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# Characteristic Dynamic Strength Ratio as the Limiting Parameter in Shock Design

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## ABSTRACT

The dynamic strength ratio of a structure is defined as the ratio of the dynamic yield strength (a number determined by multiplying the static yield strength times a dynamic loading factor) of the structural material to the maximum fiber stress in the structure when it is subject to the acceleration of gravity alone (one g). Dynamic strength ratios for bending and buckling are derived and plotted versus frequency for three types of beams, respectively modeled as single degree of freedom systems. The strength ratio plots are superimposed on sample shock spectra to illustrate which regions of the spectra are appropriate for the design of each type of beam. This comparison of the strength ratio curves and the shock spectra indicates that a specific shock spectrum amplitude is suitable for the elastic design of a given type of structure only if the dynamic strength ratio is greater than or equal to the spectrum amplitude.


## PROBLEM STATUS

This is an interim report on a continuing problem.

## AUTHORIZATION


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SYMBOLS

| | |
|---|---|
| $A$ | dynamic strength ratio ( $\sigma_F/\sigma_s$) |
| $A_o$ | upper limiting dynamic strength ratio |
| $A_L$ | lower limiting dynamic strength ratio |
| $b$ | width of beam cross section |
| $c$ | distance from neutral axis to outermost fiber, $(\frac{h}{2})$ |
| $E$ | modulus of elasticity |
| $f$ | frequency |
| $g$ | acceleration of gravity (386 in/sec$^2$) |
| $G$ | modulus of rigidity |
| $h$ | depth of cross section |
| $I$ | area moment of inertia |
| $k$ | spring constant of beam |
| $L$ | beam length |
| $m$ | total mass |
| $m_{cr}$ | critical concentrated load (mass) which causes lateral buckling |
| $m_p$ | vertical concentrated load (mass) |
| $m_V$ | total static load (mass) in shear |
| $m_W$ | mass of a real beam of cross section bh |
| $M$ | moment |
| $M_F$ | moment (associated with $m_{cr}$) which causes lateral buckling |
| $M_s$ | static moment |
| $\alpha$ | fraction of $m_W$ used to model equivalent massless cantilever of uniform section |

| | |
|---|---|
| $\beta$ | fraction of $m_W$ used to model equivalent massless cantilever of uniform strength |
| $\gamma$ | fraction of $m_W$ used to model equivalent massless simply supported beam |
| $\alpha_o$ | value of $\alpha$ in the upper limit ($m_P = 0$) |
| $\beta_o$ | value of $\beta$ in the upper limit ($m_P = 0$) |
| $\gamma_o$ | value of $\gamma$ in the upper limit ($m_P = 0$) |
| $\rho$ | mass density |
| $\sigma_F$ | dynamic yield strength (static yield strength times a dynamic factor) |
| $\sigma_S$ | maximum normal stress caused by static load (or by a one g equivalent static acceleration in the case of non-vertical excitation) |
| $\tau_F$ | dynamic shearing yield strength (see $\sigma_F$ above) |
| $\tau_S$ | maximum shearing stress caused by static load |

# CHARACTERISTIC DYNAMIC STRENGTH RATIO AS THE LIMITING PARAMETER IN SHOCK DESIGN

## INTRODUCTION

Increasing environmental and performance demands on military structures and equipment have caused present day designers to place increased emphasis on dynamic analysis. In particular, there is greater interest in, and use of, shock design criteria.

One shock severity criterion, the shock spectrum,* may be used to provide a concise description of the effect of an expected shock environment upon equipments which can be modeled as single degree of freedom systems.† However, cursory examination of a shock spectrum does not reveal (a) which regions of the spectrum, if any, are applicable for the design of a certain type of structure, (b) what variables (as natural frequency, mass, etc.) can be used to define such regions of interest, or (c) whether similar regions exist for different types of structures.

For example, examination of the bending stress equation for a single degree of freedom cantilever (Appendix A) reminds one that, even though the maximum stress in the beam is directly proportional to the acceleration shock spectrum magnitude, it is also a function of parameters which represent the beam's geometry and load. A statement of the effect of these and other pertinent variables in equation form should enable a designer to employ a shock spectrum with greater facility.

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*Essentially, a shock spectrum is a plot of the maximum response of a set of single degree of freedom oscillators to a given shock motion versus oscillator natural frequency. It is assumed in this report that the mass and stiffness of the oscillator do not affect the shock motion input and that there is no damping.

†The spectrum also specifies an input for modal analysis of multi-degree of freedom systems.

The purpose of this report is to demonstrate that only a certain region of a shock spectrum is suitable for the elastic design of a given type of structure. This region is designated by employing a quantity called dynamic strength ratio, which is a function of a structure's material, load, geometry, and frequency.

Dynamic strength ratios for three different types of beams, modeled as single degree of freedom systems, are derived and plotted versus frequency. The ratio plots are superimposed on shock spectra to illustrate the appropriate region (range of frequencies or corresponding spectrum amplitudes) for elastic design of each type of beam. The beams are: cantilevers of uniform cross section (point loaded at the free end), cantilevers of uniform strength (point loaded at the free end), and simply supported beams of uniform cross section (point loaded at midspan).

Although the derivations and discussion which follow are limited to the dynamic strength ratios for the three types of structures mentioned above, there is no apparent reason why other structures that may be modeled as single degree of freedom systems cannot be similarly treated.

## DERIVATION OF DYNAMIC STRENGTH RATIOS OF POINT LOADED BEAMS

Before the strength ratios of the beams are derived, it is necessary that the meaning of dynamic strength ratio itself be made clear.

The evolution of the definition of dynamic strength ratio begins with the selection of a commonly employed static strength parameter that can be adapted to dynamic conditions. Yield strength, usually taken at 0.2% offset on a statically determined stress-strain curve, is such a parameter. When this static yield strength is multiplied by a factor to account for the effect of dynamic loading, it becomes dynamic yield strength. (The dynamic factor is determined by the designer for a particular state of dynamic loading. The dynamic loading is considered to be the application of a unidirectional acceleration field of greater intensity than one g.) Dynamic yield strength will be used herein as the failure criterion for dynamic loading.*

An indication of the dynamic strength relative to the static strength of a given type of structure may be gained by comparing the dynamic yield strength of the structural material with the maximum stress existing

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*It is assumed in the report that stress remains directly proportional to strain until the dynamic yield strength is reached.

in the structure under static conditions. It is convenient, then, to define the dynamic strength ratio, A, of a structure as the ratio of the dynamic yield strength $\sigma_F$ (in bending, shear, or buckling) to the maximum static stress $\sigma_s$; that is, $A = \sigma_F/\sigma_s$. It can be shown that the quantity A is the equivalent static acceleration, or "number of g's," which will cause the maximum fiber stress to equal the dynamic yield strength.* Also, A, as subsequently derived, is a function of frequency. Thus, the dynamic strength ratio is in suitable form for later comparison with shock spectra.

For the initial application of this dynamic strength ratio, consider the cantilever of uniform cross section represented by the single degree of freedom model in Fig. 1. It has length L, width b, depth h, and area moment of inertia $I = bh^3/12$. It is massless, and supports a concentrated mass $m = m_P + \alpha m_W$ at the free end. The mass of a real cantilever of cross section bh and length L is $m_W$, while $\alpha$ is the fraction of this mass added to the load $m_P$ at the free end to produce the equivalent massless cantilever.† Other pertinent quantities are listed on page iii.

In general, the natural frequency of an undamped single degree of freedom system is given by

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \quad . \tag{1}$$

The spring constant of a cantilever beam may be written as

$$k = \frac{3EI}{L^3} \quad .$$

When this expression is substituted into Eq. (1), and the beam-mass relationship

$$m_W = bhL\rho \tag{2}$$

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*Note that the dynamic strength ratio and equivalent static acceleration are both pure numbers.

