N R L REPORT NO R-3281

THEORY AND DESIGN OF RESONANT TRANSFORMER-COUPLED LOOP-ANTENNA INPUT SYSTEMS FOR VLF RECEPTION

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NAVAL RESEARCH LABORATORY

WASHINGTON, D.C.

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S. V. Fratianni

April 28, 1948

Approved by:

T. McL. Davis, Head, Radio Techniques Section
L. A. Gebhard, Superintendent, Radio Division II

NAVAL RESEARCH LABORATORY
CAPTAIN H. A. SCHADE, USN, DIRECTOR
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Units and Symbols Used

(Throughout this report, both self and mutual inductances are expressed in henries, capacitances in farads, and resistances and reactances in ohms.)

C = Capacitance of the variable (tuning) capacitor.

E = Induced voltage in series with the transformer-coupled loop collector.

E_g = Voltage developed across the tuning capacitor.

E_s = Voltage induced in transformer secondary due to the current in the transformer primary circuit.

E_{eg} = Equivalent noise voltage generator of the tube and circuits following the receiver tuning capacitor.

E_n = Root-mean-square (rms) noise voltage of a resistor.

G = Voltage amplification expressed as E_g/E.

i_1 = Current flowing in the transformer primary circuit.

i_2 = Current flowing in the transformer secondary circuit.

k = Circuit coefficient of coupling.

k_1 = Transformer coefficient of coupling.

L = Inductance of the direct-connected loop collector.

L_o = Inductance of the transformer-coupled loop collector.

L_p = Transformer primary inductance.

L_2 = Transformer secondary inductance.

L' = Inductance reflected into the secondary circuit due to the primary circuit.

L'' = Total secondary circuit inductance with the primary circuit considered.

L_T = Secondary inductance required for resonance at the desired frequency with C.

L_1 = Total primary circuit inductance.

M = Transformer mutual inductance.

N = Turns ratio of the direct-connected to the transformer-coupled loop collectors.

Q = Figure of merit of the direct-connected loop collector $\left(X_L/r\right)$

Q_o = Figure of merit of the transformer-coupled loop collector (X_{Lo}/r_o).

Q_p = Figure of merit of the transformer primary. (X_{Lp}/r_p).

Q_1 = Total figure of merit of the primary circuit (X_{L1}/R_1).

Q_2 = Figure of merit of the transformer secondary (X_{L2}/r_2).

Q' = Figure of merit of the transformer secondary circuit with the primary circuit connected (X'_{L2}/r'_2).

r = Series resistance of the direct-connected loop collector.

R = Equivalent parallel resistance of the direct-connected loop collector at resonance.

r_o = Resistance of the transformer-coupled loop collector.

r_p = Resistance of the transformer primary.

R_1 = Total resistance of the primary circuit.

r_2 = Resistance of the transformer secondary.

r' = Resistance reflected into the secondary circuit due to the primary circuit.

r_{res} = Combined parallel resonant resistance of the loop collector and transformer.

r_{eq} = Equivalent resistance of an equivalent noise voltage generator.

T = Notation for the transformer.

X_L = Inductive reactance of the direct-connected loop collector (ωL).

X_c = Capacitive reactance of the variable capacitor $(-1/\omega c)$.

X_{Lo} = Reactance of the transformer-coupled loop collector (ωL_o).

X_{Lp} = Inductive reactance of the transformer primary (ωL_p).

X_{L1} = Total inductive reactance of the primary circuit (ωL_1).

X_{L2} = Inductive reactance of the transformer secondary (ωL_2).

X_M = Inductive reactance of the transformer mutual (ωM).

ω = Angular velocity: 2π times frequency at which the variable capacitor resonates its attached circuit.

Δf = Pass band of the entire receiver in cps.

δ = Voltage-amplification factor.

ψ = Signal-to-noise ratio at the grid of the first radio-frequency amplifier.

γ = Signal-to-noise ratio factor.

ABSTRACT

Loop antennas are, in general, the most effective small collectors for very-low-frequency radio energy. Methods of effectively coupling such loops to receivers are discussed and theoretical treatment of the transformer-coupled loop case is included. Mathematical analysis of the transformer-coupled case indicates that design for maximum voltage amplification differs considerably from design for optimum signal-to-noise ratio at the grid of the first tube in the receiving equipment, but that compromise designs are usually feasible. Examples of performance of typical direct-connected loops as compared to the transformer-coupled loops are also included. For most practical applications in the naval service, a suitable loop coupled to the grid of the first tube of the receiver through a highly efficient transformer is generally superior to direct coupling in the VLF range, when all the important factors are taken into account. This is also true for higher frequencies, when coverage of more than one tuning band by the same loop, ganged tuning with other receiver circuits, and considerable length of loop cable are required.

