

In considering the energy storage in the electromagnetic field, it becomes apparent that the most natural viewpoint is the one which appears in Eq. (3). That is, insofar as the stored energy is concerned, the flux linkage and displacement should be regarded as independent variables, and the current and force as dependent quantities. Using functional notation, this conclusion can be expressed in the form

$$i' = i'(\lambda', x')$$

and

$$f' = f'(\lambda', x').$$

This choice of functional dependence is not as radical as it might appear at first glance. In the classical analysis the force relation which is obtained directly from the fundamental force law actually is in terms of the flux density vector $\vec{B}$. The current is then introduced into this force equation by a separate consideration whereby the flux density vector is related to the current in a coil. Therefore, the choice of flux (or flux linkage) and displacement as the independent variables for the force function is actually a natural system to use even in the classical development of the force relation.

The decision to consider that the current depends on the flux linkage, rather than vice versa, can be viewed simply as a mathematical manipulation in which the classically derived relation

$$\lambda' = \lambda'(i', x')$$

is solved for $i'$ in terms of $\lambda'$ and $x'$.

Now focusing attention once again on Eq. (3) it can be seen that $dw$ depends only on $\lambda'$, $x'$, and the changes $d\lambda'$ and $dx'$. That is, since $i'$ and $f'$ can be expressed as functions of $\lambda'$ and $x'$, then Eq. (3) can be rewritten more explicitly as

$$dw = i'(\lambda', x')\ d\lambda' + f'(\lambda', x')\ dx' \qquad (3)$$

and therefore the stored energy $w$ is only a function of $\lambda'$ and $x'$. Thus the energy stored in the magnetic field is a function only of the particular state of the field, as specified by $(\lambda', x')$, and does not depend on the process by which this state is reached. Any function which has this property is referred to as a state function. The conclusion that the energy is a state function of $(\lambda', x')$ can be stated mathematically with the functional notation

$$w = w(\lambda', x').$$

As a result of this state function property of $w$, it is possible to express any incremental variation of $w$ in the general form

$$dw = \frac{\partial w}{\partial \lambda'}\, d\lambda' + \frac{\partial w}{\partial x'}\, dx'$$

and comparing this expression (which is based on the state function properties of $w$) with Eq. (3) (which is based on the energy inputs to the field) the current and force functions can be expressed in terms of the energy stored in the field. The resulting identities are

$$\begin{aligned} i'(\lambda', x') &= \frac{\partial w(\lambda', x')}{\partial \lambda'} \\ f'(\lambda', x') &= \frac{\partial w(\lambda', x')}{\partial x'}. \end{aligned} \qquad (4)$$

Inserting these relations into Eqs. (1) and (2) the operating equations of this device (Fig. 1) are obtained. From Eqs. (1),

$$i_0' = i' = \frac{\partial w(\lambda', x')}{\partial \lambda'}$$

$$R_w\, i_0' + \frac{d\lambda'}{dt} = v_0(t)$$

and from Eqs. (2),

$$M\ddot{x}_0' + B\dot{x}_0' + Kx_0' + \frac{\partial w(\lambda', x')}{\partial x'} = f_0(t)$$

$$x_0' = x.$$

Combining these two pairs of equations to obtain a set of relations involving only $\lambda'$, $x'$, $v_0$, and $f_0$ as variables,

$$\begin{aligned} R_w \frac{\partial w(\lambda', x')}{\partial \lambda'} + \frac{d\lambda'}{dt} &= v_0(t) \\ M\ddot{x}_0' + B\dot{x}_0' + Kx_0' + \frac{\partial w(\lambda', x')}{\partial x'} &= f_0(t) \end{aligned} \qquad (5)$$

are obtained as the operating equations of this device.

The possibility of solving this completely general set of operating equations cannot be discussed until the particular functional dependence of the stored energy, $w(\lambda', x')$, is known. Thus in order to utilize Eqs. (5), as well as Eq. (3), it is necessary to study the energy function in greater detail.

## THE ENERGY FUNCTION

In principle the energy function can be obtained through a straightforward integration of the expression for the differential of energy. Referring to the general expression for $dw$,

$$\begin{aligned} dw(\lambda', x') = i'(\lambda', x')\ d\lambda' \\ + f'(\lambda', x')\ dx' \end{aligned} \qquad (3)$$

it is seen that it is necessary to know both $i(\lambda', x')$ and $f'(\lambda', x')$ before this expression can be integrated. That is, if Eq. (3) is integrated from a state of zero energy (represented by $\Lambda_0$, $X_0$) to the state $(\Lambda, X)$, the result is the pair of integrals

$$\begin{aligned} w(\Lambda, X) = \int_{\Lambda_0, X_0}^{\Lambda, X} dw = \int_{\Lambda_0, X_0}^{\Lambda, X} i'\ (\lambda', x')\ d\lambda' \\ + \int_{\Lambda_0, X_0}^{\Lambda, X} f'\ (\lambda', x')\ dx' \end{aligned} \qquad (6)$$

where the integration must be carried out along some specified path given by a relation of the form

$$\psi(\lambda', x') = 0$$

and where the functions $i'(\lambda', x')$ and $f'(\lambda', x')$ must be known for the general path.

Although Eq. (6) is a general expression for the energy function at a given state, it is not particularly practical, for two reasons. First, the requirement that both $i'$ and $f'$ be completely known functions is a rather detailed requirement of prior knowledge for the device. Second, if these two functions are already known, then there is no reason, other than academic curiosity, to pursue the energy analysis. In order to be able to judge the possible usefulness of the energy approach, it is necessary to digress for a moment to discuss what should be considered as a realistic prior knowledge of $i'$ and $f'$ from a practical point of view.

There are two basic methods for evaluating $f'(\lambda', x')$ and $i'(\lambda', x')$: experimental or theoretical. The theoretical development for $f'$ by the classical method illustrates that the gap force can be considered as a function of the gap flux and that the gap flux can be related to the flux linkage of the isolated coil. Thus the gap force can be evaluated in principle as a function of $\lambda'$ and $x'$ upon purely theoretical grounds. However, any attempt at an explicit evaluation of this relationship for a practical situation requires numerous approximations for purely computational reasons, and the resulting expression for $f'(\lambda', x')$ loses its generality. Thus it does not seem realistic to assume that an explicit general relation for $f'(\lambda', x')$ can be obtained from theoretical considerations. This leaves the alternative of experimental determination.

It is not possible to measure $\lambda'$ directly; therefore any experimental determination of $f'(\lambda', x')$ must be based on indirect measurements of $\lambda'$. One method would be to measure $f'$, $x'$, and $i'$ simultaneously and then to utilize a known relation of $\lambda' = \lambda'(i', x')$ to eliminate $i'$ from the experimental results. Thus in principle it is possible to determine $f'(\lambda', x')$ experimentally, provided that the function $\lambda'(i', x')$ is known. This flux function, however, must itself be obtained either theoretically or experimentally.

The problem of evaluating $\lambda'(i', x')$ is equivalent to finding the function $i'(\lambda', x')$. Thus the preceding discussion leads to the result that the experimental determination of $f'(\lambda', x')$ is contingent on a prior knowledge of $i'(\lambda', x')$. From the classical development it is known that the flux linkage can be considered to be a function of $i'$ and $x'$. In attempting to evaluate this dependence theoretically, it is necessary to make certain simplifying approximations, and the resulting theoretical computation for $\lambda'(i', x')$ is not general.

Therefore it is again necessary to resort to an experimental approach. The flux linkage cannot usually be measured directly, but an indirect measure can be obtained by measuring the voltage difference induced by the rate of change of $\lambda'$ (*i.e.*, by measuring the inductance). If the transducer under consideration is magnetically linear, then it will suffice to evaluate the inductance once for each possible displacement of the mass. However, if it is not magnetically linear, then it will be necessary to perform a more complex series of measurements, for each fixed $x'$, in order to evaluate $i'(\lambda', x')$.

The purpose of this discussion has been to determine what constitutes a reasonable prior knowledge of the current and force relations of a transducer. The conclusion is that it is realistic to assume that $i'(\lambda', x')$ can be obtained experimentally and that once this function is known a separate series of measurements can be utilized to obtain $f'(\lambda', x')$.