†A massless cantilever loaded with a point mass $m_P$ and a mass $\alpha m_W$--both placed at its free end--has the same frequency as a cantilever loaded with a point mass $m_P$ at the free end and a mass $m_W$ uniformly distributed along its length, provided that $m_P > m_W$. The factor $\alpha$ is derived by the Rayleigh Method in Reference 1 (values are listed in Table 1).

is employed,

$$f = \frac{h}{4\pi L^2}\sqrt{\frac{m_w E}{\rho(m_p + \alpha m_w)}} \quad . \tag{3}$$

This may be rearranged as

$$h = 4\pi L^2 f\sqrt{\frac{\rho(m_p + \alpha m_w)}{m_w E}} \quad . \tag{4}$$

The maximum static bending stress, which occurs at the fixed end of the beam, is given by

$$\sigma_s = \frac{M_s c}{I} \quad .$$

At the fixed end, the static moment is

$$M_s = (m_p + \alpha\, m_w)\, g L \quad ;$$

thus,

$$\sigma_s = (m_p + \alpha m_w)\, g L \frac{c}{I}$$

or

$$\sigma_s = \frac{6(m_p + \alpha\, m_w) g L}{b h^2} \quad .$$

Substituting

$$b h = \frac{m_w}{L\rho} \quad , \tag{cf 2}$$

then

$$\sigma_s = \frac{6 L^2 \rho\, g (m_p + \alpha\, m_w)}{m_w h} \quad . \tag{5}$$

As defined above, the dynamic strength ratio is $A = \sigma_F / \sigma_s$. Thus, from Eq. (5),

$$A = \frac{m_w h \sigma_F}{6 L^2 \rho\, g (m_p + \alpha\, m_w)} \quad .$$

After substituting for h (Eq. (4)), the dynamic strength ratio equation for the cantilever of uniform section in bending becomes

$$A = \frac{2\pi\,\sigma_F}{3g\sqrt{\rho E}}\;\frac{f}{\sqrt{(m_P/m_W)+\alpha}}\;. \tag{6}$$

It can be seen that A will vary directly with frequency for a given material, applied load, and beam mass. Accordingly, if a steel cantilever having the properties listed in Tables 1 and 2 is chosen, then

$$A = 0.659\ f^{*}\;.$$

This result is plotted as the lowest bending curve of Fig. 2. Also shown in Fig. 2 is the bending curve for an aluminum cantilever with the same load and mass (see Tables 1 and 2). Various similar curves could be drawn simply by varying $m_P/m_W$.

An interesting case arises in the upper limit, which occurs when $m_P$ approaches zero. This is equivalent to saying that the model cantilever is uniformly loaded under its own mass, $m_W$ (rather than being point loaded by a mass $m = m_P + \alpha\, m_W$). The maximum moment at the fixed end now becomes $m_W gL/2$, so that

$$\sigma_S = \frac{3L^2\rho g}{h}\;.$$

and

$$A_0 = \frac{\sigma_F h}{3L^2\rho g}\;. \tag{7}$$

However, the natural frequency for a uniformly loaded

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*Since A is a pure number, the numerical coefficient 0.659 has units which are reciprocal of frequency units.

cantilever (1) is $$f = \frac{3.530}{2\pi}\sqrt{\frac{EI}{m_w L^3}}$$

Therefore, Eq. (3) can be put in the form

$$f = \frac{h}{4\pi L^2}\sqrt{\frac{E}{\rho\,\alpha_o}} \tag{8}$$

with $\alpha_o = 0.241$. The upper limiting dynamic strength ratio in bending (Eq. (7)) for the cantilever of uniform section now may be written

$$A_o = \left(\frac{4\pi\,\sigma_F}{3g}\sqrt{\frac{\alpha_o}{\rho E}}\right) f \tag{9}$$

The value of $A_o$ depends simply on the material and the frequency of the beam. When plotted versus frequency (Fig. 2), Eq. (9) yields straight lines having greater slopes than the corresponding lines derived from Eq. (6) for the steel and aluminum cantilevers previously examined. (Using the values in Tables 1 and 2, $A_o = 1.44f$ for steel, and $A_o = 2.11f$ for aluminum.) Note that it is impossible for a cantilever (of materials described in Table 2) to exist elastically in the region above and to the left of these limiting $A_o$ lines.

Next, consider the dynamic properties of the same cantilever beam of uniform cross section (Fig. 1) under conditions of lateral buckling. Since

$$A = \frac{\sigma_F}{\sigma_S},$$

and

$$\sigma_F = M_F\frac{c}{I},$$

$$\sigma_S = M_S\frac{c}{I},$$

then $$A = \frac{M_F}{M_S} = \frac{m_F L}{(m_P + \alpha\, m_W) L} \quad .$$

This may be expressed as

$$A = \frac{m_{cr}}{m_P + \alpha\, m_W} \quad , \tag{10}$$

where $m_{cr} = m_F$ is the critical load which will cause buckling.

The critical load for a cantilever of narrow rectangular cross section with a point load at the free end is given by Timoshenko and Gere (2) as

$$P_{cr} = m_{cr}\, g = \frac{4.013}{L^2} \sqrt{E I_\eta C} \tag{11}$$

where $$I_\eta = \frac{hb^3}{12} ,$$

$$C = \frac{1}{3} hb^3 G ,$$

and

G = modulus of rigidity.

The critical load now is formulated as

$$m_{cr} = \frac{4.013\, h b^3 \sqrt{EG}}{6 L^2 G} \quad . \tag{12}$$

Combining Eqs. (10) and (12), the buckling dynamic strength ratio becomes

$$A = \frac{0.669\sqrt{EG}}{L^2 g} \; \frac{h b^3}{(m_p + \alpha\, m_w)} \quad . \tag{13}$$

Observing that $h = \dfrac{m_w}{b L \rho}$

(cf 2)

and $f = \dfrac{h}{4\pi L^2} \sqrt{\dfrac{m_w E}{\rho (m_p + \alpha m_w)}}$ ,

(3)

then

$$b = \frac{1}{f} \left( \frac{1}{4\pi L^3} \sqrt{\frac{m_w^3 E}{\rho^3 (m_p + \alpha\, m_w)}} \right) . \tag{14}$$

The dynamic strength ratio of the uniform cantilever in buckling is, therefore,

$$A = \frac{0.0418}{\pi^2 g} \left( \frac{E^{\frac{3}{2}} G^{\frac{1}{2}}}{\rho^4} \right) \left( \frac{1}{\frac{m_p}{m_w} + \alpha} \right)^2 \frac{m_w^{\,2}}{L^9 f^2} \quad . \tag{15}$$

Reference to Tables 1 and 2 shows that

$$A = \frac{6.46 \times 10^4}{f^2} \tag{16}$$

for buckling of the steel cantilever and

$$A = \frac{7.48 \times 10^3}{f^2} \tag{17}$$

for the buckling of the aluminum cantilever. These curves intersect the corresponding bending curves as shown in Fig. 2.

An upper limit buckling curve will occur if the applied load $m_p$ approaches zero. As in the case of bending, this is equivalent to placing the cantilever under a distributed load equal to its own weight. Timoshenko and Gere (2) give the critical load for a cantilever which buckles under its own weight as

$$w_{cr} = \frac{12.85\sqrt{EI\eta C}}{L^2} \quad . \tag{18}$$

When this is compared with Eq. (11), it is seen that

$$w_{cr} = 3.2\, P_{cr} = 3.2\, m_{cr} g \quad .$$

Therefore, the limiting dynamic strength ratio for buckling is

$$A_o = \frac{2.14}{g} \frac{\sqrt{EG}}{L^2} \frac{hb^3}{m_w} \quad .$$

After substituting for h and b from Eqs. (2) and (14) respectively, the upper limiting dynamic strength ratio in buckling for the cantilever of uniform section becomes

$$A_o = \left( \frac{0.134}{\pi^2 g} \frac{E^{\frac{3}{2}} G^{\frac{1}{2}}}{\alpha_o \rho^4} \right) \frac{m_w^2}{L^9 f^2} \tag{19}$$

This limiting strength ratio is not as convenient as the corresponding one for bending because it depends not only on the material properties and the frequency, but also on the beam's geometry

$$\left[ m_w^2 = (bhL\rho)^2 \right] \quad .$$

Upper limiting buckling curves for the sample steel and aluminum beams are plotted in Fig. 2. (Referring to Tables 1 and 2,

$$A_o = \frac{2.08 \times 10^7}{f^2}$$

for aluminum,

and $$A_0 = \frac{2.41 \times 10^6}{f^2}$$

for steel.) Similar to the upper limiting bending curves, the upper limiting buckling curves lie above the corresponding dynamic strength ratio curves for beams subjected to a given static load $m_p$. The upper limiting bending and buckling curves intersect as illustrated in the figure.

The final strength property to be discussed for the uniform cantilever is shear. The maximum static stress in shear is given by

$$\tau_s = \frac{3 m_V g}{2bh} \quad . \tag{20}$$

But $m_V$ is the total static load; thus

$$\tau_s = \frac{3g(m_p + \alpha m_w)}{2bh} \quad .$$

Employing Eq. (2),

$$\tau_s = \frac{3gL\rho}{2}\left(\frac{m_p}{m_w} + \alpha\right) \quad .$$

Taking $\tau_F/\tau_s$, the dynamic strength ratio expression becomes

$$A = \frac{2\tau_F}{3gL\rho\left(\frac{m_p}{m_w} + \alpha\right)} \quad . \tag{21}$$

It is readily seen that the dynamic strength ratio in shear is independent of beam frequency, unlike the cases of bending and buckling.

