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THE CRACK-EXTENSION-FORCE FOR A CRACK AT A FREE SURFACE BOUNDARY

G. R. Irwin

Mechanics Division

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NAVAL RESEARCH LABORATORY
Washington, D.C.

ABSTRACT

In previous work it has been assumed the stress-intensity-factor $\mathcal{K}$ from two dimensional elastic theory analysis for an edge crack in tension is the same as that for a double length completely embedded crack. The influence of the side boundary free surface upon stresses near an edge crack in tension was studied and evaluated. To an accuracy of several percent the stress-intensity-factor $\mathcal{K}$ for an edge crack of length, a, is ten percent larger than the $\mathcal{K}$ value, $\sigma\sqrt{a}$, which pertains to an embedded crack of length 2a subjected to a normal tensile stress σ.

PROBLEM STATUS

This is an interim report. Work on this problem is continuing.

AUTHORIZATION

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INTRODUCTION

A fracture "fail safe" strength analysis procedure has been developed (1,2) using the concepts crack-extension-force and crack-edge stress-intensity-factor, designated $\mathcal{G}$ and $\mathcal{K}$ respectively. These parameters are associated by the equations

$$K^2 = \frac{E\mathcal{G}}{\pi}, \text{ for plane stress}$$

and

$$K^2 = \frac{E\mathcal{G}}{(1 - \nu^2)\pi}, \text{ for plane strain}$$

where E is Young's Modulus and ν is Poisson's ratio.* In the domain of applicability of the above strength analysis procedure it is permissible to ignore plastic strains. In this report only homogeneous isotropic solids obeying linear elastic stress-strain relations are considered.

Exact stress analysis solutions permitting calculation of $\mathcal{G}$ and $\mathcal{K}$ exist for systems of colinear two-dimensional cracks (3) and for an embedded crack in the shape of a circular disc (4). In the latter case, one axis of principal extensional stress remote from the crack must be perpendicular to the plane containing the crack and the other two principal stresses must be equal. Although it is possible to write general equations for the stresses and strains (assumed linear) at the edge of any embedded crack (5), exact equations relating $\mathcal{G}$ and $\mathcal{K}$ to the applied loads and boundary conditions are known only for the above restricted group of problems.

A situation frequently encountered both in laboratory test and in service experience is that of a crack extending into a solid from a free surface where the stress state is primarily one of tension parallel to the free surface. A straight crack extending inward at right angles to a free surface possesses $\mathcal{K}$ and $\mathcal{G}$ values which are thought to be similar to those of a larger embedded crack composed of the real crack plus its reflection across the free surface. This equivalence is thought to be a good approximation. Because the larger embedded crack is symmetrical about the free surface, the stress system will subject that surface only to normal forces. It is believed the contribution of these normal forces to the values of $\mathcal{K}$ and $\mathcal{G}$, if calculated, would be small since, at the crack edge, they primarily influence the magnitude of the stresses parallel to the crack whereas $\mathcal{K}$ and $\mathcal{G}$ depend only upon stresses perpendicular to the crack plane.

*Those who possess NRL Report 4956, "Crack-Extension-Force Near a Riveted Stiffener," by J. P. Romualdi, J. T. Frasier, and G. R. Irwin, October 1957, will notice that the symbol K was used with the same meaning as the symbol $\mathcal{K}$ in this present report. The symbol $\mathcal{K}$ is preferable because it emphasizes the close relation to $\mathcal{G}$ and avoids confusion with the symbol $K = \sqrt{E\mathcal{G}}$ which has been used in other reports from this Laboratory and from other laboratories.

The work reported here was done in order to provide some quantitative support for this reasoning. This report will discuss errors in $\mathcal{K}$ and $\mathcal{G}$ which might result from assuming an edge crack to be equivalent to the double size embedded crack as discussed above. It is assumed that the edge crack under consideration extends inward at right angles from a free surface, is two dimensional, and is remote from any other free surface.

