

A NEW DEFINITION OF A FLUID PART I, THE STOKESIAN FLUID

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FOREWORD

This paper forms a part of a program aimed at derivation of the correct aerodynamic equations for moderately rarefied atmospheres, such as those at altitudes of 20,000 to 100,000 feet. The basic ideas of the author's theory of fluid dynamics are here illustrated with a simple example, The Stokesian Fluid.

While this model does not represent all the desired effects, it illustrates, without elaborate calculations, the method of development of the theory and some of the phenomena encountered. In particular, some new criteria for model experiments are discussed.

PROBLEM STATUS

This is an interim report on this problem; work is continuing.

AUTHORIZATION

NRL Problem No. P10-02R

A NEW DEFINITION OF A FLUID
Part I, The Stokesian Fluid *

§1. PRELIMINARY DISCOURSE

While most recent extensions of the Navier-Stokes equations to take account of phenomena in moderately rarefied gases are derived from the kinetic theory,⁶ † I have been able to formulate a new definition of a fluid as a particular type of continuum, whence follows a more general expression for the stress than any hitherto suggested.⁹ In particular, Burnett's equations⁴ are included as a special case. A short summary of my results, without proofs, has already appeared in print.¹⁰

Since the effects which my theory describes are extremely varied and involved, the resulting formulae for the stresses are very complicated, and the intervening manipulations are necessarily elaborate. The underlying method is, however, very simple and of such a conceptual immediacy as hardly to arouse objection in any quarter. To display these ideas concisely and unencumbered by details of calculation, I am presenting here a very simple example of their use, leading to equations intermediate in generality between the classical linear expressions and the very general ones already mentioned. This intermediate theory, in fact, was the product of my first attempt to construct a rational extension of classical fluid dynamics a year ago,⁷ and my more general theory may be regarded simply as a further development of the same ideas.

The basic method, which may be recommended for establishing any branch of phenomenological mechanics upon a sound foundation, consists of two parts. First, experience with physical fluids is summarized in a formal definition of a fluid, which is neither more nor less than the specification of the functional dependence of some of the variables upon others. It need hardly be mentioned that in such a definition no special type of function, such as a linear or quadratic form, is singled out for preference. An incidental feature, not necessary to the method but very useful in the interest of simplicity, is that but a single scalar is so specified. Second, the defining function is expanded in a power series, the form of whose terms is determined by a full use of the principles of invariance, both dimensional and tensorial. From this power series, in which any number of terms may be retained at will, the complete dynamical equations may then be derived.

§2. PROPERTIES OF CONTINUOUS MEDIA

If ds^2 be the squared element of arc length and D/Dt the symbol of material differentiation, we have rigorously

$$\frac{D}{Dt} (ds^2) = 2d_{ij} dx^i dx^j, \quad (2.1)$$

* A portion of the work described here was supported by contracts (Nos. 45-47 and 53-47) from the Office of Naval Research to the Naval Ordnance Laboratory and has already been reported elsewhere.^{7,9,10}

† See references at end of report.

where the rate of deformation tensor d_{ij} is given by

$$d_{ij} \equiv \frac{1}{2} (V_{i,j} + V_{j,i}) , \quad (2.2)$$

V_i being the velocity vector. Thus the components d_{ij} are measures of the local and instantaneous rates at which the medium is suffering deformation. They are of dimension T^{-1} , where T is a unit of time. Similarly, the components ω_{ij} given by

$$\omega_{ij} \equiv \frac{1}{2} (V_{i,j} - V_{j,i}) \quad (2.3)$$

are measures of the rate of rotation (vorticity).