For the earlier mentioned steel beam the dynamic strength ratio in shear is

$$A = 178$$

and for the aluminum beam it is

$$A = 261 \quad .$$

($\tau_F$ is taken as 0.471 $\sigma_F$ according to the von Mises-Hencky energy-of-distortion condition (3)). These would appear on the A versus frequency plots of Fig. 2 as horizontal straight lines, but they are not so plotted for reasons to be explained shortly.

In the upper limit, when $m_P$ approaches zero, the expression for maximum static shearing stress becomes

$$\tau_s = \frac{3gm_w}{2bh} \quad .$$

Applying Eq. (2),

$$\tau_s = \frac{3gL\rho}{2} \quad .$$

Therefore, the upper limiting dynamic strength ratio of a cantilever of uniform section shear is

$$A_o = \frac{2\tau_F}{3gL\rho} \tag{22}$$

But recall that the expression for limiting dynamic strength ratio in bending is

$$A_o = \frac{\sigma_F h}{3L^2 \rho g} \tag{7}$$

or

$$A_o = \frac{\sigma_F}{3L\rho g}\left(\frac{h}{L}\right) \quad . \tag{23}$$

Also, remembering that $\tau_F$ (shear) is 0.471 $\sigma_F$ (bending), Eq. (22) for shear takes the form

$$A_o = \frac{\sigma_F}{3L\rho g}(0.942) \quad . \tag{24}$$

When Eq. (23) and (24) are compared, it is seen that they are identical except for the final terms. This means that, in order for $A_o$ in shear to be less than $A_o$ in bending,

$$0.942 < \frac{h}{L}$$

or

$$h > 0.942\,L \quad . \tag{25}$$

The limiting value of dynamic strength ratio in shear is always greater than the limiting value in bending unless condition (25) is met. Since it is rather unusual for a cantilever to have a depth very nearly equal to its length, the bending expression will have wider application as a limiting expression and will be used as such in subsequent discussion. In other words, it will be considered that the upper limiting shearing curve will not fall within the region already defined by the upper limiting bending and buckling curves in Fig. 2.

With the upper limiting strength ratios of the uniform cantilever now establshed, the lower strength characteristics come under scrutiny. The lower limiting strength ratio occurs when the magnitude of the static load, $m_P$, causes the beam to be stressed to the elastic limit. Then the lower limiting dynamic strength ratio* is

$$A_L = \frac{\sigma_F}{\sigma_S} = 1 \quad .$$

This result is shown as the low horizontal line intersecting both the limiting bending and buckling curves in Fig. 2 (and more clearly indicated in Figs. 6 and 7).

An analysis similar to that in the foregoing paragraphs was carried out for a cantilever of uniform strength ($\sigma_S$ constant along the length of the beam) with a point load at the free end, and for a simply supported beam of uniform cross section with a point load at midspan (Fig. 3). Tables 1,3,4, and 5 show the derived equations and numbers; Figs. 4 and 5 illustrate the corresponding graphs.

## DYNAMIC STRENGTH RATIO EXPRESSIONS AND CURVES

Examination of the dynamic strength ratio parameters of the three beams in Table 3 indicates that, except for the numerical coefficients, the bending expressions are the same, the buckling expressions are the same, and the shearing expressions are the same. These comparisons hold in Table 4 for the upper limiting dynamic strength ratios. According to Table 5, the dynamic strength ratio in bending for a given $m_W$, L, and $m_P$ increases from the simple beam of uniform section to the cantilever of uniform strength. However, for buckling and shear, the strength ratio increases in the order the beams are listed in the table--cantilever of uniform section, cantilever of uniform strength, simple beam of uniform section.

The expressions in Table 5 and the corresponding curves plotted in Figs. 2,4, and 5 all suggest that the dynamic strength ratios of the aluminum beams are greater than those of the corresponding steel beams. It is emphasized that this strictly applies only for the specific beam and material parameters listed in Table 2. Various alloying and heat treatment processes would alter most of the values in the table.

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*Also arbitrarily defined as one for non-vertical excitation.

Further examination of the graphs in Figs. 2, 4, and 5 indicates that for different types of beams the regions enclosed by each group of dynamic strength ratio curves (both upper limiting and given load) all have the same general formation. The straight line bending curve on the left intersects the second order buckling curve on the right, and both the bending and buckling curves intersect the horizontal static strength curve. This basic shape is the important feature of the ratio curves which have been derived; it is illustrated in Fig. 6.

Suppose the basic shape representation in Fig. 6 is that for upper limiting dynamic strength ratio curves. Then the area bounded by the limiting bending curve, the limiting buckling curve, and the lower limiting static curve encloses the region in which elastic structures will exist. That is, the dynamic strength ratio of an elastic single degree of freedom structure of specified mass, length, and material, as previously discussed for three sample beams, never lies outside the region enclosed by the upper limiting bending and buckling ratio curves and the lower limiting static strength ratio curve.

## DYNAMIC STRENGTH RATIO CURVES AND SHOCK SPECTRA

Dynamic strength ratio curves, plotted in the preceding section as the equivalent static acceleration of the total load ($m_P + \alpha\, m_W$) in number of g's versus natural frequency in cycles per second, may be directly compared with shock spectrum curves since it is also possible to plot shock spectra for undamped systems as equivalent static acceleration versus natural frequency (4,5). Because shock spectrum amplitude and shape are unique (over a large frequency range) for each shock motion, one may imagine a shock motion that will produce almost any desired shock spectrum plot. (Shock spectra for various shock motions are published in Refs. 5, 6, and 7.) Therefore, for present purposes, the three hypothetical shock spectra shown as curves I, II, and III in Fig. 7 shall be employed.

An upper limiting dynamic strength ratio plot (curve a) and a dynamic strength ratio plot (curve b, with specific $m_P$, $m_W$, and L) are superposed on the three shock spectra in Fig. 7. The equivalent static acceleration of the structure whose limiting dynamic strength is given by curve a can never exceed the confines of curve a, and must lie on a curve such as b.

It can be seen that no portion of curve a ever intersects or exceeds shock spectrum I. Notice in particular that if one tried to design to the lower spectrum values to the left of $f_a$, the shock motion would overstress the beam material in bending even when no load (other than that equal to the mass of the real beam) was applied. Thus, a given structure of limiting strength ratio a will not survive the shock environment described by spectrum I despite the existence of low shock spectrum values at certain frequencies.

Theoretically, a suitable static loading could be chosen so that a beam of limiting strength ratio a could withstand the dynamic loading indicated by shock spectrum II. However, this would occur only if some portion of curve b happened to fall within the region bounded by spectrum II between points 1 and 2 and the portion of curve a above spectrum II. Thus, the beam of limiting strength ratio a is still unusable because the values between 1 and 2 are only valid for lighter loads ($m_p$) than have been assumed (curve b). A lighter load on the structure enables it to withstand higher accelerations.

Now consider the milder dynamic loading specified by shock spectrum III. Limiting curve a lies well above most of spectrum III, and strength curve b intersects it at points 3 and 4 (corresponding to frequencies $f_3$ and $f_4$). The segment of spectrum III between and including points 3 and 4 is acceptable for the design of the particular type of structure with material, mass ($m_W$), load ($m_p$), and length (L) employed for curve b. At frequencies between $f_3$ and $f_4$ the type of beam of strength b has a margin of strength above that required by spectrum III, as illustrated by the difference in magnitude between curve b and spectrum III. However, curve b indicates insufficient strength for design at frequencies lower than $f_3$ or higher than $f_4$ even though the spectrum amplitude is low for these frequencies.

