NRL Report 4624

METHODS OF MEASURING THE BIREFRINGENCE OF AIRCRAFT GLAZING MATERIALS

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October 12, 1955

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ABSTRACT

Three methods of measuring the birefringence of hot-worked acrylics are described. Two of these methods, by utilizing oblique incidence of light upon the sample, avoid the necessity for cutting a specimen out of the material tested.

Equipment required for the tests is detailed, and the procedures outlined. Also included are experimental results of trials of the nondestructive tests on ten sheets of hot-worked materials.

An analysis of the precision of the nondestructive methods shows that one of them will yield an accuracy of five percent or better. The other method is shown to be suitable for special cases only.

The birefringence value computed from nondestructive measurements is related to that obtained from a destructive test. Demonstration is made of graphs, such as are used by crystallographers, for shortening the computation of the birefringence.

The oblique incidence methods will provide a means for the nondestructive testing of aircraft canopies made of hot-stretched acrylics.

PROBLEM STATUS

This is an interim report. Work on this problem is continuing.

AUTHORIZATION

NRL Problem F01-05
Project No. NR 631-050

Manuscript submitted August 23, 1955.

IDENTIFICATION OF SYMBOLS

d_0	compensator dial reading for extinction of compensator alone.
d	compensator dial reading for extinction of sample.
dW/dA	toughness parameter of a material.
F	birefringence as measured by the destructive method.
i	angle of incidence of light beam upon sample.
K	calibration constant of compensator
k	ratio of birefringence of a given section to the maximum birefringence
n	average index of refraction
P	probable error of a measured or computed quantity.
r	angle of refraction.
t	thickness of sample.
t'	path length of light beam within sample.
V	true optic angle.
V'	apparent optic angle as measured in air.
α	minimum index of refraction of a sample.
β	intermediate index of refraction of a sample.
γ	maximum index of refraction of a sample.
δ	any given index of refraction of a sample.
θ, θ'	angles between the light beam within a sample and the optic axes of the sample.
α_o, β_o, γ_o	angles between an optic axis and the α, β, and γ axes, respectively.
α_N, β_N, γ_N	angles between the normal to a section and the α, β, and γ axes.

METHODS OF MEASURING THE BIREFRINGENCE OF AIRCRAFT GLAZING MATERIALS

BACKGROUND

Interest in the hot working of glazing materials has been stimulated by the enhancement of shatter resistance* which it produces. Today it is not unusual to find such commercially produced acrylic sheets with six times the resistance to crack propagation of the as-cast material. At the same time the work of Wolock and Kline at the National Bureau of Standards has demonstrated that hot-working increases resistance of the material to crazing. With stretching techniques now advanced to the stage where these gains may be had without sacrifice of the optical properties of the material, it seems worthwhile to turn attention to other properties of the material which may furnish a clue to the nature of the stretching process.

The birefringence produced by hot stretching of acrylics offers a simple means of study. Although the origins of the birefringence on the molecular scale are not clearly understood, it does appear that the birefringence and the stretching process are related in a simple fashion. It is not the purpose of this report to discuss the possibility of correlation of this birefringence with the mechanical properties of the material. Rather the subject will be limited to methods of measuring the birefringence.

INTRODUCTION

When an acrylic material is stretched while warm and then cooled while still held under the stretching stress, it will retain the property of being birefringent or doubly refracting after the stress is removed. The stretching process produces an optical anisotropy in the material, similar to that found in many crystals, so that if polarized light is sent through the material, its speed of transmission (and, by the same token, its index of refraction) will vary as the plane of polarization is changed.

A uniaxially stretched material may be characterized by two indices of refraction, one for light polarized parallel (i.e. with its electric vector vibrating parallel) to the stretching direction, and another for light polarized in any direction transverse to the stretching direction. The birefringence of such a material is defined to be the difference between these two indices of refraction.

Biaxial or multiaxial stretching, however, is of more practical interest than uniaxial stretching. Ideally, the mechanical properties of a stretched sheet should be the same in

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*The NRL test for the shatter resistance or toughness of a material may not be known to the reader. The dW/dA test, as it is familiarly known, measures the work required per unit fracture area for catastrophic failure of a specimen. The test consists in starting a crack in the center of the specimen and then loading it in uniaxial tension, transversely to the direction of crack propagation, until the crack becomes unstable. A detailed description of this test will be found in NRL Memorandum Report No. 237, "A Method of Evaluating the Shatter Resistance of Aircraft Canopy Materials," by J. A. Kies.

all directions parallel to the surface of the sheet, but such perfection is not required; in general there will be a small difference in the amount of stretching in directions parallel to the surface of the sheet. As a result, the sheet will have three principal indices of refraction, one for each stretching direction, and one for light vibrating normal to the surface of the sheet. The sheet will also have three characteristic birefringence values, corresponding to the three differences between pairs of the three indices. Let us call the three principal indices of refraction, α, β, and γ, and follow the crystallographic convention, $\alpha < \beta < \gamma$. The corresponding birefringence values will be $\beta - \alpha$, $\gamma - \alpha$, and $\gamma - \beta$; thus $\gamma - \alpha$ will be the maximum birefringence of the sheet.

The question then arises: which birefringence value is the quantity one wishes to measure? In the past,[1] the resistance to crack propagation has been compared to the birefringence of that section of the sheet normal to the direction of crack propagation. We shall see that this birefringence is $\gamma - \alpha$, or a value close to it.

Three methods of measuring the birefringence will be described. The first, or destructive method, requires that a section be cut from the sheet. The two nondestructive methods involve the calculation of $\gamma - \alpha$ from measurements made while viewing the sheet at an oblique angle of incidence. If the sample is of nonuniform birefringence through its thickness, only the destructive method can differentiate between the birefringence at the center and the edges. The nondestructive methods will average these values.

APPARATUS

The birefringence values generally encountered in hot-stretched acrylics are of the order of 10^{-4}; that is to say, α, β, and γ differ from the average index of refraction, 1.5, by a few parts in ten thousand. Thus, we cannot measure α, β, and γ by conventional methods of measuring refractive index, and expect to find significant differences between them. Instead, we must resort to some method of measuring birefringence directly.

The use of a quartz compensator is a simple and, at the same time, a very sensitive method. In this laboratory, the Babinet-Soleil compensator is preferred, because of its uniform coloration over the field of view. Other apparatus needed in conjunction with the compensator include: an optical bench, a source of light (either white or monochromatic), an analyzer and a polarizer (circular and/or plane), a suitable mounting for the sample, and a low-power microscope.

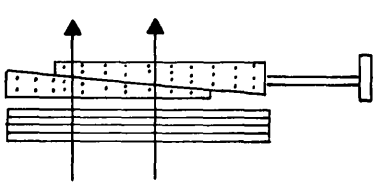

Fig. 1 - The Babinet-Soleil compensator

The Babinet-Soleil compensator, shown schematically in Fig. 1, is comprised of two quartz wedges with their optic axes aligned, together with a flat plate of quartz with its axis perpendicular to the axes of the wedges. Quartz is birefringent, so that polarized light falling on the wedges will be broken up into two plane-polarized components, with different indices of refraction. Consequently, the two components will be out of phase as they emerge from the wedges. The flat plate of quartz is so set that the component having the higher index of refraction in the wedges will have the lower index of refraction in the plate. By sliding one of the quartz wedges over the other by means of a micrometer screw, the relative thicknesses of the wedges and plate may be varied to give resultant phase differences from zero to several wavelengths.

[1] Kramer, M. J., and Post, D., "The Birefringence of Hot Worked Acrylics," NRL Memorandum Report No. 321, June 1954

The calibration of the compensator is comprised of two parts: finding the zero position (the position of the movable wedge for zero phase difference), and finding the number of turns of the micrometer screw required to advance the wedge to a position giving a phase difference of an integral number of wavelengths.

The compensator is set on the bench with its face normal to the path of the light beam. One of its axes is parallel to the direction of motion of the micrometer screw; the other is parallel to the face of the instrument and perpendicular to the first axis. It is most convenient to set the axes on the horizontal and the vertical. Crossed polaroids, with their axes at 45° to the axes of the compensator, are placed on either side of the compensator. The micrometer screw of the compensator is then turned until the field becomes black. This position is recorded as the compensator zero, denoted as d_0. If white light is used, one will see, as the screw is turned in either direction from the zero position, the following succession of colors: grayish white, white, yellow, orange, red, violet, blue, green, yellow, orange, etc. The violet color, called the tint of passage, marks the boundary between the red of the first order and the blue of the second order. For the average of white light, 5500 A.U., it corresponds to a phase difference of one wavelength, hereinafter called one fringe. The difference in micrometer readings between the zero and the tint of passage is the calibration constant of the compensator, called K. This measurement may be checked by advancing the wedge to the second tint of passage, between the red of the second order and the blue of the third order. The second tint of passage will correspond to a phase difference of two wavelengths. Beyond the third order, the colors become so mixed that it is difficult to distinguish the tint of passage. This does not constitute a problem, however, since the flatness of the wedge faces requires that the orders be equally spaced.