The principle of conservation of mass is expressed by Euler's equation,

$$\frac{D \log \rho}{Dt} + v^i_{,i} = 0, \quad (2.4)$$

where ρ is the density, whose dimensions are ML^{-3} , M being a unit of mass and L a unit of length. The principle of conservation of momentum is expressed by Cauchy's equations,

$$T^{ij}_{,j} + f^i = \rho \frac{DV^i}{Dt}, \text{ and } T^{ij} = T^{ji}, \quad (2.5)$$

where T^{ij} is the stress tensor and f^i is the extraneous force. The dimensions of T^i_j are $ML^{-1}T^{-2}$. The mean pressure p_m is given by

$$p_m \equiv -\frac{1}{3} T^i_i. \quad (2.6)$$

Let ϵ be the internal energy (per unit mass). Then the principle of conservation of energy implies that the rate of change of the total energy (internal plus kinetic) of any material volume must equal the sum of the rate at which the extraneous force does work upon it, the rate at which the stresses upon its boundary do work, and the rate at which thermal energy flows into it from the exterior, in terms of the heat flux vector q^i . The resulting equation may be simplified by (2.5), yielding finally

$$\rho \frac{D\epsilon}{Dt} = T^i_j d^j_i - q^i_{,i}. \quad (2.7)$$

This equation may be taken as a definition of the internal energy if we please, for from it the value of ϵ for each particle may be calculated up to an arbitrary constant if the other variables be known. The dimensions of ϵ are $L^2 T^{-2}$; those of q^i are MT^{-3} .

It will further be assumed that each particle of the continuum constitutes a thermodynamic system: that is, regardless of the state of motion each particle has an equation of state,

$$\epsilon = \epsilon(\sigma, \eta), \quad (2.8)$$

where ϵ is the internal energy (per unit mass), σ is the specific volume ($\sigma \equiv 1/\rho$), and η is the entropy (per unit mass). The form of (2.8) may vary from one particle to another, but it is to remain constant in time; that is, the medium may be heterogeneous, but changes of phase, chemical reactions, etc. are excluded. Then if the pressure p and temperature θ be defined by

$$p \equiv - \left(\frac{\partial \epsilon}{\partial \sigma} \right)_{\eta}, \quad \theta \equiv \left(\frac{\partial \epsilon}{\partial \eta} \right)_{\sigma}, \quad (2.9)$$

we have

$$\frac{D\epsilon}{Dt} = \theta \frac{D\eta}{Dt} - p \frac{D\sigma}{Dt}. \quad (2.10)$$

The dimensions of p so defined are $ML^{-1}T^{-2}$, the same as the dimensions of T_j^i and of p_m . While we are here considering the entropy as a primitive variable, in accord with common physical practice we shall take a unit of temperature θ as the fundamental thermal dimension; then the dimensions of η are $L^2T^{-2}\theta^{-1}$.

If now the extra stress W_j^i be defined by

$$W_j^i \equiv p\delta_j^i + T_j^i, \quad (2.11)$$

it follows from (2.5) that

$$W^{ij} = W^{ji}. \quad (2.12)$$

Combination of (2.7), (2.10), and (2.11) yields

$$\rho \theta \frac{D\eta}{Dt} = \Phi - q^i_{,i}, \quad (2.13)$$

where the dissipation function Φ is given by

$$\Phi \equiv W_j^i dj_i. \quad (2.14)$$

Thus Φ , whose dimensions are $ML^{-1}T^{-3}$, is the rate at which mechanical work is being transformed into thermal energy. From (2.14) it follows that

$$\Phi = 0 \text{ if } dj_j^i = 0. \quad (2.15)$$

The decomposition (2.11) is of primary physical significance, for it resolves the stress into two portions, the first, $-p\delta_j^i$, being that whose work is always mechanically recoverable, and the second, W_j^i , being that whose work † in deforming the medium is irretrievably lost as heat, which may be stored in terms of entropy or conducted away. It cannot be too strongly emphasized that (1) no resolution of the stress other than (2.11) can yield the simple energy equation (2.13) and the consequent simple physical interpretation for Φ , (2) both p and p_m are defined variables whose equations of definition show them to represent quite different physical concepts and there is no reason whatever a priori to suppose them equal or even related to each other, and (3) the dissipation function Φ is a defined quantity which exists in all continua and requires no special hypotheses.§ The equations presuppose the medium to be compressible; the final equation (2.13) and the end results of this paper can be shown to be valid for incompressible fluids, but it would diminish clarity to explain here the necessary adjustments in the intermediate steps.

† It would not be correct to conclude that from $W_j^i \neq 0$ it necessarily follows that dissipation takes place, for it is possible that W_j^i does no work, i.e., $W_j^i dj_i = 0$. An example is mentioned at the end of §5.