It is interesting to observe that the apex of curve b denotes the maximum dynamic strength ratio $A_b$ and corresponding frequency $f_b$ of a given type of beam with specified mass and length and under a given load. $A_b$ and $f_b$ indicate the design which most efficiently utilizes the material and geometry of the given type of beam. By equating the bending and buckling expressions in Table 3, the frequency expression which identifies the maximum strength ratio is found to be

$$f_b^3 = 0.126 \left( \frac{E^2 G^{\frac{1}{2}}}{\sigma_F \rho^{\frac{7}{2}}} \right) \frac{(m_W)^{\frac{7}{2}}}{L^9 (m_p + a\, m_W)^{\frac{3}{2}}} \tag{26}$$

for the cantilevers of uniform section,

$$f_{\underline{b}}^{3} = 0.913 \left( \frac{E^2 G^{\frac{1}{2}}}{\sigma_F \rho^{\frac{7}{2}}} \right) \frac{(m_W)^{\frac{7}{2}}}{L^9 (m_p + \beta m_W)^{\frac{3}{2}}} \tag{27}$$

for the cantilevers of uniform strength and

$$f_{\underline{b}}^{3} = 52.8 \left( \frac{E^2 G^{\frac{1}{2}}}{\sigma_F \rho^{\frac{7}{2}}} \right) \frac{(m_W)^{\frac{7}{2}}}{L^9 (m_p + \gamma m_W)^{\frac{3}{2}}} \tag{28}$$

for the simply supported beams. Once $f_{\underline{b}}$ is known, $A_{\underline{b}}$ may be calculated from the appropriate bending expression (Table 3).

The preceding comments concerning the superposition of dynamic strength ratio curves on three hypothetical spectra may assume more practical significance if sample dynamic strength ratio curves are superposed on published shock spectrum curves. Consider the two typical shock spectrum bound curves for earth motion resulting from nuclear detonation derived by Newmark, Hansen and Associates and shown as curves I and II in Fig. 8 (see Fig. 5-5 of (7)). These curves are plotted on a log-log scale with a pseudo-velocity ordinate (circular frequency times the relative displacement of the oscillator) and a natural frequency abscissa. Each consists of maximum displacement, maximum velocity, and maximum acceleration regions. Curve I applies for surface or near surface conditions, and curve II for 100 ft. depth.

Because of the log-log scale and the pseudo-velocity ordinate, dynamic strength ratio curves plotted in Fig. 8 assume an appearance different from that seen in previous figures. For example, curve a represents the upper limiting dynamic strength ratio for the steel cantilever of uniform cross section, and curve b gives the dynamic strength ratio of the same cantilever loaded with $m_p g$ = 244 lb. The bending segments of these strength curves now appear as horizontal lines of constant pseudo-velocity, and the buckling portions become straight lines of negative slope. (It is emphasized that curve a applies only for the specially selected steel cantilever of the

parameters listed in Table 2.) The lower limit (static) line of one g applies strictly for vertical excitation and arbitrarily for excitation in other directions.

Figure 9 illustrates the dynamic strength ratio and spectrum bound curves of Fig. 8 on a linear plot with equivalent static acceleration as ordinate. The dynamic strength ratio curves a and b are easily recognized as being the same as curves III and I, respectively, in Fig. 2. The velocity portion of the spectrum bound curves is a straight line with positive slope, and the acceleration portion is a horizontal straight line. The displacement section of the spectrum bound curves lies below the one g line and is not shown.

Consider a direct comparison of the dynamic strength ratio and shock spectrum bound curves in Figs. 8 and 9. The comments pertaining to the previous comparison between dynamic strength ratio and hypothetical shock spectra of Fig. 7 apply. This means that, if a proper load was applied, the steel cantilever with upper limiting strength ratio a could theoretically withstand the shock specified by the maximum velocity section of spectrum bound curve I (surface or near surface). The steel cantilever of strength a could not survive the shock indicated by the maximum acceleration portion of curve I, regardless of the load applied. If the load selected happens to be $m_p g = 244$ lb., the dynamic strength ratio is given by curve b. In this case the cantilever will endure the shock corresponding to the maximum acceleration portion of spectrum bound curve II (100 ft. depth), provided that it is designed to have its frequency between 12 1/2 and 30 cps (frequencies corresponding to points 1 and 2).

## CONCLUDING REMARKS

The dynamic strength ratios derived for three types of beams, modeled as single degree of freedom systems, are shown to depend on mass, length, frequency, material, and static load. Curves of dynamic strength ratio plotted versus frequency illustrate that the beams exist elastically only within certain failure limits designated by bending, buckling, (shearing), and static boundaries. These boundaries have the same fundamental shape for each of the beams considered. It is expected that this fundamental shape will apply to other types of structures which can be modeled as single degree of freedom systems.

Superposition of dynamic strength ratio curves on shock spectra indicates that the strength ratio curves for a given type of single degree of freedom structure may be employed to define

the portion of a shock spectrum which is appropriate for the design of that type of structure. The superposition process points out that a specific shock spectrum amplitude is suitable for the design of a given type of structure only if the dynamic strength ratio is greater than or equal to the spectrum amplitude. Thus, shock design at very low frequencies is often not advantageous because in this region the dynamic strength ratio in bending may be expected to lie below a shock spectrum design value at the same frequency.

In general, selection of a design point from a shock spectrum without considering a dynamic strength criterion of proposed structural designs may inadvertently lead to physically unrealizable shock design requirements.

ACKNOWLEDGMENTS

The valuable discussions held with G.J. O'Hara concerning this report are gratefully acknowledged.

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Table 1
Percentage of Beam Mass Added to the Point Load

| | CANTILEVER OF UNIFORM SECTION | CANTILEVER OF UNIFORM STRENGTH | SIMPLY SUPPORTED; UNIFORM SECTION |
|---|---|---|---|
| POINT LOAD | $\alpha = 0.236$ | $\beta = 0.0905$ | $\gamma = 0.486$ |
| UPPER LIMIT | $\alpha_o = 0.241$ | $\beta_o = 0.186$ | $\gamma_o = 0.494$ |

Table 2
Values of Beam Parameters

| PARAMETER / MATERIAL | $\rho$ $\left(\frac{\text{lb-sec}^2}{\text{in.}^4}\right)$ | $\sigma_F$ (lb/in.$^2$) | $E$ (lb/in.$^2$) | $G$ (lb/in.$^2$) | $m_p$ $\left(\frac{\text{lb-sec}^2}{\text{in.}}\right)$ | $m_W$ $\left(\frac{\text{lb-sec}^2}{\text{in.}}\right)$ | $L$ (in.) | $\tau_F$ (lb/in.$^2$) |
|---|---|---|---|---|---|---|---|---|
| STEEL | $\frac{0.287}{g}$ | 40,000 | $29.6 \times 10^6$ | $11.4 \times 10^6$ | $\frac{244}{g}$ | $\frac{52}{g}$ | 50 | 18,840 |
| ALUMINUM | $\frac{0.0975}{g}$ | 20,000 | $10.1 \times 10^6$ | $3.8 \times 10^6$ | $\frac{244}{g}$ | $\frac{52}{g}$ | 50 | 9,420 |

Table 3
Dynamic Strength Ratio (A) of Sample Beams

| STRENGTH CRITERION \ BEAM TYPE | CANTILEVER OF UNIFORM SECTION | CANTILEVER OF UNIFORM STRENGTH | SIMPLY SUPPORTED; UNIFORM SECTION |
|---|---|---|---|
| BENDING | $\left(\frac{2\pi\sigma_F}{3g\sqrt{\rho E}}\right)\frac{f}{\sqrt{\frac{m_p}{m_w}+\alpha}}$ | $\left(\frac{2\pi\sigma_F}{\sqrt{3g}\sqrt{\rho E}}\right)\frac{f}{\sqrt{\frac{m_p}{m_w}+\beta}}$ | $\left(\frac{2\pi\sigma_F}{3g\sqrt{\rho E}}\right)\frac{f}{\sqrt{\frac{m_p}{m_w}+\gamma}}$ |
| BUCKLING | $\frac{0.0418}{\pi^2 g}\left[\frac{E^{3/2}G^{1/2}}{\rho^4}\right]\left(\frac{1}{\frac{m_p}{m_w}+\alpha}\right)^2\frac{m_w^2}{L^9 f^2}$ | $\frac{0.0848}{\pi^2 g}\left[\frac{E^{3/2}G^{1/2}}{\rho^4}\right]\left(\frac{1}{\frac{m_p}{m_w}+\beta}\right)^2\frac{m_w^2}{L^9 f^2}$ | $\frac{2.82}{\pi^2 g}\left[\frac{E^{3/2}G^{1/2}}{\rho^4}\right]\left(\frac{1}{\frac{m_p}{m_w}+\gamma}\right)^2\frac{m_w^2}{L^9 f^2}$ |
| SHEAR | $\frac{2\tau_F}{3gL\rho\left(\frac{m_p}{m_w}+\alpha\right)}$ | Does Not Apply | $\frac{4\tau_F}{3gL\rho\left(\frac{m_p}{m_w}+\gamma\right)}$ |

