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Normal Mode Theory for Three-Directional Motion

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ABSTRACT

Normal mode theory is applied to undamped linear elastic structures represented as lumped parameter systems undergoing translational motion in three directions. The derived equations are primarily concerned with the response of such structures subject to applied forces and base motions and the inertia forces required to calculate stress in each mode of vibration. Additional relationships are presented for special types of loading and for the effective mass acting in a given mode due to base motion. Similar equations are summarized in an appendix for structures with six directions of motion, namely, three translational directions and three rotational directions subject to prescribed assumptions.

PROBLEM STATUS

This is an interim report on one phase of the problem; work is continuing on this and other phases.

AUTHORIZATION

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SYMBOLS

A dot over a symbol in the text indicates differentiation with respect to time.

$D_a^r(t)$	Duhamel integral for base motion, direction r
$F_i^r(t)$	applied force acting on m_i, direction r
F_{ia}^r	inertial force plus applied force at m_i in mode a, direction r
I_i^r	impulse applied at m_i, direction r
M	total mass of a structure
M_a^r	apparent mass in mode a, direction r
P_a^r	participation factor in mode a, direction r
Q_{ia}^r	inertia force acting on m_i in the mode a, direction r
V_0^r	velocity step, direction r
$\underline{X}_i^r$	absolute displacement of m_i, direction r
X_i^r	relative displacement between m_i and the base, direction r
X_{ia}^r	relative displacement between m_i and the base, for mode a, direction r
$\overline{X}_{ia}^r$	normal mode shape for mode a, direction r
Z^r	base motion, direction r
m_i	ith mass
$q_a(t)$	time function for displacement
t	time
δ_{ij}^{rs}	influence coefficient
ω_a	natural frequency of mode a for an undamped multi-degree-of-freedom system.

NORMAL MODE THEORY FOR THREE-DIRECTIONAL MOTION

INTRODUCTION

In recent years normal mode theory has become more widely used and accepted as a tool for structural design and analysis. While the theory has been presented for unidirectional motion by earlier works, including NRL reports (1-3), it was felt necessary to extend the theory to structures undergoing translational motion in three dimensions. While no claim is made to originality of the essential contents of the report, many steps are included which often are not published in works dealing with the subject.

The primary concern of this report is to find the motions and inertia forces for calculating stresses of undamped linear elastic structures which are idealized as lumped parameter systems capable of undergoing translational motion in three directions. The derivation of the equations is deliberately limited in the use of mathematical methods to those which are no more complex than necessary. While this report is self-contained, Ref. 3 is especially recommended as reading material to precede this report.

NOTATION

Figure 1 shows the orientation of mass m_i with relation to the possible motions of the base of the structure. The $\bar{X}$ axes, which describe the absolute motion of a mass, are parallel to the Z axes, associated with base motions. Note that the origin of the Z axes is not located at a particular point; hence no loss of generality is made if the 1, 2, and 3 directions refer respectively to the longitudinal, vertical, and athwartships directions of a ship. It is assumed that m_i is a point mass with no rotational inertia and that there are n mass points representing the structure.

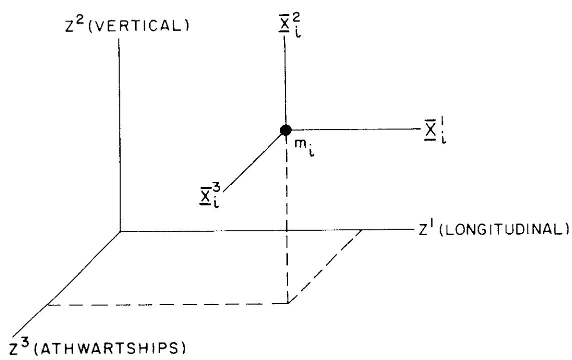

Fig. 1 - Orientation of mass m_i with relation to the orientation of a ship for the case of three-directional motion

Representation of Displacements and Forces

The general rule of notation for displacements and forces is as follows:

A subscript refers to the mass point and the superscript refers to the direction of the displacement or force. For example, $\bar{X}_j^r$ represents the absolute displacement of m_j in the rth direction; F_i^s represents the force applied at m_i in the s direction. Note that $i, j = 1, 2, \ldots, n$ while $r, s = 1, 2, 3$.

Raising a Quantity to a Power

The general rule of notation for raising a quantity to a power is as follows: Always place the quantity inside brackets before raising to the power. For example, to square ω_a the rule requires $(\omega_a)^2$.

Summations

Unless otherwise indicated, all summations on i, j, and k are taken from 1 to n. For example,

$$\sum_i = \sum_{i=1}^{n} \cdot$$

All summations on a are taken from 1 to $3n$. For example,

$$\sum_a = \sum_{a=1}^{3n} \cdot$$

All summations on p, r, and s are taken from 1 to 3. For example,

$$\sum_r = \sum_{r=1}^{3} \cdot$$

All double summations are abbreviated in the following form:

$$\sum_{j,r} = \sum_j \sum_r = \sum_{j=1}^{n} \sum_{r=1}^{3} \cdot$$

Influence Coefficients

The influence coefficient δ_{ij}^{rs} reads as follows: The deflection at i in the rth direction due to a unit force at j in the s direction. Thus, if a static force F_i^s is applied to a linear elastic structure which is attached to an immovable base, the deflection due to distortion of any point j on the structure in direction r is given by the relationship

$$\underline{\overline{X}}_j^r = X_j^r = \sum_s \delta_{ji}^{rs} F_i^s \, .$$

For applied forces at each mass point of the structure, this becomes

$$X_j^r = \sum_{i,s} \delta_{ji}^{rs} F_i^s \, . \tag{1}$$

Appendix A shows the influence coefficient written out in the form to be used for finding natural frequencies and normal mode shapes by the iteration method (4).

For linear elastic structures, Maxwell's law of reciprocal deflections (5) holds, namely, $\delta_{ij}^{rs} = \delta_{ji}^{sr}$.

FREE VIBRATIONS

Normal Modes

Assume that a weightless structure attached to a fixed base is carrying a set of n concentrated masses which are attached at the n points i. Consider its free vibrations, that is, the possible motions in the absence of external forces. This is done by D'Alembert's principle, which states that a system in motion can be considered to be in equilibrium at any instant if appropriate inertia forces $-m_i\ddot{\underline{X}}_i^r$ are applied to the system. For the case of the freely vibrating structure, simply apply these inertia forces so as to view the structure as being in a state of equilibrium. The set of forces on the structure is now treated as a problem of statics.

Recall that for an elastically distorted structure in equilibrium

$$X_j^r = \sum_{i,s} \delta_{ji}^{rs} F_i^s . \tag{1}$$

For free vibrations the only forces on the structure are the inertia forces, so

$$X_j^r = - \sum_{i,s} \delta_{ji}^{rs} m_i \ddot{\underline{X}}_i^s . \tag{2}$$

This is a set of $3n$ differential equations with constant coefficients expressing the X_i's in terms of the $\ddot{\underline{X}}_j$'s. Since there is no base motion, $\underline{X}_j^r = X_j^r$ and $\ddot{\underline{X}}_j^r = \ddot{X}_j^r$. Equation (2) is rewritten

$$X_j^r = - \sum_{i,s} \delta_{ji}^{rs} m_i \ddot{X}_i^s . \tag{3}$$

To obtain a solution try $X_j^r = \overline{X}_j^r \sin(\omega t + \beta)$, which is usually done for the single-degree-of-freedom system. Then

$$\overline{X}_j^r = (\omega)^2 \sum_{i,s} \delta_{ji}^{rs} m_i \overline{X}_i^s . \tag{4}$$

Equation (4) consists of three sets of n algebraic equations which are written out over the range on r as follows:

$$\overline{X}_j^1 = (\omega)^2 \sum_i \left(\delta_{ji}^{11} m_i \overline{X}_i^1 + \delta_{ji}^{12} m_i \overline{X}_i^2 + \delta_{ji}^{13} m_i \overline{X}_i^3\right)$$

$$\overline{X}_j^2 = (\omega)^2 \sum_i \left(\delta_{ji}^{21} m_i \overline{X}_i^1 + \delta_{ji}^{22} m_i \overline{X}_i^2 + \delta_{ji}^{23} m_i \overline{X}_i^3\right)$$

$$\overline{X}_j^3 = (\omega)^2 \sum_i \left(\delta_{ji}^{31} m_i \overline{X}_i^1 + \delta_{ji}^{32} m_i \overline{X}_i^2 + \delta_{ji}^{33} m_i \overline{X}_i^3\right) .$$

These equations can be further written out as $3n$ algebraic equations. Appendix A shows these equations written in matrix form.

If a solution is to exist other than the trivial one where all the $\overline{X}_j^r$'s equal zero (static equilibrium case), it occurs only for those values of ω which make the determinant of the coefficients of the $\overline{X}_j^r$'s equal to zero (6). This leads to an algebraic equation of degree $3n$ in $(\omega)^2$ usually called the frequency equation. Since undamped structures are considered, these roots are real and positive (6). These frequencies are called the fixed base natural frequencies of the system oscillating in the absence of external forces. For the systems where the roots of $(\omega)^2$ are all distinct, the ratio of amplitudes of the masses can be found by the back substitution solution of the set of equations, which set is defined by

$$\overline{X}_{ja}^r = (\omega_a)^2 \sum_{i,s} \delta_{ji}^{rs} m_i \overline{X}_{ia}^s . \tag{5}$$

The $\overline{X}_{ja}^{r}$ are called the normal mode shapes and are defined by Eq. (5) for each mode a in each direction r.

Those systems which have a pair or more of equal roots are called degenerate systems. Other techniques for solving such a set of equations treat them as an eigenvalue-eigenvector problem, which is a characteristic value problem with latent roots. For the degenerate systems, back substitution in Eq. (5) does not produce the set of mode shapes. Other techniques such as matrix deflation or special forms of adjoint matrices can be used. It is assumed that these mode shapes can be found in order to proceed.

