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The Magnetic Torque of Ideal Superconducting Toroids

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Abstract: Torsional experiments have been performed at this laboratory with superconducting toroids. In order to discuss the results, knowledge of the expected behavior of an ideal superconducting toroid is needed. Equations which describe this ideal behavior have been found in terms of three geometrically dependent constants, namely, the proportionality constants between components of the applied magnetic field and components of the magnetic moment of the toroid resulting from flux expulsion and any net current. Calculations of these three constants are available for toroids of circular cross sections from other NRL Reports. It is possible to estimate values for other than circular cross sections.

INTRODUCTION

In order to understand more fully the magnetic behavior of superconductors, torsional experiments have been performed with superconducting toroids (1). In these experiments the torque produced by a toroid is measured as a function of both the strength and the direction of an applied magnetic field. Essentially, such measurements determine the magnetic moment of the toroid, **M**, since the torque is given by $\mathbf{T} = \mathbf{M} \times \mathbf{B}$.

For a toroid which exhibits ideal superconducting behavior, two phenomena can produce magnetic moments: a net current around the toroid, and flux expulsion from the volume of the superconductor. Nonideal behavior (such as the phenomenon of trapped flux) might also affect the magnetic moment; however, such nonideal behavior can be discerned only as it produces deviations from the ideal.

MAGNETIC MOMENT OF A SUPERCONDUCTING TOROID

When a constant magnetic field is applied to an ideal superconducting toroid, surface currents are induced which both expel flux from the toroid's volume and conserve flux within the toroid's center opening. The distribution of this surface current depends only upon the superconductor's geometry, while the current strength at any point on the surface is directly proportional to the intensity of the applied field. If the current distribution is known, the magnetic moment can be calculated; however, deriving the current distribution for the toroidal geometry is difficult even for the simplest cases. Solutions are available only for a toroid of circular cross section in a field which is (a) parallel to the axis of the torus (2), and (b) perpendicular to the axis (3).

A general expression for the magnetic moment of a toroid can be written in terms of geometrical constants. Consider a toroid with its axis placed at an angle θ with respect to an applied magnetic field **B** (Fig. 1). The applied field can be broken into two components—one perpendicular and the other parallel to the axis of the toroid. For the parallel component of the magnetic field, magnetic

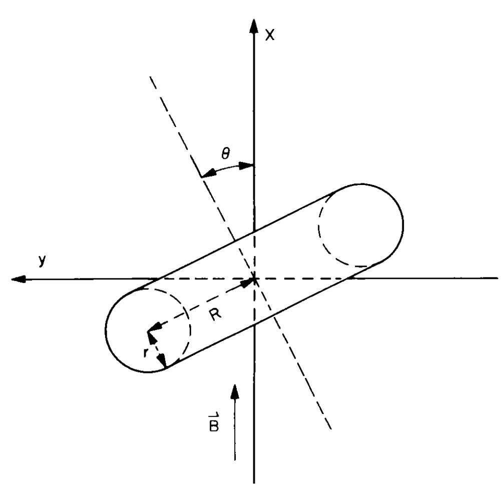


Fig. 1 — Geometry of the toroid

NRL Problem E02-02; Project RR 010-01-44-5601. This is a final report on one phase of the problem; work on the problem is continuing. Manuscript submitted March 28, 1966.

moments arise from any net current (subscript c) and from the flux expulsion or Meissner effect (subscript m). By defining appropriate geometrically dependent proportionality constants, one can write

$$M_m^{||} = -K_m^{||} B^{||}, \quad (1)$$

$$M_c^{||} = -K_c^{||} (B^{||} - B_0). \quad (2)$$

The minus signs preceding the constants indicate a direction which opposes the direction of $B^{||}$, and B_0 is the parallel component of the field in which the toroid was cooled into the superconducting state. The perpendicular component of the magnetic field produces a moment due to only the Meissner effect:

$$M_m^{\perp} = -K_m^{\perp} B^{\perp}. \quad (3)$$

For the coordinate system shown in Fig. 1 it can be shown easily that the total magnetic moment of the toroid is

$$\begin{aligned} \mathbf{M} = & [-K_m^{||} B \cos^2\theta - K_c^{||} (B \cos\theta - B_0) \cos\theta \\ & - K_m^{\perp} B \sin^2\theta] \hat{i} \\ & - [K_m^{||} B \cos\theta \sin\theta + K_c^{||} (B\cos\theta - B_0) \sin\theta \\ & - K_m^{\perp} B \cos\theta \sin\theta] \hat{j}. \end{aligned} \quad (4)$$