Returning to the original problem of evaluating the energy function, consider the following possibility. Suppose that the energy function could be obtained solely on the basis of a knowledge of $i'(\lambda', x')$. The consequences of such a result would be twofold:

1. It would remove the restrictive requirement of knowing both $i'(\lambda', x')$ and $f'(\lambda', x')$ prior to evaluating $w(\Lambda, X)$.
2. It would allow $f'(\lambda', x')$ to be found from a knowledge of $i'(\lambda', x')$ by applying Eqs. (4) to the energy function found from $i'(\lambda', x')$.

The second consequence is particularly attractive for it would provide a unifying interrelation between $i'(\lambda', x')$ and $f'(\lambda', x')$.

Referring to Eq. (6) it is apparent that the only possible method by which $w(\Lambda, X)$ can be found without knowing $f'(\lambda', x')$ is to keep the second integral identically zero. This can be done by choosing a path of integration such that either (a) $f'(\lambda', x')$ is zero when $dx'$ is not zero, or (b) $dx'$ is zero when $f'(\lambda', x')$ is not zero. Both of these possibilities will be utilized in the following development of the special expression for the energy function.

Before developing this special expression it will be useful to digress for a moment to consider the zero energy state $(\Lambda_0, X_0)$. In order to assure zero energy storage in the magnetic field it is sufficient to specify that the flux linkage be zero. Therefore $X_0$ can be chosen arbitrarily, and for convenience it will be chosen as the origin of $x'$. Thus the general zero energy state $(\Lambda_0, X_0)$ can actually be replaced by the specific state $(0, 0)$ or $\lambda' = \Lambda_0 = 0$ and $x' = X_0 = 0$.

Now returning to the evaluation of the energy function, a special form of Eq. (6) is obtained by integrating along the path consisting of the following two segments:

1. The magnetic mass is brought from $x'=0$ to its final position of $x' = X$, while holding the flux linkage at zero;

2. The mass is then held in this position and the flux linkage is increased from zero to its final value of $\lambda' = \Lambda$.

This path can be stated mathematically as

1. From $(0, 0)$ to $(0, X)$
2. From $(0, X)$ to $(\Lambda, X)$.

The advantage of this particular path is that the force function plays no part in the resultant integration for $w(\lambda', x')$. During part 1 of the integration, the field force function is identically zero, and in part 2 the incremental displacement $dx'$ is zero; thus no work is done by $f'$ on either segment of the integration path. Therefore Eq. (6) assumes the special form

$$w(\Lambda, X) = \int_0^{\Lambda} i'(\lambda', X)\, d\lambda' \tag{7}$$

which does not require any *a priori* knowledge of the force function $f'$.

Applying the identities of Eqs. (4) to this special form of the energy function, the relations

$$\begin{aligned} i' &= i'(\lambda', X) \\ f' &= \int_0^{\Lambda} \frac{\partial i'(\lambda', X)}{\partial X}\, d\lambda' \end{aligned} \tag{8}$$

are obtained, of which the first equation is an obvious identity resulting from the special form of Eq. (7) and the second relation allows a determination of $f'$ solely from a knowledge of $i'(\lambda', x')$.

Particularly simple forms of Eqs. (7) and (8) are obtained by assuming the existence of a reluctance function which is dependent only on $x'$. This assumption leads to a form of the function $i'(\lambda', x')$:

$$i'(\lambda', x') = \frac{\lambda' R(x')}{N^2}$$

where

$$R(x') = \text{reluctance function}$$

$$N = \text{number of turns on coil.}$$

Inserting this relation into Eq. (7) the energy expression becomes

$$w(\lambda', x') = \frac{(\lambda')^2 R(x')}{2N^2}. \tag{9}$$

Applying Eqs. (8) the force and current relations become

$$\begin{aligned} f'(\lambda', x') &= \frac{(\lambda')^2}{2N^2} \frac{dR(x')}{dx'} \\ i'(\lambda', x') &= \frac{\lambda' R(x')}{N^2} \end{aligned} \tag{10}$$

which specifies the force and current functions in terms of the reluctance, the flux linkage, and the number of turns on the coil. Equations (9) and (10) can also be written in terms of the total inductance. Recalling the definition of self-inductance for an isolated coil,

$$L(x') = \frac{\lambda'}{i'} = \frac{N^2}{R(x')}$$

and inserting this into the expression for energy given by Eq. (9) and into Eqs. (10), then

$$w(\lambda', x') = \frac{(\lambda')^2}{2L(x')}$$

$$i'(\lambda', x') = \frac{\lambda'}{L(x')}$$

$$f'(\lambda', x') = -\frac{(\lambda')^2}{2L^2(x')}\frac{dL(x')}{dx'}$$

become the relations specifying the energy, current, and force functions in terms of the indtictance parameter.

The inductance-based form of these relations has been included for the sake of completeness. In the subsequent treatment only the reluctance based forms given in Eqs. (9) and (10) will be utilized.

The purpose of the preceding discussion and development has been to determine the functional dependence of the stored energy, $w(\lambda', x')$. This dependence is of interest because it determines the form and properties of the set of general operating equations, Eqs. (5). Two special expressions for the energy function are available: Eqs. (7) and (9). In addition, it has been shown that the force function $f'(\lambda', x')$, can be developed solely from a knowledge of the current function, $i'(\lambda', x')$. The general relation between these two functions is given in Eqs. (8), and a special form appears in Eqs. (10).

Having accomplished the objective of determining the form of $w(\lambda', x')$, as well as the significant by-product of relating $f'$ and $i'$, it is now possible to return to a consideration of the operating equations, Eqs. (5).

## OPERATING EQUATIONS AND LINEARIZATION

Through the use of the energy function a general set of operating equations have been derived for the simple variable reluctance transducer under consideration. These equations are repeated for reference purposes:

$$\begin{aligned} R_w \frac{\partial w(\lambda', x')}{\partial \lambda'} + \frac{d\lambda'}{dt} &= v_0(t) \\ M\ddot{x}' + B\dot{x}' + Kx' + \frac{\partial w(\lambda', x')}{\partial x'} &= f_0(t). \end{aligned} \tag{5}$$

It was not previously possible to discuss these equations in any detail because the form of the functional dependence of $w(\lambda', x')$ was not known. However, now that this has been developed and discussed, it is possible to reconsider these equations and their general properties.

Inserting the special form of the stored energy given in Eq. (7) into Eqs. (5) the result is

$$\begin{aligned} R_w\, i'(\lambda', x') + \frac{d\lambda'}{dt} &= v_0(t) \\ M\ddot{x}' + B\dot{x}' + Kx' + \int_0^{\lambda'} \frac{\partial i'(\lambda', x')}{\partial x'}\, d\lambda' &= f_0(t) \end{aligned} \tag{11}$$

as a particular form of the operating equations.

Of primary interest to the analyst is whether these equations are linear or nonlinear in $\lambda'$ and $x'$. An examination of the form of the first reveals that it will be linear in $\lambda'$ and $x'$ provided that $i'(\lambda', x')$ is linear in these variables. And a consideration of the second leads to the conclusion that it will be linear only if $\partial i'/\partial x'$ is constant and independent of both $\lambda'$ and $x'$. This last condition is the more restrictive since it in essence requires that $i'(\lambda', x')$ be independent of $\lambda'$ and linearly dependent on $x'$. Such a condition is clearly a physical impossibility, and thus it becomes clear that in general the operating equations will be nonlinear in $\lambda'$ and $x'$ as dependent variables.

In order to determine the degree of nonlinearity to be expected, the special expression for the energy which is based on the reluctance function can be used. Inserting Eq. (9), or more directly Eqs. (10), into Eqs. (5) the result is

$$\begin{aligned} R_w \frac{\lambda' R(x')}{N^2} + \frac{d\lambda'}{dt} &= v_0(t) \\ M\ddot{x}' + B\dot{x}' + Kx' + \frac{(\lambda')^2}{2N^2}\frac{dR(x')}{dx'} &= f_0(t) \end{aligned} \tag{12}$$

as a second special form of the operating equations. This formulation indicates quite clearly the general aspects of nonlinearity which can be expected to appear in these equations. The only way in which the first could be made linear would be for $R(x')$ to be constant, but this would remove all electromechanical terms from the second equation. Similarly, the second would be linear only if $dR(x')/dx'$ were constant, but this would make the first nonlinear, since $R(x')$ would then be linear in $x'$. Thus the general conclusions concerning nonlinearity which were obtained from Eqs. (11) are strikingly confirmed by the form of

Eqs. (12). The problem, therefore, becomes one of solving a set of nonlinear differential equations with all the mathematical problems attendant thereto.