§ Φ is not to be confused with Rayleigh's dissipation function which, for continuous media, could better be called a "stress potential" and which exists only in very special circumstances.

§3. PHENOMENOLOGICAL PROPERTIES OF FLUIDS

The phenomenological properties of fluids are familiar to everyone and may be summarized in a few words. First, in a compressible fluid suffering no deformation the only stress is that arising from the pressure p as computed from (2.9). Hence, by (2.11),

$$W^i_j = 0 \text{ if } d^i_j = 0. \quad (3.1)$$

Secondly, a fluid offers resistance to change of shape while that change is being effected, but no resistance to permanent deformation (without change of volume) or to rigid rotation. When in motion, a fluid tends to adhere to solid boundaries, and hence by (2.1) stresses W^i_j must arise. We then expect these viscous stresses W^i_j to be functions of the rate of deformation d^i_j alone of all kinematic variables; in particular, W^i_j must be independent of the rotation ω_{ij} and the strain with respect to any initial configuration. From (2.14) we then have

$$\Phi = \Phi(d_{ij}). \quad (3.2)$$

Thus far we have essentially repeated the classical inferences of Stokes^{1, 2} except that he (in effect) immediately assumed the function (3.2) to be quadratic. It is far preferable to carry full generality as long as possible, reserving approximations until after the structure is complete.

Now the phenomenon of resistance requires a nearer analysis. Consider the conceptual experiment of Maxwell, in which a viscous fluid is confined between infinite parallel plane boundaries which are in constant relative motion at a velocity v . Then the walls experience a retarding force according to the formula proposed by Newton as a hypothesis:

$$\text{force} = \mu \frac{v}{\text{distance between walls}} \cdot (\text{area in contact}), \quad (3.3)$$

where μ is a quantity, called the coefficient of viscosity, which is independent of other magnitudes in the formula but is dependent upon the particular fluid. The dimensions of μ are $ML^{-1}T^{-1}$. We are not concerned here with the exactness of (3.3); it is sufficient that a very crude actual experiment shows it to be a fair approximation, for our only purpose at the moment is to indicate that a quantity μ of dimensions $ML^{-1}T^{-1}$ should enter the definition of a fluid.

The resistance offered by fluids such as oil and molasses is a manifest function of the temperature θ , and more generally may be supposed to be a function of the thermodynamic state:

$$\mu = \mu(p, \theta). \quad (3.4)$$

But this relation connects three quantities composed of the three independent dimensions $ML^{-1}T^{-1}$, T , and θ , and hence it cannot subsist unless some of the variables are struck out or other quantities composed of the same dimensions are inserted. The only alternative to $\mu = \text{const.}$, then, is to introduce new quantities. At first sight it would seem reasonable to propose

$$\frac{\mu}{\mu_0} = f\left(\frac{p}{p_0}, \frac{\theta}{\theta_0}\right), \quad (3.5)$$

where μ_0 , p_0 , and θ_0 are certain reference values of μ , p , and θ characteristic of the particular fluid. But if there be quantities μ_0 and p_0 representing properties of the fluid, then the quantity μ_0/p_0 must also represent a property of the fluid. Now μ_0/p_0 is of the

dimension T, so that such a fluid would necessarily exhibit time dependent phenomena, such as relaxation effects. These effects are expressly excluded by our initial statement that it is contrary to experience for a fluid to respond to strain from a past configuration. Hence, we may either adopt the usual expedient of excluding p altogether from (3.4), as is done implicitly at the outset in the kinetic theory of gases, or, embracing a more liberal view, we may seek another quantity of the dimension of p, which is readily found in the mean pressure p_m given by (2.6). In place of (3.5) we then have

$$\frac{\mu}{\mu_0} = f \left(\frac{p}{p_m}, \frac{\theta}{\theta_0} \right) . \quad (3.6)$$

All the remarks of this section are heuristic, serving only to motivate the formal definition of a fluid which follows now.