When polarized light enters a doubly refracting sample, it is broken up into two plane-polarized components. The directions of vibration are parallel to the directions of the principal stresses at the time the sample became frozen. Because the index of refraction varies with the plane of polarization, the two components will emerge from the sample out of phase. The first step in measuring birefringence is to find the axes of polarization. The second step is to align these axes with those of the compensator, and then to move the compensator wedge until the phase difference introduced by the compensator nullifies that introduced by the sample.

To find the axes of polarization, we observe the sample between crossed plane polaroids. The sample is rotated about an axis parallel to the light beam until the coloration caused by its birefringence is removed. At this position, one of its axes is parallel to the axis of the analyzer, and the other is parallel to the axis of the polarizer. The light emerging from the polarizer travels through the sample with its polarization unchanged, since there is no component of the incident light in the direction of the second axis. This light is then extinguished by the analyzer, and the sample appears dark. The axes having been found, the sample is then aligned with the compensator axes.

At this point, there is a choice of using plane or circular polarizers on the bench. Circular polarizers have the advantage that the axes of the sample and compensator need not be set in the bench in any particular orientation. However, since plane polarizers are needed for finding the axes of the sample, it may be simpler to leave them in place on the bench for the remainder of the operation. If plane polarizers are used, it is best to set their axes at 45° to the axes of the sample and compensator, since this results in the most brilliant coloration of the field.

The sample mount must allow for rotation of the sample about an axis parallel to the light beam. If the nondestructive methods are to be used, the sample mount must also allow for rotation about an axis perpendicular to the light beam, and parallel to one of the compensator axes. The mount must allow for measurement of these rotations to 5° or better.

A low-power microscope, focused on the sample, will increase considerably the ease and accuracy of measuring. It is also convenient to have a bench which allows swinging the various parts of the polariscope out of the light path during the intervals in which they are not needed.

In this laboratory, white light has been used for measuring the birefringence of glazing materials. Hence the measurements tabulated in this report should be referred to a wave length of 5500A, the average for white light.

THE DESTRUCTIVE METHOD

This method, the simplest of all, requires that a section be cut from the sample.[2] The machining must be done carefully, with minimum heating of the sample, and without the use of organic coolants.

It is convenient to use a sample cut to a width of about 0.10 in and a length of one or more inches. For correlation purposes, the long edges should be perpendicular to the direction of crack propagation used in a dW/dA measurement. The sample is placed in the bench so that the light beam is parallel to the surface of the sheet and to the direction of propagation of the crack. Thus, the light beam "sees" the same section of sheet that the head of the advancing crack has "seen."

The cut surfaces viewed in the measurement need not be polished, but they should be flat, and parallel to about 0.001 in.

The axes of the sample must first be found and aligned with those of the compensator. This step has already been described. Once alignment is completed, the micrometer of the compensator is turned until the black is seen (the sample now being in place). This reading is recorded, and the thickness of the sample between the cut faces is measured with a micrometer.

The birefringence of the section may then be calculated from the formula:

$$F = \frac{d - d_0}{Kt} \tag{1}$$

where d is the compensator reading for the black with the sample in place, d_0 is the compensator zero, K is the compensator calibration constant, and t is the thickness of the sample in inches.* The determination of d_0 and K has been described earlier in this report. F will be the birefringence in fringes per inch. If it is desired to convert to units of refractive index difference, we may use the conversion factor

$$1 \text{ fringe/inch} = \frac{1 \text{ fringe}}{1 \text{ inch}} \cdot \frac{5500 \text{ A}}{1 \text{ fringe}} \cdot \frac{1 \text{ cm}}{1 \times 10^8 \text{ A}} \cdot \frac{1 \text{ inch}}{2.54 \text{ cm}} = 2.17 \times 10^{-5}.$$

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*Occasionally it will be found that the birefringence of a sample is so high that it is out of the range of a single compensator. In this case, two compensators may be used in series, and the birefringence of the sample found from the formula:

$$F = \frac{d_1 - d_{0_1}}{K_1 t} + \frac{d_2 - d_{0_2}}{K_2 t}$$

where the subscripts 1 and 2 refer to the readings of the first and second compensators.

[2]This method is discussed in greater detail in NRL Memorandum Report No. 321.

NONDESTRUCTIVE METHODS

The methods to be described here are adaptations of methods which have long been used by crystallographers. Treatments of them are given in standard reference works, such as Wright, F. E., "The Methods of Petrographic-Microscopic Research," Carnegie Institute Publication 158, Washington, D. C., 1911; Johannsen, A., "Manual of Petrographic Methods," McGraw-Hill Book Co., 1914; and Wahlstrom, E. E., "Optical Crystallography," John Wiley & Sons, 1943.

The derivation of the equations used in these methods will be found in Appendix A of this report.

Method I

The first method utilizes the equation

$$\gamma - \alpha = \frac{\beta - \alpha}{\sin^2 V}. \tag{2}$$

The quantity $(\gamma - \alpha)$, as we have seen above, is the maximum birefringence which would be measured if, using the destructive method, one were to cut a specimen from the sheet so that the light beam would be transverse to the direction of greatest stretching. The quantity $(\beta - \alpha)$ is the birefringence measured with the light beam normal to the original surface of the sheet; it is a measure of the inequality of the stretching along the directions of principal stress. The quantity, V, is the optic angle of the sheet to be defined below.

To measure $(\beta - \alpha)$, the uncut sheet is placed on the bench with the light beam at normal incidence upon its surface. Its polarizing axes are aligned with those of the compensator, as described above, and the compensator is adjusted to the black. Then $(\beta - \alpha)$ is calculated in the same fashion as the birefringence of a cut sample is calculated by

$$\beta - \alpha = \frac{d - d_0}{Kt}$$

where t is the thickness of the plate.

The compensator is then removed, and the sample is rotated about a vertical axis (i.e. one of its polarizing axes) intersecting the light beam, until the position is found at which the sheet loses its color and becomes black. The angle between the light beam and normal to the surface of the sheet is the apparent optic angle, V'. This extinction position will be found on both sides of the normal to the sheet.

If the extinction is not found upon rotating about a vertical axis, one must then rotate about the horizontal axis intersecting the light beam (i.e., the second of the two polarizing axes of the sample). An alternative is to rotate the sample 90° about the light beam as axis, and then to rotate about the vertical.

The formula for method one refers to the optic angle as measured within the sample. As the measurement described is made in air, correction must be made for refraction of the light beam in traversing the specimen. This is best seen by reference to Fig. 2, a view of the uncut sheet, looking down upon the bench.

We may correct for this refraction, using Snell's law

$$n = \frac{\sin i}{\sin r} = \frac{\sin V'}{\sin V}$$

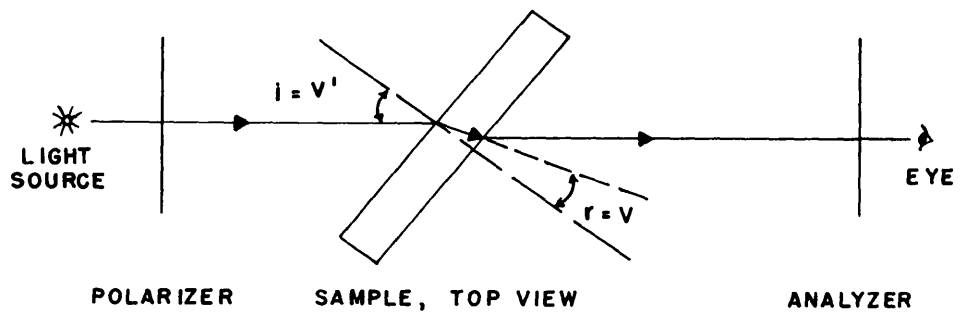

Fig. 2 - Oblique incidence, Method I

where n is the average refractive index of the acrylic sheet, i is the angle of incidence, r is the angle of refraction, V' is the apparent optic angle measured in air, and V is the true optic angle.