In general the solution of such a set of nonlinear equations requires the use of graphical, numerical, and series approximation techniques. Having found a solution through these laborious techniques for a particular set of inputs, it is usually not possible to use this solution for other inputs. That is, it is normally necessary to obtain a completely separate solution for every change, even of the simplest character, in the inputs. These characteristics of the solution of nonlinear equations are most undesirable from the analyst's point of view, and thus the question naturally arises as to whether or not the equations can be "linearized" for some particular mode of operation which is of practical interest. Fortunately the answer is in the affirmative, and the remainder of this section will be devoted to the techniques of linearization which can be utilized.

The derivation of a set of linearized equations for the current and force functions requires the introduction of the concept of "incremental variables." This approach is frequently referred to as "small signal analysis" by electrical engineers, but it will be referred to in this report as "incremental analysis."

The basic technique is to consider the incremental excursions of the variables of interest in the vicinity of a particular state. The particular state will determine the values of the "incremental parameters" in the linearized equations which result. To begin this analysis, a basic set of incremental variables is defined for excursions about the state $(\Lambda, X)$:

$\lambda$ = incremental variation of $\lambda'$ around $\Lambda$

$x$ = incremental variation of $x'$ around $X$

and for $i'$ and $f'$:

$i$ = incremental variation of $i'$ around $I$

$f$ = incremental variation of $f'$ around $F$

where $I$ and $F$ are the values of $i'$ and $f'$ corresponding to the state $(\Lambda, X)$.

The word definitions of the incremental variables can be stated in mathematical form as

$$\lambda = d\lambda'; \; x = dx'$$

$$i = di'; \; f = df'$$

where the differentials are with respect to the state $(\Lambda, X)$.

Expanding $i'$ and $f'$ in Taylor series around the state $(\Lambda, X)$, and neglecting higher order terms in $d\lambda'$ and $dx'$,

$$i' \approx i'(\Lambda, X) + \left.\frac{\partial i'}{\partial \lambda'}\right|_0 d\lambda' + \left.\frac{\partial i'}{\partial x'}\right|_0 dx'$$

$$f' \approx f'(\Lambda, X) + \left.\frac{\partial f'}{\partial \lambda'}\right|_0 d\lambda' + \left.\frac{\partial f'}{\partial x'}\right|_0 dx'$$

where the zero subscript on the partial derivatives indicates that they are evaluated at the state $(\Lambda, X)$.

The quantities $I$ and $F$ can be identified as

$$\begin{aligned} I &= i'(\Lambda, X) \\ F &= f'(\Lambda, X) \end{aligned} \tag{13}$$

and the incremental variables $i$ and $f$ can be seen to be

$$\begin{aligned} i &\approx \left.\frac{\partial i'}{\partial \lambda'}\right|_0 \lambda + \left.\frac{\partial i'}{\partial x'}\right|_0 x \\ f &\approx \left.\frac{\partial f'}{\partial \lambda'}\right|_0 \lambda + \left.\frac{\partial f'}{\partial x'}\right|_0 x \end{aligned} \tag{14}$$

on the basis of the definitions of the incremental variables as differentials.

Equations (14) can be written in a somewhat more concise form by considering the partial derivatives evaluated at $(\Lambda, X)$ to be a set of incremental parameters. Defining two of these partial derivatives in terms of the familiar concepts of inductance and spring constant, and the other two by introducing the concept of coupling parameters between the electrical and mechanical variables, leads to a set of definitions for the incremental parameters:

$$\frac{1}{l} = \left.\frac{\partial i'}{\partial \lambda'}\right|_0 ; \; l = \text{incremental inductance}$$

$$g_1 = \left.\frac{\partial i'}{\partial x'}\right|_0 ; \; g_1 = \text{incremental coupling parameter}$$

$$g_2 = \frac{\partial f'}{\partial \lambda'}\bigg|_0 ; \; g_2 = \text{incremental coupling parameter}$$

$$h = \frac{\partial f'}{\partial x'}\bigg|_0 ; \; h = \text{incremental spring constant.}$$

Through the use of this shorthand notation Eqs. (14) can be written in the form

$$\begin{aligned} i &= \frac{1}{l}\,\lambda + g_1\,x \\ f &= g_2\,\lambda + h\,x. \end{aligned} \tag{15}$$

Equations (13) and (15) would seem to be adequate for an analysis of the energy conversion process during small variations around a given state. However, Eq. (15) is not in its simplest form, because it implies that there are four unrelated and unequal incremental parameters. A detailed examination of these parameters will reveal that the two electromechanical coupling parameters are actually equal. In order to prove this statement, and to develop more specifically the dependence of these incremental parameters, it will now be necessary to express them as functions of the stored energy $w(\lambda', x')$.

## INCREMENTAL PARAMETERS

The incremental parameters can be defined in terms of the stored energy by recalling

$$\begin{aligned} i' &= \frac{\partial w}{\partial \lambda'} \\ f' &= \frac{\partial w}{\partial x'} \end{aligned} \tag{4}$$

and by applying the definitions of the incremental parameters to these general forms of $i'$ and $f'$. The results are

$$\frac{1}{l} = \frac{\partial^2 w}{\partial (\lambda')^2}\bigg|_0 ; \; g_1 = \frac{\partial^2 w}{\partial x' \partial \lambda'}\bigg|_0$$

$$g_2 = \frac{\partial^2 w}{\partial \lambda' \partial x'}\bigg|_0 ; \; h = \frac{\partial^2 w}{\partial (x')^2}\bigg|_0 .$$

Now the state function property of $w(\lambda', x')$ insures that the mixed second derivative is independent of the order of differentiation; therefore,

$$g_1 = g_2 = g$$

which reduces the number of parameters to three.

Thus Eqs. (15) can be rewritten in the form

$$\begin{aligned} i &= \frac{1}{l}\,\lambda + g\,x \\ f &= g\,\lambda + h\,x \end{aligned} \tag{16}$$

where

$$\begin{aligned} \frac{1}{l} &= \frac{\partial i'}{\partial \lambda'}\bigg|_0 = \frac{\partial^2 w}{\partial (\lambda')^2}\bigg|_0 \\ g &= \frac{\partial i'}{\partial x'}\bigg|_0 = \frac{\partial f'}{\partial \lambda'}\bigg|_0 = \frac{\partial^2 w}{\partial \lambda' \partial x'}\bigg|_0 \\ h &= \frac{\partial f'}{\partial x'}\bigg|_0 = \frac{\partial^2 w}{\partial (x')^2}\bigg|_0 \end{aligned} \tag{17}$$

are the general definitions of the incremental parameters.

Some interesting insight into the behavior of these parameters can be obtained by applying the general definitions of Eqs. (17) to the reluctance form of the energy function given by Eq. (9). The results are

$$\frac{1}{l} = \frac{R(x')}{N^2}\bigg|_0 = \frac{R(X)}{N^2}$$

$$g = \frac{\lambda'}{N^2}\frac{dR(x')}{dx'}\bigg|_0 = \frac{\Lambda}{N^2}\frac{dR(x')}{dx'}\bigg|_{x'=X}$$

$$h = \frac{(\lambda')^2}{2N^2}\frac{d^2R(x')}{d(x')^2}\bigg|_0 = \frac{\Lambda^2}{2N^2}\frac{d^2R(x')}{d(x')^2}\bigg|_{x'=X}$$

and defining

$$\begin{aligned} R_0 &= R(x')\bigg|_{x'=X} \\ R_0' &= \frac{dR(x')}{dx'}\bigg|_{x'=X} \\ R_0'' &= \frac{d^2R(x')}{d(x')^2}\bigg|_{x'=X} \end{aligned} \tag{18}$$

the special form of the incremental parameters can be written more concisely as

$$l = \frac{N^2}{R_0}$$
$$g = \frac{\Lambda R_0'}{N^2} \qquad (19)$$
$$h = \frac{\Lambda^2 R_0''}{2N^2}$$

where the parameters are completely given if the form of the reluctance function $R(x')$ is known.