Orthogonality of the Normal Modes

To establish the orthogonality of the normal mode shapes, multiply both sides of Eq. (5) by $m_j \overline{X}_{jb}^{r}$ and sum on j and r. This gives

$$\sum_{j,r} m_j \overline{X}_{jb}^{r} \overline{X}_{ja}^{r} = (\omega_a)^2 \sum_{j,r} m_j \overline{X}_{jb}^{r} \sum_{i,s} \delta_{ji}^{rs} m_i \overline{X}_{ia}^{s}$$

$$= (\omega_a)^2 \sum_{i,s} m_i \overline{X}_{ia}^{s} \sum_{j,r} \delta_{ij}^{sr} m_j \overline{X}_{jb}^{r} \tag{6}$$

since $\delta_{ij}^{rs} = \delta_{ji}^{sr}$. Also,

$$\overline{X}_{ib}^{s} = (\omega_b)^2 \sum_{j,r} \delta_{ij}^{sr} m_j \overline{X}_{jb}^{r}$$

by Eq. (5). Equation (6) becomes

$$\sum_{j,r} m_j \overline{X}_{jb}^{r} \overline{X}_{ja}^{r} = \left(\frac{\omega_a}{\omega_b}\right)^2 \sum_{i,s} m_i \overline{X}_{ia}^{s} \overline{X}_{ib}^{s} .$$

Since i and j are now dummy subscripts as well as r and s,

$$\left[1 - \left(\frac{\omega_a}{\omega_b}\right)^2\right] \sum_{j,r} m_j \overline{X}_{jb}^{r} \overline{X}_{ja}^{r} = 0 .$$

There are two possible cases: $b = a$, or $b \neq a$. When $b = a$ the term in the brackets becomes zero and the summation becomes

$$\sum_{j,r} m_j \left(\overline{X}_{ja}^{r}\right)^2 .$$

This is a series of positive terms which cannot be zero. When $b \neq a$, the term in the brackets is not zero, so that the summation term must be zero. This yields the orthogonality conditions

$$\sum_{j,r} m_j \left(\overline{X}_{ja}^{r}\right)^2 \neq 0 \tag{7}$$

$$\sum_{j,r} m_j \overline{X}_{jb}^{r} \overline{X}_{ja}^{r} = 0 , \quad a \neq b . \tag{8}$$

Note that these orthogonality relationships include a double summation, that is, the usual summation of all mass points as experienced in the unidirectional system and the additional summation over the three possible directions of motion.

Type of Normal Mode Solution

The distortion of the structure is completely described if the set of X_i^r's is found. Let the time mode response at point i be X^r_{ia}. The total response X_i^r can be found by superposition, that is,

$$X_i^r = \sum_a X^r_{ia} .$$

At each i in each mode a there is a relative amplitude of $\overline{X}^r_{ia}$. There must be a function which converts the $\overline{X}^r_{ia}$ to X^r_{ia}. That is, a solution will be sought in the form

$$X^r_{ia} = \overline{X}^r_{ia} \, q_a$$

so that

$$X_i^r = \sum_a \overline{X}^r_{ia} \, q_a \tag{9}$$

and

$$\ddot{X}_i^r = \sum_a \overline{X}^r_{ia} \, \ddot{q}_a . \tag{10}$$

If q_a is found, the free vibration problem is solved.

Substitution of Eqs. (9) and (10) into Eq. (3) yields

$$\sum_a \overline{X}^r_{ja} \, q_a = - \sum_{i,s} \delta^{rs}_{ji} \, m_i \sum_a \overline{X}^s_{ia} \, \ddot{q}_a .$$

By transposition

$$\sum_a \left(\ddot{q}_a \sum_{i,s} \delta^{rs}_{ji} \, m_i \, \overline{X}^s_{ia} + \overline{X}^r_{ja} \, q_a \right) = 0 ,$$

and by use of Eq. (5) this becomes

$$\sum_a \left[\frac{\ddot{q}_a}{(\omega_a)^2} + q_a \right] \overline{X}^r_{ja} = 0 .$$

Multiplication of both sides by $m_j \, \overline{X}^r_{jb}$ and summation over j and r yields

$$\sum_a \left[\frac{\ddot{q}_a}{(\omega_a)^2} + q_a \right] \sum_{j,r} m_j \, \overline{X}^r_{jb} \, \overline{X}^r_{ja} = 0 .$$

There is only one case when the summation over j and r is not equal to zero: when $a = b$. The summation over the modes is then reduced to

$$\ddot{q}_a + (\omega_a)^2 \, q_a = 0 . \tag{11}$$

This has the solution

$$q_a = q_a(0) \cos \omega_a t + \frac{\dot{q}_a(0)}{\omega_a} \sin \omega_a t . \tag{12}$$

Returning to Eq. (9),

$$X_i^r = \sum_a \bar{X}_{ia}^r q_a(0) \cos \omega_a t + \sum_a \bar{X}_{ia}^r \frac{\dot{q}_a(0)}{\omega_a} \sin \omega_a t . \tag{13}$$

Initial Conditions

Assume that the general initial conditions at $t = 0$ are $X_i^r = X_i^r(0)$ and $\dot{X}_i^r = \dot{X}_i^r(0)$. Equation (13) yields

$$X_i^r(0) = \sum_a \bar{X}_{ia}^r q_a(0) . \tag{14}$$

Upon differentiating Eq. (13) and introducing the initial condition on velocity, this yields

$$\dot{X}_i^r(0) = \sum_a \bar{X}_{ia}^r \dot{q}_a(0) . \tag{15}$$

The orthogonality relationship can now be used by multiplying both sides of Eqs. (14) and (15) by $m_i \bar{X}_{ib}^r$ and summing on i and r:

$$\sum_{i,r} m_i \bar{X}_{ib}^r X_i^r(0) = \sum_a q_a(0) \sum_{i,r} m_i \bar{X}_{ib}^r \bar{X}_{ia}^r$$

$$\sum_{i,r} m_i \bar{X}_{ib}^r \dot{X}_i^r(0) = \sum_a \dot{q}_a(0) \sum_{i,r} m_i \bar{X}_{ib}^r \bar{X}_{ia}^r .$$

Therefore,

$$q_a(0) = \frac{\sum_{i,r} m_i \bar{X}_{ia}^r X_i^r(0)}{\sum_{i,r} m_i \left(\bar{X}_{ia}^r\right)^2} \tag{16}$$

$$\dot{q}_a(0) = \frac{\sum_{i,r} m_i \bar{X}_{ia}^r \dot{X}_i^r(0)}{\sum_{i,r} m_i \left(\bar{X}_{ia}^r\right)^2} . \tag{17}$$

Substitution of Eqs. (16) and (17) into Eq. (13) yields the complete normal mode solution for free vibrations:

$$X_i^r = \sum_a \frac{\bar{X}_{ia}^r \sum_{j,s} m_j \bar{X}_{ja}^s X_j^s(0)}{\sum_{j,s} m_j \left(\bar{X}_{ja}^s\right)^2} \cos \omega_a t + \sum_a \frac{\bar{X}_{ia}^r \sum_{j,s} m_j \bar{X}_{ja}^s \dot{X}_j^s(0)}{\omega_a \sum_{j,s} m_j \left(\bar{X}_{ja}^s\right)^2} \sin \omega_a t . \tag{18}$$

RESPONSE TO AN APPLIED FORCE

Consider a structure which rests on an immovable base, and suppose a force $F_k{}^r$, applied to m_k, is time dependent but independent of structural reaction. Using D'Alembert's principle and influence coefficients, the distortion of the structure is described by the $3n$ equations

$$X_j^{\,r} = -\sum_{i,s} \delta_{ji}^{rs}\, m_i\, \ddot{X}_i^{\,s} + \sum_s \delta_{jk}^{rs}\, F_k^{\,s}. \tag{19}$$

A solution of the form

$$X_j^{\,r} = \sum_a \bar{X}_{ja}\, q_a \tag{9}$$

is sought. Substituting Eq. (9) into Eq. (19) yields

$$\sum_a \bar{X}^r_{ja}\, q_a = -\sum_a \ddot{q}_a \sum_{i,s} \delta_{ji}^{rs}\, m_i\, \bar{X}^s_{ia} + \sum_s \delta_{jk}^{rs}\, F_k^{\,s}.$$

Transposing,

$$\sum_a \left(q_a\, \bar{X}^r_{ja} + \ddot{q}_a \sum_{i,s} \delta_{ji}^{rs}\, m_i\, \bar{X}^s_{ia}\right) = \sum_s \delta_{jk}^{rs}\, F_k^{\,s}.$$

Using Eq. (5) this may be written as

$$\sum_a \left[\frac{\ddot{q}_a}{(\omega_a)^2} + q_a\right] \bar{X}^r_{ja} = \sum_s \delta_{jk}^{rs}\, F_k^{\,s}. \tag{20}$$

Consider expanding the expression on the right side of Eq. (20) into a series of mode shapes. Let

$$\sum_s \delta_{jk}^{rs} F_k^{\,s} = \sum_a \bar{X}^r_{ja} \sum_p \Delta^p_{ka}. \tag{21}$$

Multiplying both sides by $m_j\, \bar{X}^r_{jb}$ and summing on j and r yields

$$\sum_s \left(\sum_{j,r} m_j\, \bar{X}^r_{jb}\, \delta^{rs}_{jk}\right) F_k^{\,s} = \sum_{a,p} \Delta^p_{ka} \left(\sum_{j,r} m_j\, \bar{X}^r_{jb}\, \bar{X}^r_{ja}\right).$$

Using $\delta^{rs}_{jk} = \delta^{sr}_{kj}$,

$$\sum_s \left(\sum_{j,r} \delta^{sr}_{kj}\, m_j\, \bar{X}^r_{jb}\right) F_k^{\,s} = \sum_{a,p} \Delta^p_{ka} \left(\sum_{j,r} m_j\, \bar{X}^r_{jb}\, \bar{X}^r_{ja}\right). \tag{22}$$

The left side is reduced by Eq. (5), that is,

$$\sum_{j,r} \delta^{sr}_{kj}\, m_j\, \bar{X}^r_{jb} = \frac{\bar{X}^s_{kb}}{(\omega_b)^2}.$$

The summation over the modes in the right side of Eq. (22) reduces to simply a by virtue of orthogonality. Thus, Eq. (22) is rewritten

$$\sum_s \frac{\overline{X}^s_{ka} F_k^s}{(\omega_a)^2} = \sum_p \Delta^p_{ka} \sum_{j,r} m_j \left(\overline{X}^r_{ja}\right)^2 .$$

Therefore,

$$\sum_p \Delta^p_{ka} = \frac{\sum_s \overline{X}^s_{ka} F_k^s}{(\omega_a)^2 \sum_{j,r} m_j \left(\overline{X}^r_{ja}\right)^2} . \tag{23}$$

Equation (21) becomes

$$\sum_s \delta^{rs}_{jk} F_k^s = \sum_a \frac{\overline{X}^r_{ja} \sum_s \overline{X}^s_{ka} F_k^s}{(\omega_a)^2 \sum_{i,p} m_i \left(\overline{X}^p_{ia}\right)^2} .$$

Since each component of F_k^s is independent, that is, the magnitudes F_k^1, F_k^2, and F_k^3 are separate and independent of each other, an expansion in s on each side of the equation leads to

$$\delta^{rs}_{jk} = \sum_a \frac{\overline{X}^r_{ja} \overline{X}^s_{ka}}{(\omega_a)^2 \sum_{i,p} m_i \left(\overline{X}^p_{ia}\right)^2} . \tag{24}$$

This defines the influence coefficient in terms of the normal mode properties of the structure for translational motion in three directions.

