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# Rigorous Treatment of the Propagation and Damping of Small-Amplitude Plasma Waves in an Initially Maxwellian Plasma

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CONTENTS




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## ABSTRACT

In the present report we make a mathematically rigorous investigation of the dispersion relation for small amplitude plasma oscillations of definite wave-number vector which are excited in an initially Maxwellian plasma. We work within the domain of the usual linearized Vlasov theory and concentrate our attention mainly on the case when the dimensionless quantity $\alpha \equiv (r_D k)^2$ is a small enough positive number, $r_D$ being the pertinent Debye length and $k$ the magnitude of the wave-number vector of the plasma oscillation of interest. More precisely, we study in the limit $\alpha \to +0$ the so-called Landau denominator or plasma dispersion function $\Delta(z) \equiv 1 + \alpha - \pi^{1/2} z e^{z^2} \operatorname{erfc}(z)$ corresponding to such a plasma. We show that in this limit $\Delta(z)$ has exactly two zeros (complex conjugates of one another) in the half-plane $\operatorname{Re} z < 0$ which lie closer to the imaginary $z$ axis than all the other zeros of $\Delta(z)$ in this half-plane. We also establish rigorous asymptotic formulas for the real and imaginary parts of these two nearest zeros for $\alpha \to +0$. As expected, these formulas agree essentially, but not exactly, with the corresponding results found by Landau some eighteen years ago. The discrepancy arises in the factor $e^{-3/2}$ which occurs in our formula for the imaginary part of these nearest zeros in the cited limit, the absolute value of this imaginary part being proportional to the Landau damping rate (under the conditions of Ref. 5). Our results for the location of the zeros in question agree with results obtained by other authors by nonrigorous methods.


## PROBLEM STATUS

This is a final report on one phase of a continuing problem.

## AUTHORIZATION

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RIGOROUS TREATMENT OF THE PROPAGATION AND DAMPING OF SMALL-AMPLITUDE PLASMA WAVES IN AN INITIALLY MAXWELLIAN PLASMA

## 1. INTRODUCTION

As is well known, the long-time behavior of plasma oscillations was treated by Landau (1) in a paper published about eighteen years ago. One of the most significant contributions of this paper was the derivation of an explicit expression for the damping of long-wavelength plasma waves excited in an initially Maxwellian plasma.

It will be recalled that Landau determined this long-time behavior by employing Laplace-transform techniques to solve the linearized Vlasov equation and the associated Poisson equation. In this way, he discovered an expression for the electric potential induced by plasma disturbances which has the form of an integral over a suitable Bromwich path in the plane of the Laplace-transform variable $s$. For an initially Maxwellian plasma, the familiar denominator of the integrand of this contour integral is proportional to the function

$$\Delta(z) \equiv 1 + \alpha - \pi^{1/2} z e^{z^2} \operatorname{erfc}(z) , \tag{1.1}$$

where $z = s/\omega_p(2\alpha)^{1/2}$ and $\alpha = (r_D k)^2$, $\omega_p$ and $r_D$ being the usual plasma frequency and Debye radius, respectively, and $k$ the wavenumber of the plasma oscillation of interest.*†

It will also be recalled that a basic preliminary step in Landau's study of the long-time dependence of the above electric potential was his famous investigation of the zeros of $\Delta(z)$. There are several easily proved properties of these zeros. For instance, it follows readily from (1.1) that $\overline{\Delta(z)} = \Delta(\bar{z})$, so that these zeros always occur in complex conjugate pairs. It can also be proved that $\Delta(z)$ has an infinite number of zeros for each choice of $\alpha$.‡ Finally, one can show that all of these zeros lie in the half-plane $\mathrm{Re}\ z < 0$ and that they are all simple.§ However, excluding the present report, we are not aware of any proof of the fact that, in the limit $\alpha \to +0$, two of the zeros of $\Delta(z)$, which are perforce complex conjugates of one another, lie closer to the imaginary $z$ axis than all the other zeros of $\Delta(z)$. As is widely known, Landau (1) did not, strictly speaking, prove this

*That the denominator of interest can be written in the form (1.1) for an initially Maxwellian plasma is shown, for example, in Ref. 2.

†Although the situation of physical interest is $\alpha > 0$, it will prove useful in this report for mathematical purposes to let $\alpha$ vanish or even to give it complex values. If no remark to the contrary is made, $\alpha$ should be understood to be a fixed nonnegative number.

‡$\Delta(z)$ has an infinite number of zeros even when $\alpha$ is taken to be an arbitrary complex number. To prove this assertion, we observe that $\Delta(z)$ is an entire function of $z$ of order 2. Hence it has an infinite number of zeros unless it is of the form $P(z)\ \exp[Q(z)]$, where $P(z)$ is a polynomial and $Q(z)$ is quadratic in $z$, by Hadamard's factorization theorem (see, for example, E. C. Titchmarsh, "Theory of Functions," Oxford University Press, London, 1939, 2nd ed., p. 250). Since this is not the case here, the present $\Delta(z)$ has an infinite number of zeros.

§For a proof that $\Delta(z) \neq 0$ when $\mathrm{Re}\ z \geq 0$ and $\alpha \geq 0$ see, for example, Sec. VI of Ref. 3. Dr. J. N. Hayes (private communication) has pointed out that the simple nature of the zeros of $\Delta(z)$ for the case $\alpha \geq 0$ can be established in a completely elementary way by evaluating $d\Delta(z)/dz$ at the zeros in question and by using the fact that none of these zeros is in the half-plane $\mathrm{Re}\ z \geq 0$ in this case.

fact. Instead, he implicitly assumed it and this assumption aided him in an essential way in actually calculating the location of these nearest zeros for $\alpha \to +0$. The reader will recall that the existence of these last zeros plays a basic role in Landau's proof of his result that the electric potential induced in an initially Maxwellian plasma by suitable initial perturbations is damped exponentially in the long-time limit, the corresponding damping rate being, as expected, proportional to the real part of the above closest zeros. Reference 4 contains detailed criticisms, on grounds of mathematical rigor, of Landau's computation of these zeros and of his conclusions concerning the cited long-time behavior.

The main contribution of this report is to confirm Landau's conclusions on the zeros of $\Delta(z)$ for $\alpha \to +0$ in all essential respects by the use of strict mathematical procedures. For rigorous treatments of the long-time behavior of the electric potential, the reader is referred to two recent studies (5,6).

It is perhaps of interest to motivate the precise mathematical discussions of the succeeding sections of this report. We begin by recalling that the closest zeros found by Landau have the following essential qualitative properties in the limit $\alpha \to +0$: their real parts tend to -0 and their imaginary parts approach the respective limits $+i\infty$ and $-i\infty$. It is also of interest in the present connection to notice that the auxiliary function $D(z)$ defined by Eq. (2.1) is equal to $\Delta(z)$ plus $\pi^{1/2} z e^{z^2}$. Heuristically, one can easily see that $D(z)$ has two complex conjugate, purely imaginary, zeros which tend to $+i\infty$ and $-i\infty$, respectively, as $\alpha \to +0$. The term $\pi^{1/2} z e^{z^2}$ is thus purely imaginary and its absolute value tends to zero exponentially at the position of these zeros of $D(z)$ in the cited limit. From this fact, one is led to expect, via simple heuristic considerations, that there are two complex conjugate zeros of $\Delta(z)$ which approach these two purely imaginary zeros of $D(z)$ as $\alpha \to +0$. On the basis of such reasoning, the present author began to believe very strongly that the two zeros of $\Delta(z)$ just mentioned were in fact the zeros of this function closest to the imaginary axis when $\alpha \to +0$, a belief which is fully confirmed in this report.