THE EXACT SOLUTION

In the Cartesian coordinate system to be employed the free surface is the y-z plane and the crack extends along the x-z plane from $x = 0$ to $x = a$. At points in the semi-infinite solid remote from the crack the stress system is assumed to be

$$\sigma_y = \sigma, \quad \sigma_x = 0, \quad \sigma_z = 0.$$

On the y-z plane

$$\sigma_y = \text{function of } y$$

$$\sigma_z = \nu \sigma_y$$

$$\sigma_x = \tau_{xy} = \tau_{yz} = 0.$$

On the crack surface

$$\sigma_x = \text{function of } x$$

$$\sigma_z = \nu \sigma_x$$

$$\sigma_y = \tau_{xy} = \tau_{xz} = 0.$$

Consider the stresses σ_y along $y = 0$ due to a pair of line forces P applied to the free surface at $y = \pm y_0$ as shown in Fig. 1. From known solutions to the problem one obtains (Appendix A)

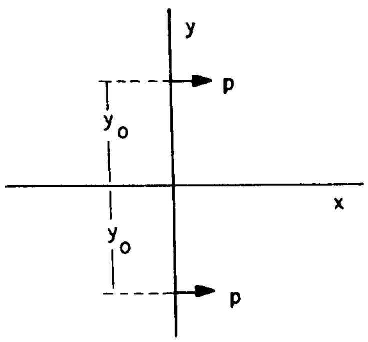

Fig. 1 - Pair of line pressures P against free surface

$$\sigma_y = -\frac{4 P x y_0^2}{\pi (x^2 + y_0^2)^2}. \tag{1}$$

Consider next the stresses σ_x along $x = 0$ due to a double pair of splitting forces Q in a crack of length 2a as shown in Fig. 2. From known solutions one obtains (Appendix B)

$$\sigma_x = \frac{2 Q y \sqrt{a^2 - b^2}}{\pi (y^2 + b^2) \sqrt{y^2 + a^2}} \left(\frac{2y^2}{y^2 + b^2} + \frac{y^2}{y^2 + a^2} - 2 \right). \tag{2}$$

The third stress system needed is the familiar one for a crack of length 2a subjected to stresses $\sigma_y = \sigma$ and $\sigma_x = 0$ at points remote from the crack as shown in Fig. 3. Along $x = 0$, from known solutions of the problem one obtains (Appendix C)

$$\sigma_x = \sigma \left[\frac{2y}{\sqrt{y^2 + a^2}} - \frac{y^3}{(y^2 + a^2)^{3/2}} - 1 \right]. \tag{3}$$