Substituting Eq. (24) into Eq. (20) yields

$$\sum_a \left[\frac{\ddot{q}_a}{(\omega_a)^2} + q_a\right] \overline{X}^r_{ja} = \sum_a \frac{\overline{X}^r_{ja} \sum_s \overline{X}^s_{ka} F_k^s}{(\omega_a)^2 \sum_{i,p} m_i \left(\overline{X}^p_{ia}\right)^2} .$$

Transposing,

$$\sum_a \left[\frac{\ddot{q}_a}{(\omega_a)^2} + q_a - \frac{\sum_s \overline{X}^s_{ka} F_k^s}{(\omega_a)^2 \sum_{i,p} m_i \left(\overline{X}^p_{ia}\right)^2}\right] \overline{X}^r_{ja} = 0 .$$

Now the orthogonality relationship is applied. Thus,

$$\sum_a \left[\frac{\ddot{q}_a}{(\omega_a)^2} + q_a - \frac{\sum_s \overline{X}^s_{ka} F_k^s}{(\omega_a)^2 \sum_{i,p} m_i \left(\overline{X}^p_{ia}\right)^2}\right] \sum_{j,r} m_j \overline{X}^r_{ja} \overline{X}^r_{jb} = 0 .$$

Therefore,

$$\ddot{q}_a + (\omega_a)^2 q_a = \frac{\sum_s \bar{X}_{ka}^s F_k^s}{\sum_{j,r} m_j \left(\bar{X}_{ja}^r\right)^2} . \tag{25}$$

Equation (25) is in the form of the equation of motion for a single-degree-of-freedom system, thus having separated each normal mode. The particular solution is written by sight, using the Duhamel integral form for the single-degree-of-freedom system. Thus

$$q_a = \frac{1}{\omega_a \sum_{j,r} m_j \left(\bar{X}_{ja}^r\right)^2} \int_0^t \left[\sum_s \bar{X}_{ka}^s F_k^s(T)\right] \sin \omega_a (t - T)\, dT .$$

The desired solution is then

$$X_j^r = \sum_a \frac{\bar{X}_{ja}^r}{\omega_a \sum_{i,p} m_i \left(\bar{X}_{ia}^p\right)^2} \int_0^t \left[\sum_s \bar{X}_{ka}^s F_k^s(T)\right] \sin \omega_a (t - T)\, dT . \tag{26}$$

If more than one force is applied at different points throughout the structure at the same time, say d points, superposition is used to solve the problem. Since the derivation assumed the force to be applied at m_k, sum the d applied forces. In this case the particular solution is

$$X_j^r = \sum_a \frac{\bar{X}_{ja}^r}{\omega_a \sum_{i,p} m_i \left(\bar{X}_{ia}^p\right)^2} \int_0^t \left[\sum_{k=1}^{d} \sum_s \bar{X}_{ka}^s F_k^s(T)\right] \sin \omega_a (t - T)\, dT . \tag{27}$$

To find the general solution add the complementary solution represented by Eq. (18) to the particular solution of Eqs. (26) or (27).

RESPONSE TO BASE MOTION

Suppose a structure initially at rest is attached to some base. Assume that this undergoes a translational motion $Z^r(t)$ which is a known time-dependent function.

Consider the equations of an elastically distorted structure:

$$X_i^r = \sum_{j,s} \delta_{ij}^{rs} F_j^s .$$

Using D'Alembert's principle, this becomes

$$X_i^r = - \sum_{j,s} \delta_{ij}^{rs} m_j \ddot{\underline{X}}_j^s . \tag{28}$$

Since $X_i^r = \underline{X}_i^r - Z^r$, Eq. (28) is written

$$X_i^r = - \sum_{j,s} \delta_{ij}^{rs} m_j \left(\ddot{X}_j^s + \ddot{Z}^s\right) \tag{29}$$

where $\ddot{X}_j^r$ represents the components of relative acceleration. Let

$$x_i^r = \sum_a \bar{X}_{ia}^r q_a$$

and substitute this into Eq. (29):

$$\sum_a \bar{X}_{ia}^r q_a = - \sum_{j,s} \delta_{ij}^{rs} m_j \sum_a \bar{X}_{ja}^s \ddot{q}_a - \sum_{j,s} \delta_{ij}^{rs} m_j \ddot{Z}^s . \tag{30}$$

With use of Eq. (24), the last term of Eq. (30) can be expanded in its normal modes, that is,

$$\sum_{j,s} \delta_{ij}^{rs} m_j \ddot{Z}^s = \sum_a \frac{\bar{X}_{ia}^r \sum_{j,s} m_j \bar{X}_{ja}^s \ddot{Z}^s}{(\omega_a)^2 \sum_{j,p} m_j \left(\bar{X}_{ja}^p\right)^2} .$$

Substituting this into Eq. (30) and rearranging terms leads to

$$\sum_a \left[\frac{\ddot{q}_a}{(\omega_a)^2} + q_a + \frac{\sum_{j,s} m_j \bar{X}_{ja}^s \ddot{Z}^s}{(\omega_a)^2 \sum_{j,p} m_j \left(\bar{X}_{ja}^p\right)^2} \right] \bar{X}_{ia}^r = 0 .$$

The orthogonality conditions give

$$\ddot{q}_a + (\omega_a)^2 q_a = - \frac{\sum_{j,s} m_j \bar{X}_{ja}^s \ddot{Z}^s}{\sum_{j,p} m_j \left(\bar{X}_{ja}^p\right)^2} . \tag{31}$$

This equation is in the form of the equation of relative motion for a single-degree-of-freedom system if there is a base motion and no applied force. The particular solution of Eq. (31) is

$$q_a = - \frac{1}{\omega_a \sum_{j,p} m_j \left(\bar{X}_{ja}^p\right)^2} \int_0^t \left[\sum_{j,s} m_j \bar{X}_{ja}^s \ddot{Z}^s(T) \right] \sin \omega_a(t - T)\, dT ,$$

which gives the relative motion of m_i in the r direction as

$$x_i^r = - \sum_a \frac{\bar{X}_{ia}^r}{\omega_a \sum_{j,p} m_j \left(\bar{X}_{ja}^p\right)^2} \int_0^t \left[\sum_{j,s} m_j \bar{X}_{ja}^s \ddot{Z}^s(T) \right] \sin \omega_a(t - T)\, dT . \tag{32}$$

The absolute motion of m_i is

$$\underline{X}_i^r = Z^r - \sum_a \frac{\bar{X}_{ia}^r}{\omega_a \sum_{j,p} m_j \left(\bar{X}_{ja}^p\right)^2} \int_0^t \left[\sum_{j,s} m_j \bar{X}_{ja}^s \ddot{Z}^s(T) \right] \sin \omega_a(t - T)\, dT . \tag{33}$$

Recall that the origin of the z axes is not necessarily located at any particular reference point. This is due to the form by which base motions or inputs are usually prescribed. The inputs may be given in one, two, or all three possible directions of motion. For example, suppose the base disturbance is prescribed in the longitudinal direction only. The motion of m_i in the vertical and athwartships directions, each of which is perpendicular to the longitudinal motion, represents absolute motion. This agrees with Eqs. (32) and (33), since each equation is the same for finding the absolute response in the vertical and athwartships directions, while Eq. (33) gives the absolute response for longitudinal motion.

SPECIAL TOPICS

Impulse

Consider the impulse I_k^r applied at mass k. The normal mode solution is

$$X_i^r = \sum_a \frac{\bar{X}_{ia}^r \sum_s \bar{X}_{ka}^s I_k^s}{\omega_a \sum_{j,p} m_j \left(\bar{X}_{ja}^p\right)^2} \sin \omega_a t$$

where I_k^1, I_k^2, and I_k^3 are the amplitudes of impulse in the 1, 2, and 3 directions, respectively. Upon differentiating,

$$\dot{X}_i^r = \sum_a \frac{\bar{X}_{ia}^r \sum_s \bar{X}_{ka}^s I_k^s}{\sum_{j,p} m_j \left(\bar{X}_{ja}^p\right)^2} \cos \omega_a t \, . \tag{34}$$

Since the structure rests on a base and the masses were assumed to be capable of independent movement, the velocity of m_i must be zero at $t = 0$, so that

$$\sum_a \frac{\bar{X}_{ia}^r \sum_s \bar{X}_{ka}^s I_k^s}{\sum_{j,p} m_j \left(\bar{X}_{ja}^p\right)^2} = 0 \, .$$

Suppose the impulse is applied only in the 1 direction, so that the above equation becomes

$$\sum_a \frac{\bar{X}_{ia}^r \bar{X}_{ka}^1 I_k^1}{\sum_{j,p} m_j \left(\bar{X}_{ja}^p\right)^2} = 0 \, .$$

Since I_k^1 is not zero, this reduces to

$$\sum_a \frac{\bar{X}_{ia}^r \bar{X}_{ka}^1}{\sum_{j,p} m_j \left(\bar{X}_{ja}^p\right)^2} = 0 \, .$$

Likewise, if the impulse were applied in the 2 and 3 directions separately, there results

$$\sum_a \frac{\bar{X}^r_{ia}\,\bar{X}^2_{ka}}{\sum_{j,p} m_j\left(\bar{X}^p_{ja}\right)^2} = 0$$

$$\sum_a \frac{\bar{X}^r_{ia}\,\bar{X}^3_{ka}}{\sum_{j,p} m_j\left(\bar{X}^p_{ja}\right)^2} = 0\,.$$

These three expressions can be represented more generally as

$$\sum_a \frac{\bar{X}^r_{ia}\,\bar{X}^s_{ka}}{\sum_{j,p} m_j\left(\bar{X}^p_{ja}\right)^2} = 0\,, \qquad i \neq k\,. \tag{35}$$

Similarly, the velocity of the mass which is struck by the impulse applied in the 1 direction is I_k^1/m_k in the 1 direction and zero in the 2 and 3 directions at $t = 0$. Therefore, with reference to Eq. (34),

$$\frac{I_k^1}{m_k} = \sum_a \frac{\bar{X}^1_{ka}\,\bar{X}^1_{ka}\,I_k^1}{\sum_{j,p} m_j\left(\bar{X}^p_{ja}\right)^2}$$

$$\sum_a \frac{\bar{X}^2_{ka}\,\bar{X}^1_{ka}\,I_k^1}{\sum_{j,p} m_j\left(\bar{X}^p_{ja}\right)^2} = 0$$

$$\sum_a \frac{\bar{X}^3_{ka}\,\bar{X}^1_{ka}\,I_k^1}{\sum_{j,p} m_j\left(\bar{X}^p_{ja}\right)^2} = 0\,.$$