A rigorous discussion of the purely imaginary zeros of $D(z)$ is carried out in Sec. 2. Results of this discussion are applied in Sec. 3 to derive an explicit asymptotic formula, valid when $\alpha \to +0$, for a certain zero of $\Delta(z)$, say $z_1(\alpha)$, which lies in the quadrant of the $z$ plane determined by the simultaneous inequalities $\mathrm{Re}\, z < 0$ and $\mathrm{Im}\, z > 0$. This formula yields an expression for the imaginary part of $z_1(\alpha)$ in agreement with the corresponding result of Landau. On the other hand, the real part of $z_1(\alpha)$ obtained from the said formula differs from this author's value by the factor $e^{-3/2}$. Hence we find for $\alpha \to +0$ that the long-time damping rate of small amplitude plasma oscillations excited in an initially Maxwellian plasma (under the conditions stated in Ref. 5) differs from Landau's damping rate by the cited factor. This is precisely the correction factor pointed out, albeit on unrigorous grounds, by other investigators.* Finally, in Sec. 4 we establish that $z_1(\alpha)$ and the zero complex conjugate to it lie closer to the imaginary axis than all the other zeros of $\Delta(z)$ when $\alpha \to +0$.

## 2. ZEROS OF $D(z)$ ON THE IMAGINARY AXIS

In this section we shall study in detail the properties of the purely imaginary zeros of

$$D(z) \equiv 1 + \alpha + \pi^{1/2} z e^{z^2} \operatorname{erf}(z) . \tag{2.1}$$

The equation $\overline{D(z)} = D(\bar{z})$, implied by (2.1), entails that the zeros of $D(z)$ always occur in complex conjugate pairs. Observe also that (1.1), (2.1), and the identity

*See, for example, page 520 of Ref. 7 or Eq. (6.3) of Ref. 2.

$$\mathrm{erf}(z) + \mathrm{erfc}(z) = 1 \tag{2.2}$$

imply the following relation between $\Delta(z)$ and $D(z)$:

$$\Delta(z) = D(z) - \pi^{1/2} z e^{z^2}. \tag{2.3}$$

For purely imaginary values $iy$ of $z$, (2.1) can be written in the form

$$D(iy) = 1 + \alpha - F(y), \tag{2.1'}$$

where

$$F(y) \equiv 2y e^{-y^2} \int_0^y e^{\eta^2} d\eta. \tag{2.4}$$

This last function has the following asymptotic behavior for $|y| \to \infty$:

$$F(y) = 1 + \sum_{m=1}^{n} \gamma_m y^{-2m} + O\left(|y|^{-2(n+1)}\right), \tag{2.5}$$

where $n = 1, 2, \cdots$ and where

$$\gamma_m \equiv 1 \cdot 3 \cdot \ldots \cdot (2m-1)/2^m \tag{2.6}$$

for $m = 1, 2, \cdots$. One can obtain (2.5) by employing (2.2) and the asymptotic formula

$$\pi^{1/2} z e^{z^2} \mathrm{erfc}(z) = 1 + \sum_{m=1}^{n} (-1)^m \gamma_m z^{-2m} + O\left(|z|^{-2(n+1)}\right), \tag{2.7}$$

which holds for each positive integer $n$, provided that $|z| \to \infty$ and that

$$|\arg z| \le \frac{3\pi}{4} - \varepsilon < \frac{3\pi}{4},$$

$\varepsilon$ being independent of $z$.*

In our study of the zeros of $D(iy)$ in this section, we shall limit ourselves to the region $y \ge 0$, in view of the occurrence of the zeros of $D(z)$ in complex-conjugate pairs.† Before discussing the existence and pertinent properties of the zeros of $D(iy)$, we shall prove the following useful lemma.

Lemma 2.1. $F(y)$ attains its maximum value at precisely one point of the range $0 \le y < \infty$, say at $y = \xi$, where $\xi > 0$. That is,

*Equation (2.7) can be derived from the following results of Ref. 8 (the symbols $z$ and $\alpha$ occurring in this footnote coincide with symbols employed in this reference; they should not be confused with the corresponding letters used in the text of the present report): (a) the connection between $\mathrm{Erfc}(x)$ $[\equiv (\pi^{1/2}/2)\, \mathrm{erfc}(x)]$ and $W_{-1/4,\,-1/4}(x)$ on page 341; (b) the asymptotic formula for $W_{k,m}(z)$ on page 343, observing that this formula holds for $|z| \to \infty$ and $|\arg z| \le (3\pi/2) - \alpha < (3\pi/2)$, as is shown in Sec. 16.4 of Ref. 8 (notice, incidentally, that $k \pm m + (1/2)$ should be replaced by $k \pm m - (1/2)$ in the last line of page 343).

†Notice, incidentally, that $D(z)$ never has a zero at $z = 0$ for any given $\alpha$ in the range $\alpha \ge 0$ of interest here. In fact, (2.1) implies that $D(0) = 1 + \alpha$, so that $D(0) \ne 0$ for any such $\alpha$.

$$F(y) < F(\xi) \tag{2.8}$$

for any $y$ which lies in this range and which is different from $\xi$. Moreover,

$$F(\xi) > 1. \tag{2.9}$$

Proof. To prove this lemma, we investigate the sign changes of $dF(y)/dy$ for $0 \leq y < \infty$. From (2.4),

$$\frac{dF(y)}{dy} = 2(1 - 2y^2)\, e^{-y^2} \int_0^y e^{\eta^2}\, d\eta + 2y. \tag{2.10}$$

Hence $dF(y)/dy \geq 0$ if $y$ is in the closed interval $[0, 2^{-1/2}]$, the equality sign holding if and only if $y = 0$.*

We shall now show that $dF(y)/dy$ has exactly one zero if $y$ is in the range $[2^{-1/2}, \infty)$. In fact, $dF(y)/dy$ has at most one zero in this range, because the first term of the right-hand side of (2.10) is strictly decreasing in $y$ in the range in question, while the second term of this right-hand side always increases with $y$. That $dF(y)/dy$ has at least one zero in $[2^{-1/2}, \infty)$, and that it therefore possesses exactly one zero in this range, follows by invoking the continuity of $dF(y)/dy$ and the fact that this derivative changes sign in this range. Indeed, we have seen that $dF(2^{-1/2})/dy > 0$ and one can also show that $dF(y)/dy < 0$ for large enough values of $y$, since $dF(y)/dy \to -0$ as $y \to \infty$. This limit property results from the equation

$$\frac{dF(y)}{dy} = -\frac{1}{y^3} + O\left(\frac{1}{|y|^5}\right), \tag{2.11}$$

which is valid for $|y| \to \infty$. Equation (2.11) follows by expressing the right-hand side of (2.10) in terms of $F(y)$ and of elementary functions and by subsequently invoking (2.5).

Let $\xi$ be the unique point of the range $[2^{-1/2}, \infty)$ at which $dF(y)/dy$ vanishes. We proceed to show that (2.8) holds for any finite $y \geq 0$ such that $y \neq \xi$. In fact, since $dF(y)/dy$ vanishes solely at the points $y = 0$ and $y = \xi$ of the range $[0, \infty)$, this derivative does not change sign in the open intervals $(0, \xi)$ and $(\xi, \infty)$ of the $y$ axis, by continuity. Hence $dF(y)/dy > 0$ on $(0, \xi)$, since $dF(y)/dy > 0$ on $(0, 2^{-1/2}]$, and $dF(y)/dy < 0$ on $(\xi, \infty)$, since $dF(y)/dy < 0$ for large enough $y > 0$. These sign properties of $dF(y)/dy$ on $(0, \xi)$ and $(\xi, \infty)$ and the continuity of this derivative imply that $F(y)$ is strictly increasing on $[0, \xi]$ and strictly decreasing on $[\xi, \infty)$, i.e., that (2.8) holds in the stated sense.