MAGNETIC TORQUE OF A SUPERCONDUCTING TOROID

The magnetic torque of the superconducting toroid in the coordinate system of Fig. 1 is given by

$$\mathbf{T} = \mathbf{M} \times \mathbf{B} = -M_y B \hat{k}. \quad (5)$$

From Eqs. (4) and (5) the general expression for the magnetic torque of a toroid is

$$T = [(K_m^{||} + K_c^{||} - K_m^{\perp}) B \cos\theta - B_0 K_c^{||}] B \sin\theta. \quad (6)$$

In a torsional experiment, an equilibrium is attained between the force produced by the toroid and the restoring force provided by a suspension system. Since the force produced by the suspension system presumably is known, the force produced by the toroid is determined. To a very good approximation, the torque of most suspension systems can be written

$$T_s = -c(\theta - \phi), \quad (7)$$

where $\theta - \phi$ is the angular twist in the suspension and c is the torsion constant of the suspension. At equilibrium, $T + T_s$ is zero; therefore,

$$[(K_m^{||} + K_c^{||} - K_m^{\perp}) B \cos\theta - B_0 K_c^{||}] B \sin\theta = c(\theta - \phi). \quad (8)$$

From Eq. (8) the torsional properties of an ideal superconducting toroid can be found if the three K constants are known.

THE K CONSTANTS

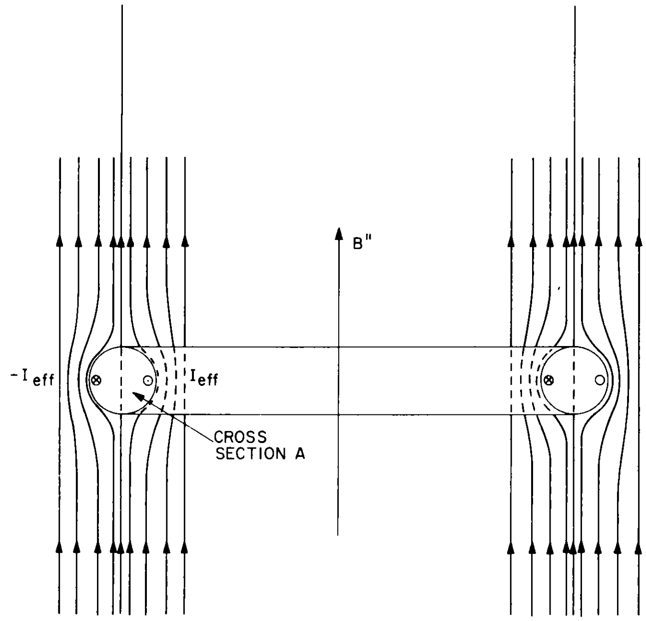
In general, the K constants must be determined by solving the electromagnetic boundary value problem for the surface current distribution and then calculating the magnetic moment. For toroids of circular cross section, $K_m^{||}$ and $K_c^{||}$ are calculated in the report by de Launay (2),* and $K_m^{\perp}$ is calculated by Wright and Peterson (3).

An approximate expression for $K_m^{\perp}$ in terms of $K_m^{||}$ can be found for the toroid of circular cross section. Cooling the torus in the presence of a parallel field $B^{||}$ induces currents around the outer and inner rims of the torus. These induced currents must be equal and opposite in order that they produce no net current. Then this current distribution can be reduced to two concentric current loops having currents $\mp I_{eff}$ and radii $R \pm r$, where, as shown in Fig. 1, R is the mean radius of the torus and r is the radius of the torus' circular cross section. Thus, the magnetic moment due to a field in the parallel direction can be written