Table 4
Upper Limiting Dynamic Strength Ratio ($A_o$) of Sample Beams

| STRENGTH CRITERION \ BEAM TYPE | CANTILEVER OF UNIFORM SECTION | CANTILEVER OF UNIFORM STRENGTH | SIMPLY SUPPORTED; UNIFORM SECTION |
|---|---|---|---|
| BENDING | $\left(\frac{4\pi\sigma_F}{3g}\sqrt{\frac{\alpha_o}{\rho E}}\right)f$ | $\left(\frac{5\pi\sigma_F}{\sqrt{3g}}\sqrt{\frac{\beta_o}{\rho E}}\right)f$ | $\left(\frac{4\pi\sigma_F}{3g}\sqrt{\frac{\gamma_o}{\rho E}}\right)f$ |
| BUCKLING | $\left(\frac{0.134\,E^{3/2}G^{1/2}}{\pi^2 g\alpha_o\rho^4}\right)\frac{m_w^2}{L^9 f^2}$ | $\left(\frac{0.450\,E^{3/2}G^{1/2}}{\pi^2 g\beta_o\rho^4}\right)\frac{m_w^2}{L^9 f^2}$ | $\left(\frac{4.72\,E^{3/2}G^{1/2}}{\pi^2 g\gamma_o\rho^4}\right)\frac{m_w^2}{L^9 f^2}$ |
| SHEAR | $\frac{2}{3}\frac{\tau_F}{gL\rho}$ | Does Not Apply | $\frac{4}{3}\frac{\tau_F}{gL\rho}$ |

Table 5
Values of Dynamic Strength Ratio when $m_w g$ = 52 lb, $L$ = 50 in., and $m_p g$ = 244 lb (See Table 2)

| STRENGTH CRITERION \ BEAM TYPE | | CANTILEVER OF UNIFORM SECTION | | CANTILEVER OF UNIFORM STRENGTH | | SIMPLY SUPPORTED, UNIFORM SECTION | |
|---|---|---|---|---|---|---|---|
| | | ALUMINUM | STEEL | ALUMINUM | STEEL | ALUMINUM | STEEL |
| BENDING | A | 0.967f | 0.659f | 1.70f | 1.16f | 0.944f | 0.643f |
| | $A_o$ | 2.11f | 1.44f | 4.01f | 2.73f | 3.02f | 2.06f |
| BUCKLING | A | $\frac{6.46 \times 10^4}{f^2}$ | $\frac{7.48 \times 10^3}{f^2}$ | $\frac{1.39 \times 10^5}{f^2}$ | $\frac{1.61 \times 10^4}{f^2}$ | $\frac{3.96 \times 10^6}{f^2}$ | $\frac{4.59 \times 10^5}{f^2}$ |
| | $A_o$ | $\frac{2.08 \times 10^7}{f^2}$ | $\frac{2.41 \times 10^6}{f^2}$ | $\frac{8.85 \times 10^7}{f^2}$ | $\frac{1.02 \times 10^7}{f^2}$ | $\frac{3.58 \times 10^8}{f^2}$ | $\frac{4.15 \times 10^7}{f^2}$ |
| SHEAR | A | 261 | 178 | Does Not Apply | | 498 | 338 |
| | $A_o$ | 1290 | 875 | | | 2580 | 1750 |

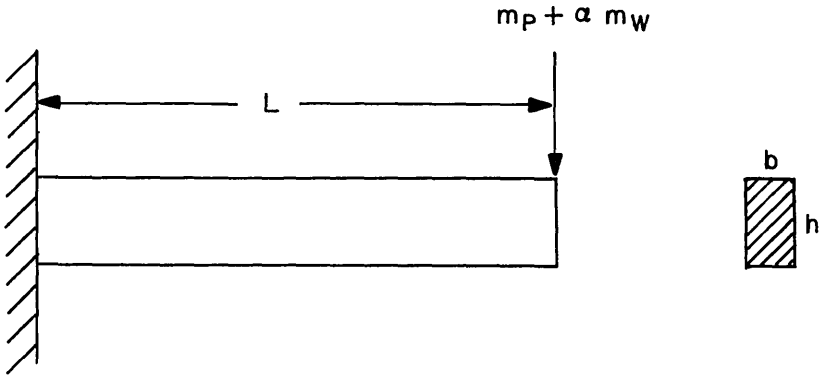


Fig. 1 - Massless cantilever of uniform section loaded at free end

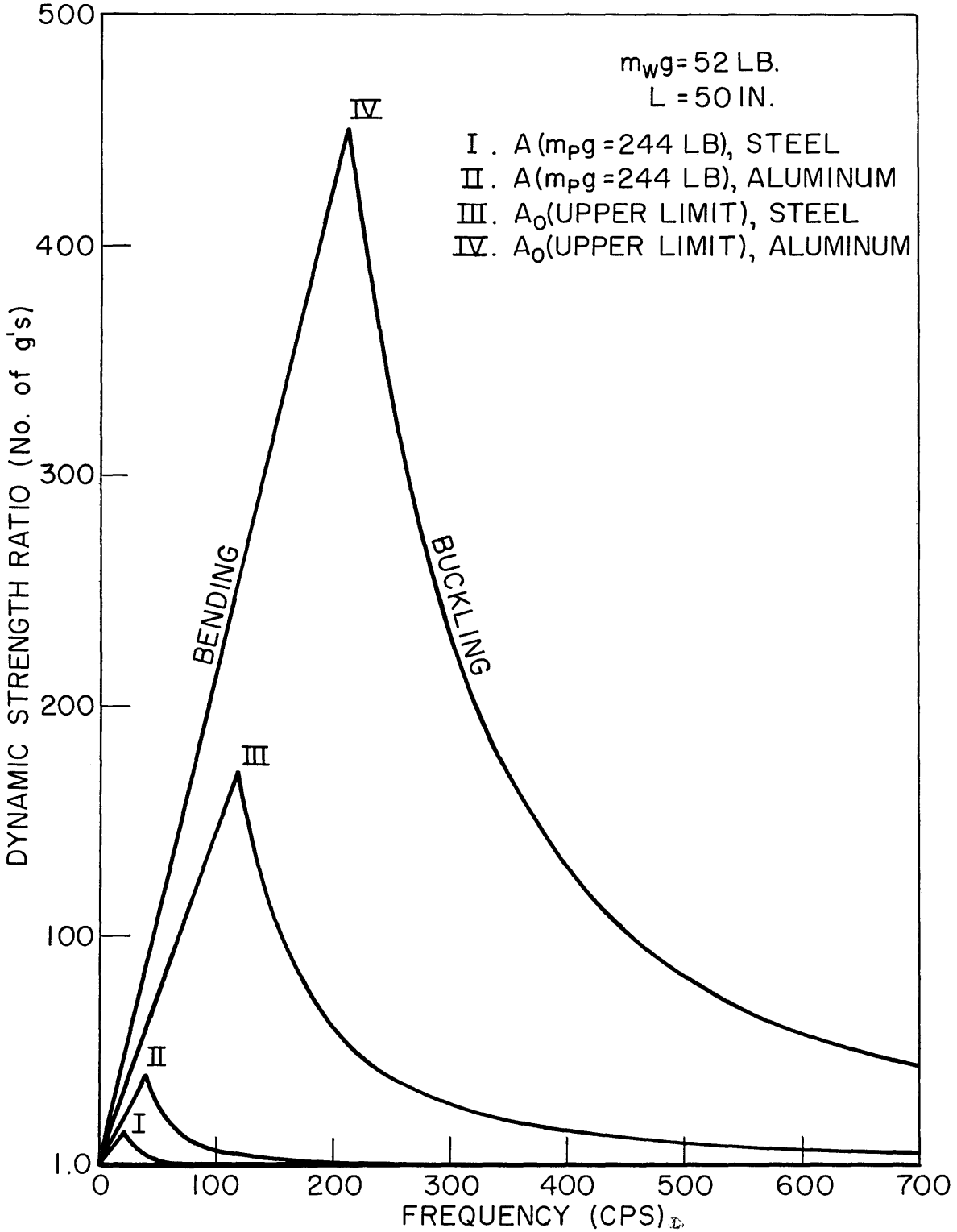


Fig. 2 - Dynamic strength ratio vs frequency for cantilever of uniform cross section