Let us prove (2.9), thus completing the proof of the lemma. Observe first that the strictly decreasing nature of $F(y)$ on $[\xi, \infty)$ entails that $F(\xi) - F(y)$ is strictly increasing on $[\xi, \infty)$. Observe also that $F(y) \to 1$ as $y \to \infty$, by virtue of (2.5). From (2.8) and these remarks, we find for any $y > \xi$:

$$\begin{aligned} 0 &< F(\xi) - F(y) \\ &< \lim_{\eta \to \infty} [F(\xi) - F(\eta)] = F(\xi) - 1, \end{aligned} \tag{2.9'}$$

so that (2.9) does indeed hold.

*As usual, $[a, b]$, $[a, b)$, $(a, b]$, and $(a, b)$ stand for the respective intervals $a \leq y \leq b$, $a \leq y < b$, $a < y \leq b$, and $a < y < b$.

Evidently, (2.9) implies that only one of the following three cases can occur when $\alpha > 0$:

$$\alpha > F(\xi) - 1\,, \tag{2.12a}$$

$$\alpha = F(\xi) - 1\,, \tag{2.12b}$$

$$0 < \alpha < F(\xi) - 1\,. \tag{2.12c}$$

The case (2.12c) will be the only one of significance in this report. However, for the sake of completeness, we shall also consider the cases (2.12a) and (2.12b) in the following lemma.

Lemma 2.2. $D(iy)$ has the following roots in the range $0 \leq y < \infty$ when the respective inequalities (2.12a), (2.12b), and (2.12c) hold: no roots, exactly one root (at $y = \xi$), and exactly two roots, say $y_0(\alpha)$ and $y_1(\alpha)$, with $0 < y_0(\alpha) < y_1(\alpha)$. Moreover, when (2.12c) is satisfied, the following inequalities hold:

$$y_0(\alpha) < \xi\,, \tag{2.13a}$$

$$y_1(\alpha) > \hat{y}(\alpha) \equiv \left[\left(1 + \frac{1}{\alpha}\right)\Big/2\right]^{1/2}. \tag{2.13b}$$

Proof. Since $F(\xi)$ is the maximum value of $F(y)$ in the range $[0,\infty)$, we obtain from (2.1') in this range:

$$\begin{aligned} D(iy) &= [\alpha - F(\xi) + 1] + |F(\xi) - F(y)| \\ &\geq \alpha - F(y) + 1\,. \end{aligned} \tag{2.14}$$

From (2.14), one sees that $D(iy)$ has no zeros when $0 \leq y < \infty$ if (2.12a) holds. When (2.12b) obtains, the fact that (2.8) holds over the stated range and (2.14) imply that $y = \xi$ is the unique nonnegative root of $D(iy)$.

Let us now consider the situation when the condition (2.12c) holds. Write (2.12c) in the form $1 < 1+\alpha < F(\xi)$. It should then be clear from the strictly increasing nature of $F(y)$ on $[0,\xi]$, from the fact that $F(0) = 0$, and from the continuity of $F(y)$ that $F(y)$ attains the value $1+\alpha$ at exactly one point in $[0,\xi]$ when (2.12c) is satisfied, say at the point $y = y_0(\alpha)$, where $0 < y_0(\alpha) < \xi$. Similarly, the strictly decreasing nature of $F(y)$ in $[\xi,\infty)$, its continuity, and the property that $F(y) \to 1$ as $y \to \infty$ show that $F(y)$ assumes the value $1+\alpha$ precisely once in $[\xi,\infty)$, say at $y = y_1(\alpha) > \xi$, provided that (2.12c) holds. By (2.1'), $y_0(\alpha)$ and $y_1(\alpha)$ are the only nonnegative roots of $D(iy)$.

To finish the proof of the lemma, we shall prove that (2.13b) holds when (2.12c) does. For this purpose, we define

$$\begin{aligned} G(y) &\equiv \frac{e^{y^2}}{y} D(iy) \\ &= (1+\alpha)\,\frac{e^{y^2}}{y} - 2\int_0^y e^{\eta^2}\,d\eta\,. \end{aligned} \tag{2.15}$$

Equation (2.15) yields:

$$\frac{dG(y)}{dy} = 2\alpha\,\frac{e^{y^2}}{y^2}\left[y^2 - \hat{y}^2(\alpha)\right]. \tag{2.16}$$

From (2.16), $G(y)$ is strictly increasing in $[0, \hat{y}(\alpha)]$ and strictly decreasing in $(\hat{y}(\alpha), \infty)$. Hence $G(y)$ has at most one zero in each of these two intervals. But, from (2.15), $G(y)$ has the positive zeros $y_0(\alpha)$ and $y_1(\alpha)$ when (2.12c) is fulfilled, since $D(iy)$ possesses these zeros in this case. In particular, it is thus evident that the largest of these two zeros of $G(y)$, i.e., $y_1(\alpha)$, lies in $(\hat{y}(\alpha), \infty)$. Therefore (2.13b) obtains under the desired restriction (2.12c).

The asymptotic behavior of $y_1(\alpha)$ in the limit $\alpha \to +0$ is of paramount interest to us. We shall arrive at explicit asymptotic results for $y_1(\alpha)$ in this limit (Lemma 2.3) by studying the corresponding behavior of the unique positive root of the equation

$$\alpha - \sum_{m=1}^{n} \gamma_m y^{-2m} = 0 \tag{2.18}$$

for $\alpha > 0$ and for any positive integer $n$. This last root will be called $y_1^{(n)}(\alpha)$.

The existence of this unique positive root of (2.18) for $\alpha > 0$ results from the fact that the function

$$\sum_{m=1}^{n} \gamma_m y^{-2m}$$

is a strictly increasing function of $y$ when $0 < y < \infty$ and that this function tends to $+\infty(0)$ as $y$ tends to $+0(\infty)$, by virtue of the positive nature of all the coefficients $\gamma_m$.

Lemma 2.3.* For any positive integer $n$ and any $\alpha$ satisfying (2.12c),

$$y_1^{(n)}(\alpha) = (2\alpha)^{-1/2} w^{(n)}(\alpha) . \tag{2.19}$$

Here

$$w^{(n)}(\alpha) = \sum_{m=0}^{n-1} c_m \alpha^m + O(\alpha^n) \tag{2.20}$$

for $n \geq 1$ in the limit $\alpha \to 0$, the coefficients $c_m$ being independent of $\alpha$. In particular,

$$c_0 = 1 . \tag{2.21}$$

Proof. For each $n \geq 1$ and each $\alpha \geq 0$, the equation

$$w^{2n} - \sum_{m=1}^{n} 2^m \gamma_m \alpha^{m-1} w^{2(n-m)} = 0 \tag{2.22}$$

has a unique positive root, say $w^{(n)}(\alpha)$. For $\alpha = 0$, this statement is implied immediately by (2.22) since this equation yields