In F. E. Wright's book, mentioned above, will be found charts giving graphical solutions of Snell's law as well as the equation $\gamma - \alpha = \beta - \alpha/\sin^2 V$. These are reproduced in Appendix C. The scale of the chart for the birefringence equation was not useful for most of the samples tested. The graphical solution to Snell's law is sufficiently precise so as not to warrant calculation of the solution unless measurement of the optic angle is accurate to a degree.

To summarize, the steps in the use of Method I are as follows:

1. Place the sample in the bench with light at normal incidence to the surface of the sheet.

2. Align its axes with those of the compensator.

3. Measure the birefringence, $\beta - \alpha$, for light normal to the surface of the sheet.

4. Remove the compensator and rotate the sheet about a vertical (or horizontal) axis until its color is extinguished. Record the apparent optic angle, V'.

5. Find the true optic angle, V, using Snell's law, and its sine squared.

6. Substitute in the equation for the maximum birefringence of the sheet, $\gamma - \alpha = \beta - \alpha/\sin^2 V$.

Method II

The second method employs the equation

$$\gamma - \alpha = \delta - \beta/\sin\theta \sin\theta'. \tag{4}$$

The quantity $(\gamma - \alpha)$ is, as before, the maximum birefringence of the sheet. In this method, the sheet is set at an arbitrary angle to the light beam; the birefringence measured at this angle is $\delta - \beta$. The angles, θ and θ', are the angles between the light beam and the optic axes.

We digress here to distinguish between the polarizing axes and the optic axes of sheet. Any plane in the sheet will contain two mutually perpendicular polarizing axes. These are the directions of vibration of the two polarized components into which incident light is separated. They are found by setting the sample between crossed polaroids and rotating it about the normal to the plane until an extinction position is found. For a 360° rotation, there will be four such extinction positions, at which the polarizing axes are parallel to the axes of the crossed polaroids. The polarizing axes are the directions of the principal stresses in the given plane, at the time the sample was "frozen."

There are two optic axes for the sheet. Light travelling along an optic axis is not doubly-refracted. The index of refraction is the same for all planes of polarization. The optic axes may be found by first locating the polarizing axes in the plane of the surface of the sheet, and then rotating the sheet about one of the polarizing axes to the two extinction positions. If provision is now made for rotating the sheet about the optic axes thus found, it will be seen that the sheet will appear black at all orientations through a 360° rotation.

To use Method II, first set the sheet for normal incidence of the light beam and align its polarizing axes with the vertical and horizontal. Rotate the sheet to find its optic angle, as in Method I. Continue rotating the sheet about the same axis until the angle of incidence is as large as is convenient. It will be seen by inspection that the birefringence increases, the larger the angle. Hence the accuracy of the measurement improves with larger angles of incidence. Record this angle of incidence and with the compensator measure the birefringence of the sheet at this position. The birefringence, $\delta - \beta$, is calculated from the formula

$$\delta - \beta = \frac{d - d_0}{Kt'} .$$

Here the quantities d, d_0, and K are the same as before. The quantity t' is the path length of the light beam traversing the sample, a quantity greater than the thickness of the sheet. From Fig. 3, it can be seen that

$$t' = \frac{t}{\cos r}$$

where t is the thickness of the sheet and r is the angle between the light beam and the normal to the sheet. Of course, the angle i is the angle measured and the angle r must be computed from Snell's law:

$$n = \frac{\sin i}{\sin r} .$$

By inspection of Fig. 3, it may be seen that

$$\theta = r + V \tag{5}$$

and

$$\theta' = r - V \tag{6}$$

where V is again the true optic angle. Note that these simple relationships between θ, θ', r, and V will hold only if the plane in which the angle r lies is the same plane as the one in which the angle V lies. The equation

$$\gamma - \alpha = \frac{\delta - \beta}{\sin \theta \sin \theta'}$$

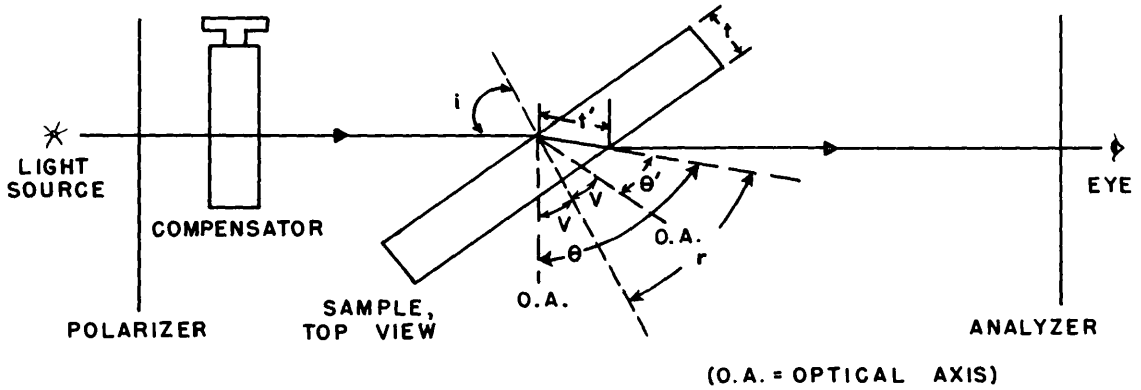

Fig. 3 - Oblique incidence, Method II

holds true regardless of the plane of incidence, but the measurement of θ and θ' will be unnecessarily complicated by setting the angle r in some other plane. It is for this reason, that in the procedure outlined above it is recommended that, after the optic axis is found, the investigator should not change the axis of rotation in setting the angle r.

A very convenient chart for the solution of Eq. (4) is given in Wright's test. It has been reproduced in Appendix C. The solution is given in the form

$$\frac{\gamma' - \alpha'}{\gamma - \alpha} = k = \sin\theta \sin\theta'$$

In our notation, this would be written:

$$\frac{\alpha - \beta}{\gamma - \alpha} = k$$

In outline, the procedure for using Method II is as follows:

1. Remove the compensator from the bench.

2. Set the sheet in the polariscope for normal incidence of the light, and find the polarizing axes by rotating about the normal to the sheet.

3. With one of the polarizing axes vertical, rotate the sheet about the vertical until extinction is found. Record the angle of rotation as the apparent optic angle, V'.

4. Continue rotating the sheet about the vertical to as large an angle as is convenient. Record the total rotation as i, the angle of incidence.

5. Replace the compensator on the bench with its axes vertical and horizontal, and with it find the number of turns of the micrometer required to compensate the birefringence of the sample. Record the micrometer reading as d.

6. Measure the thickness of the sample.

7. Using Snell's law, compute the angle r from the angle i, and the angle V from the angle V'.

8. Compute θ and θ' from Eqs. (5) and (6).

9. Find $\delta - \beta$ from the relationship.

$$\delta - \beta = \frac{(d - d_0) \cos r}{Kt} .$$

10. Find the sines of θ and θ', and using Eq. (4) find $(\gamma - \alpha)$, or

10a. Using the chart in Appendix C find $k = \delta - \beta/\gamma - \alpha$ from θ and θ', and divide to find $\gamma - \alpha$.

Corollary to Method II

In the particular case where a sheet is equally biaxially stretched (or equally multiaxially stretched), the index of refraction α will be equal to the index of refraction β. Here, Method I will not be applicable. The optic axis is now the normal to the sheet, and V is zero. Therefore, Eq. (4) reduces to

$$\gamma - \alpha = \frac{\delta - \beta}{\sin^2 r} . \tag{7}$$

The procedure becomes as follows:

1. Rotate the sample to an arbitrary angle, i.

2. Use Snell's law to compute the angle r from the angle i.

3. Use the compensator to measure $\delta - \beta$:

$$\delta - \beta = \frac{(d - d_0) \cos r}{Kt} .$$

TESTS OF THE NONDESTRUCTIVE METHODS

Ten samples were chosen for tests of the nondestructive methods. Eight of these were flat sheets of biaxially stretched polymethyl methacrylate. Two were sheets of stretched polymethyl alpha chloracrylate, which were warmed-formed to a radius of curvature of 15 in. subsequent to stretching.

The purpose of the test was to see how well the birefringence computed by the nondestructive methods compared to the birefringence directly measured by the destructive method. To this end, samples 0.20 in. wide were cut from the sheet to be used for all measurements. By this means it was assured that both the nondestructive and the destructive type of readings were taken at the same location in the sheet.

Table 1 contains a summary of the results. Because the section viewed by the destructive test was not usually the section of maximum birefringence, the results of the destructive test were corrected for this difference. It would have been more logical to have corrected the $(\gamma - \alpha)$ value computed from the nondestructive methods, but the alternative was chosen as shorter. The method of correction is discussed below.