Similar equations can be obtained for the impulse applied in the 2 and 3 directions. The resulting equations in general form are

$$\sum_a \frac{\bar{X}^r_{ka}\,\bar{X}^s_{ka}}{\sum_{j,p} m_j\left(\bar{X}^p_{ja}\right)^2} = 0\,, \qquad r \neq s \tag{36}$$

$$= \frac{1}{m_k}\,, \qquad r = s\,. \tag{37}$$

Sudden Motion of the Base

Consider the response of a structure initially at rest to a step change in the base velocity. Let $\dot{z}_0^1$, $\dot{z}_0^2$, and $\dot{z}_0^3$ comprise the components of this step change. The normal mode solution is

$$\dot{X}_i^{\,r} = -\sum_a \frac{\overline{X}^r_{i\,a} \sum_{j,s} m_j \overline{X}^s_{j\,a} \dot{Z}_0^{\,s}}{\sum_{j,p} m_j \left(\overline{X}^p_{j\,a}\right)^2} \cos \omega_a t \,. \tag{38}$$

Thus, if the step change in base velocity occurs only in the 1 direction, Eq. (38) becomes

$$\dot{X}_i^{\,r} = -\sum_a \frac{\overline{X}^r_{i\,a} \sum_{j} m_j \overline{X}^1_{j\,a} \dot{Z}_0^{\,1}}{\sum_{j,p} m_j \left(\overline{X}^p_{j\,a}\right)^2} \cos \omega_a t \,.$$

At $t = 0$, the absolute velocity of each mass is zero, so that the velocity relative to the base is $-\dot{Z}_0^{\,1}$. Therefore,

$$\sum_a \frac{\overline{X}^1_{i\,a} \sum_{j} m_j \overline{X}^1_{j\,a}}{\sum_{j,p} m_j \left(\overline{X}^p_{j\,a}\right)^2} = 1$$

$$\sum_a \frac{\overline{X}^2_{i\,a} \sum_{j} m_j \overline{X}^1_{j\,a}}{\sum_{j,p} m_j \left(\overline{X}^p_{j\,a}\right)^2} = 0$$

$$\sum_a \frac{\overline{X}^3_{i\,a} \sum_{j} m_j \overline{X}^1_{j\,a}}{\sum_{j,p} m_j \left(\overline{X}^p_{j\,a}\right)^2} = 0 \,.$$

Similar expressions can be obtained for a step change in base velocity in the 2 and 3 directions. The general equations which result are

$$\sum_a \frac{\overline{X}^r_{i\,a} \sum_{j} m_j \overline{X}^s_{j\,a}}{\sum_{j,p} m_j \left(\overline{X}^p_{j\,a}\right)^2} = 0, \qquad r \neq s \tag{39}$$

$$= 1, \qquad r = s \,. \tag{40}$$

Define as the participation factor

$$P_a^{\,s} = \frac{\sum_{j} m_j \overline{X}^s_{j\,a}}{\sum_{j,p} m_j \left(\overline{X}^p_{j\,a}\right)^2} \,, \tag{41}$$

so that Eqs. (39) and (40) can be rewritten as

$$\sum_a \overline{X}^r_{ia} P^s_a = 0 , \qquad r \neq s \tag{42}$$

$$= 1 , \qquad r = s . \tag{43}$$

Now sum over r in Eq. (43), so that

$$\sum_{a,r} \overline{X}^r_{ia} P^r_a = 3 .$$

Equivalent Forces for Base Motion

As a final special topic let $F_k^r = -m_k C^r(t)$, in which the force on a mass is proportional to that mass. Assume that such forces are applied to each mass and $C^r(t)$ is not a function of k. Then Eq. (27) becomes

$$X_j^r = -\sum_a \frac{\overline{X}^r_{ja}}{\omega_a \sum_{i,p} m_i \left(\overline{X}^p_{ia}\right)^2} \int_0^t \left[\sum_{k,s} m_k \overline{X}^s_{ka} C^s(T)\right] \sin \omega_a(t-T)\, dT . \tag{44}$$

This is precisely Eq. (32), if $C^s(T) = \ddot{Z}^s(T)$. Therefore, the displacement response for many applied forces can be converted to the relative displacement response due to base motion by substituting $F_k^s(T) = -m_k \ddot{Z}^s(T)$ and summing over all k.

INERTIA FORCES

Single Applied Force

In order to calculate for stress, it is convenient to determine the inertia loadings that the masses apply to the structure. It has been shown that each normal mode acts as a single-degree-of-freedom system with certain characteristics. If the absolute acceleration of each mass point m_i is found, the inertia forces can be added to the structure as a loading by D'Alembert's principle.

Consider the case of an applied force at m_k with no base motion. The q_a equation is

$$\ddot{q}_a + (\omega_a)^2 q_a = \frac{\sum_s \overline{X}^s_{ka} F_k^s}{\sum_{j,r} m_j \left(\overline{X}^r_{ja}\right)^2} .$$

Solving for $\ddot{q}_a$,

$$\ddot{q}_a = \frac{\sum_s \overline{X}^s_{ka} F_k^s}{\sum_{j,r} m_j \left(\overline{X}^r_{ja}\right)^2} - (\omega_a)^2 q_a .$$

Since

$$\ddot{X}_i^r = \sum_a \bar{X}_{ia}^r \ddot{q}_a$$

then

$$\ddot{X}_i^r = \sum_a \frac{\bar{X}_{ia}^r \sum_s \bar{X}_{ka}^s F_k^s}{\sum_{j,r} m_j \left(\bar{X}_{ja}^r\right)^2} - \sum_a (\omega_a)^2 \bar{X}_{ia}^r q_a . \tag{45}$$

Rewrite Eq. (45) using Eqs. (35), (36), and (37). Thus, for any mass but the kth mass (where the force is applied)

$$\ddot{\underline{X}}_i^r = \ddot{X}_i^r = -\sum_a (\omega_a)^2 \bar{X}_{ia}^r q_a$$

and for the kth mass

$$\ddot{\underline{X}}_k^r = \ddot{X}_k^r = \frac{F_k^r}{m_k} - \sum_a (\omega_a)^2 \bar{X}_{ka}^r q_a .$$

The inertia loadings are

$$Q_i^r = \sum_a (\omega_a)^2 m_i \bar{X}_{ia}^r q_a , \quad (i \neq k)$$

$$Q_k^r = -F_k^r + \sum_a (\omega_a)^2 m_k \bar{X}_{ka}^r q_a .$$

These forces describe the inertial loadings for each mass point. At m_k, there is an applied force F_k^r. The sum of the forces on m_k is the net applied force

$$Q_k^r + F_k^r = \sum_a (\omega_a)^2 m_k \bar{X}_{ka}^r q_a .$$

The structure is therefore loaded in mode a by a force system of the form

$$F_{ia}^r = (\omega_a)^2 m_i \bar{X}_{ia}^r q_a , \quad \text{for all } i \tag{46}$$

where

$$q_a = \frac{1}{\omega_a \sum_{j,p} m_j \left(\bar{X}_{ja}^p\right)^2} \int_0^t \left[\sum_s \bar{X}_{ka}^s F_k^s(T)\right] \sin \omega_a (t-T)\, dT .$$

This force system described by Eq. (46) may be used to calculate stresses in the structure for each mode.

Many Applied Forces

Consider the case where there are many applied forces acting on a structure which vary as different functions of time. The $\ddot{q}_a$ equation is

$$\ddot{q}_a = \frac{\sum_{k=1}^{d} \sum_{s} \overline{X}^s_{k\,a} F^s_k}{\sum_{j,p} m_j \left(\overline{X}^p_{j\,a}\right)^2} - (\omega_a)^2 q_a ,$$

so that

$$\ddot{\underline{X}}^r_{i\,a} = \ddot{X}^r_{i\,a} = \frac{\overline{X}^r_{i\,a} \sum_{k=1}^{d} \sum_{s} \overline{X}^s_{k\,a} F^s_k}{\sum_{j,p} m_j \left(\overline{X}^p_{j\,a}\right)^2} - (\omega_a)^2\, \overline{X}^r_{i\,a}\, q_a$$

$$= \frac{\overline{X}^r_{i\,a}}{\sum_{j,p} m_j \left(\overline{X}^p_{j\,a}\right)^2} \left(\sum_s \overline{X}^s_{1\,a} F^s_1 + \sum_s \overline{X}^s_{2\,a} F^s_2 + \cdots + \sum_s \overline{X}^s_{d\,a} F^s_d \right) - (\omega_a)^2\, \overline{X}^r_{i\,a}\, q_a .$$

Upon summing over the modes all terms in the series expression are zero except when $r = s$ and $d = i$, according to Eqs. (36) and (37). Therefore,

$$\ddot{X}^r_i = \sum_a \ddot{X}^r_{i\,a} = \frac{F^r_i}{m_i} - \sum_a (\omega_a)^2\, \overline{X}^r_{i\,a}\, q_a .$$

The inertia loadings are

$$Q^r_i = -F^r_i + \sum_a (\omega_a)^2\, m_i\, \overline{X}^r_{i\,a}\, q_a .$$

The net force acting on each mass is

$$Q^r_i + F^r_i = \sum_a (\omega_a)^2\, m_i\, \overline{X}^r_{i\,a}\, q_a .$$

The structure is therefore loaded in mode a by a force system of the form

$$F^r_{i\,a} = (\omega_a)^2\, m_i\, \overline{X}^r_{i\,a}\, q_a , \quad \text{for all } i \tag{47}$$

where

$$q_a = \frac{1}{\omega_a \sum_{j,p} m_j \left(\overline{X}^p_{j\,a}\right)^2} \int_0^t \left[\sum_{k=1}^{d} \sum_s \overline{X}^s_{k\,a}\, F^s_k(T) \right] \sin \omega_a (t - T)\, dT .$$

This force system described by Eq. (47) may be used to calculate stresses in the structure for each mode in the case of many applied forces.