§4. FORMAL DEFINITION OF A STOKESIAN COMPRESSIBLE FLUID

Since the qualities of fluids discussed in the preceding section are the same as those considered by Stokes, a material possessing these qualities in their most general form may well be called a Stokesian fluid. Accordingly, a Stokesian compressible fluid is a continuous medium such that

- I. There exist material constants, μ_0 and θ_0 , called respectively the reference viscosity and the reference temperature, whose dimensions are

$$\dim \mu_0 = ML^{-1}T^{-1} \quad (4.1)$$

and

$$\dim \theta_0 = \theta . \quad (4.2)$$

- II. The dissipation function, Φ , is of the functional form

$$\Phi = \Phi (\mu_0, \theta_0, p_m, p, \theta, d_{ij}) \quad (4.3)$$

and is an analytic function of the components d_{ij} .

- III. If $d_{ij} = 0$ then (4.4)

$$W^j_k = 0 .$$

A fluid will be said to be isotropic if Φ is invariant in form as a function of the components d_{ij} . For isotropic fluids then, Φ must depend upon the d_{ij} only through their invariants, a fundamental set of which is D_1, D_2, D_3 , where

$$D_1 \equiv d^i_i, D_2 \equiv d^i_j d^j_i, D_3 \equiv d^i_j d^j_k d^k_i . \quad (4.5)$$

For isotropic fluids (4.3) becomes then

$$\Phi = \Phi (\mu_0, \theta_0, p_m, p, \theta, D_1, D_2, D_3) . \quad (4.6)$$

It is by no means essential to the present method, and in fact rather complicates than simplifies the ensuing calculations, to restrict attention to isotropic fluids, but there is little physical interest in the more general case.

§5. POWER SERIES FOR THE DISSIPATION FUNCTION

Since Φ is assumed to be an analytic function of the d_{ij} , (4.6) may be put into the form

$$\begin{aligned} \Phi = & \alpha + \beta D_1 + \lambda D_1^2 + 2\mu D_2 \\ & + TD_1^3 + UD_1 D_2 + VD_3 + \dots, \end{aligned} \quad (5.1)$$

where the coefficients $\alpha, \beta, \lambda, \mu, T, U, V, \dots$ are functions of μ_0, θ_0, p_m, p , and θ , and where the terms are written in the order of the powers of the components d_{ij} which they contain. From (2.15) it follows that $\alpha=0$. From (2.14) we then have

$$\begin{aligned} W_j^i = & \beta \delta_j^i + \lambda d_k^k \delta_j^i + 2\mu d_j^i \\ & + T (d_k^k)^2 \delta_j^i + (U - W) d_k^l d_l^k \delta_j^i \\ & + W d_k^k d_j^i + V d_k^i d_j^k + \dots \end{aligned} \quad (5.2)$$

By (4.4) it follows that $\beta=0$. It is interesting to observe that one portion of the stress, namely $W(d_k^k d_j^i - d_k^l d_l^k \delta_j^i)$, can do no work in any motion. The three terms whose coefficients are λ, T , and $U - W$ act simply as modifications of the static pressure p .

§6. THE FORM OF THE COEFFICIENTS

Comparing dimensions in (5.2) we have at once

$$\dim \lambda, \mu = ML^{-1}T^{-1}, \quad (6.1)$$

$$\dim T, U, V, W = ML^{-1}. \quad (6.2)$$

Hence, by (4.1) λ/μ_0 and μ/μ_0 are dimensionless. But

$$\lambda = f(\mu_0, p, p_m, \theta, \theta_0). \quad (6.3)$$

This relation connects six quantities composed of the three fundamental dimensions $ML^{-1}T^{-1}$, T , and θ and hence must yield a relation among $6 - 3 = 3$ dimensionless ratios of these quantities, viz.,

$$\frac{\lambda}{\mu_0} = f\left(\frac{p}{p_m}, \frac{\theta}{\theta_0}\right). \quad (6.4)$$

A like relation holds for μ .