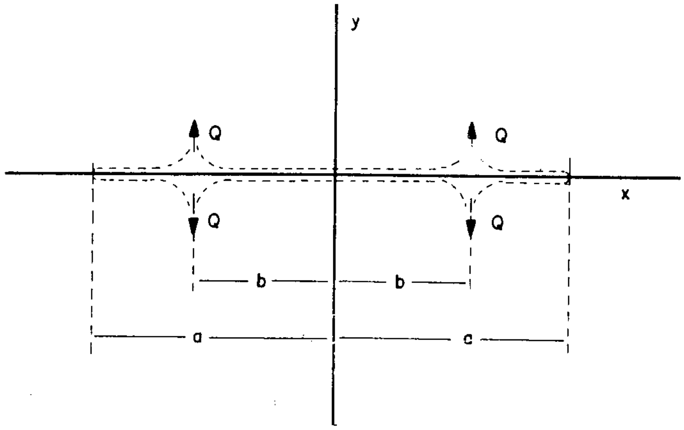

Fig. 2 - A double pair of splitting forces Q within a crack

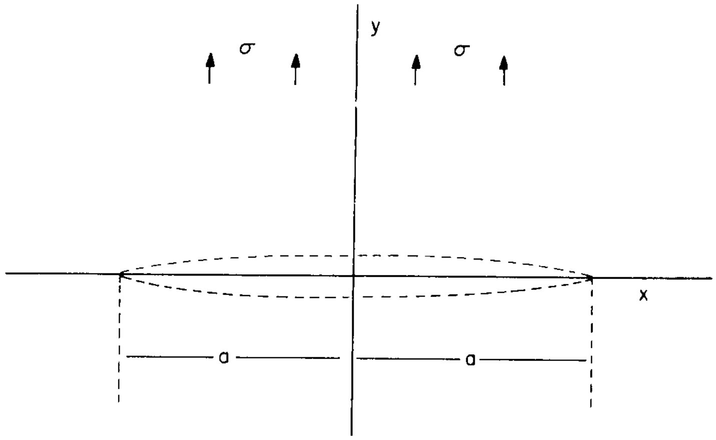

Fig. 3 - A central crack subjected to uniaxial tension

By suitable superposition of stress systems similar to those of Figs. 1, 2, and 3 it is conceptually possible to satisfy free surface conditions on $x = 0$ and on the crack plane for $|x| < a$ while having a constant stress $\sigma_y = \sigma$ applied remote from the crack. To do this it is necessary to assume a system of line pressures $Q(x)$ in the crack and $P(y)$ on $x = 0$ such that

$$\int_0^a Q(b) \frac{2\sqrt{a^2 - b^2}\, y}{\pi (y^2 + b^2)\sqrt{y^2 + a^2}} \left(\frac{2y^2}{y^2 + b^2} + \frac{y^2}{y^2 + a^2} - 2 \right) db + \sigma \left(\frac{2y}{\sqrt{y^2 + a^2}} - \frac{y^3}{(y^2 + a^2)^{3/2}} - 1 \right) = P(y) \quad (4)$$

$$\int_0^\infty P(y) \frac{(-4)\, b\, y^2}{\pi (b^2 + y^2)^2}\, dy = Q(b). \quad (5)$$

To see that this pair of integral equations solves the problem, consider superposition of the stress σ at ∞ as in Fig. 3 and of the pressures $P(y)$ on the plane $x = 0$. Superposition of these two stress systems must result in the development of normal stresses equal to $Q(x)$ along the crack. If now one adds a third stress system by putting pressures in the crack equal to $Q(x)$, the normal stresses along the crack will vanish. At the same time stresses equal to

$$P(y) - \sigma\left[\frac{2y}{\sqrt{y^2 + a^2}} - \frac{y^3}{(y^2 + a^2)^{3/2}} - 1\right]$$

will be added on $x = 0$. Thus the normal stresses originally applied on $x = 0$ also are cancelled. Only normal stresses are considered in meeting free surface boundary conditions, because τ'_{xy} is zero on the coordinate axes in the above stress systems.

CALCULATION OF STRESS-INTENSITY-FACTOR $\mathcal{K}$

Only stress systems which possess a singularity at the end of the crack contribute to the $\mathcal{K}$ and $\mathcal{G}$ values. The contribution to the $\mathcal{K}$ value of the stress σ at infinity as in Fig. 3 is $\sigma\sqrt{a}$. The pressures $P(y)$ on $x = 0$ add nothing to the $\mathcal{K}$ value. The contribution from the pressures $Q(x)$ in the crack is

$$\mathcal{K}' = \int_0^a \frac{2Q(b)\sqrt{a}}{\pi\sqrt{a^2 - b^2}}\,db\,. \tag{6}$$

The total $\mathcal{K}$ value then becomes

$$\mathcal{K} = \sigma\sqrt{a} + \mathcal{K}'\,. \tag{7}$$

Thus it is the value of $\mathcal{K}'$ relative to $\sigma\sqrt{a}$ which is needed for the purpose of this report.