Base Motion

Consider the equation of motion in one direction for a single-degree-of-freedom system subject only to base motion in that direction:

$$\ddot{X} + (\omega)^2 X = -\ddot{Z}.$$

The absolute acceleration is therefore

$$\ddot{\underline{X}} = \ddot{X} + \ddot{Z} = -(\omega)^2 X.$$

In an analogous manner, the absolute acceleration in mode a for the type of structure under investigation is

$$\ddot{\underline{X}}^r_{i\,a} = -(\omega_a)^2 X^r_{i\,a}$$

and the inertia forces are

$$Q^r_{i\,a} = (\omega_a)^2\, m_i\, X^r_{i\,a} = (\omega_a)^2\, m_i\, \bar{X}^r_{i\,a}\, q_a$$

where

$$q_a = -\frac{1}{\omega_a \sum\limits_{j,p} m_j\left(\bar{X}^p_{j\,a}\right)^2} \int_0^t \left[\sum_{j,s} m_j\, \bar{X}^s_{j\,a}\, \ddot{Z}^s(T)\right] \sin \omega_a(t-T)\, dT$$

$$= -\frac{1}{\omega_a} \int_0^t \left[\sum_s P_a^{\,s}\, \ddot{Z}^s(T)\right] \sin \omega_a(t-T)\, dT \tag{48}$$

by Eq. (41). Let

$$D_a^{\,r} = \omega_a \int_0^t \ddot{Z}^r(T) \sin \omega_a(t-T)\, dT. \tag{49}$$

Equation (48) becomes

$$q_a = -\frac{1}{(\omega_a)^2} \sum_s P_a^{\,s}\, D_a^{\,s}.$$

The inertia forces in mode a, which are the net effective forces for calculating stress, can be rewritten

$$F^r_{i\,a} = Q^r_{i\,a} = -m_i\, \bar{X}^r_{i\,a} \sum_s P_a^{\,s}\, D_a^{\,s}. \tag{50}$$

EFFECTIVE MASS WITH BASE MOTION

To determine the effective mass present in a given mode of vibration for a structure subject to base motion, consider the net effective force in mode a at m_i:

$$F_{ia}^{r} = Q_{ia}^{r} = -m_i \bar{X}_{ia}^{r} \sum_s P_a^s D_a^s . \tag{50}$$

The total force acting in mode a in the r direction is

$$F_a^r = -\sum_i m_i \bar{X}_{ia}^{r} \sum_s P_a^s D_a^s . \tag{51}$$

For the single-degree-of-freedom system this becomes

$$F = -MD$$

where M is the mass of the structure and

$$D = \omega \int_0^t \ddot{Z}(T) \sin \omega(t - T)\, dT .$$

Equation (51) is now expanded to give

$$F_a^r = -\sum_i m_i \bar{X}_{ia}^{r} P_a^1 D_a^1 - \sum_i m_i \bar{X}_{ia}^{r} P_a^2 D_a^2 - \sum_i m_i \bar{X}_{ia}^{r} P_a^3 D_a^3 . \tag{52}$$

Since the components of the base motion are independent of each other, Eq. (52) gives the mass acting in the r direction due to motion in the s direction for mode a as

$$M_a^{rs} = \sum_i m_i \bar{X}_{ia}^{r} P_a^s . \tag{53}$$

For example, a base motion in the longitudinal direction causes mass to act in a mode in each direction as follows:

$$M_a^{11} = \sum_i m_i \bar{X}_{ia}^{1} P_a^1 ,$$

mass in the longitudinal direction;

$$M_a^{21} = \sum_i m_i \bar{X}_{ia}^{2} P_a^1 ,$$

mass in the vertical direction;

$$M_a^{31} = \sum_i m_i \bar{X}_{ia}^{3} P_a^1 ,$$

mass in the athwartships direction.

The latter two terms, namely M_a^{21} and M_a^{31}, might be called the cross-mass terms. It can be shown that

$$M_a^{rs} = M_a^{sr} , \quad r \neq s . \tag{54}$$

If the masses are summed over the modes, there results

$$M^{11} = \sum_i m_i \sum_a \bar{X}_{ia}^{1} P_a^1 = \sum_i m_i = M$$

$$M^{21} = \sum_i m_i \sum_a \bar{X}_{ia}^2 P_a^1 = 0$$

$$M^{31} = \sum_i m_i \sum_a \bar{X}_{ia}^3 P_a^1 = 0$$

using Eqs. (42) and (43).

Similar statements can be made for base motions in the vertical and athwartships directions. The summation over the modes of vibration can therefore be generalized as

$$M^{rr} = M \tag{55}$$

$$M^{rs} = M^{sr} = 0. \tag{56}$$

Equation (55) indicates that the sum of all the effective masses acting in the r direction due to motion in the r direction for the total number of modes is equal to the total mass of the actual structure. Since M_a^{rr} is always a positive quantity, a calculation of the amount of mass remaining in the higher modes can be made after the lower modes have been found.

SUMMARY

The essential relationships have been derived for studying translational motion of three-directional lumped parameter systems based on normal mode theory. The approach used to develop these expressions is an extension of an earlier report (3) on unidirectional normal mode theory. Appendix B summarizes the equations of normal mode theory for the case of each mass having six directions of motion, that is, three translational directions and three rotational directions.

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Appendix A

MATRIX FORM OF LUMPED PARAMETER SYSTEMS

The relationship between normal mode shapes and fixed base natural frequencies is

$$\bar{X}^{r}_{ja} = (\omega_a)^2 \sum_{i,s} \delta^{rs}_{ji}\, m_i\, \bar{X}^{s}_{ia} \,. \tag{5}$$

The mode shapes in each of the three directions of motion are related as follows:

$$\frac{\bar{X}^{1}_{ja}}{(\omega_a)^2} = \sum_i \delta^{11}_{ji}\, m_i\, \bar{X}^{1}_{ia} + \sum_i \delta^{12}_{ji}\, m_i\, \bar{X}^{2}_{ia} + \sum_i \delta^{13}_{ji}\, m_i\, \bar{X}^{3}_{ia} \tag{A1}$$

$$\frac{\bar{X}^{2}_{ja}}{(\omega_a)^2} = \sum_i \delta^{21}_{ji}\, m_i\, \bar{X}^{1}_{ia} + \sum_i \delta^{22}_{ji}\, m_i\, \bar{X}^{2}_{ia} + \sum_i \delta^{23}_{ji}\, m_i\, \bar{X}^{3}_{ia} \tag{A2}$$

$$\frac{\bar{X}^{3}_{ja}}{(\omega_a)^2} = \sum_i \delta^{31}_{ji}\, m_i\, \bar{X}^{1}_{ia} + \sum_i \delta^{32}_{ji}\, m_i\, \bar{X}^{2}_{ia} + \sum_i \delta^{33}_{ji}\, m_i\, \bar{X}^{3}_{ia} \,. \tag{A3}$$

Each of these expressions has a range of n, so that there are a total of $3n$ equations represented by Eqs. (A1)-(A3). They can be written in matrix form as

$$\frac{1}{(\omega_a)^2}\{\bar{X}^1\} = [\delta^{11}][m]\{\bar{X}^1\} + [\delta^{12}][m]\{\bar{X}^2\} + [\delta^{13}][m]\{\bar{X}^3\} \tag{A4}$$

$$\frac{1}{(\omega_a)^2}\{\bar{X}^2\} = [\delta^{21}][m]\{\bar{X}^1\} + [\delta^{22}][m]\{\bar{X}^2\} + [\delta^{23}][m]\{\bar{X}^3\} \tag{A5}$$

$$\frac{1}{(\omega_a)^2}\{\bar{X}^3\} = [\delta^{31}][m]\{\bar{X}^1\} + [\delta^{32}][m]\{\bar{X}^2\} + [\delta^{33}][m]\{\bar{X}^3\} \tag{A6}$$

where, for example,

$$\{\bar{X}^1\} = \begin{bmatrix} \bar{X}^{1}_{1a} \\ \bar{X}^{1}_{2a} \\ \vdots \\ \bar{X}^{1}_{na} \end{bmatrix}$$

$$[\delta^{11}] = \begin{bmatrix} \delta^{11}_{11} & \delta^{11}_{12} & \cdots & \delta^{11}_{1n} \\ \delta^{11}_{21} & \delta^{11}_{22} & \cdots & \delta^{11}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \delta^{11}_{n1} & \delta^{11}_{n2} & \cdots & \delta^{11}_{nn} \end{bmatrix}$$

$$[m] = \begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & m_n \end{bmatrix}$$

Equations (A4)-(A6) can also be written in matrix form as

$$\frac{1}{(\omega_a)^2} \begin{bmatrix} \{\bar{X}^1\} \\ \{\bar{X}^2\} \\ \{\bar{X}^3\} \end{bmatrix} = \begin{bmatrix} [\delta^{11}] & [\delta^{12}] & [\delta^{13}] \\ [\delta^{21}] & [\delta^{22}] & [\delta^{23}] \\ [\delta^{31}] & [\delta^{32}] & [\delta^{33}] \end{bmatrix} \begin{bmatrix} [m] & 0 & 0 \\ 0 & [m] & 0 \\ 0 & 0 & [m] \end{bmatrix} \begin{bmatrix} \{\bar{X}^1\} \\ \{\bar{X}^2\} \\ \{\bar{X}^3\} \end{bmatrix}$$

This expression written out in its entirety is as follows on the next page.

$$
\left[\begin{array}{c}
\overline{X}^1_{1a}\\ \overline{X}^1_{2a}\\ \vdots\\ \overline{X}^1_{na}\\ \hdashline
\overline{X}^2_{1a}\\ \overline{X}^2_{2a}\\ \vdots\\ \overline{X}^2_{na}\\ \hdashline
\overline{X}^3_{1a}\\ \overline{X}^3_{2a}\\ \vdots\\ \overline{X}^3_{na}
\end{array}\right]
= (\omega_a)^2
\left[\begin{array}{cccc:cccc:cccc}
\delta^{11}_{11} & \delta^{11}_{12} & \cdots & \delta^{11}_{1n} & \delta^{12}_{11} & \delta^{12}_{12} & \cdots & \delta^{12}_{1n} & \delta^{13}_{11} & \delta^{13}_{12} & \cdots & \delta^{13}_{1n}\\
\delta^{11}_{21} & \delta^{11}_{22} & \cdots & \delta^{11}_{2n} & \delta^{12}_{21} & \delta^{12}_{22} & \cdots & \delta^{12}_{2n} & \delta^{13}_{21} & \delta^{13}_{22} & \cdots & \delta^{13}_{2n}\\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots\\
\delta^{11}_{n1} & \delta^{11}_{n2} & \cdots & \delta^{11}_{nn} & \delta^{12}_{n1} & \delta^{12}_{n2} & \cdots & \delta^{12}_{nn} & \delta^{13}_{n1} & \delta^{13}_{n2} & \cdots & \delta^{13}_{nn}\\ \hdashline
\delta^{21}_{11} & \delta^{21}_{12} & \cdots & \delta^{21}_{1n} & \delta^{22}_{11} & \delta^{22}_{12} & \cdots & \delta^{22}_{1n} & \delta^{23}_{11} & \delta^{23}_{12} & \cdots & \delta^{23}_{1n}\\
\delta^{21}_{21} & \delta^{21}_{22} & \cdots & \delta^{21}_{2n} & \delta^{22}_{21} & \delta^{22}_{22} & \cdots & \delta^{22}_{2n} & \delta^{23}_{21} & \delta^{23}_{22} & \cdots & \delta^{23}_{2n}\\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots\\
\delta^{21}_{n1} & \delta^{21}_{n2} & \cdots & \delta^{21}_{nn} & \delta^{22}_{n1} & \delta^{22}_{n2} & \cdots & \delta^{22}_{nn} & \delta^{23}_{n1} & \delta^{23}_{n2} & \cdots & \delta^{23}_{nn}\\ \hdashline
\delta^{31}_{11} & \delta^{31}_{12} & \cdots & \delta^{31}_{1n} & \delta^{32}_{11} & \delta^{32}_{12} & \cdots & \delta^{32}_{1n} & \delta^{33}_{11} & \delta^{33}_{12} & \cdots & \delta^{33}_{1n}\\
\delta^{31}_{21} & \delta^{31}_{22} & \cdots & \delta^{31}_{2n} & \delta^{32}_{21} & \delta^{32}_{22} & \cdots & \delta^{32}_{2n} & \delta^{33}_{21} & \delta^{33}_{22} & \cdots & \delta^{33}_{2n}\\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots\\
\delta^{31}_{n1} & \delta^{31}_{n2} & \cdots & \delta^{31}_{nn} & \delta^{32}_{n1} & \delta^{32}_{n2} & \cdots & \delta^{32}_{nn} & \delta^{33}_{n1} & \delta^{33}_{n2} & \cdots & \delta^{33}_{nn}
\end{array}\right]
\left[\begin{array}{cccc:cccc:cccc}
m_1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\
0 & m_2 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots\\
0 & 0 & \cdots & m_n & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\ \hdashline
0 & 0 & \cdots & 0 & m_1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\
0 & 0 & \cdots & 0 & 0 & m_2 & \cdots & 0 & 0 & 0 & \cdots & 0\\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots\\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & m_n & 0 & 0 & \cdots & 0\\ \hdashline
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & m_1 & 0 & \cdots & 0\\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & m_2 & \cdots & 0\\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots\\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & m_n
\end{array}\right]
\left[\begin{array}{c}
\overline{X}^1_{1a}\\ \overline{X}^1_{2a}\\ \vdots\\ \overline{X}^1_{na}\\ \hdashline
\overline{X}^2_{1a}\\ \overline{X}^2_{2a}\\ \vdots\\ \overline{X}^2_{na}\\ \hdashline
\overline{X}^3_{1a}\\ \overline{X}^3_{2a}\\ \vdots\\ \overline{X}^3_{na}
\end{array}\right]
$$