Taking up now the terms of higher order, we similarly conclude that the equation,

$$T = T(\mu_0, p, p_m, \theta, \theta_0), \quad (6.5)$$

must reduce to relation among three dimensionless ratios. By (6.2), one such ratio is Tp/μ_0^2 . Hence the counterpart of (6.4) is now

$$\frac{Tp}{\mu_0^2} = f\left(\frac{p}{p_m}, \frac{\theta}{\theta_0}\right), \quad (6.6)$$

or

$$T = \frac{\mu_0^2}{p} f\left(\frac{p}{p_m}, \frac{\theta}{\theta_0}\right). \quad (6.7)$$

Like relations hold for U, V, and W. Hence we obtain as the final form for (5.2)

$$\begin{aligned}
 W_j^i = & \lambda d_k^k \delta_j^i + 2\mu d_j^i \\
 & + \frac{\mu \nu^2}{p} \left[A \left(d_k^k \right)^2 \delta_j^i + B d_k^l d_l^k \delta_j^i \right. \\
 & \left. + C d_k^k d_j^i + D d_k^i d_j^k \right] + \dots, \quad (6.8)
 \end{aligned}$$

where λ and μ are the ordinary coefficients of viscosity given by (6.4), while A, B, C, D are dimensionless second order coefficients of viscosity which are functions of p/p_m and θ/θ_0 alone.

Terms of still higher order in (6.8) may be written down at will; the first term omitted above, for example, is $\mu_0^3 E (d_k^k)^3 \delta_j^i / p^2$, where E is one of the six dimensionless third order coefficients of viscosity. Thus we have the full expansion for the stresses according to Stokes' theory.**

§7. NEW SIMILARITY PARAMETERS

If the expression (6.8) for the stress be adopted, in addition to the usual dimensionless parameters governing dynamic similarity, there are four more, viz.,

$$\frac{\mu B \delta}{p}, \frac{A}{B}, \frac{C}{B}, \frac{D}{B} \quad (7.1)$$

where δ is a rate of deformation, frequency, expansion, vorticity, or reciprocal of a time which is characteristic for the class of motions being compared. Now in the classical linear theory no such quantity occurs in the complete set of parameters for local dynamic similarity (although for certain motions such a quantity may occur in parameters governing similarity at boundaries). In a general Stokesian fluid however, either a time, a frequency, an expansion, or a vorticity must always be taken into account in considerations of similarity. And furthermore this quantity is the only new dimensional parameter which need be employed no matter how many higher order terms in (6.8) are retained, for the only new characteristic numbers to which these give rise are ratios of the several dimensionless higher order coefficients, viz., E/B , etc. Thus the classical linear theory altogether neglects one effect of primary physical significance. There are, in fact, two qualities of the classical theory which are very striking. The first is that, while the vorticity is expressly excluded from the defining formulae for the stress, it is nevertheless of the greatest dynamical importance, occurring quite analogously to the expansion in the Navier-Stokes equations and in the basic energy equation. The second is that, although the theory represents fluid dynamics, all the variables which occur in the parameters governing dynamic similarity are static quantities (in the usual Eulerian description), so that it would seem that time rates are of secondary importance in the resulting structure even though they alone are considered in its basic dynamical laws. Both these strange qualities, as

** Stokes began¹ by formulating a principle equivalent to our (4.6). By a certain argument regarding the ultimate molecules of the fluid, he then concluded that the principal components of W_j^i should be linear functions of the principal components of d_{ij} . The present theory, being purely phenomenological, consists in a full elaboration of Stokes' principle through employment of the method of dimensions, a method not developed in Stokes' time.

we now see, result only from omitting the higher order terms in (6.8) and thus are not proper to the Stokesian concept but are mere accidents of the classical linear theory. From a physical point of view we may say that, in the general theory here presented, time rates play the dominant part we should expect from the concept of fluidity and that, moreover, this part is essentially the same no matter how many terms in the series for the stresses are retained so long as they go past the linear terms.