A first approximation $\mathcal{K}_1'$ to the value of $\mathcal{K}'$ may be obtained by assuming $Q(b)$ to be zero in the first of the pair of integral equations. Then

$$P_1(y) = \sigma\left[\frac{2y}{\sqrt{y^2 + a^2}} - \frac{y^3}{(y^2 + a^2)^{3/2}} - 1\right] \tag{8}$$

$$Q_1(b) = \int_0^\infty P_1(y)\,\frac{(-4)\,by^2}{\pi(b^2 + y^2)^2}\,dy \tag{9}$$

and

$$\mathcal{K}_1' = \int_0^a \frac{2Q_1(b)\sqrt{a}\,db}{\pi\sqrt{a^2 - b^2}}\,. \tag{10}$$

After carrying out the integration with respect to b one finds

$$\mathcal{K}_1' = -\frac{4\sigma\sqrt{a}}{\pi^2}\int_0^\infty \frac{du}{(1 + u^2)}\left[1 + \frac{u^2}{\sqrt{1 + u^2}}\sinh^{-1}\frac{1}{u}\right]\left[\frac{u(2 + u^2)}{(1 + u^2)^{3/2}} - 1\right]. \tag{11}$$

Estimation of this integral by numerical procedures gave the value 0.2337. Thus

$$\mathcal{K}_1' = (0.2337)\left(\frac{4}{\pi^2}\right)\sigma\sqrt{a}$$

$$= (0.0947)\,\sigma\sqrt{a}\,.$$

Figure 4 shows the function $Q_1(x)$. The stress-intensity-factor corresponding to this distribution of pressures in a crack of length $2a$ is $\mathcal{K}_1'$. The effectiveness of $Q_1(x)$ upon stresses near $x = a$ is equal to that of a constant pressure σ extending from $x = -0.15a$ to $x = 0.15a$ as shown by the dashed lines in Fig. 4.

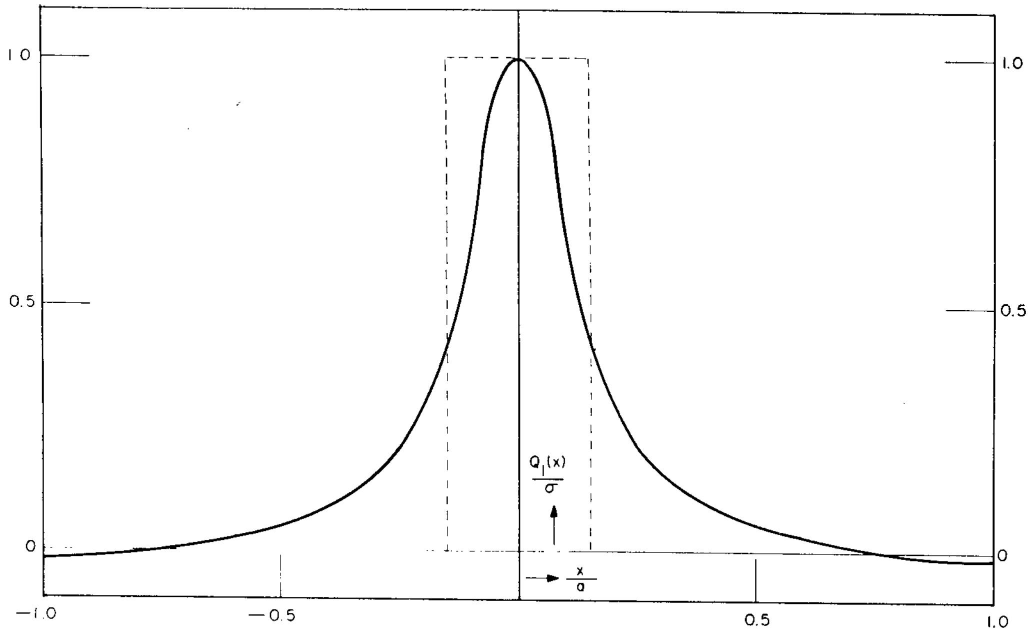

Fig. 4 - The pressure distribution $Q_1(x)$ as a function of position in the crack.

As one might anticipate, the introduction of a free surface bisecting a crack of length 2a results in a moderate increase of the stress-intensity-factor $\mathcal{K}$. The amount of the increase from an exact solution of the problem would not be expected to differ from $\mathcal{K}_1'$ by more than 10 percent.