Appendix B

EQUATIONS FOR SIX-DIRECTIONAL NORMAL MODE THEORY

NOTATION AND ASSUMPTIONS

1. The structure is attached to a fixed base and is represented by n lumped masses, each mass being capable of translational motion along three mutually perpendicular axes and rotational motion about each of these axes.

2. Each mass has dimensions, so that it has rotational inertia.

3. Figure B1 shows the orientation of the ship by the primed coordinates (fixed axes) and the axes of orientation for mass m_i of the structure (moving axes). In addition to the usual translational motions given by the 1, 2, and 3 axes, the 4, 5, and 6 directions represent the angular motions about each axis, respectively. Thus, X_i^1, X_i^2, and X_i^3 are the components of translational motion of m_i, while X_i^4, X_i^5, and X_i^6 are the components of rotational motion of m_i.

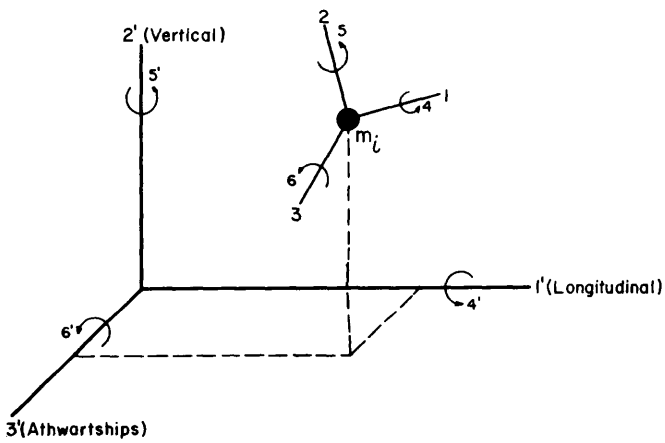

Fig. B1 - Reference axes for orientation of a ship and mass m_i for the case of six-directional motion

4. For the purpose of developing the equations of motion of a body about a fixed point, let $\theta_i^1 = X_i^4$, $\theta_i^2 = X_i^5$, and $\theta_i^3 = X_i^6$.

5. Let I_i^{rr} be the moment of inertia of m_i about its r axis and I_i^{rs} be the product of inertia of m_i about its r and s axes.

6. It can be shown* that the angular momenta, H_i^r, of a body with respect to its moving axes are

*S. Timoshenko and D.H. Young, "Advanced Dynamics," New York:McGraw-Hill Book Company, Inc., p. 332, 1948.

$$
\begin{aligned}
H_i^1 &= I_i^{11}\,\dot\theta_i^1 - I_i^{12}\,\dot\theta_i^2 - I_i^{13}\,\dot\theta_i^3 \\
H_i^2 &= I_i^{21}\,\dot\theta_i^1 - I_i^{22}\,\dot\theta_i^2 - I_i^{23}\,\dot\theta_i^3 \\
H_i^3 &= I_i^{31}\,\dot\theta_i^1 - I_i^{32}\,\dot\theta_i^2 - I_i^{33}\,\dot\theta_i^3 .
\end{aligned}
\tag{B1}
$$

7. Select the 1, 2, and 3 axes of m_i as the principal axes so that the products of inertia are zero and Eq. (B1) reduces to

$$
\begin{aligned}
H_i^1 &= I_i^{11}\,\dot\theta_i^1 \\
H_i^2 &= I_i^{22}\,\dot\theta_i^2 \\
H_i^3 &= I_i^{33}\,\dot\theta_i^3
\end{aligned}
\tag{B2}
$$

8. The principle of angular momentum states that the rate of change of the angular momentum of a body rotating about a fixed point is equal to the moment of all forces acting on the body with respect to the same point. After taking into account the rate of change of the angular momenta with respect to the 1, 2, and 3 axes and the fact that the 1, 2, and 3 axes are also rotating about a fixed point, the following equations result:

$$
\begin{aligned}
\frac{dH_i^1}{dt} - \dot\theta_i^2\,H_i^3 - \dot\theta_i^3\,H_i^2 &= N_i^1 \\
\frac{dH_i^2}{dt} - \dot\theta_i^3\,H_i^1 - \dot\theta_i^1\,H_i^3 &= N_i^2 \\
\frac{dH_i^3}{dt} - \dot\theta_i^1\,H_i^2 - \dot\theta_i^2\,H_i^1 &= N_i^3
\end{aligned}
\tag{B3}
$$

where N_i^r is the moment of all forces acting on the body about the r axis. Substitute Eq. (B2) into (B3) to get

$$
\begin{aligned}
I_i^{11}\,\frac{d\dot\theta_i^1}{dt} + \left(I_i^{33} - I_i^{22}\right)\dot\theta_i^2\,\dot\theta_i^3 &= N_i^1 \\
I_i^{22}\,\frac{d\dot\theta_i^2}{dt} + \left(I_i^{11} - I_i^{33}\right)\dot\theta_i^1\,\dot\theta_i^3 &= N_i^2 \\
I_i^{33}\,\frac{d\dot\theta_i^3}{dt} + \left(I_i^{22} - I_i^{11}\right)\dot\theta_i^1\,\dot\theta_i^2 &= N_i^3 .
\end{aligned}
\tag{B4}
$$

These equations are called the Euler equations of motion.

10. For small oscillations assume that the terms containing the product of the $\dot\theta$'s are small compared with the other terms in Eq. (B4), so that

$$I_i^{11}\,\ddot{\theta}_i^{1} = N_i^{1}$$

$$I_i^{22}\,\ddot{\theta}_i^{2} = N_i^{2}$$

$$I_i^{33}\,\ddot{\theta}_i^{3} = N_i^{3}.$$

Using the notation of Fig. B1 for m_i, these equations become

$$I_i^{4}\,\ddot{X}_i^{4} = N_i^{4}$$

$$I_i^{5}\,\ddot{X}_i^{5} = N_i^{5} \tag{B5}$$

$$I_i^{6}\,\ddot{X}_i^{6} = N_i^{6}$$

where, for example, I_i^4 is the moment of inertia of m_i in the 4 direction (or about the 1 axis) and $\ddot{X}_i^4$ is the angular acceleration of m_i in the 4 direction. Equations (B5) represent the relationships between the inertia torques and the applied torques for equilibrium about a fixed point. These inertia torques along with the inertia forces will be used for the free vibration problem of the structure under investigation.

11. Since the axes of each mass are the principal axes, the axes of different masses are therefore not necessarily parallel to each other. This requires a new definition of the influence coefficient δ_{ij}^{rs} as the motion (deflection or rotation) of m_i in the r direction of m_i due to a unit load (force or torque) at m_j in the s direction of m_j. For example, δ_{ij}^{11} is the deflection of m_i in the 1 direction of m_i due to a unit force at m_j in the 1 direction of m_j. Note that the 1 direction associated with m_i is not necessarily parallel with the 1 direction of m_j.

12. It is assumed that the change in geometry of the structure is small during its dynamic response under the action of external forces and torques. This means that the influence coefficients calculated for statical loads on the structure with respect to the principal axes of each mass are used to find the dynamic motions while the axes are permitted to translate and rotate with each mass.

13. Define the direction cosine between the r axis and r' axis at m_i as

$$\ell_i^{rr'} = \cos_i(r, r').$$

Assume that during the structure's dynamic response the direction cosines remain constant.

NORMAL MODE EQUATIONS

The equations of motion for the free vibrations of the assumed structure are written using D'Alembert's principle and treating the inertia forces and torques as the applied loads.

$$X_j^r = -\sum_i \sum_{s=1}^{3} \delta_{ji}^{rs} m_i \ddot{X}_i^s - \sum_i \sum_{s=4}^{6} \delta_{ji}^{rs} I_i^s \ddot{X}_i^s$$

$$= -\sum_i \sum_{s=1}^{6} \delta_{ji}^{rs} I_i^s \ddot{X}_i^s, \qquad r = 1, \ldots, 6 \tag{B6}$$

where $I_i^s = m_i$ for $s = 1$, 2, and 3.