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*Let the $\gamma_m$ not be required to fulfill (2.6) but solely to have the following properties: $\gamma_1 > 0$ and $\gamma_m \geq 0$ $(m \geq 2)$. The existence of a unique positive root $y_1^{(n)}(\alpha)$ $(n \geq 1)$ of (2.18) when $\alpha > 0$ for the case of these new $\gamma_m (m \geq 1)$ can evidently be established by the pertinent argument of the text given previously. It should be clear from the proof of Lemma 2.3 that this lemma holds when the $\gamma_m$ and $y_1^{(n)}(\alpha)$ are understood in this new context, provided that $(2\alpha)^{-1/2}$ is replaced by $(\gamma_1/\alpha)^{-1/2}$ in (2.19).

$$w^{(n)}(0) = 2\gamma_1 = 1 . \tag{2.23}$$

To prove the cited statement for the case $\alpha > 0$, we notice that in this case (2.21) implies and is implied by (2.18) via the transformation

$$w = (2\alpha)^{1/2} y . \tag{2.24}$$

From this observation and the structure of (2.24), it is obvious for any $\alpha > 0$ that the number of roots of (2.22) of a given sign is the same as the number of roots of (2.18) with that sign. Hence (2.22) has a unique positive root when $\alpha > 0$.* Therefore, if $\alpha > 0$, $w^{(n)}(\alpha)$ exists and is related to $y_1^{(n)}(\alpha)$ by the special case (2.19) of the transformation (2.24).

The root $w^{(n)}(\alpha)$ is simple for each $\alpha \geq 0$, as can be shown by the usual elementary differentiation rule and by invoking the positive nature of all the $\gamma_m$. Therefore, if we regard $\alpha$ in (2.22) as a complex variable, a well-known theorem† about simple roots of algebraic equations with analytic coefficients informs us that $w^{(n)}(\alpha)$ can be expanded in the complex $\alpha$ plane about any fixed $\alpha \geq 0$ in a Taylor series with a positive radius of convergence.

Consider the Taylor expansion of $w^{(n)}(\alpha)$ about $\alpha = 0$. It follows from (2.23) that the coefficient $c_0$ of $\alpha^0$ in this series in powers of $\alpha$ is unity, in agreement with (2.21). Hence

$$w^{(n)}(\alpha) = 1 + O(\alpha) , \tag{2.20'}$$

so that (2.20) holds in the limit $\alpha \to 0$ for the special case $n = 1$.

We proceed to complete the proof of the lemma by showing that (2.20) also holds for $n > 1$. To establish this, we shall prove the result that for any fixed positive integers $n$ and $n'$, such that $n' > n$, the function

$$w^{(n')}(\alpha) - w^{(n)}(\alpha) \tag{2.25}$$

has a zero of order $n$ at $\alpha = 0$. That this last result implies (2.20) for $n \geq 1$, and is in fact stronger than (2.20) for each such $n$, is an immediate consequence of the aforementioned representability of each $w^{(p)}(\alpha)$ $(p = 1, 2, \ldots)$ by a convergent series in powers of $\alpha$ for $|\alpha|$ small enough.

To establish (2.25), we start by remarking that (2.23) and the isolated nature of the zeros of analytic functions in their domains of regularity imply that $w^{(n)}(\alpha) \neq 0$ for $\alpha$ in the interior of a circle of nonzero radius centered at $\alpha = 0$ in the complex $\alpha$ plane. Hence, for all $\alpha$ in the interior of this circle, we can divide by $[w^{(n)}(\alpha)]^{2n}$ the equation obtained from (2.22) when we replace $w$ by $w^{(n)}(\alpha)$ therein, thus obtaining:

$$\sum_{m=1}^{n} 2^m \gamma_m \alpha^{m-1} \left[w^{(n)}(\alpha)\right]^{-2m} = 1 . \tag{2.22'}$$

Subtracting (2.22') from the parallel equation obeyed by $w^{(n')}(\alpha)$, we find for such $\alpha$:

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*Of course, the mere existence of a unique positive root of (2.22) for $\alpha \geq 0$ can also be proved directly from the facts that $\gamma_1 > 0$ and that $\gamma_m \alpha^{m-1} \geq 0$ $(m \geq 2)$ in the case of interest.

†See, for example, Sec. 14, Ch. 5, of Ref. 9, especially pages 125 and 126.

$$\sum_{m=1}^{n} 2^m \gamma_m \alpha^{m-1} \left\{ \left[ w^{(n')}(\alpha) \right]^{-2m} - \left[ w^{(n)}(\alpha) \right]^{-2m} \right\} = - \sum_{m=n+1}^{n'} 2^m \gamma_m \alpha^{m-1} \left[ w^{(n')}(\alpha) \right]^{-2m} . \quad (2.26)$$

For fixed values of $n$ and of $n' > n$, let $k$ be the order of the zero of the function (2.25) at $\alpha = 0$, so that

$$w^{(n')}(\alpha) - w^{(n)}(\alpha) = \lambda_{nn'} \alpha^k + O(\alpha^{k+1}) \quad (2.27)$$

in the limit $\alpha \to 0$, $\lambda_{nn'}$ being a nonzero constant.

With the aid of (2.20'), (2.26), and (2.27), we conclude for any such $n$ and $n'$ in this limit:

$$\lambda_{nn'} \alpha^k + O(\alpha^{k+1}) = 2^n \gamma_{n+1} \alpha^n + O(\alpha^{n+1}) . \quad (2.28)$$

Since $\lambda_{nn'}$ and $\gamma_{n+1}$ are nonzero, (2.28) implies that $k = n$, and thus that the function (2.25) does indeed possess a zero of order $n$ at $\alpha = 0$ for $n' > n$.

The determination of the $c_m$ by means of (2.6), (2.20), and (2.22) is obvious. In particular, one finds from these equations that

$$c_1 = \frac{3}{2}, \qquad c_2 = \frac{15}{8} . \quad (2.29)$$

An asymptotic formula giving $y_1(\alpha)$ in terms of $y_1^{(n)}(\alpha)$ is furnished by the next theorem. In this theorem and in the subsequent work of this report, $O_+(A)$ should be understood to mean "$O(A)$ in the limit $\alpha \to +0$."

Theorem 2.1.* For each positive integer $n$,

$$y_1(\alpha) = y_1^{(n)}(\alpha) + O_+\left(\alpha^{n-\frac{1}{2}}\right) . \quad (2.30)$$

Proof. Since $y_1(\alpha)$ is a zero of $D(iy)$ which is $O_+(\alpha^{-1/2})$ by (2.13b), we obtain from (2.1') and (2.5):

$$\alpha - \sum_{m=1}^{n} \gamma_m \left[ y_1^{(n)}(\alpha) \right]^{-2m} = O_+\left(\alpha^{-(n+1)}\right) . \quad (2.31)$$

For any $\alpha$ obeying (2.12c) we define the positive quantity

$$w_1(\alpha) \equiv (2\alpha)^{1/2} y_1(\alpha) . \quad (2.32)$$

Employing (2.31) and (2.32), we see that $w_1(\alpha)$ satisfies the equation

*It is easily seen that the method used to prove Theorem 2.1 can be employed to prove the following generalization of this theorem. Let the $\gamma_m (m \geq 1)$ and $y_1^{(n)}(\alpha)$ $(n \geq 1)$ be as specified in the first footnote to Lemma 2.3. Let $F(y)$ be a function having the asymptotic representation (2.5) for $|y| \to \infty$ in terms of these last $\gamma_m$. Moreover, let $F(y)$ be such that the function $D(iy)$, which we define by (2.1') in terms of this $F(y)$, has a positive zero $y_1(\alpha)$ tending to zero as $\alpha \to +0$. Then (2.30) holds.