The overall selectivity comparison of direct and transformer-coupled loop systems, and criteria determing optimum loop inductance will be dealt with in subsequent reports.

PROBLEM STATUS

This is an interim report; work on other phases of the problem is continuing.

AUTHORIZATION

NRL Problem No. R10-43R(BuShips Problem No. S1083.1).

THEORY AND DESIGN OF RESONANT TRANSFORMER-COUPLED LOOP-ANTENNA INPUT SYSTEMS FOR VLF RECEPTION

INTRODUCTION

Optimum transfer and utilization of collected energy from an antenna system is of vital concern in receiving systems.[1]. The following study of loop antennas was undertaken primarily for application to the problem of submerged reception, but is generally applicable to other frequency ranges and types of operation. It considers mainly the transformer problem; subsequent reports will deal with the overall selectivity comparison of direct and transformer-coupled loop systems, and criteria determining optimum loop inductance.

Underwater reception of radio waves depends essentially on refraction effects, which can be considered as resulting in deflection of a portion of the vertically advancing wave front at the surface downward into the water. The resultant horizontal electric vector is attenuated with increase in depth, an effect which increases rapidly with frequency.[2]. The useful frequency range for underwater reception is thereby largely confined to the very low radio frequencies.

"Inductive" antennas (loops) have thus far been the most effective collectors for underwater radio energy at very low frequencies. The output voltage appearing at the terminals of such collectors is the resultant of the difference in the voltages respectively induced in the equivalent upper and lower horizontal members of the loop. These two voltage components differ in magnitude, due to the attenuation of the sea water between the equivalent horizontal members, and consequently the output voltage is mainly the result of amplitude rather than phase difference. It should be noted in passing that, even though the output is substantially independent of phase difference, wave-front propagation effects result in figure-8 patterns of loop output vs underwater azimuthal rotation,[3], closely resembling the patterns obtained in air.

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1 NRL Interim Report C-SS/7 (1223:SVF), C-1220-38/46: Modification and Developments

2 NRL Report R-1669, 2 December 1940: The Submerged Reception of Radio Frequency Signals

3 NRL Report R-1717, 1 April 1941: Test of Underwater Reception of Low Frequency Radio Signals.

The problems involved in transferring the loop-induced energy to the receiver with minimum loss and maximum effectiveness are considered in this report. The discussion is limited to the transformer-coupled and the direct-connected loop cases. The initial work performed for the Bureau of Ships under Problem S1083.1[4] indicated that the transformer-coupled loop collector was superior to other devices for underwater reception in submarines. Another possible method of utilizing the energy collected by a loop is the so-called "high-impedance" or direct-connected loop system, which, if practicable, would avoid the insertion loss of a transformer. The relative merits of each system are discussed herein.

GENERAL CONSIDERATIONS IN VLF UNDERWATER RADIO RECEPTION

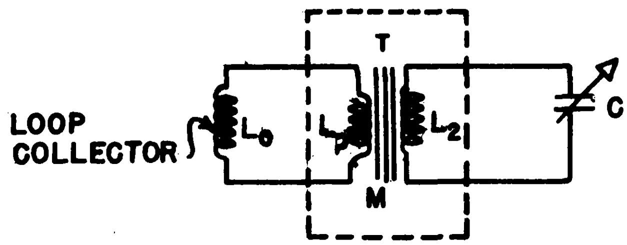
Much of the current theoretical treatment of loop collectors is based on conditions much more nearly ideal than those normally met in the field of underwater radio reception. First, the practicable underwater reception frequencies are very low, generally in the range of approximately 15 to 30 kc. Second, submarine installations almost invariably require placing the loop at a considerable distance from the receiving equipment proper, and this involves the use of very long lengths of relatively high-capacity transmission cable. Most of the literature available indicates that the "high-impedance" type of loop system (a loop directly tuned by a variable capacitor and directly connected to the grid of an amplifier) is always superior by as much as 8 or 9 db to the so-called "low-impedance" type of loop system (a much lower inductance loop coupled to the tuning capacitor through an efficient transformer). It is, of course, obvious that the addition of a component, such as a transformer, into a resonant circuit will introduce loss. But it is also important to recognize that the use of a suitable transformer will, under the proper conditions, result in overall advantages which may more than counteract the loss in power efficiency.