Three distinct possibilities for the parameters at a given state $(\Lambda, X)$ can be distinguished:

1. $R(x')$ has a nonzero first and second derivative, and therefore all of the parameters are nonzero;
2. $R(x')$ has a nonzero first derivative and a zero second derivative, and therefore $h$ is zero and the other two parameters are nonzero;
3. $R(x')$ has a nonzero second derivative and a zero first derivative, and therefore $g$ is zero and $h$ and $l$ are nonzero.

A fourth possibility would occur if both the first and second derivative were zero, but since this would render both $g$ and $h$ zero it is of no interest in the study of an electromechanical transducer.

In order to give the general expressions in Eqs. (19) a more concrete meaning, it is useful to consider specific types of energy conversion processes. The analysis of two particular classes of magnetic fields are presented below in the form of illustrative examples.

### Example 1

In the first example the incremental energy conversion process will be analyzed for a magnetic field in which the mechanical motion is in the direction of the air gap. A sketch of a possible pole configuration for such a field is shown in Fig. 3, where the origin of $x'$ is chosen so that it represents the displacement from the equilibrium separation, $d$, for the state $(\Lambda, X)$ under consideration.

We can express the reluctance of the magnetic path of the coil in a power series of the form,

$$R(x') = R_0 \left[ 1 + \alpha_1(x') + \alpha_2(x')^2 + \ldots + \alpha_n(x')^n + \ldots \right] \qquad (20)$$

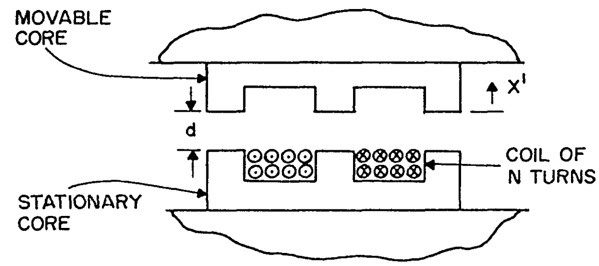


Fig. 3 - Magnetic field with mechanical motion in the direction of the air gap

where the constant $\alpha_n$ are dependent on the state $(\Lambda, X)$ and the geometry and composition of the two $E$ cores, and where $R_0$ is the reluctance for $x' = 0$.

If we apply the expressions for the incremental parameters, Eqs. (19), to this function, we obtain

$$l = \frac{N^2}{R_0}$$
$$g = \alpha_1 \frac{\Lambda R_0}{N^2} \qquad (21)$$
$$h = \alpha_2 \frac{\Lambda^2 R_0}{N^2}.$$

Thus for this type of magnetic field device the coefficient $\alpha_2$ directly determines the magnitude of the incremental magnetic spring constant $h$. Making the usual approximate assumptions, namely, that the reluctance of the air gap will be linearly dependent on $x'$ and that the reluctance of the cores can be neglected, the reluctance can be expressed as

$$R = \frac{d + x'}{\mu_0 A_0} = \frac{d}{\mu_0 A_0}\left(1 + \frac{x'}{d}\right) = R_0\left(1 + \frac{x'}{d}\right) \qquad (22)$$

where

$R_0 = \dfrac{d}{\mu_0 A_0}$ = equilibrium reluctance

$d$ = separation of the $E$ cores at equilibrium

$\mu_0$ = permeability of the air gap

$A_0$ = effective area of the air gap.

Since there is no term in $(x')^2$ in Eq. (22), the value of $h$ will be zero, and the approximate value of $\alpha_1$ will be

$$\alpha_1 = \frac{1}{d}.$$

which means that the resulting approximate value of $g$ will be

$$g = \frac{\Lambda R_0}{dN^2} \tag{23}$$

which is linearly dependent on the flux linkage of the state $(\Lambda, X)$ under study.

### Summary of Example 1

The purpose of this example was to investigate the form of $h$, $g$, and $l$ for a magnetic field in which the mechanical motion is in the direction of the air gap.

The results indicate that if the reluctance of the air gap can be legitimately represented as being linearly dependent on $x'$, the incremental magnetic spring constant $h$ will be zero. That is, as long as Eq. (22) is a valid expression for the reluctance of the air gap, then $h$ will be zero, and $g$ will be approximately the function given in Eq. (23).

### Example 2

In the second example the incremental energy conversion process will be considered for a magnetic field in which the mechanical motion is perpendicular to the direction of the air gap. The representation of a possible system of the class is shown in Fig. 4, where $x'$ is the displacement from the symmetrical position of the movable $E$ core. The reluctance of the magnetic path of the $N$-turn coil can be expressed as

$$R(x') = R_0 \,[1 + \beta_1(x')^2 + \beta_2(x')^4 + \dots + \beta_n(x')^{2n} + \dots] \tag{24}$$

where the $\beta_n$ are functions of the geometry and composition of the poles, and $R_0$ is the minimum reluctance of the magnetic path. The absence of odd powers of $x'$ in the expansion of Eq. (24) is a result of the symmetry of the pole arrangement with respect to positive and negative values of displacement, which insures that $R(x')$ will be an even function of $x'$.

Inserting this reluctance expression into Eqs. (19), the following values for the incremental parameters are obtained,

$$l = \frac{N^2}{R_0}$$

$$g = 0 \tag{25}$$

$$h = \beta_1 \frac{\Lambda^2 R_0}{N^2}.$$

Therefore the results indicate that this class of magnetic field devices will in general exhibit an incremental magnetic spring constant $h$ but not an incremental electromechanical coupling parameter $g$.

A very crude approximation for the reluctance variation of the air gap can be derived in the following manner. For a movement $x'$ of the upper $E$ core the "effective length" $b$ of the air gap (see Fig. 4) can be approximated as

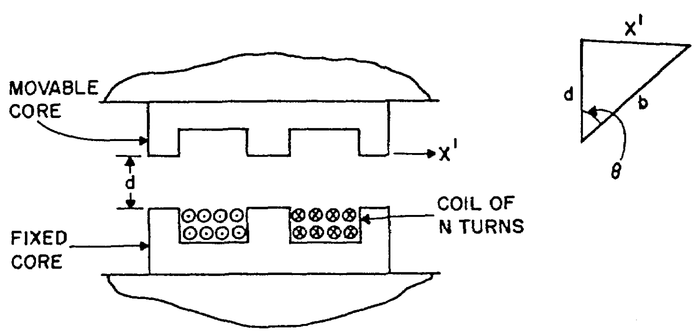


Fig. 4 - Magnetic field with mechanical motion perpendicular to the direction of the air gap

$$b = \sqrt{d^2 + (x')^2}$$

where

$$d = \text{separation of the } E \text{ cores}$$

and the "effective area" $A$ of the air gap can be approximated as (see Fig. 4)

$$A = A_0 \cos\theta = \frac{A_0 d}{\sqrt{d^2 + (x'_f)^2}}$$

where $A_0$ = effective area of the air gap for $x' = 0$

$\cos\theta$ = correction factor to account for the reduction in the cross-sectional area normal to the flux lines in the air gap

$\theta$ = angle by which the movable core is shifted.

Thus the reluctance of the air gap $R_g$ can be expressed, in a very rough approximation, as

$$R = \frac{b}{\mu_0 A} = \frac{d}{\mu_0 A_0}\left[1 + \left(\frac{x'}{d}\right)^2\right] \tag{26}$$

which, neglecting the reluctance of the $E$ cores, is an approximate equation for the total reluctance $R(x')$. The values of $\beta_1$ and $R_0$ for this particular reluctance function are

$$\beta_1 = \frac{1}{d^2}\,;\; R_0 = \frac{d}{\mu_0 A_0}$$

and the "magnetic spring constant" $h$ is

$$h = \frac{\Lambda^2 R_0}{d^2 N^2} \tag{27}$$

which is dependent on the square of the flux linkage $\Lambda$ of the state $(\Lambda, 0)$ under consideration.