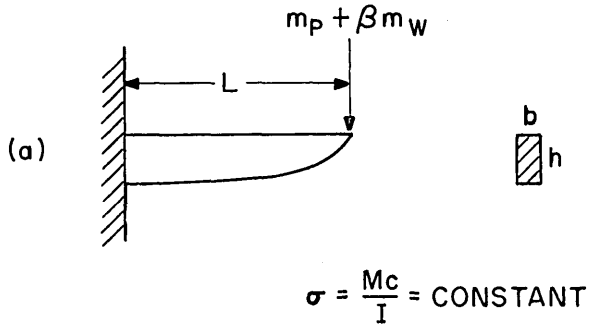


Fig. 3(a) - Massless cantilever of uniform strength with point load at free end

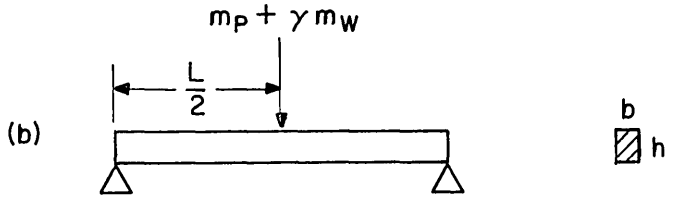


Fig. 3(b) - Massless simply supported beam of uniform cross section with point load at center

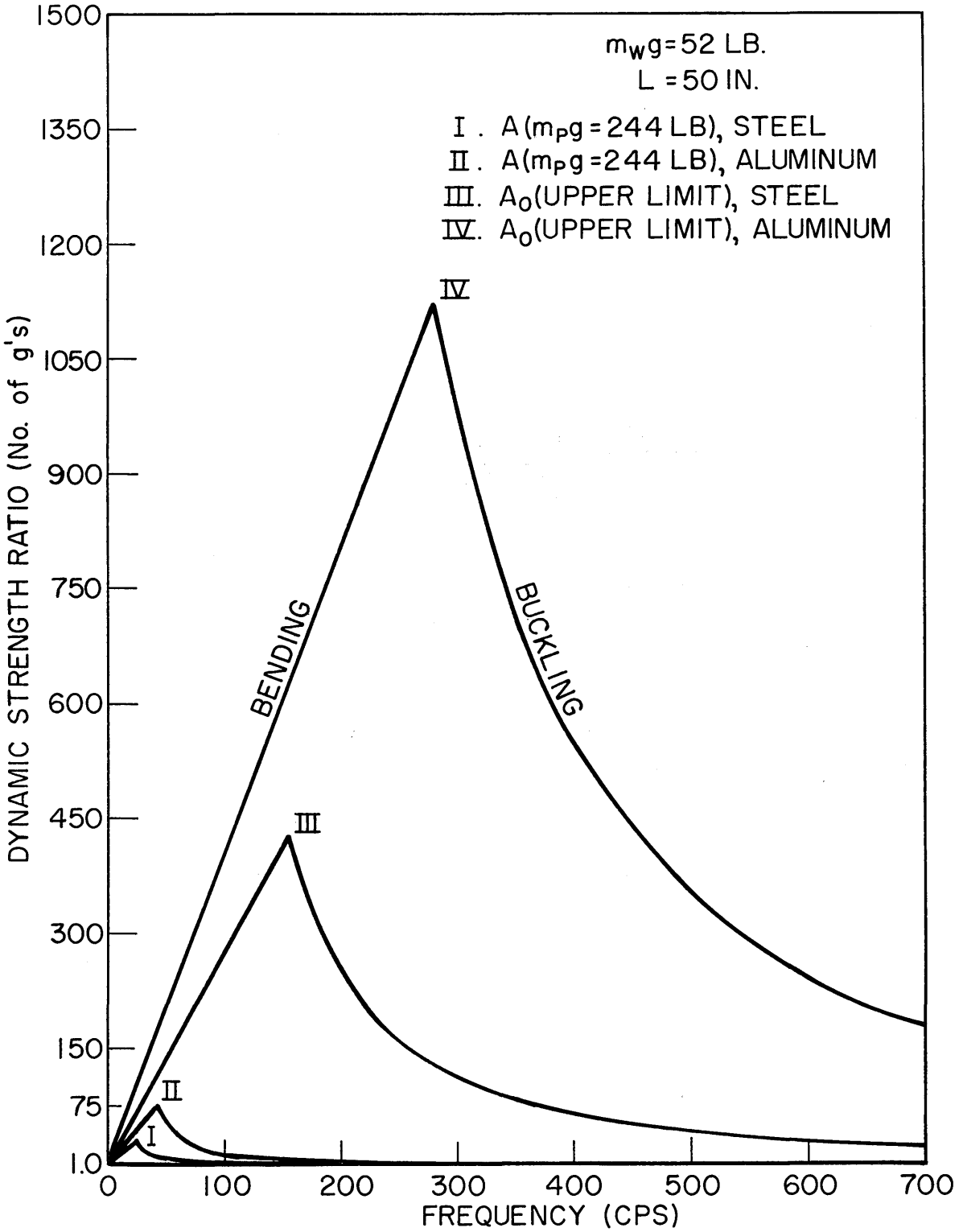


Fig. 4 - Dynamic strength ratio vs frequency for cantilever of uniform strength

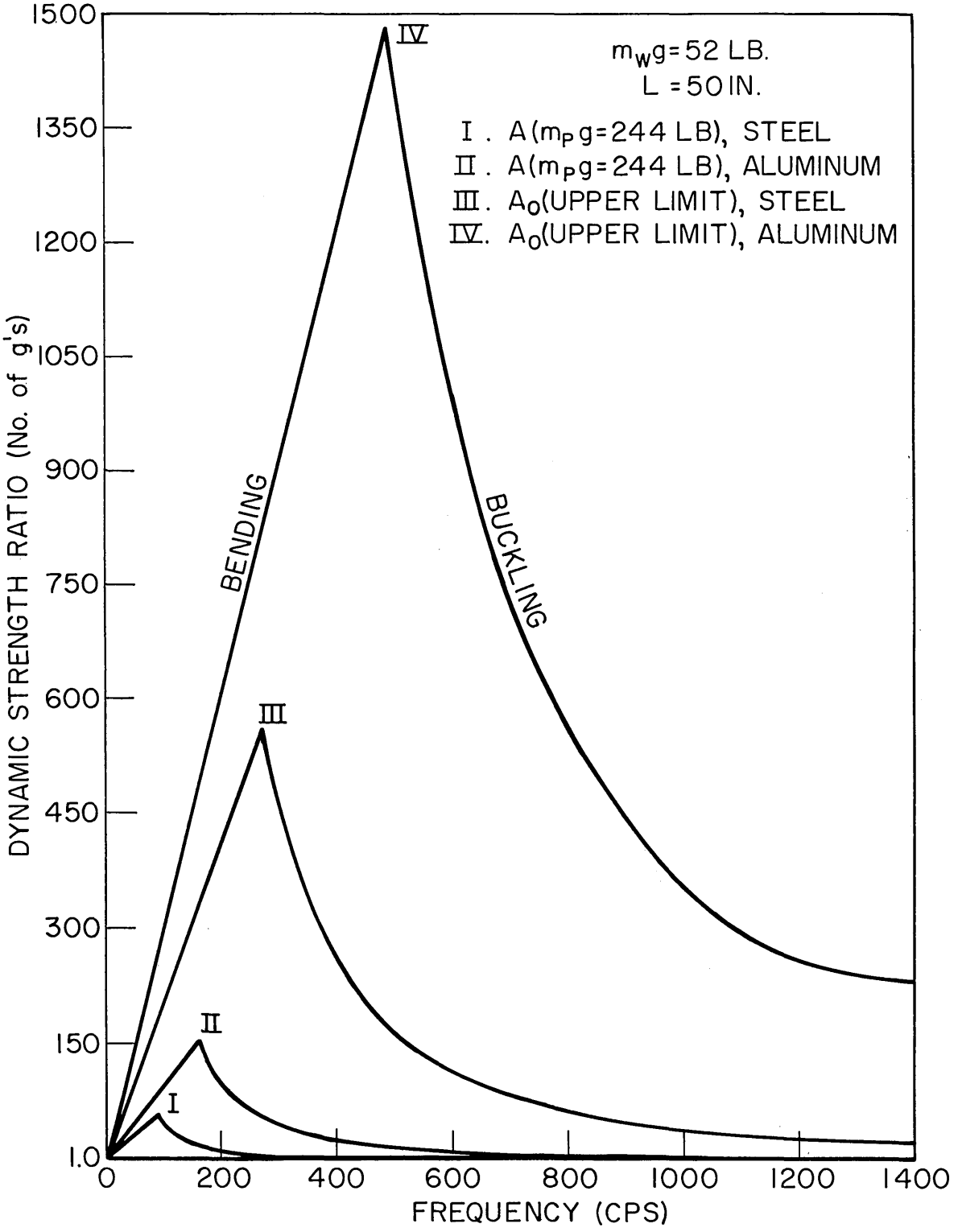


Fig. 5 - Dynamic strength ratio vs frequency of simply supported beam of uniform cross section