TABLE 1
Summary of Results

Sample No.	Sample Description	$\gamma - \alpha$ by method:				$\gamma - \alpha$ by Destructive Method
		I	IA	II	IIA	
1	Biax. Plex II UVA	14.6		18.4	18.2	1/2, 14.8; 1/2, 18.5
2	" "	30.6		14.2	14.0	14.4
3	" "	11.3		15.8	15.8	15.9
4	" "	14.8		15.7	15.4	15.8
5	" "	12.7		16.6	16.0	16.1
6	" "	26.8		16.7	16.7	15.9
7	" "	15.1		16.7	16.7	17.1
8	" "	14.5		17.2	17.1	16.9
9	Multiax PMAC, curved	78.1	79	78.1	76.5	C, 70.2; E, 78 and 87.9
10	" " "	78.6	80	85.4	84.8	C, 714; E, 90.8

Note: All units are in fringes/inch.
IA and IIA were found by use of charts.
C and E denote center and edges of specimen.

Tables 2 and 3 give the data used for the computations with Methods I and II, respectively. Table 4 gives the data for correcting the destructive test result to $\gamma - \alpha$.

It will be seen by examination of Table 1, that Method II will give results of about 5% accuracy. Method I is generally poorer. The reason for this is obvious. The manufacturer aims to produce a sheet that is uniformly stretched, so that $\beta - \alpha$ is usually a small quantity. Very small birefringences are measured with poor accuracy by the compensator. Method II, however, provides for making the quantity measured by the compensator a reasonably large one, so the accuracy may be improved. Notice that for the PMAC samples where $\beta - \alpha$ is a relatively large number, Method I is quite as accurate as Method II. A more rigorous analysis of the accuracy of the methods is given in Appendix B.

Notice also that the curved specimens introduced no difficulty. No doubt this is due to the fact that the field of view in the bench was limited. All quartz compensators provide a small field.

FURTHER OBSERVATIONS WITH THE NONDESTRUCTIVE METHODS

In finding the optic axes of a stretched sheet, the sheet is rotated about one of the polarizing axes in the plane of its surface, until that angle of incidence is found for which the sheet is no longer doubly-refracting. We shall see that the axis of rotation must then be the β axis, the axis of the intermediate index of refraction. Perpendicular to it is the plane containing the α and γ axes. Therefore, when one locates the optic axes, at the

TABLE 2
Tests of Method I: $\gamma - \alpha = (\beta - \alpha)/(\sin^2 V)$

Sample No.	Sheet Thickness (in.)	$\beta - \alpha$		$\sin^2 V$			$\gamma - \alpha$
		$d - d_o$	$\frac{d - d_o}{kt}$	V'	V	$\sin^2 V$	
1	0.222	200	1.62	30°	19.5°	0.111	14.6
2	0.209	54	0.46	15	10	0.015	30.6
3	0.214	20	0.17	15	10	0.015	11.3
4	0.211	90	0.77	20	13	0.052	14.8
5	0.196	72	0.66	20	13	0.052	12.7
6	0.200	45	0.40	15	10	0.015	26.8
7	0.193	76	0.71	19	12.5	0.047	15.1
8	0.205	131	1.15	25	16.5	0.079	14.5
9	0.206	2324	20.3	50	30.5	0.26	78.1
10	0.206	1999	17.5	45	28	0.222	78.6

same time he determines the β axis and also the section of maximum birefringence. It is then apparent that the maximum birefringence, $\gamma - \alpha$, will be seen in a destructive test when the light beam travels parallel to the β axis.

The next problem is that of relating the maximum birefringence to the birefringence of any other section. This situation is illustrated in Fig. 4. A case in point is the problem of finding the birefringence of a section normal to the direction of crack propagation in a dW/dA test, once the maximum birefringence has been determined. More often than not, it is found that the polarizing axes in the surface of the sheet are not parallel and perpendicular to the direction of crack propagation.

The solution is found in the second nondestructive method presented above. However, it will now be used in the reverse order. The $(\gamma - \alpha)$ quantity having been computed, it may be employed to compute the birefringence in another section. The only problem involved is that of determining the angles between the optic axes and the normal to the section of interest; these are the angles θ and θ' in the equation

$$\frac{\gamma' - \alpha'}{\gamma - \alpha} = \sin\theta \sin\theta'.$$

To find the angle between two lines we use the formula from analytic geometry:

$$\cos\theta = \cos\alpha_o \cos\alpha_N + \cos\beta_o \cos\beta_N + \cos\gamma_o \cos\gamma_N$$

where the subscripts o and N refer to an optic axis and the normal to the section, respectively, and the angles α, β, and γ are the angles with the α, β, and γ axes respectively.

TABLE 3

Tests of Method II: $\gamma - \alpha = (\delta - \beta)/(\sin\theta \sin\theta')$

Sample No.	Sheet Thickness (in.)	r			V	θ	$\sin\theta$	θ'	$\sin\theta'$	K from chart	$\delta - \beta$		$\gamma - \alpha$
		i	r	cos r							$d - d_o$	$\frac{d - d_o}{kt/\cos r}$	
1	0.222	45°	28°	0.882	19.5°	47.5°	0.737	8.5°	0.148		280	2.01	18.4
		45	28							0.11			18.2
2	0.209	50	30.5	0.860	10	40.5	0.649	20.5	0.350		435	3.22	14.2
		50	30.5							0.23			14.0
3	0.214	60	35	0.817	10	45	0.707	25	0.423		687	4.73	15.8
		60	35							0.30			15.8
4	0.211	55	33	0.838	13	46	0.719	20	0.342		540	3.86	15.7
		55	33							0.25			15.4
5	0.196	40	25.5	0.904	13	38.5	0.623	12.5	0.216		270	2.24	16.6
		40	25.5							0.14			16.0
6	0.200	60	35	0.817	10	45	0.707	25	0.423		680	5.01	16.7
		60	35							0.30			16.7
7	0.193	50	30.5	0.860	12.5	43	0.682	18	0.309		437	3.51	16.7
		50	30.5							0.21			16.7
8	0.205	60	35	0.817	16.5	51.5	0.783	18.5	0.317		595	4.27	17.2
		60	35							0.25			17.1
9	0.206	30	19.5	0.943	30.5	49.5	0.760	11.0	0.191		1394	11.47	78.1
		30	19.5							0.15			76.5
10	0.206	30	19.5	0.943	28	47.5	0.737	8.5	0.148		1131	9.33	85.4
		30	19.5							0.11			84.8

Note: The second set of data given for each sample is that found by the use of charts.

TABLE 4
Conversion of Birefringence by Destructive Method to $\gamma - \alpha$

Sample No.	Axis Observed	$\cos \alpha_N$			$\cos \alpha_o$			$\cos \theta$	$\theta = \theta'$	K (from chart)	F	$\gamma - \alpha$
		Tilt to Vertical	α_N	$\cos \alpha_N$	V	α_o	$\cos \alpha_o$					
1	α	12°	78°	0.208	19.5°	70.5°	0.334	0.0695	86°	0.99	14.7 18.4	14.8 18.5
2	β	55	55	0.574	10	80	0.174	0.099	84	0.98	14.1	14.4
3	α	27	63	0.454	10	80	0.174	0.079	85.5	0.99	15.8	15.9
4	α	0	90	0							15.8	15.8
5	β	3	3	0.999	13	77	0.225	0.225	77	0.95	15.3	16.1
6	β	2	2	0.999	10	80	0.174	0.174	80	0.97	15.4	15.9
7	β	42	42	0.743	12.5	77.5	0.216	0.160	81	0.97	16.7	17.1
8	β	7	7	0.993	16.5	73.5	0.284	0.282	73.5	0.92	15.6	16.9
9	α	5	85	0.087	30.5	59.5	0.508	0.044	87.5	0.99	70.2	70.2
10	β	2	2	0.999	28	62	0.469	0.468	62	0.78	55.7 70.8	71.4 90.8