### Summary of Example 2

The purpose of this example was to illustrate the form of $h$, $g$, and $l$ for a magnetic field in which the mechanical motion is perpendicular to the direction of the air gap.

The results indicate that such a field will exhibit an incremental magnetic spring constant $h$ but no incremental electromechanical coupling parameter $g$. An approximate form of the reluctance, given by Eq. (26), leads to the approximate value of $h$ given by Eq. (27).

This and the preceding example treated two specific types of magnetic field and derived the applicable incremental energy conversion parameters. Of course, it must be realized that the most general possibility is when $h$, $g$, and $l$ are all nonzero, and that all other possibilities are only special cases.

## LINEAR EQUIVALENT NETWORKS

Now that the incremental parameters have been considered in detail, it is meaningful to return to a consideration of the linearized current and force equations and some alternative forms of these energy conversion relations.

Up to this point the analysis has been confined to the use of the original set of field variables $\lambda'$, $x'$, $i'$, and $f'$ and the related incremental variables. In the following discussion the consequences of introducing the velocity will be considered. In addition, for the first time the concept of the incremental equivalent network (or "small signal equivalent network") will be utilized. These networks do not contribute any new information to the analysis, but they do provide a useful schematic representation of the energy conversion process.

The general energy conversion relations of Eqs. (4) have been linearized into the form

$$\begin{aligned} i &= \frac{1}{l}\lambda + g\,x \\ f &= g\,\lambda + h\,x \end{aligned} \tag{16}$$

and these equations can be represented by the incremental equivalent network shown in Fig. 5.

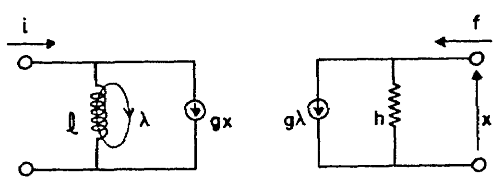


Fig. 5 - Incremental equivalent network representing Eqs. (16). The symbol ⏀ indicates a current generator (left side) and a force generator (right side).

Introducing the two new incremental variables

$v$ = incremental voltage variable

$u$ = incremental velocity variable

a somewhat more conventional set of equations, and equivalent network, results. Letting the new incremental variables be defined as

$$u = \frac{dx}{dt} \; ; \; v = \frac{d\lambda}{dt} \tag{28}$$

Eqs. (16) can be rewritten in terms of $v$ and $u$ in the form

$$\begin{aligned} i &= \frac{1}{l}\int v\,dt + g\int u\,dt \\ f &= g\int v\,dt + h\int u\,dt \end{aligned} \tag{29}$$

and the equivalent network shown in Fig. 6 can be constructed. Equations (29) will be referred to as the node-node form because both relations are in the form of node equations.

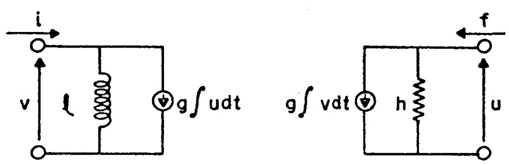


Fig. 6 - Equivalent network for Eqs. (29)

Another form of the energy conversion equations can be obtained by choosing to solve Eqs. (29) for $v$ and $f$ in terms of $i$ and $u$. After some manipulation the following set of equations result:

$$v = l\frac{di}{dt} - (gl)u$$

$$f = (gl)i + (h - g^2 l)\int u\,dt.$$

These are in the form of an electrical mesh equation and a mechanical node equation.

Defining two secondary incremental parameters as

$$\begin{aligned} g^* &= gl \\ h^* &= h - g^2 l \end{aligned} \tag{30}$$

it is possible to rewrite the mesh-node form of the equations as

$$\begin{aligned} v &= l\frac{di}{dt} - g^* u \\ f &= g^* i + h^* \int u\,dt \end{aligned} \tag{31}$$

and these equations can be represented by the equivalent network shown in Fig. 7. As an illustration of possible values for the secondary parameters defined by Eqs. (30), the results from the two examples presented during the discussion of the primary incremental parameters can be utilized.

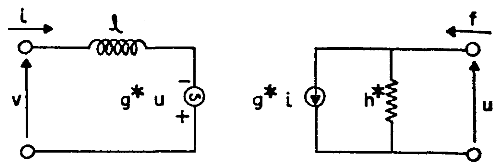


Fig. 7 - Equivalent network for Eqs. (31). The symbol ⊖ indicates a voltage generator.

## Sample Calculations

For the case of motion parallel to the air gap the value of $l$ from the first of Eqs. (21) can be used, and the approximate values of $h$ and $g$ associated with the reluctance expression of Eq. (22) can be used—the approximate value of $g$ being given by Eq. (23). That is, the values

$$l = \frac{N^2}{R_0} \; ; \; g = \frac{\Lambda R_0}{N^2 d} \; ; \; h = 0$$

will be used. Inserting these in Eqs. (30), the results are

$$\begin{aligned} g^* &= \frac{\Lambda}{d} \\ h^* &= -\frac{\Lambda^2 R_0}{N^2 d^2} \end{aligned} \tag{32}$$

as the secondary incremental parameters for the magnetic field of Fig. 3. It must be recognized, of course, that these specific values are based on the reluctance expression of Eq. (22), which is only an approximation. Thus they are valid only as long as Eq. (22) is a valid approximation for the reluctance.

In the case of motion perpendicular to the air gap, the values of $l$ and $g$ from Eqs. (25) will be used, together with the values of $h$ from Eq. (27) resulting from the approximate reluctance formula of Eq. (26). Thus the values will be

$$l = \frac{N^2}{R_0} \; ; \; g = 0 \; ; \; h = \frac{\Lambda^2 R_0}{N^2 d^2}$$

and this yields for $g^*$ and $h^*$ the expressions

$$\begin{aligned} g^* &= 0 \\ h^* &= h = \frac{\Lambda^2 R_0}{N^2 d^2} \end{aligned} \tag{33}$$

which are general results except for the special value of $h$ based on the approximate reluctance formula of Eq. (26).

The two alternative formulations of the linear equations, Eqs. (29) and (31), are equally valid relations. However, because of the lack of direct dependence of the $g^*$ and $h^*$ parameters on the stored energy function, it is felt that the form of Eqs. (29) is the more physically meaningful of these two possibilities. As a consequence, it seems logical to introduce Eqs. (31) only when there are mathematical advantages in the subsequent analysis.

### Remarks

The basic technique for quantitative description of the energy conversion process in a magnetic field has been presented. It is based on a consideration of the energy storage in the magnetic field, and the interaction of this energy with external electrical and mechanical sources. This general technique has been discussed in the context of the simplest possible type of transducer, but its applicability is not limited to such cases. This vehicle was chosen only to prevent obscuring the basic techniques involved in the analysis.

In order to illustrate the general applicability of this technique a more complex example will now be presented involving two magnetic fields and two movable magnetic masses.

## AN ILLUSTRATIVE APPLICATION

An example is included to illustrate the general applicability of the energy conversion technique presented in the foregoing discussion. The transducer to be analyzed employs two magnetic fields and has a mechanical construction which includes two movable magnetic masses. A representation of the transducer is shown in Fig. 8.

Since the electromechanical energy conversion process has already been discussed in detail for the simplest possible case, the following will be limited to the essential steps of the general technique as applied to the system of Fig. 8. The first step will be to define the symbology which is used in the example.