Equation (B6) is precisely the same as Eq. (3) except that the range on the direction of motion includes six independent coordinates for each mass. The normal mode equations for this case are now summarized from earlier results of this report, where summations on r, s, and p are from 1 through 6 unless otherwise indicated and the summation on a is from 1 to $6n$:

Mode Shapes and Natural Frequencies

$$\overline{X}_{ja}^r = (\omega_a)^2 \sum_{i,s} \delta_{ji}^{rs} I_i^s \overline{X}_{ia}^s, \qquad a = 1, 2, \ldots, 6n. \tag{B7}$$

Orthogonality of Normal Modes

$$\sum_{j,r} I_j^r \left(\overline{X}_{ja}^r\right)^2 \neq 0 \tag{B8}$$

$$\sum_{j,r} I_j^r \overline{X}_{ja}^r \overline{X}_{jb}^r = 0, \qquad a \neq b. \tag{B9}$$

Response for Free Vibrations

$$X_i^r = \sum_a \frac{\overline{X}_{ia}^r \sum_{j,s} I_j^s \overline{X}_{ja}^s X_j^s(0)}{\sum_{j,p} I_j^p \left(\overline{X}_{ja}^p\right)^2} \cos \omega_a t + \sum_a \frac{\overline{X}_{ia}^r \sum_{j,s} I_j^s \overline{X}_{ja}^s \dot{X}_j^s(0)}{\omega_a \sum_{j,p} I_j^p \left(\overline{X}_{ja}^p\right)^2} \sin \omega_a t. \tag{B10}$$

Response to an Applied Force and Torque at m_k

$$X_j^r = \sum_a \frac{\overline{X}_{ja}^r}{\omega_a \sum_{i,p} I_i^p \left(\overline{X}_{ia}^p\right)^2} \int_0^t \left[\sum_{s=1}^{3} \overline{X}_{ka}^s F_k^s(T) + \sum_{s=4}^{6} \overline{X}_{ka}^s N_k^s(T)\right] \sin \omega_a(t-T)\, dT. \tag{B11}$$

Response to Many Applied Forces and Torques

Assume that there are d applied forces and h applied torques; therefore,

$$X_j^r = \sum_a \frac{\overline{X}_{ja}^r}{\omega_a \sum_{i,p} I_i^p \left(\overline{X}_{ia}^p\right)^2} \int_0^t \left[\sum_{k=1}^{d} \sum_{s=1}^{3} \overline{X}_{ka}^s F_k^s(T) + \sum_{k=1}^{h} \sum_{s=4}^{6} \overline{X}_{ka}^s N_k^s(T)\right] \sin \omega_a(t-T)\, dT. \tag{B12}$$

Response to Base Motion

Consider the case where translational base motion is the prescribed input referred to the axes of the ship, namely, the primed axes shown in Fig. B1. If each mass of the structure is loaded with the set of forces $-m_k\ddot{Z}^{r'}$, the response of each mass is the same as for the case of many applied forces. However, this response is now the relative motion between each mass and the base.

Figure B2 shows the forces acting on m_i oriented with respect to the primed reference axes. To transform these forces along the principal axes of m_i, use the direction cosines between the axes r and r' at m_i. The transformed forces at m_i in the r direction are

$$- \sum_{r'=1'}^{3'} m_i \, \ell_i^{rr'} \, \ddot{Z}^{r'} . \tag{B13}$$

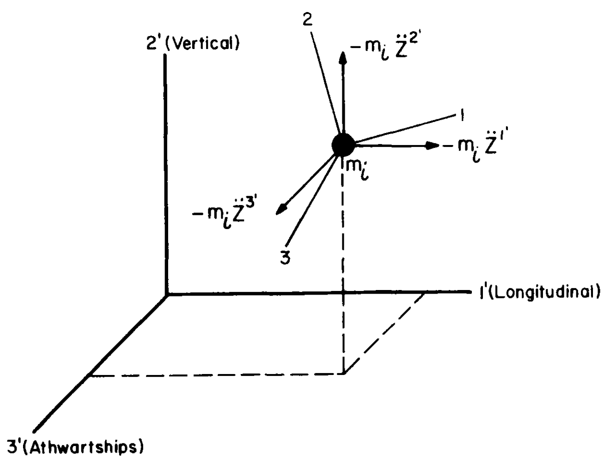

Fig. B2 - Forces acting on m_i for finding the response due to base motion for the case of six-directional motion

Use this summation to represent the forces acting on the structure. With reference to Eq. (B12), the relative motion between m_i and the base in the r direction is therefore

$$X_j^r = - \sum_a \frac{\overline{X}_{ja}^r}{\omega_a \sum_{i,p} I_i^p \left(\overline{X}_{ia}^p\right)^2} \int_0^t \left[\sum_k \sum_{s=1}^{3} \sum_{s'=1'}^{3'} m_k \, \overline{X}_{ka}^s \, \ell_k^{ss'} \, \ddot{Z}^{s'}(T) \right] \sin \omega_a (t - T) \, dT . \tag{B14}$$

Special Topic — Sudden Motion of the Base

Consider the response of a structure initially at rest to a step change in the translational velocity of the base. It follows from Eq. (B14) that

$$\dot{X}_j^r = - \sum_a \frac{\overline{X}_{ja}^r \sum_k \sum_{s=1}^{3} \sum_{s'=1'}^{3'} m_k \overline{X}_{ka}^s \ell_k^{ss'} \dot{Z}_0^{s'}}{\sum_{i,p} I_i^p \left(\overline{X}_{ia}^p\right)^2} \cos \omega_a t \tag{B15}$$

where $\dot{Z}_0^{1'}$, $\dot{Z}_0^{2'}$, and $\dot{Z}_0^{3'}$ represent the amplitudes of the base translational velocity in the 1', 2', and 3' directions, respectively. The six components of velocity defined by Eq. (B15) are now treated as two separate groups, namely, the three translational components and the three rotational components.

The translational components of velocity response are treated first. At $t = 0$, the absolute translational velocity of each mass is zero. To find the initial relative translational velocity of each mass with respect to the base motion, transform the base motion into the direction of the principal axes of each mass as follows:

$$\dot{X}_j^r(0) = - \sum_{r'=1'}^{3'} \ell_j^{rr'} \dot{Z}_0^{r'}, \qquad r = 1, 2, 3 . \tag{B16}$$

Substitute Eq. (B16) into the left side of Eq. (B15) for $t = 0$:

$$\sum_{r'=1'}^{3'} \ell_j^{rr'} \dot{Z}_0^{r'} = \sum_a \frac{\overline{X}_{ja}^r \sum_k \sum_{s=1}^{3} \sum_{s'=1'}^{3'} m_k \overline{X}_{ka}^s \ell_k^{ss'} \dot{Z}_0^{s'}}{\sum_{i,p} I_i^p \left(\overline{X}_{ia}^p\right)^2} . \tag{B17}$$

Since the components of the base velocity are independent of each other, it follows that

$$\ell_j^{rr'} = \sum_a \frac{\overline{X}_{ja}^r \sum_k \sum_{s=1}^{3} m_k \overline{X}_{ka}^s \ell_k^{sr'}}{\sum_{i,p} I_i^p \left(\overline{X}_{ia}^p\right)^2}, \qquad r = 1, 2, 3 . \tag{B18}$$

Let

$$P_{ka}^s = \frac{m_k \overline{X}_{ka}^s}{\sum_{i,p} I_i^p \left(\overline{X}_{ia}^p\right)^2}, \tag{B19}$$

so that Eq. (B18) becomes

$$\ell_j^{rr'} = \sum_a \overline{X}_{ja}^r \sum_k \sum_{s=1}^{3} P_{ka}^s \ell_k^{sr'}, \qquad r = 1, 2, 3 . \tag{B20}$$

In the case of rotational motion, the rotation of the base is zero, so that the initial relative rotational velocity of each mass is also zero; that is,

$$\dot{X}_j^r(0) = \dot{\underline{X}}_j^r(0) = 0, \qquad r = 4, 5, 6 . \tag{B21}$$

Referring to Eq. (B15) at $t = 0$,

$$\sum_a \frac{\bar{X}^r_{ja} \sum_k \sum_{s=1}^{3} \sum_{s'=1'}^{3'} m_k X^s_{ka} \ell^{ss'}_k \dot{Z}^{s'}_0}{\sum_{i,p} I^p_i \left(\bar{X}^p_{ia}\right)^2} = 0 .$$

Since the components of the base velocity are independent of each other, this becomes

$$\sum_a \bar{X}^r_{ja} \sum_k \sum_{s=1}^{3} P^s_{ka} \ell^{ss'}_k = 0 , \quad r = 4, 5, 6; \; s' = 1', 2', 3' \tag{B22}$$

using Eq. (B19).

STRESS CALCULATIONS

To calculate stresses in the structure, apply the net effective forces and torques at each mass point for each normal mode of vibration. With these loads acting in each mode, the stresses can be calculated for each mode throughout the structure, and the final stresses are obtained by superposing over the modes. This approach is the same as followed earlier in the case of translational motion in three directions only. The net effect of forces and torques for special cases are summarized as follows.

Single Applied Force and Torque at m_k

$$F^r_{ia} = (\omega_a)^2 I^r_i \bar{X}^r_{ia} q_a \tag{B23}$$

where

$$q_a = \frac{1}{\omega_a \sum_{j,p} I^p_j \left(\bar{X}^p_{ja}\right)^2} \int_0^t \left[\sum_{s=1}^{3} \bar{X}^s_{ka} F^s_k(T) + \sum_{s=4}^{6} \bar{X}^s_{ka} N^s_k(T) \right] \sin \omega_a (t - T) \, dT .$$

Many Applied Forces (d) and Torques (h)

$$F^r_{ia} = (\omega_a)^2 I^r_i \bar{X}^r_{ia} q_a \tag{B24}$$

where

$$q_a = \frac{1}{\omega_a \sum_{j,p} I^p_j \left(\bar{X}^p_{ja}\right)^2} \int_0^t \left[\sum_{k=1}^{d} \sum_{s=1}^{3} \bar{X}^s_{ka} F^s_k(T) + \sum_{k=1}^{h} \sum_{s=4}^{6} \bar{X}^s_{ka} N^s_k(T) \right] \sin \omega_a (t - T) \, dT .$$

Base Motion

$$F^r_{ia} = (\omega_a)^2 I^r_i \bar{X}^r_{ia} q_a \tag{B25}$$

where

$$q_a = -\frac{1}{\omega_a \sum_{j,p} I_j^p \left(\bar{X}_{ja}^p\right)^2} \int_0^t \left[\sum_k \sum_{s=1}^{3} \sum_{s'=1'}^{3'} m_k \, \ell_k^{ss'} \, \bar{X}_{ka}^s \, \ddot{Z}^{s'}(T)\right] \sin \omega_a(t-T)\, dT\,.$$

EFFECTIVE MASS WITH BASE MOTION

To find how much effective mass is acting in each normal mode of vibration, first consider the effective force in mode a at m_i from Eq. (B25):

$$F_{ia}^r = (\omega_a)^2 \, m_i \, \bar{X}_{ia}^r \, q_a\,, \qquad r = 1, 2, 3 \tag{B26}$$

where, using Eq. (B19),

$$q_a = -\frac{1}{(\omega_a)^2} \sum_k \sum_{s=1}^{3} \sum_{s'=1'}^{3'} \ell_k^{ss'} \, P_{ka}^s \, D_a^{s'} \tag{B27}$$

in which

$$D_a^{s'} = \omega_a \int_0^t \ddot{Z}^{s'}(T) \sin \omega_a(t-T)\, dT\,. \tag{B28}$$

These forces are transformed to the primed axes (orientation of the ship's motion) as follows:

$$F_{ia}^{r'} = \sum_{r=1}^{3} \ell_i^{r'r} \, F_{ia}^r\,, \qquad r' = 1', 2', 3'\,. \tag{B29}$$