$$\sum_{m=1}^{n} 2^m \gamma_m \alpha^{m-1}\left[w_1(\alpha)\right]^{-2m} = 1 + O_+(\alpha^n) . \tag{2.33}$$

Setting $n = 1$ in this equation, we deduce that

$$w_1(\alpha) = 1 + O_+(\alpha) . \tag{2.34}$$

Subtraction of (2.22') from (2.33) yields:

$$\sum_{m=1}^{n} 2^m \gamma_m \alpha^{m-1}\left\{\left[w_1(\alpha)\right]^{-2m} - \left[w^{(n)}(\alpha)\right]^{-2m}\right\} = O_+(\alpha^n) . \tag{2.35}$$

In view of (2.34) and of the fact that this last equation implies that $w_1(\alpha) - w^{(n)}(\alpha) = O_+(\alpha)$ we conclude:

$$\begin{aligned}\left[w_1(\alpha)\right]^{-2m} - \left[w^{(n)}(\alpha)\right]^{-2m} &= -2m\left[1 + O_+(\alpha)\right]\left[w_1(\alpha) - w^{(n)}(\alpha)\right] + O_+\left(\left[w_1(\alpha) - w^{(n)}(\alpha)\right]^2\right) \\ &= -2m\left[w_1(\alpha) - w^{(n)}(\alpha)\right]\left[1 + O_+(\alpha)\right] .\end{aligned} \tag{2.36}$$

Because of (2.35) and (2.36),

$$\begin{aligned} w_1(\alpha) - w^{(n)}(\alpha) &= O_+(\alpha^n)\Big/\sum_{m=1}^{n} 2^m \gamma_m \alpha^{m-1}(-2m)\left[1 + O_+(\alpha)\right] \\ &= O_+(\alpha^n) .\end{aligned} \tag{2.37}$$

Equations (2.19), (2.32), and (2.37) imply the assertion (2.30) of the theorem.

Collecting the results (2.19) to (2.21), (2.29), and (2.30), we see that

$$\begin{aligned} y_1(\alpha) &= (2\alpha)^{-1/2}\left[1 + \sum_{m=1}^{n-1} c_m \alpha^m + O_+(\alpha^n)\right] \\ &= (2\alpha)^{-1/2}\left[1 + \frac{3}{2}\alpha + \frac{15}{8}\alpha^2 + O_+(\alpha^3)\right] ,\end{aligned} \tag{2.30'}$$

where the first equality holds, of course, for each $n \geq 2$.

## 3. EXISTENCE OF ZEROS OF LANDAU TYPE

The proof, announced in the Introduction, of the existence of a zero of Landau type in the second quadrant of the complex $z$ plane and the explicit asymptotic formula for this zero mentioned there will be presented in this section (Theorem 3.1) on the basis of two lemmas (Lemmas 3.1 and 3.2).

Consider the equation

$$w = f(z) \tag{3.1}$$

where

$$\begin{aligned} f(z) &\equiv D(z)/\pi^{1/2} z e^{z^2} \\ &= \left[\Delta(z) + \pi^{1/2} z e^{z^2}\right]\Big/\pi^{1/2} z e^{z^2} .\end{aligned} \tag{3.2}$$

Because of (3.2) and of the fact that neither $D(z)$ nor $\Delta(z)$ vanish at $z = 0$ for $\alpha \geq 0$, we see that the roots of (3.1) for the case $w = 1$ and for such an $\alpha$ are the same as those of the equation

$$\Delta(z) = 0 \tag{3.3}$$

for the same $\alpha$. Hence it is natural to expect that "solving" (3.1) for $z$ in terms of $w$, when $w$ is in a suitable neighborhood of $w = 1$ in the complex $w$ plane, would lead to useful results on the zeros of (3.3). This is indeed the case, as we proceed to demonstrate.

We denote by $C(\alpha, \beta)$ a circle in the $z$ plane centered at $z = iy_1(\alpha)$ and having a radius $\beta y_1(\alpha)$, where $\beta$ is a number independent of $\alpha$ and $z$ such that

$$0 < \beta \leq [2(1-\delta)]^{1/2} - 1, \tag{3.4}$$

$\delta$ being also independent of $\alpha$ and $z$ and such that $0 < \delta < 1$.

Lemma 3.1. One has*

$$\begin{aligned} f'(iy_1(\alpha)) &= \frac{\left\{1 + \alpha\left[1 - 2y_1^2(\alpha)\right]\right\} e^{y_1^2(\alpha)}}{\pi^{1/2} y_1^2(\alpha)} \\ &= -\frac{4\alpha^2}{\pi^{1/2}}\left[1 + O_+(\alpha)\right] e^{[1+O_+(\alpha)]/2\alpha}, \end{aligned} \tag{3.5}$$

where the first equality holds whenever (2.12c) does. Moreover, for each fixed $\beta$ then exists a positive number $K(\beta)$ independent of $\alpha$ and of $z$ and such that

$$|f(z)| \leq K(\beta)\, \alpha^{3/2} e^{(1+\beta)^2/2\alpha} \tag{3.6}$$

for every sufficiently small $\alpha > 0$ at each point $z$ of $C(\alpha, \beta)$.

Proof. The first equality in (3.5) follows directly from (3.2), (2.1), and the fact that $D(z)$ has the zero $z = iy_1(\alpha)$ whenever (2.12c) holds. This first equality and (2.30') entail the second equality in (3.5).

To establish (3.6), let us notice that (3.2), (1.1), and (2.7) imply:

$$f(z) = \frac{e^{-z^2}}{\pi^{1/2} z}\left\{\alpha + \frac{1}{2z^2} + O\left(\frac{1}{|z|^4}\right)\right\} \tag{3.7}$$

for $|z| \to \infty$ and $(\pi/4) + \varepsilon' \leq \arg z \leq (3\pi/4) - \varepsilon'$, where $\varepsilon'$ is positive and independent of $z$ and $\alpha$, and can be chosen arbitrarily small.

Because of (3.4), each circle $C(\alpha, \beta)$ lies completely in the region $\pi/4 < \arg z < 3\pi/4$. Furthermore, for each fixed $\beta$ and for $\alpha \to +0$, the distance of each point on $C(\alpha, \beta)$ from $z = 0$ tends to infinity, since $y_1(\alpha) = O_+(\alpha^{-1/2})$. Hence we may apply (3.7) to the case when $\beta$ is fixed, $\alpha \to +0$, and $z \in C(\alpha, \beta)$. Doing this and invoking elementary inequalities, we find in this case:

---

* $f'(z) \equiv \dfrac{df(z)}{dz}$.

$$\begin{aligned}|f(z)| &\le \frac{e^{-x^2+y^2}}{\pi^{1/2}|z|}\left[\alpha + \frac{1}{2|z|^2} + O_+\left(\frac{1}{|z|^4}\right)\right] \\ &\le \frac{e^{(1+\beta)^2 y_1^2(\alpha)}}{\pi^{1/2}(1-\beta)\,y_1(\alpha)}\left[\alpha + \frac{1}{2(1-\beta)^2 y_1^2(\alpha)} + O_+\left(\frac{1}{y_1^4(\alpha)}\right)\right].\end{aligned} \tag{3.8}$$

The desired inequality (3.6) results by combining (3.8) with (2.30').