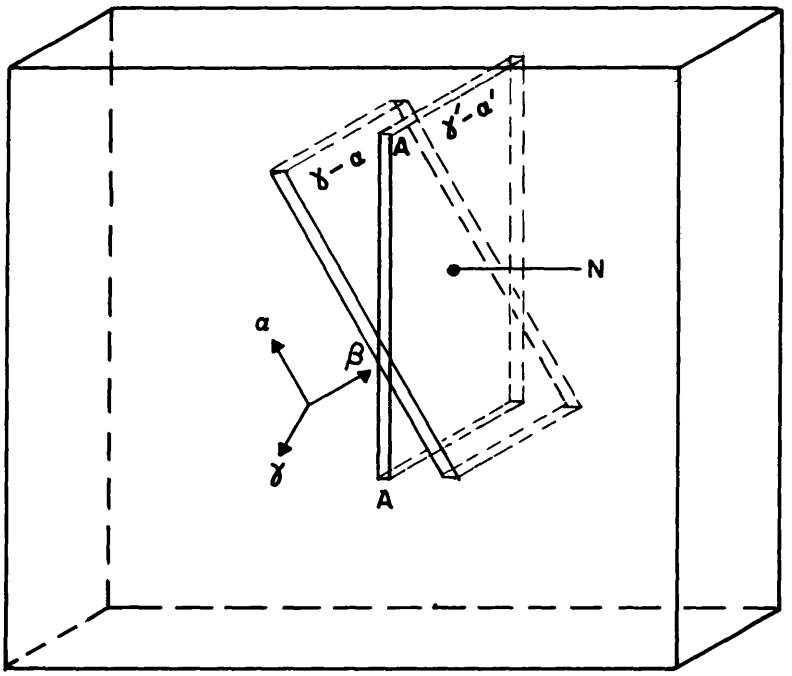

Fig. 4 - Birefringence of a section inclined to the γ - α section

Since we are interested only in birefringence values measured parallel to the surface of the sheet, the normal to the section will always be in the $\alpha\beta$ plane. (See Figure 5.) Therefore

$$\cos \gamma_N = 0 .$$

We also know that the optic axes always lie in the $\gamma\alpha$ plane. Therefore

$$\cos \beta_o = 0 .$$

(See Figure 5.) Hence our equation reduces to

$$\cos \theta = \cos \alpha_o \cos \alpha_N . \tag{8}$$

The optic angle V has been measured between the optic axes and the γ axis. Hence

$$\alpha_o = 90 - V .$$

To find $\cos \alpha_N$, we merely note the inclination to the vertical of the polarizing axes in the surface of the sheet, and the position of the β axis, found in the course of measuring V.

In Table 4, the birefringences directly measured by the destructive method have been corrected to γ - α values. The appropriate charts from Wright's text were used to shorten the calculations. Note that, by symmetry, $\theta = \theta'$ in this case.

It is clear that this correction is usually a small one. The birefringence measured parallel to the surface of the sheet can only vary between the limits γ - β and γ - α. If α and β are nearly equal, as indicated by the smallness of β - α, the birefringence measured

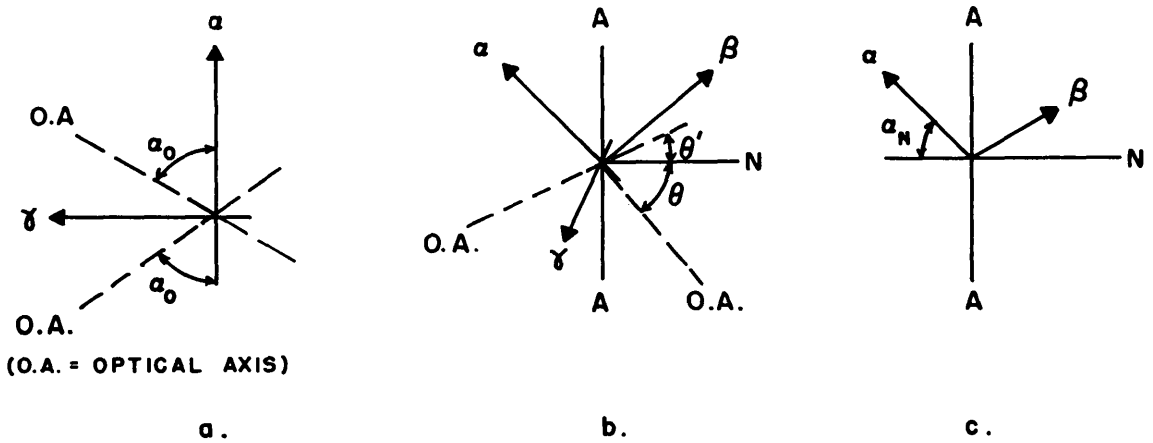

Fig. 5 - Finding the angles θ and θ'

normal to the sheet, then γ - β and γ - α will be nearly equal. In this, the usual case, the above computation will no doubt be of only academic interest. For this reason, it was stated in the introduction to this report that for purposes of correlation with mechanical properties, the γ - α value will usually suffice.

ACKNOWLEDGMENT

It is the pleasure of the author to thank Mr. Roland Kahler, formerly of the Mechanics Division, NRL, for having first suggested to her the use of oblique incidence for solving the problem of nondestructive measurement of the birefringence.

* * *

APPENDIX A
Derivation of Equations for Nondestructive Methods

We determine our rectangular coordinate system as follows: In the sample there are two directions such that the indices of refraction for light polarized parallel to these directions will be a maximum and a minimum. These directions are at right angles to each other. We take these two directions as the Z and X axes; the third, or Y axis, is normal to the X and Z axes. The indices of refraction for light vibrating parallel to these directions are related by a triaxial ellipsoid, the indicatrix, shown graphically in Fig. A.1. In this figure, the semi-axes of the ellipsoid OX, OY, and OZ are equal respectively to α, β, and γ, the three indices of refraction.

The relationship of the triaxial ellipsoid to the biaxially stretched sheet is shown in Fig. A.2. Here the OX and OY axes are parallel to the nominal stretching directions, and the OZ axis is normal to the surface of the sheet, i.e. OZ is along the thickness direction. The problem considered here is the computation of $(\gamma - \alpha)$ from measurements taken with light falling on the surface of the sheet. Let us then incline the specimen to the axis of the polariscope so that the light falls on the specimen at right angles to the OY direction and at some given angle of inclination to the OX and OZ directions. This arrangement was shown in Fig. 3 of the main report, a top view of the sheet in the polariscope. Here the OY axis is vertical, i.e., normal to the plane of the page.

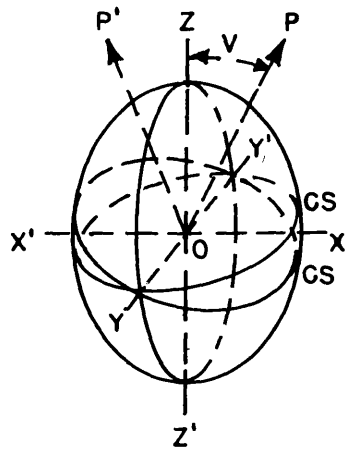

Fig. A.1 - The triaxial ellipsoid

In passing through the specimen, the light will be broken up into two waves, one vibrating parallel to the OY axis with an index of refraction β, and one vibrating in the XZ plane at the angle r to the OX direction with an index of refraction intermediate between α and γ.

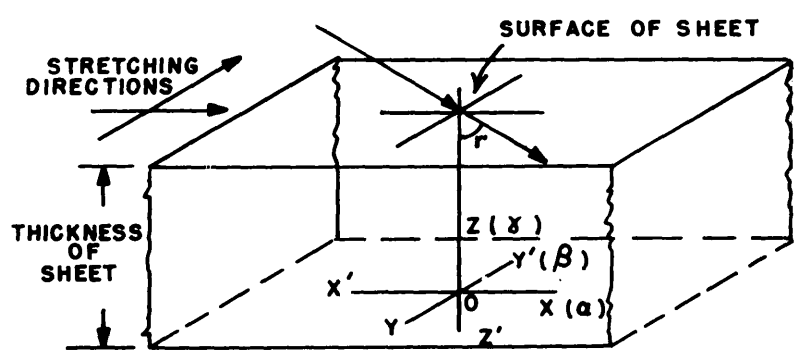

Fig. A.2 - Position of axes in sheet

We must find this second index of refraction in terms of α, γ and r. For this purpose we may confine our attention to the XZ plane of the indicatrix. This section is an ellipse; an enlarged view is shown in Fig. A.3.

The light wave is shown traversing the specimen at the angle r to the Z axis. Its vibration is along OT, at the angle r to the X axis. The index of refraction for the vibration along OT is simply the length of OT. The equation for an ellipse in polar coordinates is

$$r^2 = \frac{a^2b^2}{a^2 \sin^2 \theta + b^2 \cos^2 \theta}.$$

Letting

$$|OT| = \delta$$

and substituting the appropriate quantities, we obtain

$$\delta^2 = \frac{\alpha^2\gamma^2}{\alpha^2 \sin^2 r + \gamma^2 \cos^2 r}. \quad \text{(A.1)}$$

Therefore the birefringence, measured with the specimen in the position of Fig. 3, is

$$\delta - \beta = \frac{\alpha\gamma}{\sqrt{\alpha^2 \sin^2 r + \gamma^2 \cos^2 r}} - \beta. \quad \text{(A.2)}$$

This expression is not usable in practice.