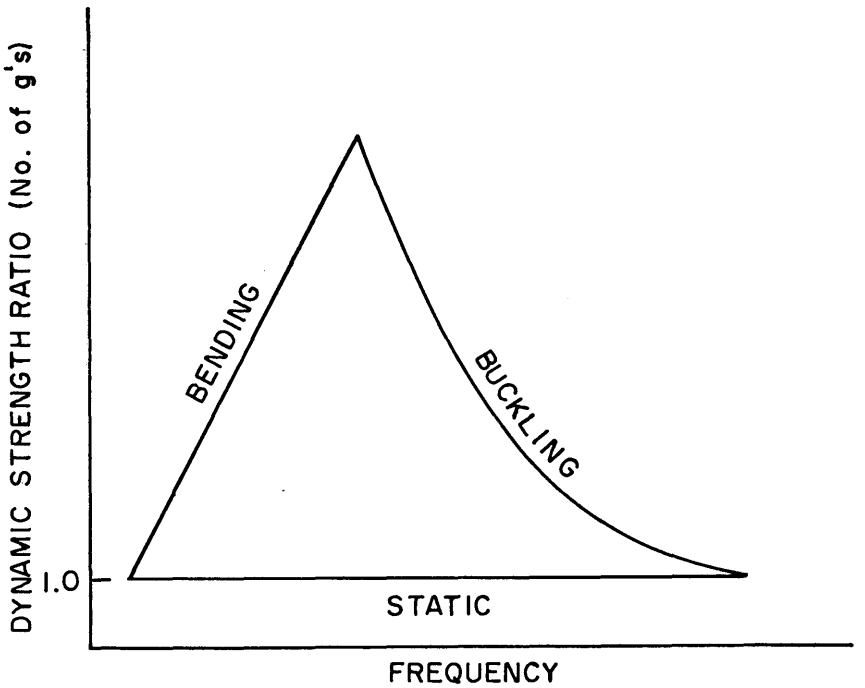


Fig. 6 - Shape of dynamic strength ratio boundary curves

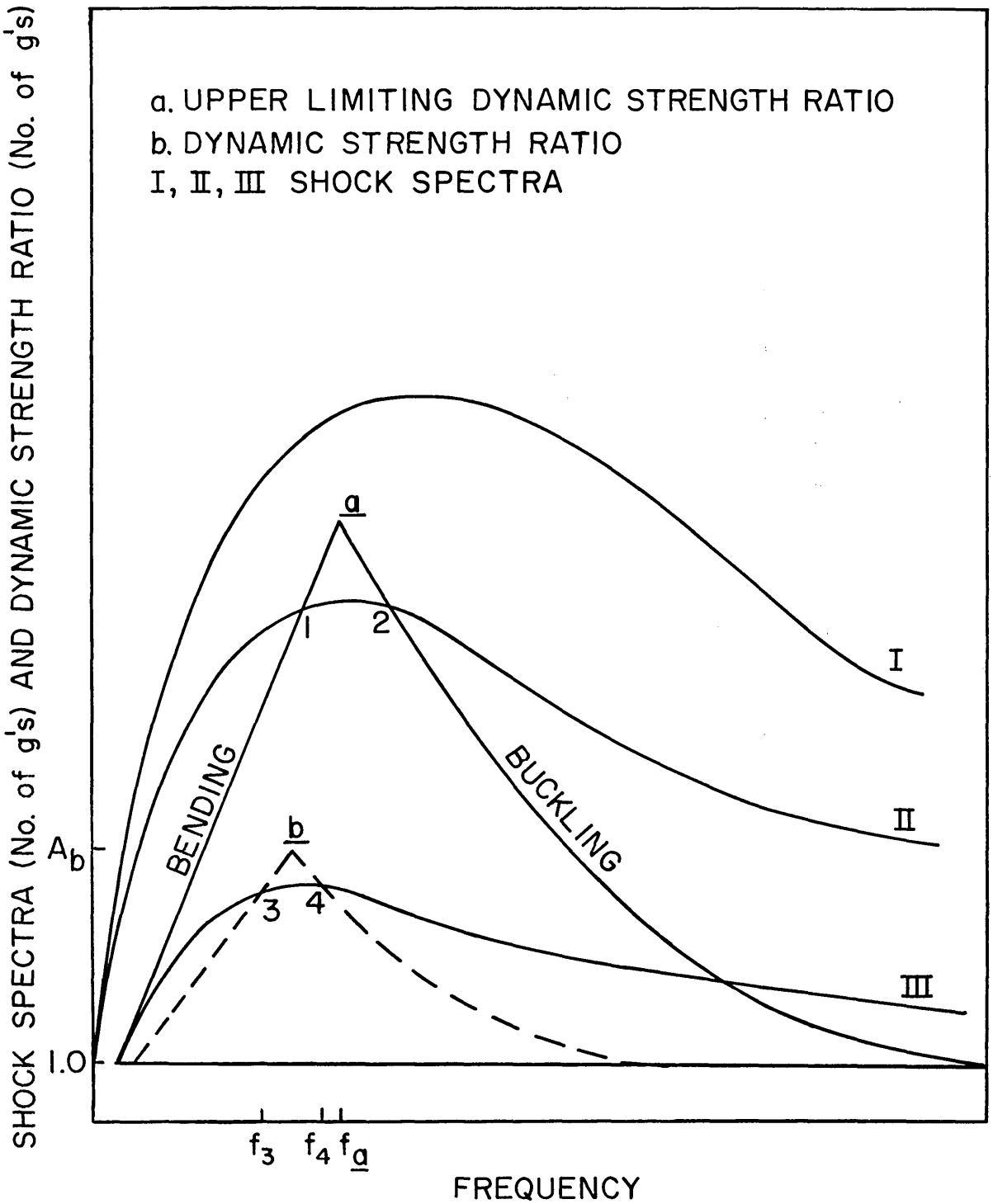


Fig. 7 - Comparison of hypothetical shock spectra (curves I, II, and III) with limiting dynamic strength ratio (a) and dynamic strength ratio (b)