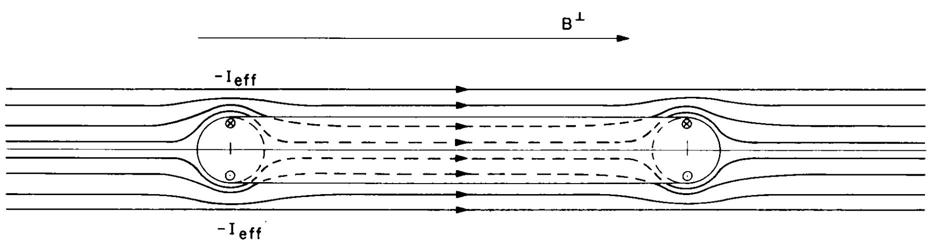
$$M_m^{||} = -K_m^{||} B^{||} = -4\pi r R\, I_{eff}.$$

Now, suppose that the torus is in a perpendicular field, $B^{\perp}$, equal in magnitude to $B^{||}$. When R is substantially larger than r, the field distribution,

*$K_m^{||}$ and $K_c^{||}$ are the proportionality constants for M^I and M^{III} respectively of Ref. 2 and are given by $(4/3)\pi R^3$ times the appropriate value in the table on page 35 of Ref. 2.



(a) Parallel applied field



(b) Perpendicular applied field

Fig. 2 – Field distribution and currents in the xy plane

and thus the current distribution, at cross section A of Fig. 2 will be essentially identical to the field and current distribution produced by $B^{\|}$ except for a 90° rotation. Therefore, the current distribution at A due to a parallel field can be replaced by currents $-I_{eff}$ at the outer surfaces as shown in Fig. 2. The magnetic moment due to $B^{\perp}$ can be written, then,

$$M_m^{\perp} = -K_m^{\perp} B^{\perp} = -2A_{eff}^{\perp} I_{eff},$$

where the area $A_{eff}^{\perp}$ is undetermined and the factor 2 is needed to include both halves of the torus. The problem of determining $A_{eff}^{\perp}$ is not simple; however, as a first approximation, $A_{eff}^{\perp}$ can be taken as the torus' projected area in the perpendicular direction. Then,

$$M_m^{\perp} = -K_m^{\perp} B^{\perp} \approx -8r[R + (\pi r/4)]\ I_{eff}.$$

The ratio of $M_m^{\perp}$ and $M_m^{\|}$ then gives the desired expression

$$\frac{M_m^{\perp}}{M_m^{\|}} = \frac{K_m^{\perp}}{K_m^{\|}} \approx \frac{8r[R + (\pi r/4)]}{4\pi r R}$$

or

$$K_m^{\perp} \approx \frac{2}{\pi}\left(1 + \frac{\pi}{4}\frac{r}{R}\right) K_m^{\|}.$$

As an example, for a torus having a radius ratio of $R/r = 7$, $K_m^{\parallel} = 0.71$ dyne-cm/gauss and $K_m^{\perp} \simeq 0.79\, K_m^{\parallel} = 0.56$ dyne-cm/gauss. The exact value from Ref. 3 is 0.52.

For toroids of other than circular cross section, no calculations are available; however, reasonable estimates of the K constants might be obtained by comparison with toroids of circular cross section. In any case, torsional experiments allow only $K_c^{\parallel}$ and the difference between $K_m^{\parallel}$ and $K_m^{\perp}$ to be determined. The latter is true because $K_m^{\parallel}$ and $K_m^{\perp}$ are linearly dependent in Eq. (8). If it is desirable to determine $K_m^{\parallel}$ and $K_m^{\perp}$ independently, an additional relationship between them must be found. In come cases, it might be possible to obtain such a relationship by considering projected areas as described for the circular-cross-section case.

SUMMARY

General expressions have been found for the magnetic moment, torque, and torsion of ideal superconducting toroids in terms of three geometrically determined constants. These constants have been discussed in detail for the toroid of circular cross section, and suggestions for handling toroids of other than circular cross section have been given.

ACKNOWLEDGMENT

I wish to thank J. de Launay for a very helpful discussion of the superconducting toroid problem.

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3. Wright, E.B., and Peterson, P.L., "Potential and Magnetic Moment of a Superconducting Torus in a Nonaxial Uniform Magnetic Field, NRL Report 6429 (in preparation)

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