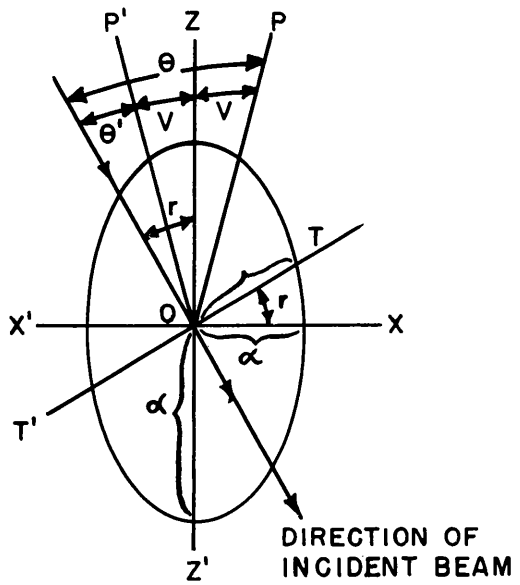

Fig. A.3 - Relationship of δ to γ and α

Simplification results if the specimen is viewed along an optic axis. Two sections of the ellipsoid will be circular; the directions perpendicular to these circular sections are called the optic axes. They are shown in Fig. A.1 as OP and OP'. Light incident along an optic axis will have the same index of refraction for all vibration directions; hence the light will not be doubly refracted. Let V be the angle between the optic axis and OZ, the normal to the sheet.

Since β is intermediate between α and γ, the circular sections must include β. Therefore OP must be perpendicular to OY, and hence OP (and OP') must be in the XZ plane. Taking one vibration along OY, we then have a second vibration in the XZ plane at an angle V to the X axis. The index of refraction for the second vibration must be equal to β. We then have:

$$\delta = \beta = \frac{\alpha\gamma}{\sqrt{\alpha^2 \sin^2 V + \gamma^2 \cos^2 V}}.$$

Squaring, we obtain

$$\beta^2\alpha^2 \sin^2 V + \beta^2\gamma^2 \cos^2 V = \alpha^2\gamma^2.$$

Substituting $(1 - \sin^2 V = \cos^2 V)$, we have

$$\sin^2 V = \frac{\gamma^2(\beta^2 - \alpha^2)}{\beta^2(\gamma^2 - \alpha^2)}. \quad \text{(A.3)}$$

If the birefringence is small, we may write

$$\sin^2 V = \frac{\beta - \alpha}{\gamma - \alpha}$$

or

$$(\gamma - \alpha) = \frac{\beta - \alpha}{(\sin^2 V)} . \tag{A.4}$$

If the effects of hot stretching are equal in both the OX and OY directions, then $\alpha = \beta$, and Eq. (A.4) cannot be used.

We return to Eq. (A.1) and rewrite it in terms of different angles. Call the angles between the direction of the light beam and the optic axes, θ and θ' (Fig. A.3). Then

$$r = V + \theta' = \theta - V.$$

Substitute

$$\sin^2 r = \tfrac{1}{2} - \tfrac{1}{2} \cos 2r = \tfrac{1}{2} - \tfrac{1}{2} \cos (\theta + \theta'),$$

and

$$\cos^2 r = \tfrac{1}{2} + \tfrac{1}{2} \cos 2r = \tfrac{1}{2} + \tfrac{1}{2} \cos (\theta + \theta').$$

Equation (A.1) then reduces to

$$\delta^2 = \frac{\alpha^2\gamma^2}{\frac{1}{2}(\gamma^2 + \alpha^2) + \frac{1}{2}(\gamma^2 - \alpha^2)[\cos (\theta + \theta')]} .$$

Substituting

$$\cos \theta \cos \theta' = \cos (\theta - \theta') - \sin \theta \sin \theta'$$

and

$$\cos (\theta - \theta') = \cos 2V = 1 - 2 \sin^2 V$$

we obtain:

$$\delta^2 = \frac{\alpha^2\gamma^2}{\frac{1}{2}(\gamma^2 + \alpha^2) + \frac{1}{2}(\gamma^2 - \alpha^2)[1 - 2 \sin^2 V - 2 \sin \theta \sin \theta']} .$$

But from Equation (A.3)

$$\sin^2 V = \frac{\gamma^2(\beta^2 - \alpha^2)}{\beta^2(\gamma^2 - \alpha^2)},$$

thus

$$\delta^2 = \frac{\alpha^2\gamma^2}{\gamma^2 - \frac{\gamma^2}{\beta^2}(\beta^2 - \alpha^2) - (\gamma^2 - \alpha^2) \sin \theta \sin \theta'} .$$

Dividing the numerator and denominator by $\alpha^2\gamma^2$

$$\delta^2 = \frac{1}{\frac{1}{\beta^2} - \left(\frac{1}{\alpha^2} - \frac{1}{\gamma^2}\right) \sin\theta \sin\theta'} .$$

Then

$$\frac{1}{\delta^2} - \frac{1}{\beta^2} = \left(\frac{1}{\gamma^2} - \frac{1}{\alpha^2}\right) \sin\theta \sin\theta'.$$

If the birefringence is small, we may write[3]

$$(\delta - \beta) = (\gamma - \alpha) \sin\theta \sin\theta'. \tag{A.5}$$

In the case where $\alpha = \beta$,

$$V = 0,$$

$$\theta = \theta' = r,$$

and Eq. (A.5) reduces to

$$(\delta - \beta) \equiv (\delta - \alpha) = (\gamma - \alpha) \sin^2 r \equiv (\gamma - \beta) \sin^2 r. \tag{A.6}$$

---

[3] This equation is derived in a different manner in Johannsen, "Manual of Petrographic Methods," where it is also shown that the incident beam need not be limited to the *XZ* plane. However, the use of any plane other than the *XZ* will unduly complicate the measurement of angles in the present application.

APPENDIX B
Analysis of Precision of Nondestructive Methods

It is of interest to know the cumulative effect upon the determination of $(\gamma - \alpha)$ of the errors in measurement of the component quantities used in calculation by the nondestructive methods. An analysis of sources of error is also useful for determining which are the controlling sources of error, or, conversely, what precision of measurement of the various components is required to achieve a given precision of the computed birefringence.

For these purposes the probable errors in measurements with equipment in use in this laboratory have been measured using the definition

$$P = 0.675\sqrt{\frac{(x - \bar{x})^2}{n}}$$

where P is the probable error, x is an individual reading, $\bar{x}$ is the mean value of a series of readings, and n is the total number of readings in the series. The probable errors we measured are as follows:

P_d = 1.5 divisions
P_{d_0} = 1.4 divisions
P_K = 1.7 divisions
P_t = 0.0001 inch
$P_i = P_{V'} = 1^\circ$ = 0.0175 rad.

The probable error of index of refraction n was not measured but a value of 0.01 was assumed. No doubt this is large. The optical bench was not devised for high precision measurements; the probable error in i and V' may easily be reduced in other equipment. The error in the measurement of thickness is actually less than 0.0001 inch; however, we shall see that this error does not have a significant effect upon the error in $(\gamma - \alpha)$.

To find the probable error in $(\gamma - \alpha)$, we use the method of the propagation of precision indices.[4] If a quantity, U, is a function of several variables, x, y, z, ..., the probable error in U is related to the probable error of the variables in the following manner:

$$P_u^2 = \left(\frac{\partial U}{\partial x}\right)^2 P_x^2 + \left(\frac{\partial U}{\partial y}\right)^2 P_y^2 + \left(\frac{\partial U}{\partial z}\right)^2 P_z^2.$$

In the following two sections we have applied this method to the equations used in the two nondestructive methods.

It should be pointed out that this treatment does not take into account errors of construction of the instruments, such as irregularity of angular scales.