Theorem 3.1. In the limit $\alpha \to +0$, $\Delta(z)$ has a zero located at the point

$$z_+(\alpha) = iy_1(\alpha) + \frac{\pi^{1/2} y_1^2(\alpha)\, e^{-y_1^2(\alpha)}}{\left\{1 + \alpha\left[1 - 2y_1^2(\alpha)\right]\right\}} + O_+(e^{-c/2\alpha}) \tag{3.9}$$

of the $z$ plane, where $c$ is a number independent of $\alpha$ and $c > 1$.

Notice that (2.30') and (3.9) yield:

$$\operatorname{Re} z_+(\alpha) = -\frac{\pi^{1/2}}{4\alpha^2}\left[1 + O_+(\alpha)\right] e^{-3/2} e^{-1/2\alpha} + O_+(e^{-c/2\alpha}). \tag{3.9'}$$

Except for the precise order terms in (3.9'), the damping decrement computed from this equation agrees with the one given in Refs. 2 and 7.

Proof: We start by recalling some basic facts about the Lagrange method of reverting power series.*

For the moment, let us regard $f(z)$ in (3.1) as an arbitrary function of $z$ analytic in the circle

$$|z - z_0| \le R. \tag{3.10}$$

Let $M$ be an upper bound of $f(z)$ on this circle and let

$$w_0 = f(z_0). \tag{3.11}$$

If $f'(z_0) \neq 0$, then Eq. (3.1) has the root

$$z = z_0 + \sum_{n=1}^{\infty} b_n (w - w_0)^n \tag{3.12}$$

provided that

*See, for example, Secs. 6.22 and 6.23 of Ref. 10. With the exception of (3.14b), the facts on the Lagrange method cited in the text of this report are either stated and proved in these sections of Ref. 10 or follow, via trivial transformations of variables, from results in the sections in question. Inequality (3.14b) results by combining the first inequality on page 124 of Ref. 10 with the contour-integral formula for $b_n$ on that page. Notice that the $a$ of this reference is to be identified with the number $|f'(z_0)|$ of the present report. Notice also that our $M$ and the $M$ of the quoted sections of Ref. 10 are identical because of the maximum modulus theorem.

$$|w - w_0| < \frac{1}{6}\,\frac{R^2|f'(z_0)|^2}{M}\,. \tag{3.13}$$

For $n = 1, 2, \ldots,$ the coefficients $b_n$ are given by

$$b_n = \frac{1}{n!}\lim_{z\to z_0}\frac{d^{n-1}}{dz^{n-1}}\left[\frac{z - z_0}{f(z) - f(z_0)}\right] \tag{3.14a}$$

and satisfy the inequality

$$|b_n| \le (R^2|f'(z_0)|/4Mn) \times \left(6M/R^2|f'(z_0)|^2\right)^n \tag{3.14b}$$

Henceforth we shall set

$$z_0 = iy_1(\alpha)\,, \tag{3.15}$$

so that

$$w_0 = 0\,. \tag{3.16}$$

We shall also set

$$\begin{aligned} R &= \text{radius of } C(\alpha,\beta) \\ &= \beta y_1(\alpha)\,. \end{aligned} \tag{3.17}$$

Lemma 3.1 then allows us to put

$$M = K(\beta)\,\alpha^{3/2}\,e^{(1+\beta)^2/2\alpha} \tag{3.18}$$

whenever $\alpha > 0$ is sufficiently small.

For such values of $\alpha$, (3.5) implies that $f'(z_0) \neq 0$ when (3.15) holds. Moreover, for the present choice of $z_0$, $R$, and $M$, (3.13) holds for $w = 1$ in the limit $\alpha \to +0$. In fact, (3.5), (3.15), (3.18), and (2.30'), in conjunction with (3.4), entail the existence of a positive number $K'(\beta)$ independent of $\alpha$ and such that

$$\frac{R^2|f'(z_0)|^2}{M} > K'(\beta)\,\alpha^{3/2}\,e^{\delta/\alpha} \tag{3.19}$$

for each sufficiently small $\alpha > 0$.

Hence (3.12), (3.15), and (3.16) allow us to conclude that, for any such $\alpha$,

$$z_1(\alpha) = iy_1(\alpha) + \sum_{n=1}^{\infty} b_n \tag{3.20}$$

is a root of (3.1) for $w = 1$, i.e., that $z_1(\alpha)$ is a root of $\Delta(z)$ for each $\alpha$ of this type.

Employing (3.14a), (3.15), and (3.5), we find that

$$b_1 = \frac{1}{f'(z_0)} = \frac{\pi^{1/2} y_1^2(\alpha)\, e^{-y_1^2(\alpha)}}{\left\{1 + \alpha\left[1 - 2y_1^2(\alpha)\right]\right\}} \tag{3.21}$$

if (2.12c) holds.

It is plain from (3.20) and (3.21) that the desired result (3.9) is true if

$$\sum_{n=2}^{\infty} b_n = O_+(e^{-c/2\alpha}) , \tag{3.22}$$

with $c$ as specified in the theorem.

To prove (3.22), let us notice first that

$$\frac{M}{R^2|f'(z_0)|^3} = O_+\left(\alpha^{-7/2} e^{-(1+2\delta)/2\alpha}\right) , \tag{3.23}$$

as can be proved by reasoning similar to that used to obtain (3.19). We find from (3.14b), (3.23), and the fact, implied by (3.19), that $M/R^2|f'(z_0)|^2 \to 0$ as $\alpha \to +0$:

$$\begin{aligned} \left|\sum_{n=2}^{\infty} b_n\right| &\le \sum_{n=2}^{\infty} |b_n| < \left(\frac{9}{2}\right)\left(\frac{M}{R^2|f'(z_0)|^3}\right) \sum_{m=0}^{\infty}\left(\frac{6M}{R^2|f'(z_0)|^2}\right)^m \\ &= \left(\frac{9}{2}\right)\left(\frac{M}{R^2|f'(z_0)|^3}\right)\left[1 - \frac{M}{R^2|f'(z_0)|^2}\right]^{-1} \\ &= O_+\left(\alpha^{-7/2} e^{-(1+2\delta)/2\alpha}\right) . \end{aligned} \tag{3.24}$$

Comparing (3.24) with (3.22), it is evident that (3.22) obtains, in particular, for any number $c$ independent of $\alpha$ and such that $1 < c < 1 + 2\delta$. Since such a $c$ has all the properties required by the theorem, our proof of the latter is complete.

## 4. PROOF THAT PRECISELY TWO ZEROS OF $\Delta(z)$ IN THE HALF-PLANE $\mathrm{Re}\ z \le 0$ LIE CLOSEST TO THE IMAGINARY AXIS FOR $\alpha \to +0$

We shall require three lemmas to show that $\Delta(z)$ has precisely two zeros in the half-plane $\mathrm{Re}\ z \le 0$ which are closer to the imaginary axis than all the other zeros of $\Delta(z)$ in this half-plane in the limit $\alpha \to +0$ .

Lemma 4.1. There exists a positive number $\rho$, independent of $\alpha$ and $z$, such that $\Delta(z)$ has no zeros in the circle $|z| \le \rho$.

Proof. Since the function $ze^{z^2}\,\mathrm{erfc}(z)$ vanishes at $z = 0$ and is continuous at each $z$, there exists a positive number $\rho$ of the type specified in the theorem such that

$$\pi^{1/2}|ze^{z^2}\,\mathrm{erfc}(z)| < 1 \tag{4.1}$$

whenever $|z| \le \rho$.

But (1.1) and (4.1) entail for each $\alpha \geq 0$ that

$$\Delta(z) \geq 1 + \alpha - \pi^{1/2} \left| z e^{z^2} \operatorname{erfc}(z) \right| > 0 , \tag{4.2}$$

which proves the lemma.