To grasp the significance of this new effect, consider the case in which the higher order terms in (6.8) are not negligible, yet act only as small perturbations. Then if motions of two different Stokesian fluids be compared, it is not unreasonable to suppose that the effect of different values of the higher order coefficients A, B, \dots for the two fluids may be neglected, so that the number \mathcal{V} given by

$$\mathcal{V} \equiv \frac{\mu \delta}{p} \quad (7.2)$$

becomes the only characteristic parameter which need be added to govern the effects of the perturbing terms. Suppose now we have two identical vessels, containing specimens of the same fluid, which we place in a vacuum at the same altitude above sea level. Let cocks be opened so as to permit the fluids to escape, and at suitable corresponding points in each issuing jet let measurements be taken. Suppose the velocity, density, and temperature at these corresponding points are found to be the same. Then all the criteria for dynamic similarity in the classical theory are satisfied. And while in the absence of a uniqueness theorem we cannot actually prove, yet we may with some confidence expect, that the two flows will be altogether identical. In the case of the Stokesian fluid, in order to gain the same expectation, we must first verify that, say, the corresponding vorticities are equal, or that the total times of efflux are equal, etc. In other words, two samples of a Stokesian fluid, each issuing from one of two identical vessels, and under identical static conditions at a typical point, may require quite different times of efflux. This greater freedom arises mathematically from the fact that the dynamical equations for the Stokesian fluid are of higher order than are the Navier-Stokes equations.

More generally, a condition for the classical linear expression for W^i_j to be valid is that, locally,

$$\mathcal{V} \equiv \frac{\mu \sqrt{A^2 + B^2 + C^2 + D^2} \sqrt{d^i_j d^j_i}}{p} \ll 1. \quad (7.3)$$

Thus, for a given Stokesian fluid the Navier-Stokes equations become a poorer approximation as the pressure is lowered or as the rate of deformation is increased. A criterion for discarding terms of higher order than those actually written down in (6.8) is

$$\mathcal{V} \frac{\sqrt{E^2 + \dots}}{\sqrt{A^2 + \dots}} \ll 1. \quad (7.4)$$

§8. MEAN PRESSURE AND THERMODYNAMIC PRESSURE

Most treatment of fluid dynamics employ the assumption $p = p_m$, although it is supported neither by sound theory nor by experiment and effects but slight simplification of the resulting equations. By (6.4) and (6.7) an immediate consequence is that the coefficients of viscosity $\lambda, \mu, A, B, \dots$ all become independent of pressure. It is sometimes said to be proved in the kinetic theory of gases that μ is independent of p . Since the basic definitions of that theory, at least as usually employed, imply $p = p_m$ at the outset, the result is not surprising.

In the general case, by (2.6), (2.11), and (6.8) we have

$$\begin{aligned}
 3(p - p_m) &= (3\lambda + 2\mu)d^k_k \\
 &+ \frac{\mu^2}{p} \left[(3A + C)(d^k_k)^2 \right. \\
 &\left. + (3B + D) d^i_j d^j_i \right] + \dots
 \end{aligned} \tag{8.1}$$

In the classical linear theory we have $p \geq p_m$ in an expanding motion and $p \leq p_m$ in a contracting motion; but (8.1) shows that if the number \mathcal{V} , as given by (7.2), be sufficiently large, it is possible that p can exceed p_m by a very large amount even in a motion of contraction except in the case when $p = p_m$ always. Necessary and sufficient conditions for $p = p_m$ are

$$\begin{aligned}
 3\lambda + 2\mu &= 0, \\
 3A + C &= 0, \\
 3B + D &= 0, \dots
 \end{aligned} \tag{8.2}$$

of which the first is the classical Stokes relation.