An exact stress analysis is known for an edge crack with rounded corners from the work of Neuber (6). To obtain values of $\mathcal{K}$ for stress systems around grooves simulating cracks from notch stress studies it is useful to note that for a notch of nearly zero flank angle

$$\mathcal{K} = \lim_{\rho \to 0} \frac{1}{2} \sigma_m \sqrt{\rho}$$

where σ_m is the largest tensile stress at the root of the notch and ρ is the notch root radius of curvature. Applying this relation to the expression for σ_m given by Neuber as representing a shallow external notch under tension one finds

$$\mathcal{K} = \frac{3}{2\sqrt{2}} \sigma \sqrt{a} = 1.061\, \sigma \sqrt{a}.$$

For comparison purposes this result is only of qualitative value because the flank angle for Neuber's shallow external notch with rounded corners is not small. Neuber reasoned that the stress analysis for a narrow groove at right angles to the free surface may be assumed to be the same as that of a double-length embedded slot along the same lines as those discussed in the introduction of this report. It would appear, however, from the calculation of $\mathcal{K}_1'$ that the stress intensity from elastic theory near the end of such an edge crack is underestimated both by the double-length embedded-crack approximation and the Neuber rounded-corner shallow-notch example.

CONCLUSION

Fracture strength estimates and experiments are such that uncertainty in $\mathcal{K}$ and $\mathcal{G}$ values of 3 and 6 percent respectively would scarcely handicap the work. From the considerations discussed it is concluded the $\mathcal{K}$ value for an edge crack of length a subjected to tensile stress σ can be assumed to be

$$\mathcal{K} = 1.1\, \sigma \sqrt{a}$$

with sufficient accuracy for most experimental purposes. The corresponding plane stress value of the crack extension force is

$$\mathcal{G} = \frac{1.2\, \pi \sigma^2 a}{E}.$$

REFERENCES

1. Irwin, G. R., and Kies, J. A., "Fracturing and Fracture Dynamics," Welding Journal, Res. Suppl., 31:95-s (February 1952)

2. Irwin, G. R., "Fracture Strength," Report of NRL Progress, November 1957, p. 10

3. Irwin, G. R., "Analysis of Stresses and Strains Near the End of a Crack Traversing a Plate," J. Appl. Mechanics 24 (No. 3):361 (1957)

4. Sneddon, I. N., "The Distribution of Stress in the Neighborhood of a Crack in an Elastic Solid," Proc. Roy. Soc. (London) 187:229 (1946)

5. Irwin, G. R., "Fracture," in "Encyclopedia of Physics," Vol. VI, Berlin:Springer (forthcoming)

6. Neuber, H., "Kerbspannungslehre: Grundlagen für genaue Spannungsrechnung," Berlin:Springer, 1937

APPENDIX A

Derivation of Eq. (1)

We will use the Westergaard equations for stresses derived from the Airy stress function, Re $\bar{\bar{Z}}$ + y Im $\bar{Z}$, (where $\bar{Z}$, Z, and Z′ are successive derivatives of $\bar{\bar{Z}}$, a function of x + iy):

$$\sigma_x = \mathrm{Re}\ Z - y\ \mathrm{Im}\ Z', \qquad \sigma_y = \mathrm{Re}\ Z + y\ \mathrm{Im}\ Z', \qquad \tau_{xy} = -\ y\ \mathrm{Re}\ Z'.$$

If we choose

$$Z = \frac{P}{i\pi\,(\zeta - x_0)}$$

where

$$Z = Z(\zeta), \qquad \zeta = x + iy$$

then the physical situation represented is one of a localized pressure P (lb/in. in the z direction) pressing upward against the lower free surface of a semi-infinite solid consisting of $y > 0$ (Fig. A1). If we choose

$$Z = \frac{P}{i\pi}\left(\frac{1}{\zeta - x_0} + \frac{1}{\zeta + x_0}\right) = \frac{2P\zeta}{i\pi\,(\zeta^2 - x_0^2)}$$