Lemma 4.2. Let $R_1$ and $R_2$ be arbitrary fixed positive numbers such that $R_1 \geq R_2$. Then there exists a positive number $\mu(R_1,R_2)$, independent of $\alpha$ and $z$, such that $\Delta(z) \neq 0$ for each real $\alpha$ and for each $z$ which fulfills the two inequalities

$$R_1 \leq |z| \leq R_2 , \tag{4.3}$$

$$|\operatorname{Re} z| \leq \mu(R_1,R_2) . \tag{4.4}$$

Proof. From (1.1) and (2.2), we find for any real $\alpha$,

$$\begin{aligned} g(x,y) &\equiv -\pi^{-1/2} \operatorname{Im}[\Delta(z)] \\ &= \operatorname{Im}\left\{ z e^{z^2} [1 - \operatorname{erf}(z)] \right\} , \end{aligned} \tag{4.5}$$

so that $g(x,y)$ does not depend on $\alpha$. Notice that (4.5) and the purely imaginary character of $\operatorname{erf}(iy)$ mean that

$$g(0,y) = y e^{-y^2} . \tag{4.5'}$$

We introduce the numbers $M(R_1,R_2)$ and $\mu(R_1,R_2)$ as follows. $M(R_1,R_2)$ is taken to be an upper bound of $|\partial g(x,y)/\partial x|$ in the closed region (4.3). Obiously, such an upper bound is positive and can be taken to be independent of $\alpha$, a choice which will be made here. Let $\mu(R_1,R_2)$ be any positive number independent of $\alpha$ and $z$ and such that

$$\mu(R_1,R_2) < \frac{R_1}{\left[1 + M^2(R_1,R_2)\, e^{2R_2^2}\right]^{1/2}} . \tag{4.6}$$

Let us assume that $\Delta(z)$ has a zero located at a point $z = \tilde{z}(\alpha)$ lying in the intersection of the regions (4.3) and (4.4), with $M(R_1,R_2)$ and $\mu(R_1,R_2)$ as just specified. It will now be shown that this assumption leads to a contradiction and hence to a proof of the lemma.

Denote the real and imaginary parts of $\tilde{z}(\alpha)$ by $\tilde{x}(\alpha)$ and $\tilde{y}(\alpha)$, respectively. Then (4.5) yields:

$$g(\tilde{x}(\alpha), \tilde{y}(\alpha)) = 0. \tag{4.7}$$

Employing (4.7), the differentiability of $g(x,y)$ with respect to $x$, and the definition of $M(R_1,R_2)$, we conclude that

$$\left| g\left(0, \tilde{y}(\alpha)\right) \right| = \left| g\left(\tilde{x}(\alpha), \tilde{y}(\alpha)\right) - g\left(0, \tilde{y}(\alpha)\right) \right| \leq |\tilde{x}(\alpha)|\, M(R_1,R_2) . \tag{4.8}$$

The assumed location of $\tilde{z}(\alpha)$ and the inequality $\mu(R_1,R_2) < R_1$ implied by (4.6) entail:

$$\left[R_1^2 - \mu^2(R_1,R_2)\right]^{1/2} \leq |\tilde{y}(\alpha)| \leq R_2 . \tag{4.9}$$

Combining (4.8) and (4.5') and (4.9), it follows that

$$|\tilde{x}(\alpha)| \geq \frac{|\tilde{y}(\alpha)| e^{-\tilde{y}^2(\alpha)}}{M(R_1, R_2)}$$

$$\geq \frac{\left[R_1^2 - \mu^2(R_1, R_2)\right]^{1/2} e^{-R_2^2}}{M(R_1, R_2)}. \tag{4.10}$$

On the other hand,

$$|\tilde{x}(\alpha)| \leq \mu(R_1, R_2) < \frac{|R_1^2 - \mu^2(R_1, R_2)|^{1/2} e^{-R_2^2}}{M(R_1, R_2)}, \tag{4.11}$$

where the first inequality in (4.11) holds because of the assumed location of $\tilde{z}(\alpha)$ and the second inequality therein is implied by (4.6). The incompatibility of (4.10) and (4.11) proves that no such zero $\tilde{z}(\alpha)$ exists.

In the proof of the next lemma, it is useful to write (1.1) in the form

$$\Delta(z) = \alpha + \frac{1}{2z^2} + h(z), \tag{4.12}$$

employing the $\alpha$-independent function

$$h(z) \equiv 1 - \frac{1}{2z^2} - \pi^{1/2} e^{z^2} \operatorname{erfc}(z). \tag{4.13}$$

If $\arg z$ lies in the closed angular interval $[-(3\pi/4) + \varepsilon, (3\pi/4) - \varepsilon]$, we find, for example from (4.13), (2.6), and (2.7), that $h(z) = O(|z|^{-4})$ for $|z| \to \infty$. This order property of $h(z)$ and the closed nature of the said angular interval imply, via an elementary argument, that there exist two positive numbers, $C = C(\varepsilon)$ and $R_0 = R_0(\varepsilon)$, independent of $z$ and such that

$$|h(z)| < \frac{C}{|z|^4} \tag{4.14}$$

when $|z| \geq R_0$ and $|\arg z| \leq (3\pi/4) - \varepsilon < (3\pi/4)$.

<u>Lemma 4.3.</u> <u>For any given $\varepsilon$ in the interval $(0, \pi/4)$, choose $R_3$ so that</u>

$$R_3 \geq \max\left\{R_0, 2\left(\frac{C}{|\cos 2\varepsilon|}\right)^{1/2}\right\} \tag{4.15}$$

<u>and subsequently select $\alpha$ so that</u>

$$0 < \alpha \leq \frac{1}{4R_3^2}. \tag{4.16}$$

<u>Then $\Delta(z)$ has exactly one root in the region determined by the simultaneous inequalities</u>

$$R_3 \leq z < \infty, \tag{4.17}$$

$$\frac{\pi}{2} < \arg z \leq \frac{3\pi}{4} - \varepsilon. \tag{4.18}$$

Proof. Let $R_4$ be a positive number independent of $\alpha$ and $z$ and such that, for any choice of $\alpha > 0$,

$$R_4 \geq \alpha^{-1/2}. \tag{4.19}$$

If $\alpha$ is chosen in conformity with (4.16), we evidently have

$$R_3 < R_4. \tag{4.20}$$

Hence, under the hypotheses of the present lemma, there exists a nonempty region $S = S(\varepsilon, R_3, R_4)$ of the $z$ plane specified by the simultaneous inequalities

$$R_3 \leq |z| \leq R_4, \tag{4.21}$$

$$\frac{\pi}{4} + \varepsilon \leq \arg z \leq \frac{3\pi}{4} - \varepsilon. \tag{4.22}$$

To prove the lemma, it is sufficient to show that, if the constants $\varepsilon$, $R_3$, $\alpha$, and $R_4$ are selected as specified therein and in (4.19), then the inequality

$$\left| \alpha + \frac{1}{2z^2} \right| > h(z) \tag{4.23}$$

holds for each point $z$ of the boundary $ABCB'A'C'A$ of the corresponding region $S$ (see Fig. 1). In fact, if these four constants are so chosen, one finds trivially from (4.16), (4.19), (4.21), and (4.22) that the sole zero possessed by the function $\alpha + (1/2z^2)$ in the region $S$ is the zero located at the interior point $i(2\alpha)^{-1/2}$ of this region. Hence we conclude from (4.12) and Rouché's theorem* that, if (4.23) obtains on $ABCB'A'C'A$ for this choice of constants, then $\Delta(z)$ has exactly one zero in $S$, this zero being in the interior of $S$. Moreover, the fact that $\Delta(z)$ has no zeros in the half-plane $\mathrm{Re}\ z \geq 0$ when