### Glossary of Symbols

The primary field variables are:

$x_1'$ = displacement of the inner mass

$x_2'$ = displacement of the outer mass

$f_1$ = that portion of the external force exerted on the inner mass which is associated with

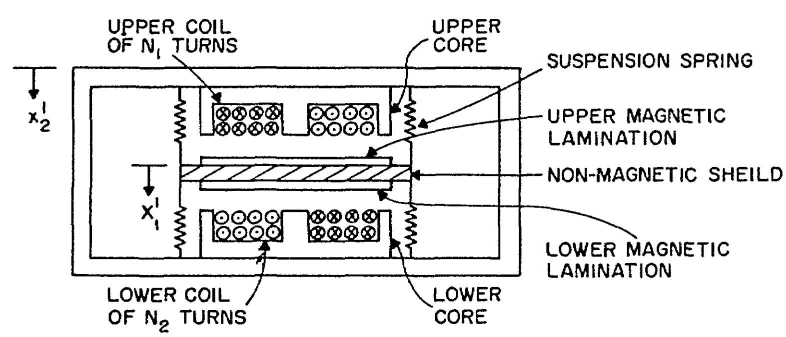


Fig. 8 - Transducer utilizing two magnetic fields and two movable magnetic masses

the upper and lower magnetic fields, where the positive direction of $f'_1$ is the same as that of $x'_1$

$f'_2$ = that portion of the external force exerted on the outer mass which is associated with the upper and lower magnetic fields, where the positive direction of $f'_2$ is the same as that of $x'_2$

$\lambda'_1$ = flux linkage of the upper coil

$\lambda'_2$ = flux linkage of the lower coil

$i'_1$ = current in the upper coil

$i'_2$ = current in the lower coil

$w_1$ = energy stored in the upper magnetic field

$w_2$ = energy stored in the lower magnetic field

The auxiliary field variables are:

$f'_{11}$ = that portion of $f'_1$ associated with the upper field

$f'_{12}$ = that portion of $f'_1$ associated with the lower field

$f'_{21}$ = that portion of $f'_2$ associated with the upper field

$f'_{22}$ = that portion of $f'_2$ associated with the lower field

$u'_1$ = velocity of the inner mass

$u'_2$ = velocity of the outer mass

$v'_1$ = voltage difference associated with the upper field

$v'_2$ = voltage difference associated with the lower field.

It follows that

$$\begin{aligned} f'_1 &= f'_{11} + f'_{12} \\ f'_2 &= f'_{21} + f'_{22} \end{aligned} \tag{34}$$

and that

$$v'_1 = \frac{d\lambda'_1}{dt} \; ; \; v'_2 = \frac{d\lambda'_2}{dt} \tag{35}$$

where the positive directions of the flux linkage and voltage are taken so that the associated current enters the positive terminal as defined by the assumed voltage difference.

The incremental variables are:

$x_1$ = incremental variation of $x'_1$ around $X_1$

$x_2$ = incremental variation of $x'_2$ around $X_2$

$f_1$ = incremental variation of $f'_1$ around $F_1$

$f_2$ = incremental variation of $f'_2$ around $F_2$

$\lambda_1$ = incremental variation of $\lambda'_1$ around $\Lambda_1$

$\lambda_2$ = incremental variation of $\lambda'_2$ around $\Lambda_2$

$i_1$ = incremental variation of $i'_1$ around $I_1$

$i_2$ = incremental variation of $i'_2$ around $I_2$

$f_{11}$ = incremental variation of $f'_{11}$ around $F_{11}$

$f_{12}$ = incremental variation of $f'_{12}$ around $F_{12}$

$f_{21}$ = incremental variation of $f'_{21}$ around $F_{21}$

$f_{22}$ = incremental variation of $f'_{22}$ around $F_{22}$

$v_1$ = incremental variation of $v'_1$ around $V_1$

$v_2$ = incremental variation of $v'_2$ around $V_2$

$u_1$ = incremental variation of $u'_1$ around $U_1$

$u_2$ = incremental variation of $u'_2$ around $U_2$

where the state of the upper and lower magnetic fields are $(\Lambda_1, X_1, X_2)$ and $(\Lambda_2, X_1, X_2)$ respectively, and the other capital symbols are the values of the respective variables corresponding to these states.

## Energy Analysis

The incremental increase in the stored energy of either field can be expressed as the sum of the incremental electrical and mechanical input energies. The resulting relations are

$$
\begin{aligned}
dw_1 &= i'_1\, d\lambda'_1 + f'_{11}\, dx'_1 + f'_{21}\, dx'_2 \\
dw_2 &= i'_2\, d\lambda'_2 + f'_{12}\, dx'_1 + f'_{22}\, dx'_2
\end{aligned} \tag{36}
$$

where the second and third terms on the right represent the total mechanical input energy to each field.

The energy functions for the upper and lower fields are state functions of the respective flux linkages, and of the displacement variables. The functional notation is

$$
\begin{aligned}
w_1 &= w_1(\lambda'_1, x'_1, x'_2) \\
w_2 &= w_2(\lambda'_1, x'_1, x'_2).
\end{aligned} \tag{37}
$$

Thus the incremental changes in $w_1$ and $w_2$ are

$$
\begin{aligned}
dw_1 &= \frac{\partial w_1}{\partial \lambda'_1}\, d\lambda'_1 + \frac{\partial w_1}{\partial x'_1}\, dx'_1 + \frac{\partial w_1}{\partial x'_2}\, dx'_2 \\
dw_2 &= \frac{\partial w_2}{\partial \lambda'_2}\, d\lambda'_2 + \frac{\partial w_2}{\partial x'_1}\, dx'_1 + \frac{\partial w_2}{\partial x'_2}\, dx'_2.
\end{aligned} \tag{38}
$$

Equating Eqs. (38) to Eqs. (36) the results are the general current and force relations

$$
\begin{aligned}
i'_1(\lambda'_1, x'_1, x'_2) &= \frac{\partial w_1(\lambda'_1, x'_1, x'_2)}{\partial \lambda'_1} \\
f'_{11}(\lambda'_1, x'_1, x'_2) &= \frac{\partial w_1(\lambda'_1, x'_1, x'_2)}{\partial x'_1} \\
f'_{21}(\lambda'_1, x'_1, x'_2) &= \frac{\partial w_1(\lambda'_1, x'_1, x'_2)}{\partial x'_2}
\end{aligned} \tag{39}
$$

and

$$
\begin{aligned}
i'_2(\lambda'_2, x'_1, x'_2) &= \frac{\partial w_2(\lambda'_2, x'_1, x'_2)}{\partial \lambda'_2} \\
f'_{12}(\lambda'_2, x'_1, x'_2) &= \frac{\partial w_2(\lambda'_2, x'_1, x'_2)}{\partial x'_1} \\
f'_{22}(\lambda'_2, x'_1, x'_2) &= \frac{\partial w_2(\lambda'_2, x'_1, x'_2)}{\partial x'_2}
\end{aligned} \tag{40}
$$

expressed in terms of the energy functions.

The energy functions can be evaluated by integrating Eqs. (36) for a general path, resulting in

$$
\begin{aligned}
w_1(\Lambda_1, X_1, X_2) &= \int_0^{\Lambda_1} i'_1\, d\lambda'_1 + \int_0^{X_1} f'_{11}\, dx'_1 + \int_0^{X_2} f'_{21}\, dx'_2 \\
w_2(\Lambda_2, X_1, X_2) &= \int_0^{\Lambda_2} i'_2\, d\lambda'_2 + \int_0^{X_1} f'_{12}\, dx'_1 + \int_0^{X_2} f'_{22}\, dx'_2
\end{aligned} \tag{41}
$$

or they can be evaluated in the special form

$$
\begin{aligned}
w_1(\Lambda_1, X_1, X_2) &= \int_0^{\Lambda_1} i'_1(\lambda'_1, X_1, X_2)\, d\lambda'_1 \\
w_2(\Lambda_2, X_1, X_2) &= \int_0^{\Lambda_1} i'_2(\lambda'_2, X_1, X_2)\, d\lambda'_2
\end{aligned} \tag{42}
$$

resulting from an integration path which brings the masses to their final positions $(X_1, X_2)$ while the flux linkages are zero and then increases the flux linkages to $(\Lambda_1, \Lambda_2)$.

Although the relations of Eqs. (39) and (40) are completely general, they are not in their simplest form. An examination of Fig. 8 reveals that, insofar as the magnetic fields are concerned, a displacement of $x'_1 = +\delta$ (with $x'_2$ constant) is completely equivalent to a displacement of $x'_2 = -\delta$ (with $x'_1$ constant).