PRELIMINARY ANALYSIS OF TRANSFORMER CIRCUITS FOR LOOP COUPLING

If an ideal transformer is postulated (i.e., one which has 100 percent coupling between windings and no copper and iron losses) with an infinite inductance and a finite turns ratio of one to one, the effect of the transformer in the circuit would be unnoticeable when compared to the direct-connected loop (see Figure 1). That is, the transformer-coupled loop would act as though the terminals of the loop were connected directly across the tuning capacitor, C.

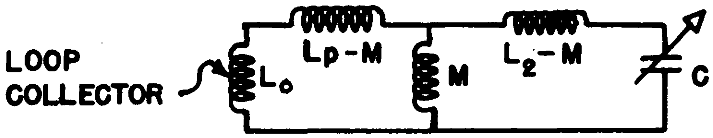
In Figure 1, L_p, L_2 and M represent the primary, secondary, and mutual inductance, respectively, of transformer T. L_o is the loop collector inductance, and C is the variable tuning capacitor (which may be located in the associated receiver proper). Assuming the ideal transformer above, circuits (A), (B), and (C) of Figure 1 are equivalent because $M = L_2 = L_p = \infty$.

[4] NRL Report R-2872, 31 December 1946: A Study of Various Loop Coupling Methods



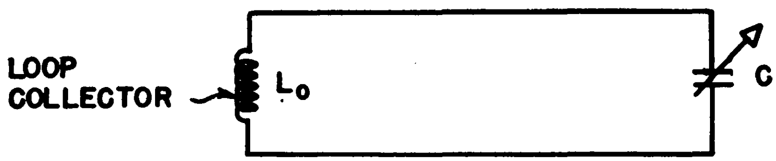
(A)

TRANSFORMER-CONNECTED LOOP COLLECTOR



(B)

EQUIVALENT OF CIRCUIT (A)



(C)

EQUIVALENT DIRECT- CONNECTED LOOP CONNECTOR

Figure 1

The L_p-M and L_2-M terms are eliminated, leaving a small inductance (L_o) shunted across an infinite inductance (M). The equivalent parallel inductance will therefore be just the loop inductance, connected across the terminals of the capacitor C, as shown in Figure 1 (C). These conditions make the transformer-coupled loop case equivalent to the direct-connected case, provided that the value of C and L_o are the same for both cases. With a practical transformer (one with finite values for L_p, L_2, and M) the reflected inductance of the primary circuit will reduce the inductance of the transformer secondary by an amount dependent on the transformer coupling coefficient k_1 and the values of L_p and L_o. If such a transformer had a one-to-one turns ratio and was resonated to a given frequency by the tuning capacitor with the primary circuit open, it would no longer tune to the same resonant frequency with the primary circuit closed, because the secondary inductance would be reduced. The secondary inductance would have to be increased in order to tune to the original frequency by adding secondary turns, and the transformer would then have a "step-up" ratio. Assuming a transformer primary inductance equal to the loop inductance, and Q values of greater than 10, the effective value of the secondary inductance will be:

$$L_2 = L_T \left(\frac{2}{2-k_1^2}\right). \qquad (6)$$

Where L_2 is the transformer secondary inductance with the primary circuit open, L_T is the inductance needed to resonate with the tuning capacitor at the desired frequency, and k_1 is the coefficient of coupling of the transformer proper. The derivation of Equation (6) is shown in Appendix I. This equation indicates that as k_1 approaches unity, the value of secondary inductance approaches twice the value of inductance needed to resonate directly at the desired frequency with a given value of C. As k_1 becomes less than unity, L_2 approaches the value of L_T as a limit. It is evident, however, that there will be some relationship between the several variables of loop inductance, transformer primary and secondary inductance, transformer coefficient of coupling, and circuit coefficient of coupling which will give optimum performance in maximum voltage across the tuning capacitor or best signal-to-noise ratio of the circuit. This is not necessarily the condition of $L_o = L_p$.

DESIGN CRITERIA FOR OPTIMUM VOLTAGE AMPLIFICATION IN A TRANSFORMER-COUPLED LOOP SYSTEM

A paper entitled "Loop Antenna Coupling Transformer Design" by W. S. Bachman[5] gives the necessary theoretical treatment for the design of loop-coupling transformers with optimum voltage amplification. It is, however, desirable to revise the final equations of this paper so that voltage amplification is expressed in terms of ratios of inductance and figures of merit (Q). This form is shown in Equation (36), with the circuit parameters corresponding to those indicated in Figure 2. The derivation of this equation is given in Appendix II.