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[4]Worthing, A. G., and Geffner, J., "Treatment of Experimental Data," New York: John Wiley, 1943

METHOD I

The equation used in Method I is

$$\gamma - \alpha = \beta - \alpha/\sin^2 V.$$

We substitute in this equation the quantities actually measured

$$\gamma - \alpha = \frac{(d - d_0)(n^2)}{Kt \sin^2 V'}.$$

We take the partial derivatives of $(\gamma - \alpha)$ with respect to d, d_0, n, K, t, and V', to find the square of the probable error in $(\gamma - \alpha)$. We then divide by $(\gamma - \alpha)^2$, to obtain the expression for the square of the percentage probable error in $(\gamma - \alpha)$.

$$\frac{(P_{\gamma - \alpha})^2}{(\gamma - \alpha)^2} = \frac{P_d^{\,2} + P_{d_0}^{\,2}}{(d - d_0)^2} + \frac{P_t^{\,2}}{t^2} + \frac{P_K^{\,2}}{K^2} + \frac{4}{n^2} P_n^{\,2} + 4 \operatorname{ctn}^2 V' P_{V'}^{\,2}.$$

As an example, we use the data for sample #3, given in Table II.

$$\frac{(P_{\gamma - \alpha})^2}{(\gamma - \alpha)^2} = \frac{(1.5)^2 + (1.4)^2}{(20)^2} + \frac{(.0001)^2}{(.214)^2} + \frac{(1.7)^2}{(555)^2} + \frac{4(.01)^2}{(1.50)^2} + 4(3.73)^2(.0175)^2,$$

$$\frac{P_{\gamma - \alpha}}{\gamma - \alpha} \times 100 = 17\%.$$

The first and last terms comprise the major portion of this error. If the last term, the error due to measurement of V', could be eliminated completely, the combined error would still be about 11%. This large error is caused by the small magnitude of $(d - d_0)$, which is the measure of $(\beta - \alpha)$. As was pointed out in the main body of this report, Method I is not suitable for use with samples which are very nearly equally biaxially stretched. In contrast, the data for sample 10, where $(\beta - \alpha)$ is quite large, shows a probable error of only 4% for $(\gamma - \alpha)$, for a probable error in V' of 1°.

METHOD II

The equation for Method II is

$$(\gamma - \alpha) = (\delta - \beta)/\sin \theta \sin \theta'.$$

In terms of the quantities directly measured, the equation becomes

$$(\gamma - \alpha) = \frac{(d - d_0)}{Kt} \left(1 - \frac{\sin^2 i}{n^2}\right)^{\frac{1}{2}} \left(\frac{n^2}{\sin^2 i - \sin^2 V'}\right).$$

We again apply the method of the propagation of precision indices, to obtain

$$\frac{(P_{\gamma - \alpha})^2}{(\gamma - \alpha)^2} = \frac{P_d^{\,2} + P_{d_0}^{\,2}}{(D - D_0)^2} + \frac{P_K^{\,2}}{K^2} + \frac{P_t^{\,2}}{t^2} + \left(\frac{2n^2 - \sin^2 i}{n^3 - n \sin^2 i}\right)^2 P_n^{\,2} +$$

$$\frac{(\sin i \cos i)^2 (\sin^2 i + \sin^2 V' - 2n^2)^2}{n^4 (\sin^2 i - \sin^2 V')^2 \left(1 - \frac{\sin^2 i}{n^2}\right)^2} P_i^{\,2} + \frac{4 \sin^2 V' \cos^2 V'}{(\sin^2 i - \sin^2 V')^2} P_v^{\,2}.$$

If we use the probable errors given above, together with the data for sample 3, given in Table 3, we arrive at a probable error in $(\gamma - \alpha)$ of less than 4%. The largest contribution to this error is made by the measurement of i, the angle of incidence. It is reasonable to expect that with equipment allowing more precise angular measurement than ours, the probable error in $(\gamma - \alpha)$ could be reduced to about 2%.

APPENDIX C
Graphical Solutions

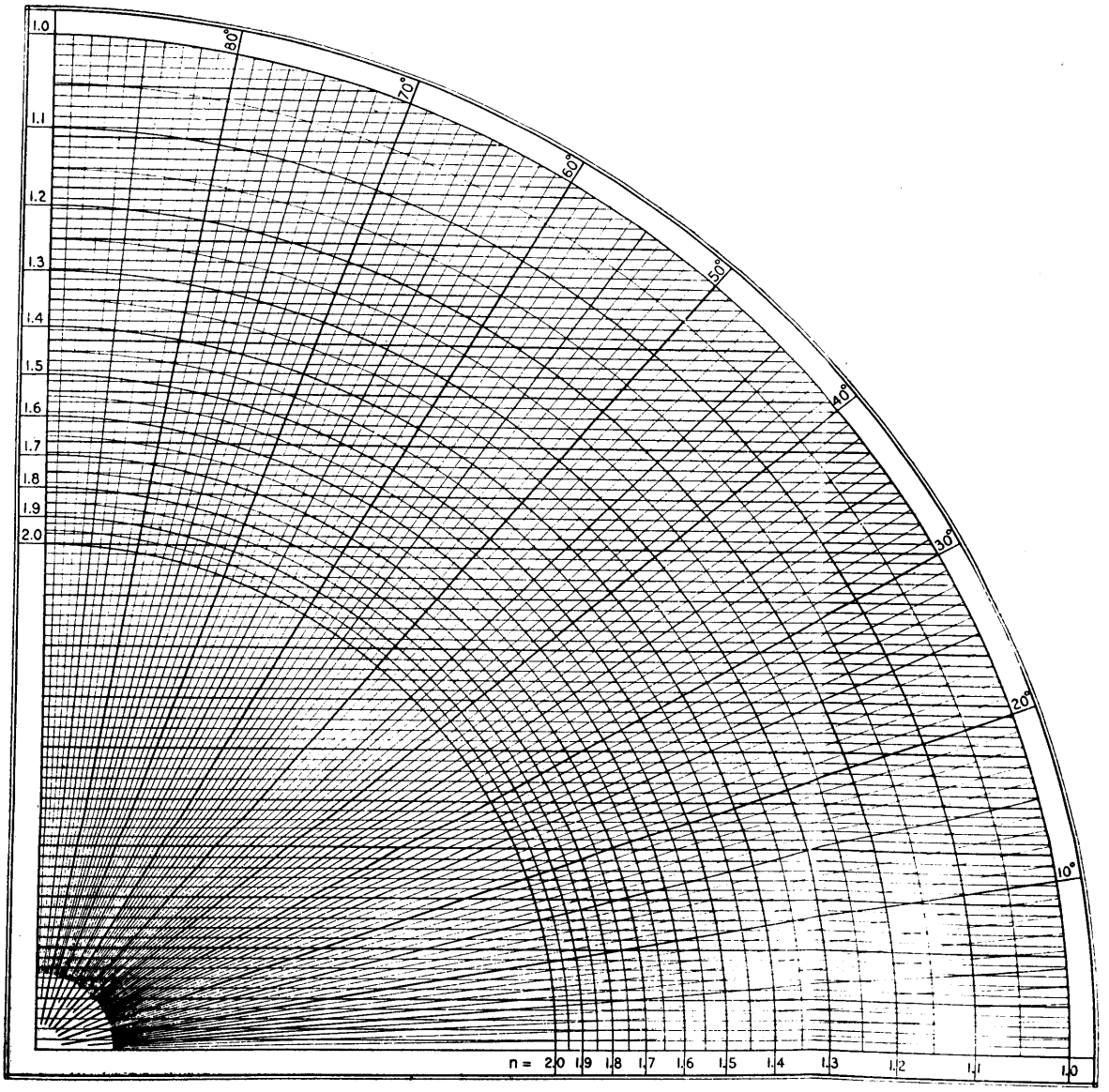

Chart 1 - Graphical solution of the equation sin i = n sin r

In this plate i = angle of incidence, r = angle of refraction, and n = the average index of refraction of any given mineral plate. This plate is used especially in the measurement of optic axial angles for reducing observed angles in air to equivalent angles within the crystal plate. The modus operandi is the following: to find r when n and i are given, pass along the radius i^0 ($i = 58^0$) (for a particular case let $i = 58^0$ and $n = 1.60$) until it intercepts the n circle ($n = 1.60$). Pass from this point along the horizontal line to the intersection with the outer circle at 32^0 which is the desired angle r. In case n and r are given, i can be found by the reverse course of procedure.