This symmetry can be stated mathematically in the form

$$
\begin{aligned}
\frac{\partial w_1}{\partial x'_1} &= -\frac{\partial w_1}{\partial x'_2} \\
\frac{\partial w_2}{\partial x'_2} &= -\frac{\partial w_2}{\partial x'_1}.
\end{aligned}
$$

In terms of the force functions this becomes

$$
f'_{11} = -f'_{21}\,;\ f'_{22} = -f'_{12}
$$

and utilizing Eqs. (34),

$$
f'_1 = -f'_2 = f'_{11} - f'_{22} \tag{43}
$$

which are the constraints on the force functions due to the mechanical symmetry of the transducer being considered.

This general relationship between the partial derivatives of each energy function can be written in a formal operator notation as

$$\frac{\partial}{\partial x'_1} = -\frac{\partial}{\partial x'_2} \tag{44}$$

which is valid for all differential operations on either of the energy functions, $w_1$ or $w_2$.

Anticipating the need of such relations in the discussion of incremental parameters, some useful equalities among second derivatives of the energy functions can be derived by formally applying Eq. (44) to $w_1$ and $w_2$. The results are

$$\frac{\partial^2 w_1}{\partial(x'_1)^2} = \frac{\partial^2 w_1}{\partial(x'_2)^2} = -\frac{\partial^2 w_1}{\partial x'_1 \partial x'_2} = -\frac{\partial^2 w_1}{\partial x'_2 \partial x'_1}$$

$$\frac{\partial^2 w_1}{\partial x'_1 \partial \lambda'_1} = -\frac{\partial^2 w_1}{\partial x'_2 \partial \lambda'_1}$$

$$\frac{\partial^2 w_2}{\partial(x'_2)^2} = \frac{\partial^2 w_2}{\partial(x'_1)^2} = -\frac{\partial^2 w_2}{\partial x'_1 \partial x'_2} = -\frac{\partial^2 w_2}{\partial x'_2 \partial x'_1}$$

$$\frac{\partial^2 w_2}{\partial x'_2 \partial \lambda'_2} = -\frac{\partial^2 w_2}{\partial x'_1 \partial \lambda'_2}$$

as a consequence of the mechanical symmetry of the magnetic fields.

From the state function properties of $w_1$ and $w_2$ it follows that

$$\frac{\partial^2 w_1}{\partial x'_1 \partial x'_2} = \frac{\partial^2 w_1}{\partial x'_2 \partial x'_1}$$

$$\frac{\partial^2 w_1}{\partial x'_1 \partial \lambda'_1} = \frac{\partial^2 w_1}{\partial \lambda'_1 \partial x'_1}$$

$$\frac{\partial^2 w_1}{\partial x'_2 \partial \lambda'_1} = \frac{\partial^2 w_1}{\partial \lambda'_1 \partial x'_2}$$

and

$$\frac{\partial^2 w_2}{\partial x'_1\ x'_2} = \frac{\partial^2 w_2}{\partial x'_2 \partial x'_1}$$

$$\frac{\partial^2 w_2}{\partial x'_1 \partial \lambda'_2} = \frac{\partial^2 w_2}{\partial \lambda'_2 \partial x'_1}$$

$$\frac{\partial^2 w_2}{\partial x'_2 \partial \lambda'_2} = \frac{\partial^2 w_2}{\partial \lambda'_2 \partial x'_2}$$

are constraints on mixed second derivatives of the energy functions.

Both these sets of relations can be summarized by the four equalities

$$\begin{aligned}
\frac{\partial^2 w_1}{\partial(x'_1)^2} &= -\frac{\partial^2 w_1}{\partial x'_1 \partial x'_2} = -\frac{\partial^2 w_1}{\partial x'_2\, \partial x'_1} = \frac{\partial^2 w_1}{\partial(x'_2)^2} \\
\frac{\partial^2 w_1}{\partial x'_1\ \lambda'_1} &= \frac{\partial^2 w_1}{\partial \lambda'_1\ x'_1} = -\frac{\partial^2 w_1}{\partial \lambda'_1 \partial x'_2} = -\frac{\partial^2 w_1}{\partial x'_2 \partial \lambda'_1} \\
\frac{\partial^2 w_2}{\partial(x'_2)^2} &= -\frac{\partial^2 w_2}{\partial x'_1 \partial x'_2} = -\frac{\partial^2 w_2}{[illegible]} = \frac{\partial^2 w_2}{\partial(x'_1)^2} \\
\frac{\partial^2 w_2}{\partial x'_2\ \lambda'_2} &= \frac{\partial^2 w_2}{\partial \lambda'_2 \partial x'_2} = [illegible] \frac{\partial^2 w_2}{[illegible]\lambda'_2}
\end{aligned} \tag{45}$$

which interrelate si[illegible] of the eig[illegible]en possible second derivatives of $w_1$ and $w_2$. The only two derivatives which do not occur in Eqs. (45) are the pure second derivative for $\lambda'_1$ and $\lambda'_2$.

### Incremental Analysis

Making a Taylor series expansion of Eqs. (39) and (40) about the states ($\Lambda_1, X_1, X_2$) and ($\Lambda_2, X_1, X_2$) respectively, the incremental currents and forces can be expressed as

$$i_1 = \frac{\partial^2 w_1}{\partial(\lambda'_1)^2}\bigg|_0 \lambda_1 + \frac{\partial^2 w_1}{\partial x'_1 \partial \lambda'_1}\bigg|_0 x_1 + \frac{\partial^2 w_1}{\partial x'_2 \partial \lambda'_1}\bigg|_0 x_2$$

$$f_{11} = \frac{\partial^2 w_1}{\partial \lambda'_1 \partial x'_1}\bigg|_0 \lambda_1 + \frac{\partial^2 w_1}{\partial(x'_1)^2}\bigg|_0 x_1 + \frac{\partial^2 w_1}{\partial x'_2 \partial x'_1}\bigg|_0 x_2$$

$$f_{21} = \frac{\partial^2 w_1}{\partial \lambda'_1 \partial x'_2} [illegible] \frac{\partial^2 w_1}{\partial x'_1 \partial x'_2}\bigg|_0 x_1 + \frac{\partial^2 w_1}{\partial(x'_2)^2}\bigg|_0 x_2$$

where the subscript zero on the [illegible]rivatives indicates evaluation at the s[illegible]$_2$), and

$$i_2 = \frac{\partial^2 w_2}{\partial(\lambda'_2)^2}\bigg|_0 \lambda_2 + \frac{\partial^2 w_2}{\partial x'_1 \partial \lambda'_2}\bigg|_0 [illegible] \frac{\partial^2 w_2}{\partial x'_2 \partial \lambda'_2}\bigg|_0 x_2$$

$$f_{12} = \frac{\partial^2 w_2}{\partial \lambda'_2 \partial x'_1}\bigg|_0 \lambda_2 + \frac{\partial^2 w_2}{\partial(x'_1)^2}\bigg|_0 x_1 + \frac{\partial^2 w_2}{\partial x'_2 \partial x'_1}\bigg|_0 x_2$$

$$f_{22} = \frac{\partial^2 w_2}{\partial \lambda'_2 \partial x'_2}\bigg|_0 \lambda_2 + \frac{\partial^2 w_2}{\partial x'_1 \partial x'_2}\bigg|_0 x_1 + \frac{\partial^2 w_2}{\partial(x'_2)^2}\bigg|_0 x_2$$

where the subscript zero implies the state ($\Lambda_2, X_1, X_2$).

Similarly the currents and forces corresponding to the equilibrium states can be written

$$I_1 = i'_1(\Lambda_1, X_1, X_2)$$

$$I_2 = i'_2(\Lambda_2, X_1, X_2) \tag{46}$$

$$F_1 = -F_2 = f'_{11}(\Lambda_1, X_1, X_2) - f'_{22}(\Lambda_2, X_1, X_2)$$

where the last is based on Eq. (43).