Substitute Eq. (B26) and (B27) into Eq. (B29). This gives

$$F_{ia}^{r'} = -\sum_{r=1}^{3} \ell_i^{r'r} \, m_i \, \bar{X}_{ia}^r \sum_k \sum_{s=1}^{3} \sum_{s'=1'}^{3'} \ell_k^{ss'} \, P_{ka}^s \, D_a^{s'}\,.$$

The effective force acting in mode a is

$$F_a^{r'} = -\sum_i m_i \sum_{r=1}^{3} \ell_i^{r'r} \, \bar{X}_{ia}^r \sum_k \sum_{s=1}^{3} \sum_{s'=1'}^{3'} \ell_k^{ss'} \, P_{ka}^s \, D_a^{s'}\,, \tag{B30}$$

so that the effective mass acting in the r' direction due to motion in the s' direction is

$$M_a^{r's'} = \sum_i m_i \sum_{r=1}^{3} \ell_i^{r's} \, \bar{X}_{ia}^r \sum_k \sum_{s=1}^{3} \ell_k^{ss'} \, P_{ka}^s\,. \tag{B31}$$

The cross-mass terms are again symmetrical, that is,

$$M_a^{r's'} = M_a^{s'r'} , \quad r' \neq s' . \tag{B32}$$

Consider the summation of the effective masses over the modes:

$$M^{r's'} = \sum_a M_a^{r's'} = \sum_i m_i \sum_{r=1}^{3} \ell_i^{r'r} \sum_a \overline{X}_{ia}^{r} \sum_k \sum_{s=1}^{3} \ell_k^{ss'} P_{ka}^{s} .$$

With reference to Eq. (B20) this reduces to

$$M^{r's'} = \sum_i m_i \sum_{r=1}^{3} \ell_i^{r'r} \ell_i^{rs'} . \tag{B33}$$

It has been shown* that

$$\sum_{r=1}^{3} \ell_i^{rr'} \ell_i^{rs'} = 1 , \quad r' = s'$$

$$= 0 , \quad r' \neq s' .$$

Since

$$\ell_i^{r'r} = \ell_i^{rr'}$$

Eq. (B33) becomes

$$M^{r's'} = \sum_i m_i = M , \quad r' = s'$$

$$= 0 , \quad r' \neq s' . \tag{B34}$$

As in the case of three-directional motion, the total mass acting over the modes in the r' direction due to a base motion in the r' direction equals the total mass of the structure, while the summation of the cross-mass terms equals zero.

The torques due to translational base motion are now treated. It is necessary to introduce new notation for the angular acceleration and the moment of inertia. In addition, matrix notation will be used in transforming the torques from the principal axes of each mass to the ship's orientation.

For angular acceleration, let

$$\ddot{\underline{X}}_{ia}^{4} = \ddot{\theta}_{ia}^{1} , \quad \ddot{\underline{X}}_{ia}^{5} = \ddot{\theta}_{ia}^{2} , \quad \ddot{\underline{X}}_{ia}^{6} = \ddot{\theta}_{ia}^{3} . \tag{B35}$$

This is necessary since direction cosines are used which refer to the 1, 2, and 3 axes and the 1', 2', and 3' axes. Likewise, N_i^1, N_i^2, and N_i^3 now represent the torques at about the 1, 2, and 3 axes, respectively.

The direction cosines are written in matrix form as follows:

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*Herbert Goldstein, "Classical Mechanics," Cambridge:Addison-Wesley, p. 98, 1950.

$$\ell_i^{s's} = [\ell]_i = \begin{bmatrix} \ell_i(1',1) & \ell_i(1',2) & \ell_i(1',3) \\ \ell_i(2',1) & \ell_i(2',2) & \ell_i(2',3) \\ \ell_i(3',1) & \ell_i(3',2) & \ell_i(3',3) \end{bmatrix} . \tag{B36}$$

When using the direction cosines for transformations, these transformations are called orthogonal transformations, and it can be shown† that

$$[\ell]_i \; [\ell]_i^{-1} = [1]_i \tag{B37}$$

and

$$[\ell]_i^{-1} = [\ell]_i^* . \tag{B38}$$

This last equation relates the inverse matrix to the transpose matrix of the direction cosines.

The transformation of the torques due to base translational motion can now be made as follows:

$$\{N^{s'}\} = [\ell]_i \{N^s\}_i , \qquad s = 1, 2, 3 ; \quad s' = 1', 2', 3' . \tag{B39}$$

The torques referred to the unprimed axes are written in matrix form as

$$\{N^s\}_i = - [I^s]_i \{\ddot{\theta}^s\}_i \tag{B40}$$

where

$$[I^s]_i = \begin{bmatrix} I_i^{11} & 0 & 0 \\ 0 & I_i^{22} & 0 \\ 0 & 0 & I_i^{33} \end{bmatrix} . \tag{B41}$$

Premultiply both sides of Eq. (B40) by $[\ell]_i$, which gives

$$[\ell]_i \{N^s\}_i = - [\ell]_i [I^s]_i \{\ddot{\theta}^s\}_i$$

$$= - [\ell]_i [I^s]_i [\ell]_i^{-1} [\ell]_i \{\ddot{\theta}^s\}_i . \tag{B42}$$

In general, the equation for the primed system relating the torque with the inertia terms and angular acceleration is

†Robert L. Halfman, "Dynamics," Vol. I, Reading, Massachusetts:Addison-Wesley, p. 207, 1962.

$$\{N^{s'}\}_i = -[I^{s'}]_i \{\ddot{\theta}^{s'}\}_i . \tag{B43}$$

Comparing terms in Eq. (B42) and (B43) and using Eq. (B38) and (B39) leads to

$$[I^{s'}]_i = [\ell]_i [I^s] [\ell]_i^* \tag{B44}$$

$$\{\ddot{\theta}^{s'}\}_i = [\ell]_i \{\ddot{\theta}^s\}_i \tag{B45}$$

Equations (B43), (B44), and (B45) are now written in the equivalent series form as

$$N_i^{r'} = -\sum_{s'=1'}^{3'} I_i^{r's'} \ddot{\theta}_i^{s'} \tag{B46}$$

$$I_i^{r's'} = \sum_{u=1}^{3} \sum_{v=1}^{3} \ell_i^{r'u} \ell_i^{s'v} I_i^{uv} \tag{B47}$$

$$\ddot{\theta}_i^{s'} = \sum_{s=1}^{3} \ell_i^{s's} \ddot{\theta}_i^{s} . \tag{B48}$$

Rewrite Eq. (B46) for the mode a as

$$N_{ia}^{r'} = -\sum_{s'=1'}^{3'} I_i^{r's'} \ddot{\theta}_{ia}^{s'} \tag{B49}$$

or

$$N_{ia}^{r'} = -\sum_{s'=1'}^{3'} I_i^{r's'} \sum_{s=1}^{3} \ell_i^{s's} \ddot{\theta}_{ia}^{s} \tag{B50}$$

where

$$\ddot{\theta}_{ia}^{s} = -(\omega_a)^2 \overline{X}_{ia}^{s} q_a , \qquad s = 1, 2, 3 \tag{B51}$$

$$q_a = -\frac{1}{(\omega_a)^2} \sum_k \sum_{s=1}^{3} \sum_{s'=1'}^{3'} \ell_k^{s's} P_{ka}^{s} D_a^{s'} . \tag{B52}$$

It should be noted that $\overline{X}_{ia}^1$, $\overline{X}_{ia}^2$, and $\overline{X}_{ia}^3$ in Eq. (B51) are actually $\overline{X}_{ia}^4$, $\overline{X}_{ia}^5$, and $\overline{X}_{ia}^6$, respectively, due to the new notation for the transformation on the torques.

Substitute Eq. (B51) and (B52) into Eq. (B50) to get

$$N_{ia}^{r'} = -\sum_{s'=1'}^{3'} I_i^{r's'} \sum_{s=1}^{3} \ell_i^{s's} \overline{X}_{ia}^{s} \sum_k \sum_{r=1}^{3} \sum_{p'=1'}^{3'} \ell_k^{rp'} P_{ka}^{r} D_a^{p'} . \tag{B53}$$

Sum over all the mass points to get

$$N_a^{r'} = -\sum_i \sum_{s'=1'}^{3'} I_i^{r's'} \sum_{s=1}^{3} \ell_i^{s's} \bar{X}_{ia}^s \sum_k \sum_{r=1}^{3} \sum_{p'=1'}^{3'} \ell_k^{rp'} P_{ka}^r D_a^{p'} . \tag{B54}$$

From Eq. (B54) the effective moment of inertia in the r' direction in mode a due to a base translational motion in the p' direction is

$$\mathcal{J}_a^{r'p'} = \sum_i \sum_{s'=1'}^{3'} I_i^{r's'} \sum_{s=1}^{3} \ell_i^{s's} \bar{X}_{ia}^s \sum_k \sum_{r=1}^{3} \ell_k^{rp'} P_{ka}^r . \tag{B55}$$

Sum over the modes to get

$$\mathcal{J}^{r'p'} = \sum_i \sum_{s'=1'}^{3'} I_i^{r's'} \sum_{s=1}^{3} \ell_i^{s's} \sum_a \bar{X}_{ia}^s \sum_k \sum_{r=1}^{3} \ell_k^{rp'} P_{ka}^r .$$

It has been shown that for rotational motion due to a sudden translational motion of the base

$$\sum_a \bar{X}_{ja}^s \sum_k \sum_{r=1}^{3} P_{ka}^r \ell_k^{rp'} = 0 , \tag{B22}$$

so that $\mathcal{J}^{r'p'} = 0$.

For the case where the principal axes of each mass are orientated parallel to the ship's reference axes, Eq. (B55) reduces to

$$\mathcal{J}_a^{rp} = \sum_i I_i^r \bar{X}_{ia}^r P_a^p . \tag{B56}$$

Note the similarity between this equation and Eq. (53). It has been previously stated that while the range on r in Eq. (B56) is 1, 2, and 3, the $\bar{X}_{ia}^r$ terms are in fact $\bar{X}_{ia}^4$, $\bar{X}_{ia}^5$, and $\bar{X}_{ia}^6$, respectively. Therefore, Eqs. (42) and (43), which were applied to Eq. (53) after summing over the modes, cannot be used with Eq. (B56).

* * *

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13. ABSTRACT

Normal mode theory is applied to undamped linear elastic structures represented as lumped parameter systems undergoing translational motion in three directions. The derived equations are primarily concerned with the response of such structures subject to applied forces and base motions and the inertia forces required to calculate stress in each mode of vibration. Additional relationships are presented for special types of loading and for the effective mass acting in a given mode due to base motion. Similar equations are summarized in an appendix for structures with six directions of motion, namely, three translational directions and three rotational directions subject to prescribed assumptions.

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