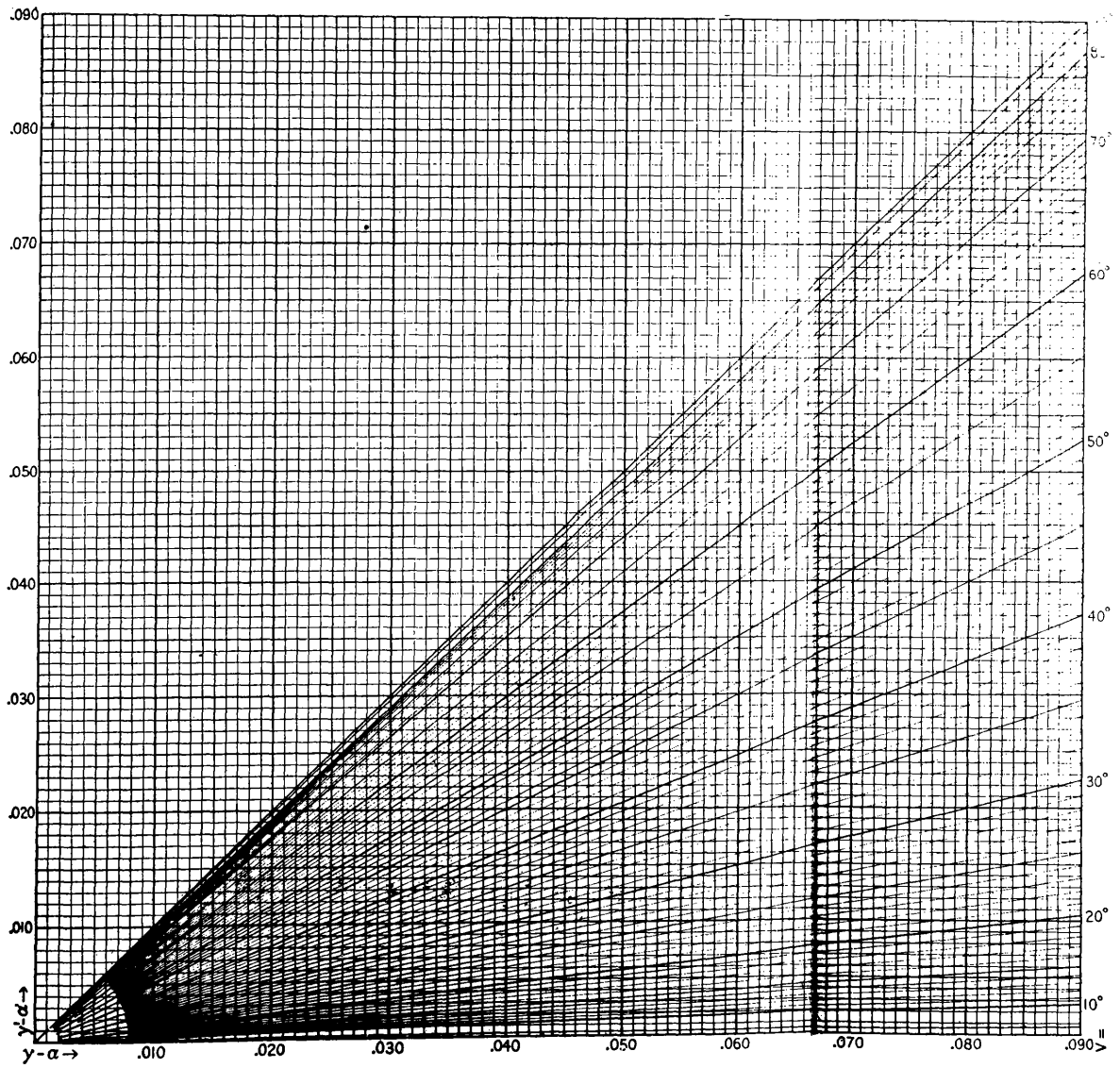

Chart 2 - Graphical solution of the equation $\sin^2 V_c = (\beta-\alpha)/\gamma-\alpha)$

In this equation V_c is the angle between the least ellipsoidal axis c and one of the optic axes: $\beta-\alpha$ and $\gamma-\alpha$, are measures of the relative birefringence of equally thick plates normal to the ellipsoidal axes c and b. In this plate the abscissae represent $\gamma-\alpha$, the ordinates represent $\beta-\alpha$ (indicated in the plate by $\gamma'-\alpha'$ so that $\gamma-\beta$ and V_α may also be used), and the curves represent the angle V_c.

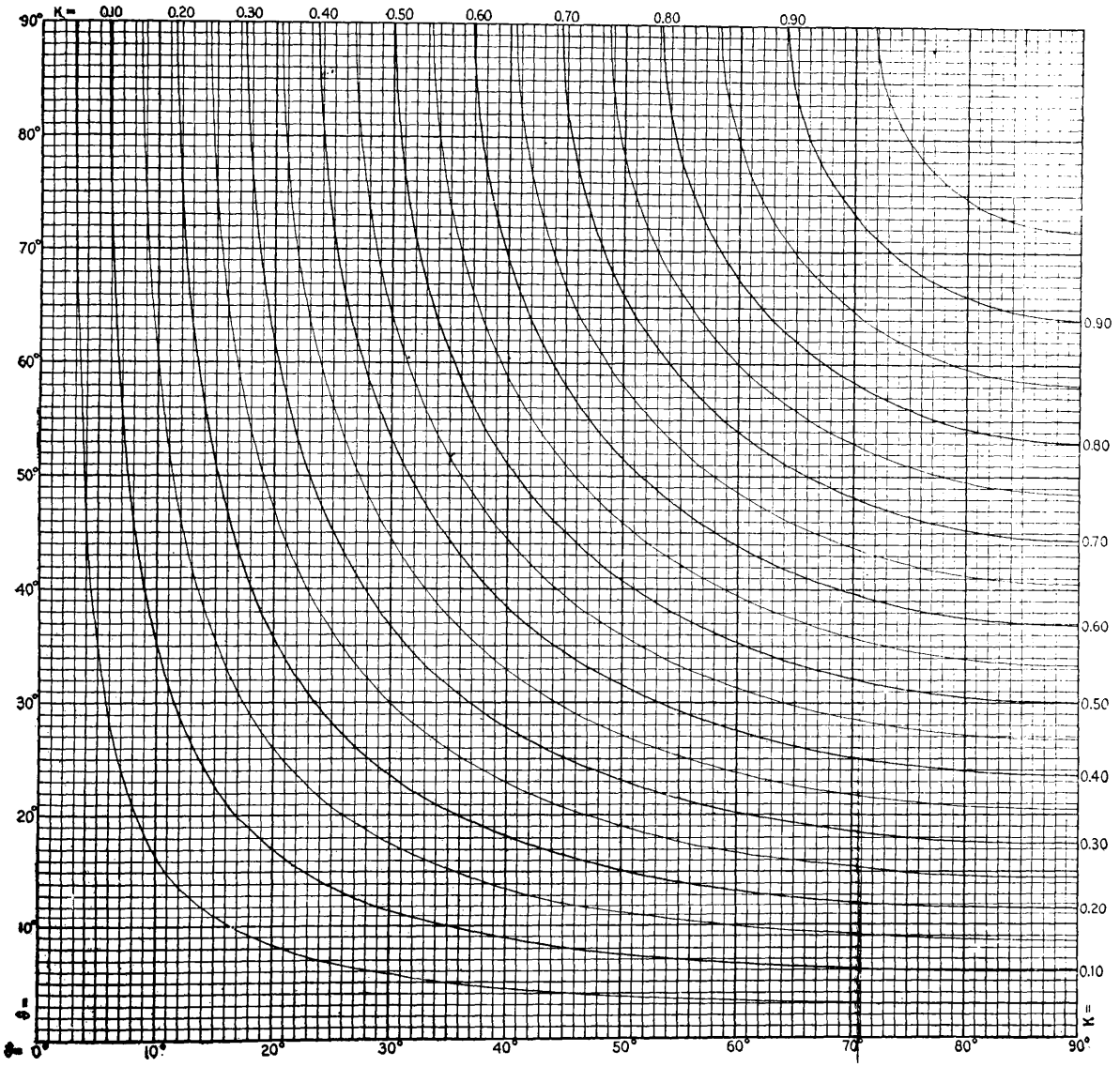

Chart 3 - Graphical solution of the birefringence formula $(\gamma'-\alpha')/(\gamma-\alpha) = \sin\theta \ \sin\theta' = k$

In this equation $(\gamma'-\alpha')/(\gamma-\alpha)$ or k is the ratio of the birefringence of any given section $(\gamma'-\alpha')$ to the maximum birefringence $(\gamma-\alpha)$ of the mineral; θ and θ' are the angles included between the normal to the given section and the two optic axes A_1 and A_2 of the mineral respectively. In this plate the abscissae are the angles θ, the ordinates are the angles θ', and the curves are the percentage birefringence $k = (\gamma'-\alpha')/(\gamma-\alpha)$.

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