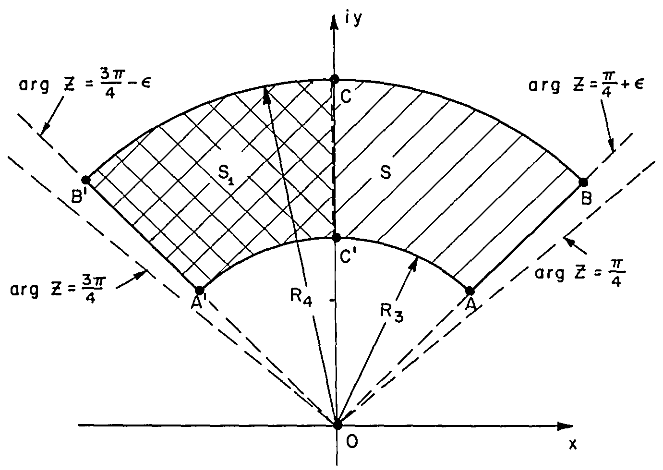


Fig. 1 - Illustrating the region $S$ (closed region bounded by $ABCB'A'C'A$) and the subregion $S_1$ (semi-open region bounded by $A'C'CB'A'$) of $S$ introduced in the proof of Lemma 4.3

*See, for instance, Sec. 6.21 of Ref. 10.

$\alpha > 0$ means that this unique zero of $\Delta(z)$ in S is located in the subregion $S_1$ of S defined by the simultaneous inequalities (4.18) and (4.21) (see Fig. 1). But this result on the zeros of $\Delta(z)$ in $S_1$ holds for arbitrarily large $R_4$. Hence, under the hypotheses of the lemma, it follows that $\Delta(z)$ has exactly one zero in the region specified by (4.17) and (4.18), provided that (4.23) holds on ABCB'A'C'A under these hypotheses.

To establish that (4.23) is true in this sense, let us notice the simple but important fact that (4.14) obtains for any $z$ on ABCB'A'C'A. Indeed, (4.15) and our definition of S inform us that (4.14) holds throughout the closed region S.

If $z$ is in the circular arc A'C'A of radius $R_3$, we find from (4.14) to (4.16) and from the inequality $|\cos 2\varepsilon| < 1$, which certainly holds when $\varepsilon$ is chosen as specified in the lemma:

$$\left|\alpha + \frac{1}{2z^2}\right| \geq \frac{1}{2R_3^2} - \alpha \geq \frac{1}{2R_3^2} - \frac{1}{4R_3^2} = \frac{1}{4R_3^2},$$
$$|h(z)| \leq \frac{C}{R_3^4} \leq \frac{C|\cos 2\varepsilon|}{4CR_3^2} < \frac{1}{4R_3^2}. \tag{4.24}$$

When $z$ is in the circular arc B'CB of radius $R_4$, we conclude with the aid of (4.14), (4.15), (4.19), and (4.20):

$$\left|\alpha + \frac{1}{2z^2}\right| \geq \alpha - \frac{1}{2R_4^2} \geq \alpha - \frac{\alpha}{2} = \frac{\alpha}{2},$$
$$|h(z)| \leq \frac{C}{R_4^4} < \frac{C}{R_3^2 R_4^2} \leq \frac{C\alpha|\cos 2\varepsilon|}{4C} < \frac{\alpha}{4}. \tag{4.25}$$

Finally, if $z$ is in either of the straight-line segments AB or A'B', we obtain from the reality of $\alpha$ and from (4.14) and (4.15):

$$\left|\alpha + \frac{1}{2z^2}\right| \geq \left|\mathrm{Im}\left(\frac{1}{2z^2}\right)\right| = \frac{|\cos 2\varepsilon|}{2|z|^2},$$
$$|h(z)| \leq \frac{C}{|z|^4} \leq \frac{C}{R_3^2|z|^2} \leq \frac{|\cos 2\varepsilon|}{4|z|^2}. \tag{4.26}$$

From (4.24) to (4.26), the desired inequality (4.23) is seen to hold on the boundary of S under the said circumstances.

Theorem 4.1. For small enough $\alpha > 0$, there are two and only two zeros of $\Delta(z)$ possessing the property of having a perpendicular distance to the imaginary $z$ axis smaller than that of all the other zeros of $\Delta(z)$. These two zeros are the zero $z_1(\alpha)$ given by Theorem 3.1 and the complex conjugate of this last zero.

Proof. We shall establish that, in the limit $\alpha \to +0$, there exists a positive constant $\nu$, independent of $\alpha$, such that the only two zeros of $\Delta(z)$ in the strip

$$-\nu < \mathrm{Re}\ z < 0 \tag{4.27}$$

are $z_1(\alpha)$ and $\overline{z_1(\alpha)}$. This will prove the theorem by virtue of the cited fact that $\Delta(z)$ has no zero in the half-plane $\mathrm{Re}\ z \geq 0$ for $\alpha > 0$.

Let us start by choosing an $\varepsilon$ in $(0,\pi/2)$. In terms of the positive constants $\rho$, $C(\varepsilon)$, and $R_0(\varepsilon)$, let us then make the following choice of the $\alpha$-independent constants $R_1$, $R_2$, and $R_3$:

$$R_1 = \min\{\rho, R_0\}, \tag{4.28}$$

$$R_2 = R_3 = \max\left\{\rho,\ R_0,\ 2\left[\frac{C}{|\cos 2\varepsilon|}\right]^{1/2}\right\}. \tag{4.29}$$

Notice that (4.28) and (4.29) imply that $0 < R_1 \le R_2$ and that $R_3$ obeys (4.15). Hence these choices of $\varepsilon$, $R_1$, $R_2$, and $R_3$ satisfy the respective hypotheses concerning these three numbers in Lemmas 4.2 and 4.3.

We set

$$\nu = \min\{\rho,\ \mu(R_1, R_2)\}, \tag{4.30}$$

thus fulfilling the conditions that $\nu$ should be an $\alpha$-independent and positive number. Finally, let us choose $\alpha > 0$ so small that (4.16) and the inequality

$$-\nu < \operatorname{Re} z_1(\alpha) < 0 \tag{4.31}$$

both hold when $\nu$, $R_1$, $R_2$, and $R_3$ are selected in accordance with (4.28) to (4.30). That these two requirements are met by a sufficiently small $\alpha > 0$ follows from (4.16), from the fact that (3.9') implies that $\operatorname{Re} z_1(\alpha) \to 0$ as $\alpha \to +0$, and from the $\alpha$-independent nature of $R_3$ and $\nu$.

Clearly, (4.31) entails that $\Delta(z)$ has <u>at least</u> two zeros in the strip (4.27), namely $z_1(\alpha)$ and $\overline{z_1(\alpha)}$.

Invoking Lemmas 4.1 to 4.3, Eqs. (4.28) to (4.30), and the circumstance that the zeros of $\Delta(z)$ always occur in complex conjugate pairs, we easily infer that $\Delta(z)$ possesses <u>at most</u> two zeros in the strip (4.27) when $\alpha$, $R_1$, $R_2$, and $R_3$ are chosen in the above way. This result and the one obtained in the previous paragraph imply that $\Delta(z)$ has <u>exactly</u> two zeros in this strip under the stated hypotheses. Our proof of the theorem is complete.

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