we add another line pressure P at $x = -x_0$. The derivative is

$$Z' = Z\left(\frac{1}{\zeta} - \frac{2\zeta}{\zeta^2 - x_0^2}\right).$$

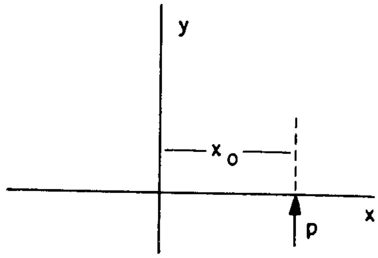

Fig. A1 - Line pressure P against lower free surface

Along $x = 0$

$$Z(iy) = -\frac{2Py}{\pi(y^2 + x_0^2)}$$

$$Z'(iy) = Zi\left(-\frac{1}{y} + \frac{2y}{y^2 + x_0^2}\right)$$

$$y\ \mathrm{Im}\ Z' = Z\left(\frac{2y^2}{y^2 + x_0^2} - 1\right)$$

$$\sigma_x = \mathrm{Re}\ Z - y\ \mathrm{Im}\ Z' = Z\left(2 - \frac{2y^2}{y^2 + x_0^2}\right).$$

Thus

$$\sigma_x = -\frac{4 P y x_0^2}{\pi (y^2 + x_0^2)^2}.$$

To convert to σ_y as in Fig. 1 we rotate the axes clockwise 90 degrees, which means we must replace $+y$ by $+x$ and $+x$ by $-y$. We also must write y_0 instead of x_0.

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APPENDIX B

Derivation of Eq. (2)

The physical situation shown in Fig. 2 can be represented in terms of

$$Z(\zeta) = \frac{2Q\zeta}{\pi(\zeta^2 - b^2)} \frac{\sqrt{a^2 - b^2}}{\sqrt{\zeta^2 - a^2}}$$

from which

$$Z'(\zeta) = Z\left(\frac{1}{\zeta} - \frac{2\zeta}{\zeta^2 - b^2} - \frac{\zeta}{\zeta^2 - a^2}\right).$$

Along $x = 0$

$$Z(iy) = -\frac{2Qy\sqrt{a^2 - b^2}}{\pi(y^2 + b^2)\sqrt{y^2 + a^2}}$$

$$Z'(iy) = Zi\left(-\frac{1}{y} + \frac{2y}{y^2 + b^2} + \frac{y}{y^2 + a^2}\right)$$

$$\sigma_x = \mathrm{Re}\, Z - y\, \mathrm{Im}\, Z' = Z\left(1 - y\frac{Z'}{iZ}\right) = Z\left(2 - \frac{2y^2}{y^2 + b^2} - \frac{y^2}{y^2 + a^2}\right).$$

Thus

$$\sigma_x = \frac{2Qy\sqrt{a^2 - b^2}}{\pi(y^2 + b^2)\sqrt{y^2 + a^2}}\left(\frac{2y^2}{y^2 + b^2} + \frac{y^2}{y^2 + a^2} - 2\right).$$

APPENDIX C

Derivation of Eq. (3)

The physical situation of Fig. 3 can be represented in terms of

$$Z(\zeta) = \frac{\sigma\zeta}{\sqrt{\zeta^2 - a^2}}$$

and by adding $-\sigma$ to σ_x to produce $\sigma_x = 0$ at points remote from the crack. From this

$$Z'(\zeta) = Z\left(\frac{1}{\zeta} - \frac{\zeta}{\zeta^2 - a^2}\right).$$

Along $x = 0$

$$Z(iy) = \frac{\sigma y}{\sqrt{y^2 + a^2}}$$

$$Z'(iy) = Zi\left(-\frac{1}{y} + \frac{y}{y^2 + a^2}\right)$$

$$\mathrm{Re}\ Z - y\,\mathrm{Im}\ Z' = Z\left(1 - \frac{yZ'}{iZ}\right) = Z\left(2 - \frac{y^2}{y^2 + a^2}\right).$$

Thus

$$\sigma_x = \mathrm{Re}\ Z - y\,\mathrm{Im}\ Z' - \sigma = \sigma\left[\frac{2y}{\sqrt{y^2 + a^2}} - \frac{y^3}{(y^2 + a^2)^{3/2}} - 1\right].$$