§9. CONTRAST WITH ELASTICITY

From the formal analogy between Hooke's Law and the classical linear terms of (6.8) it is sometimes erroneously concluded that a proper generalization of the equations of elasticity simultaneously yields a proper generalization of the Navier-Stokes equations. Since fluids and elastic solids, being the very extremes of possible types of deformable continua, embody entirely different physical concepts, such an analogy would be very surprising, and it is in fact an accident of the classical linear theories. Of the many generalizations of elasticity, we may select Murnaghan's³ as the most rational and elegant (and, not surprisingly, the best borne out by experiment). Comparison of (6.8) and Murnaghan's expression for the stress shows their entirely different form and character as soon as the first terms of higher order are considered - a difference reflecting the fact that strain is dimensionless while rate of deformation is of dimension T^{-1} . In the classical linear theories the elastic stress is the gradient of the elastic energy with respect to the strain, while the viscous stress is the gradient of Rayleigh's stress potential with respect to the rate of deformation. Neither property carries over in a rational generalization. In Murnaghan's theory of elasticity, the stress remains expressible in terms of derivatives of the elastic energy with respect to the strain but no longer as a simple gradient. In the present theory of fluids the viscous stresses are no longer derivable by differentiation, and no Rayleigh function exists. To prove this, we need only observe that, since a Rayleigh function is a scalar, it can contain (in the isotropic case) at most three terms of third order in the components d_{ij} ; for D_1^3 , $D_2 D_1$, and D_3 form a complete set of third order invariants of d_{ij} . An expression derived from a Rayleigh function by differentiation can then contain at most three third order coefficients of viscosity, while from (6.8) there are four. Our derivation of (6.8) from the scalar dissipation function (5.1) employed tensor division, a process which generally introduces new arbitrary constants each time it is employed, while differentiation at most leaves their number unchanged.

These differences of form between elasticity and fluid dynamics are only to be expected because of the different physical situations considered, and the nonlinear theories are thus of greater conceptual clarity than the classical linear theories, whose simplicity is misleading.

§10. CONTRAST WITH THE REINER-RIVLIN THEORY OF FLUIDS

There is not space here to present a full analysis of the theories of Reiner⁵ and Rivlin.⁸ Let it suffice to state now that these theories (which are essentially the same although the former is restricted to compressible, the latter to incompressible fluids) implicitly involve a relaxation time t_0 , which we expressly forbade in the heuristic formulation of the present theory (§3). Thus the Reiner-Rivlin theory and the present one embody quite different physical concepts. The present theory includes correction terms appropriate for phenomena at very low pressure, very high viscosity, or very high rates of deformation, while the Reiner-Rivlin theory includes correction terms appropriate to a departure from perfect fluidity. Both theories, of course, reduce to the classical one as a first approximation.

§11. GENERALIZED STOKESIAN FLUIDS

The Stokesian fluid as presented here is not susceptible of several phenomena observed in rarefied gases. One such is thermal creep, in which a stress arises even in a fluid at rest by the sole presence of a thermal gradient. In order to include such effects we may now extend our basic definition of §4 by permitting Φ to depend upon the various gradients

$$\theta_{,i}, \theta_{,i_1 i_2}, \dots, \theta_{,i_1 i_2 \dots i_n}; p_{,i}, p_{,i_1 i_2}, \dots, p_{,i_1 i_2 \dots i_n}; f_{i_1, i_2}, \dots, f_{i_1, i_2, \dots, i_n};$$

up to some specified order n . The analysis of §§ 5-6 may then be generalized, but it will be found (after considerable labor) that the coefficients of all terms involving these gradients are necessarily zero. The difficulty lies in our having neglected a more fundamental property of fluids which may influence the mechanism of dissipation of energy. This property is the dependence of pressure upon temperature and density, which for perfect gases assumes the form $p = R\rho\theta$, where $\dim R = L^2 T^{-2} \theta^{-1}$. More general equations of state involve a modulus of similar dimensions. If the existence of such a modulus be added in part 1 of the definition of §4, and the modulus itself be inserted as one of the arguments of Φ along with the gradients mentioned above, then the principles of §§5-6 may be applied again, this time yielding a positive result.

The formulae so derived, however, still fail to agree with Burnett's equations, in which not only the rate of deformation but also the vorticity can produce stresses. Now stresses arising from vorticity alone, or even stresses which can vary with the vorticity when the rate of deformation is kept constant, form no part of Stokes' concept of a fluid, and I must confess that in their occurrence in the results derived by Enskog's method of integrating the Maxwell-Boltzmann equation I find occasion for questioning the validity of that method, or the particular application of it. Nevertheless, if such stresses are desired, we may insert the rotation ω_{ij} into (4.3) along with the d_{ij} ; there results finally the more general theory of fluids mentioned in my opening sentence. If desired, higher velocity gradients up to any specified order may also be introduced. The calculations are very much more elaborate, but the method is quite the same. A noteworthy feature of this more general theory is that it too reduces to the classical theory as a linear approximation.

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