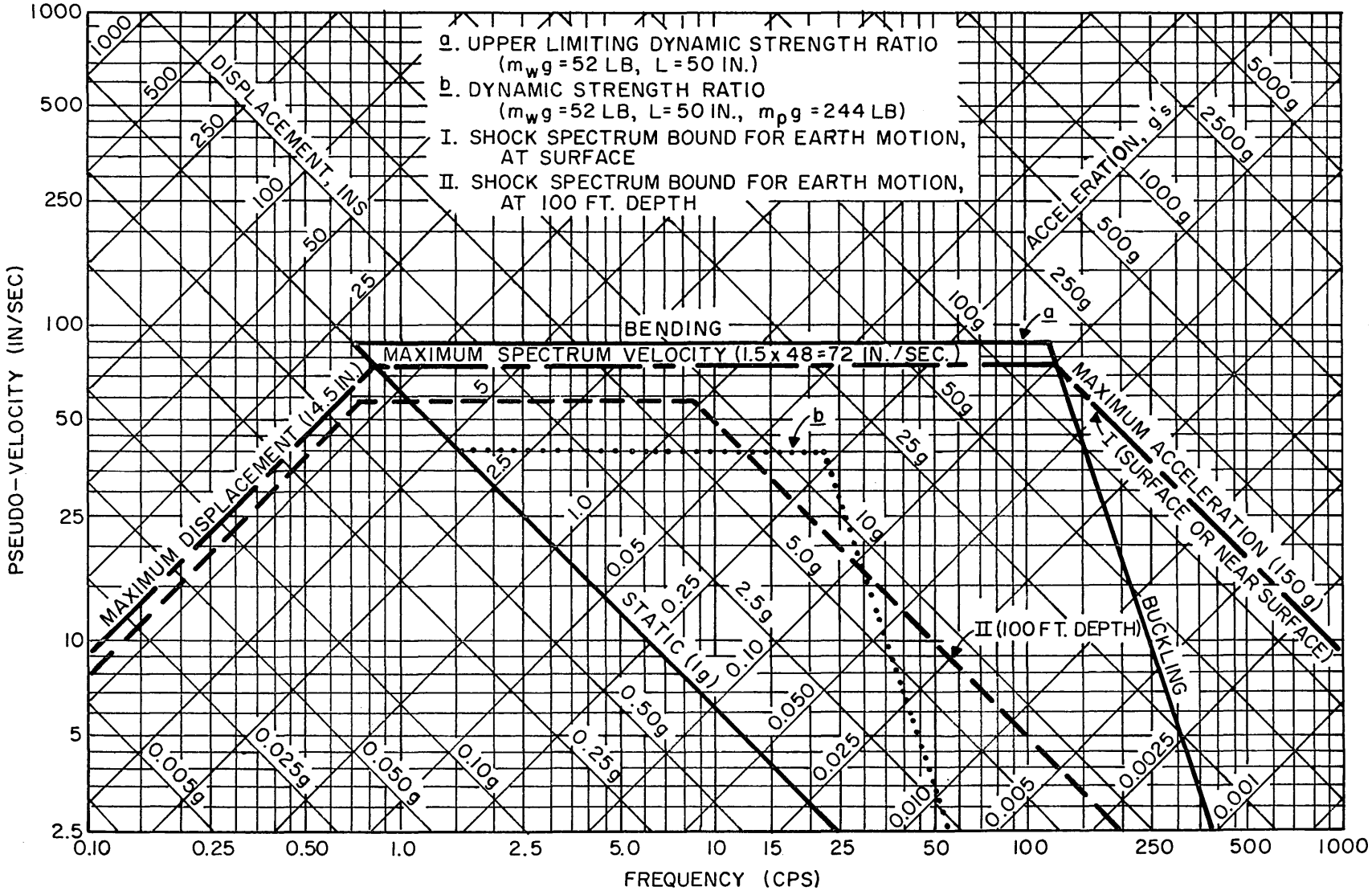


Fig. 8 - Typical shock spectrum bound for earth motion (Newmark, Hall, and Associates (7)) compared with the dynamic strength-ratio of a steel cantilever of uniform cross section (log-log plot)



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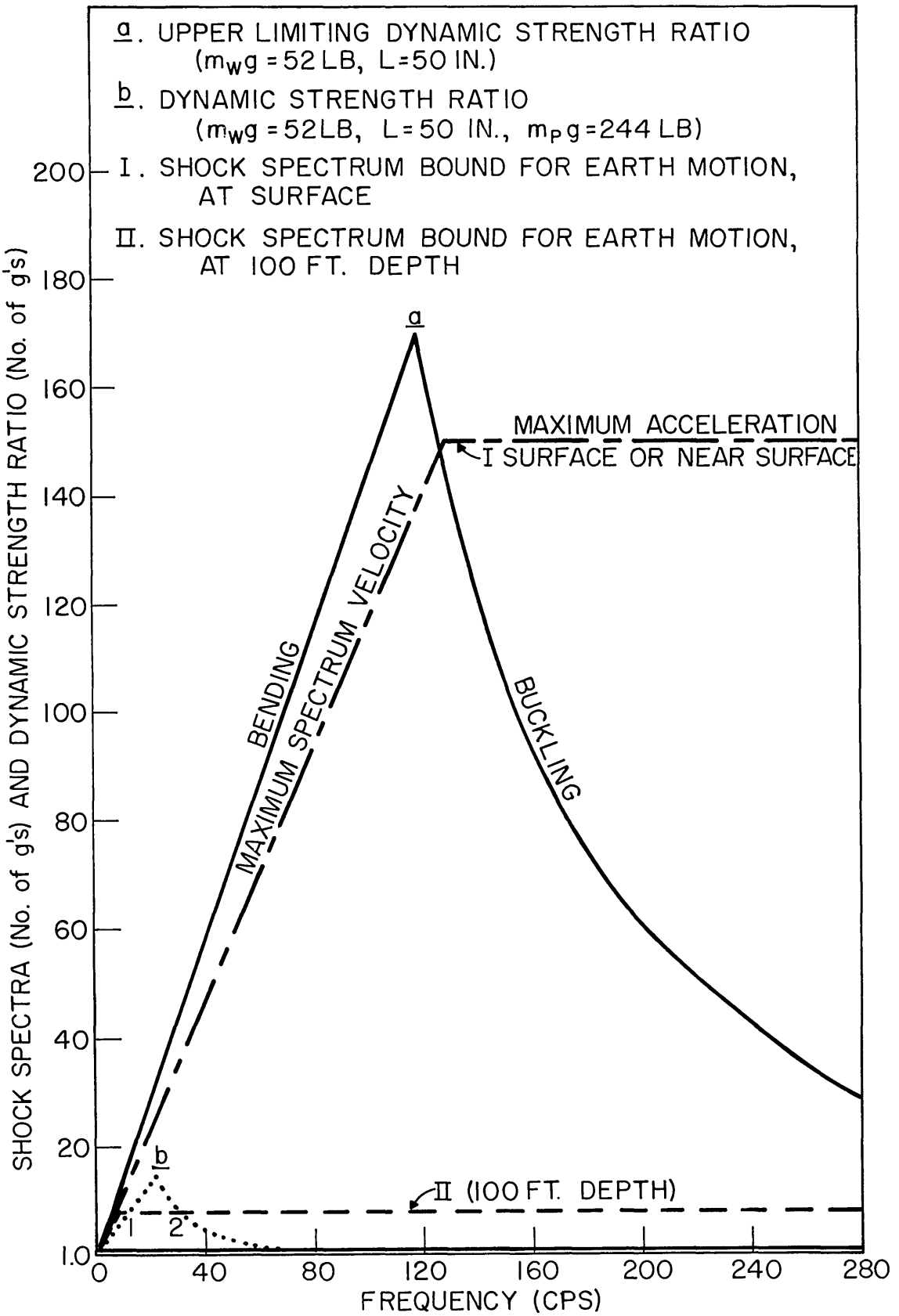


Fig. 9 - Typical shock spectrum bound for earth motion (Newmark, Hall, and Associates (7)) compared with the dynamic strength ratio of a steel cantilever of uniform cross section (linear plot)

## APPENDIX A

## BENDING STRESS AND SHOCK SPECTRUM MAGNITUDE

Figure A1 shows a massless cantilever beam of length L loaded at the free end with a mass m. Let Z be the displacement of the base, y be the displacement of the mass, and M be the moment at the fixed end. The maximum bending stress in the beam occurs at the fixed end and can be written

$$\sigma = \frac{Mc}{I} \quad .$$

If the beam is vibrating,

$$M = m\ddot{y}L$$

and

$$\sigma = m\ddot{y}L\frac{c}{I} \quad .$$

Since we can write shock spectrum magnitude (A1) as

$$S = \frac{\ddot{y}_{max}}{g} \quad ,$$

then

$$\sigma_{max} = m\left(\frac{g}{g}\right)\ddot{y}_{max}L\frac{c}{I}$$

or

$$\sigma_{max} = mgL\frac{c}{I}S \quad .$$

But

$$c = \frac{h}{2} \quad ,$$

and

$$I = \frac{bh^3}{12} \quad ,$$

so that

$$\sigma_{max} = \frac{6gmL}{bh^2} S \quad .$$

Thus, it is apparent that the maximum stress in the cantilever depends on the beam's dimensions ($b, h^2, L$) and load ($m$) as well as the shock spectrum amplitude.

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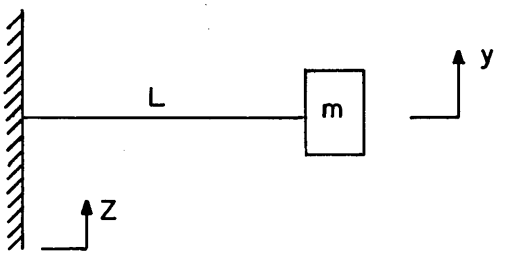


Fig. A1 - Single degree of freedom cantilever with load m

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13. ABSTRACT

The dynamic strength ratio of a structure is defined as the ratio of the dynamic yield strength (a number determined by multiplying the static yield strength times a dynamic loading factor) of the structural material to the maximum fiber stress in the structure when it is subject to the acceleration of gravity alone (one g). Dynamic strength ratios for bending and buckling are derived and plotted versus frequency for three types of beams, respectively modeled as single degree of freedom systems. The strength ratio plots are superimposed on sample shock spectra to illustrate which regions of the spectra are appropriate for the design of each type of beam. This comparison of the strength ratio curves and the shock spectra indicates that a specific shock spectrum amplitude is suitable for the elastic design of a given type of structure only if the dynamic strength ratio is greater than or equal to the spectrum amplitude.

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| Impact Shock | | | | | | |
| Shock (mechanics) | | | | | | |
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