As a result of Eqs. (45), only six of the eighteen partial derivatives in the incremental equations are independent. Thus, defining the incremental parameters as

$$\frac{1}{l_1} = \left.\frac{\partial^2 w_1}{\partial(\lambda'_1)^2}\right|_0;$$

$l_1$ = incremental inductance at $(\Lambda_1, X_1, X_2)$

$$\frac{1}{l_2} = \left.\frac{\partial^2 w_2}{\partial(\lambda'_2)^2}\right|_0;$$

$l_2$ = incremental inductance at $(\Lambda_2, X_1, X_2)$

$$h_1 = \left.\frac{\partial^2 w_1}{\partial(x'_1)^2}\right|_0;$$

$h_1$ = incremental stiffness at $(\Lambda_1, X_1, X_2)$

$$h_2 = \left.\frac{\partial^2 w_2}{\partial(x'_2)^2}\right|_0;$$

$h_2$ = incremental stiffness at $(\Lambda_2, X_1, X_2)$

$$g_1 = \left.\frac{\partial^2 w_1}{\partial x'_1 \partial\lambda'_1}\right|_0;$$

$g_1$ = incremental coupling parameter at $(\Lambda_1, X_1, X_2)$

$$g_2 = \left.\frac{\partial^2 w_2}{\partial x'_2 \partial\lambda'_2}\right|_0;$$

$g_2$ = incremental coupling parameter at $(\Lambda_2, X_1, X_2)$

and applying the identities relating the various second derivatives, the incremental equations can be rewritten as

$$i_1 = \frac{1}{l_1}\lambda_1 + g_1\,(x_1 - x_2)$$

$$f_{11} = g_1\,\lambda_1 + h_1\,(x_1 - x_2)$$

$$f_{21} = -g_1\,\lambda_1 - h_1\,(x_1 - x_2)$$

and

$$i_2 = \frac{1}{l_2}\lambda_2 - g_2\,(x_1 - x_2)$$

$$f_{12} = -g_2\,\lambda_2 + h_2\,(x_1 - x_2)$$

$$f_{22} = g_2\,\lambda_2 - h_2\,(x_1 - x_2).$$

Now since

$$f_1 = f_{11} + f_{12}, \text{ and } f_2 = f_{21} + f_{22}$$

the above six force and current equations can be combined into three equivalent expressions:

$$i_1 = \frac{1}{l_1}\lambda_1 + g_1\,(x_1 - x_2)$$

$$i_2 = \frac{1}{l_2}\lambda_2 - g_2\,(x_1 - x_2) \tag{47}$$

$$f_1 = -f_2 = g_1\,\lambda_1 - g_2\,\lambda_2 + h\,(x_1 - x_2)$$

where

$$h = h_1 + h_2 = \left.\frac{\partial^2 w_1}{\partial(x'_1)^2}\right|_0 + \left.\frac{\partial^2 w_2}{\partial(x'_2)^2}\right|_0$$

= incremental spring constant at state $(\Lambda_1, \Lambda_2, X_1, X_2)$

and where the linearized incremental equations are in their simplest form. In this set of relations it is only necessary to specify five independent parameters to describe the linearized energy conversion process.

Using these equations the equivalent network of Fig. 9 can be constructed. This network represents the energy conversion process in terms of the primary field variables of flux linkage and displacement.

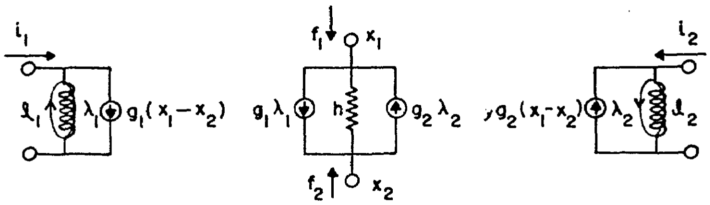


Fig. 9 - Equivalent network for Eqs. (47)

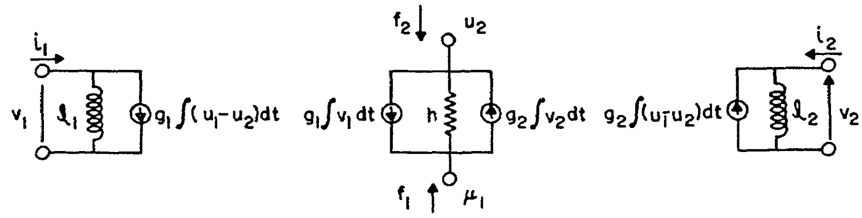


Fig. 10 - Equivalent network for Eqs. (48)

Introducing the voltage and velocity variables into Eqs. (47) these expressions become

$$
\begin{aligned}
i_1 &= \frac{1}{l_1}\int v_1\,dt + g_1\int (u_1-u_2)\,dt \\
i_2 &= \frac{1}{l_2}\int v_2\,dt - g_2\int (u_1-u_2)\,dt \qquad (48)\\
f_1 &= -f_2 = g_1\int v_1\,dt - g_2\int v_2\,dt \\
&\quad + h\int (u_1-u_2)\,dt
\end{aligned}
$$

and an equivalent network is of the form shown in Fig. 10, which will be referred to as the nodal equivalent network, since all three of Eqs. (48) are node equations.

An alternative form of the equations results from solving the nodal set, Eqs. (48), for $v_1$, $v_2$, $f_1$, and $f_2$ in terms of $i_1$, $i_2$, $u_1$, and $u_2$. The results are

$$
\begin{aligned}
v_1 &= l_1\frac{di_1}{dt} - g_1\,l_1\,(u_1-u_2) \\
v_2 &= l_2\frac{di_2}{dt} + g_2\,l_2\,(u_1-u_2) \\
f_1 &= -f_2 = (g_1l_1)i_1 - (g_2l_2)i_2 \\
&\quad + (h - g_1^2\,l_1 - g_2^2 l_2)\int (u_1-u_2)dt
\end{aligned}
$$

and by defining three new incremental parameters in the form

$$
\begin{aligned}
g^*_1 &= g_1 l_1 \\
g^*_2 &= g_2 l_2 \qquad (49)\\
h^* &= h - g_1^2 l_1 - g_2^2 l_2
\end{aligned}
$$

the equations can be rewritten

$$
\begin{aligned}
v_1 &= l_1\frac{di_1}{dt} - g^*_1(u_1-u_2) \\
v_2 &= l_2\frac{di_2}{dt} + g^*_2(u_1-u_2) \qquad (50)\\
f_1 &= -f_2 = g^*_1 i_1 - g^*_2 i_2 + h^*\int (u_1-u_2)\,dt
\end{aligned}
$$

and an equivalent network can be drawn in the form presented in Fig. 11, which will be referred to as the mesh-node form for obvious reasons.

## Remarks

This more complex example has illustrated the general applicability of the energy conversion approach to the description of physical processes in a magnetic field transducer. The results are algebraically more complex than those derived in the earlier section, but conceptually they are equally as simple.

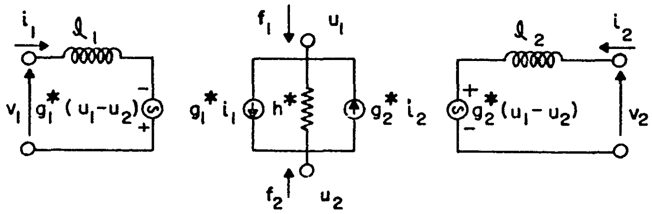


Fig. 11 - Equivalent network for Eqs. (50)

## RECAPITULATION

This report has presented a general technique for deriving the electromechanical energy conversion equations for a variable reluctance transducer. It is obvious that more general cases of energy conversion could be included simply by introducing additional mechanical and electrical degrees of freedom into our energy expressions. For example, systems having rotational modes and mutual coupling between individual coils could be treated with exactly the same technique merely by including the angular variables and mutual flux linkages in the energy equations, and proceeding in the same manner as above.

The final step in the derivation of the linear operating equations for the variable reluctance type of transducer is the use of the electrical and mechanical relations stated in Eqs. (1) and (2) to obtain the overall linear transducer equations. The solution of the resulting equations is an application of techniques which are presented in discussions of electrical network theory and mechanical vibration theory, and therefore they are not considered in this report.


## ACKNOWLEDGMENT

The energy method presented in this report was suggested to the author by Chapters 1 and 2 of "Electromechanical Energy Conversion" by D. C. White and H. H. Woodson (New York: Wiley, 1959).