$$\text{Voltage Amplification} = \frac{E_g}{E} = G$$

$$G = \left(\frac{-X_c}{X_{Lo}}\right)^{1/2} Q_2 \; \frac{k_1 \left[\dfrac{L_p/L_o}{L_p/L_o + 1}\left(1 - \dfrac{L_p/L_o}{L_p/L_o + 1}\right)\left(1 - k_1^2 \dfrac{L_p/L_o}{L_p/L_o + 1}\right)\right]^{1/2}}{1 + \dfrac{k_1^2 \, L_p/L_o}{L_p/L_o + 1}\left(1 - \dfrac{L_p/L_o}{L_p/L_o + 1}\right)\left(\dfrac{L_p}{L_o}\,\dfrac{Q_2}{Q_p} + \dfrac{Q_2}{Q_o}\right)} \qquad (36)$$

[5] Proceedings of the I.R.E., December 1945

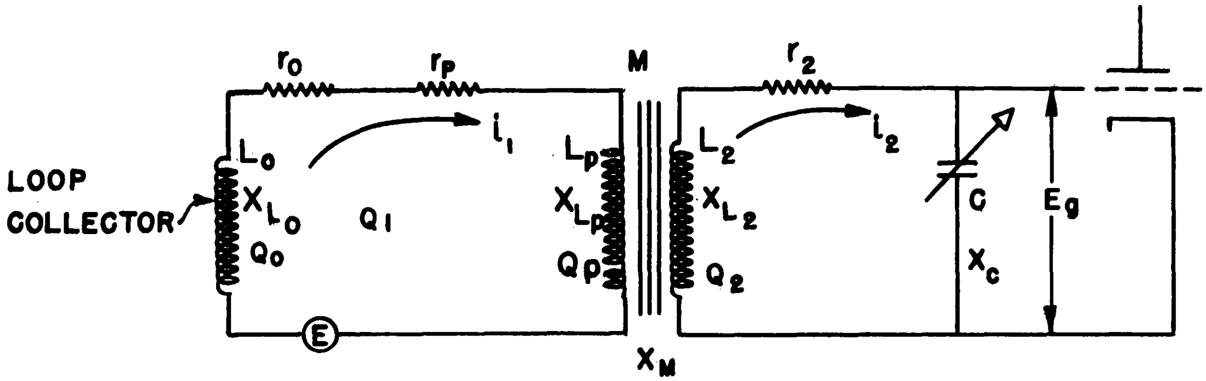


Fig. 2 - *Transformer-Coupled Loop Collector*

Although Equation (36) shows that there is a particular value for the ratio of transformer primary and loop inductance which provides an optimum value of G with given values of Q, it is useful to analyze some of the fundamental equations before proceeding with specific design information. From Equations (7), (34), and (35) of Appendix II, it is evident that when the transformer coefficient of coupling (k_1) is equal to unity,

$$X_{Lo} = X_{Lp} \left(\frac{-X_c}{X_{L2} + X_c} \right)$$

This indicates that in an optimum design, the transformer primary reactance does not necessarily equal the reactance of the loop, but has some value dependent upon the factor $-X_c/(X_{L2} + X_c)$. For values of k_1 less than unity, the expression for loop reactance becomes

$$X_{Lo} = X_{Lp} \left(\frac{-X_c + X_{L2} \left[k_1^2 - 1 \right]}{X_{L2} + X_c} \right)$$

Since the first step in design of a transformer is determination of the ratio of L_p/L_o, Equation (36) has been developed for this purpose. An expression for "voltage amplification factor" will be obtained if both sides of Equation (36) are divided by $(-X_c/X_{Lo})^{1/2} Q_2$. This factor (V.A.F.), plotted against the ratio L_p/L_o as an abscissa, is shown in Figure 3. Individual curves are shown for various conditions of Q and transformer coefficient of coupling, so that, as an intermediate step, an approximate value of the ratio L_p/L_o for optimum voltage amplification can be readily obtained for typical cases. For optimum performance under specific conditions, the voltage amplification factor V.A.F. can be plotted to determine its optimum for the particular design. Although the V.A.F. equation can be differentiated with respect to the ratio L_p/L_o and equated to zero for the value of that ratio which maximizes the expression, it will be found that a plot of the equation is a less tedious operation. From the curves of Figure 3, it is evident that for a particular condition with predetermined Q's, the inductance ratio has a definite but not critical value. If the design of the circuit is varied so that the Q's of the various circuits are made larger, the ratio of transformer primary to loop collector inductance becomes smaller. In general, the loop inductance should be larger than the transformer primary inductance, except for very low values of the transformer coupling coefficient k_1.

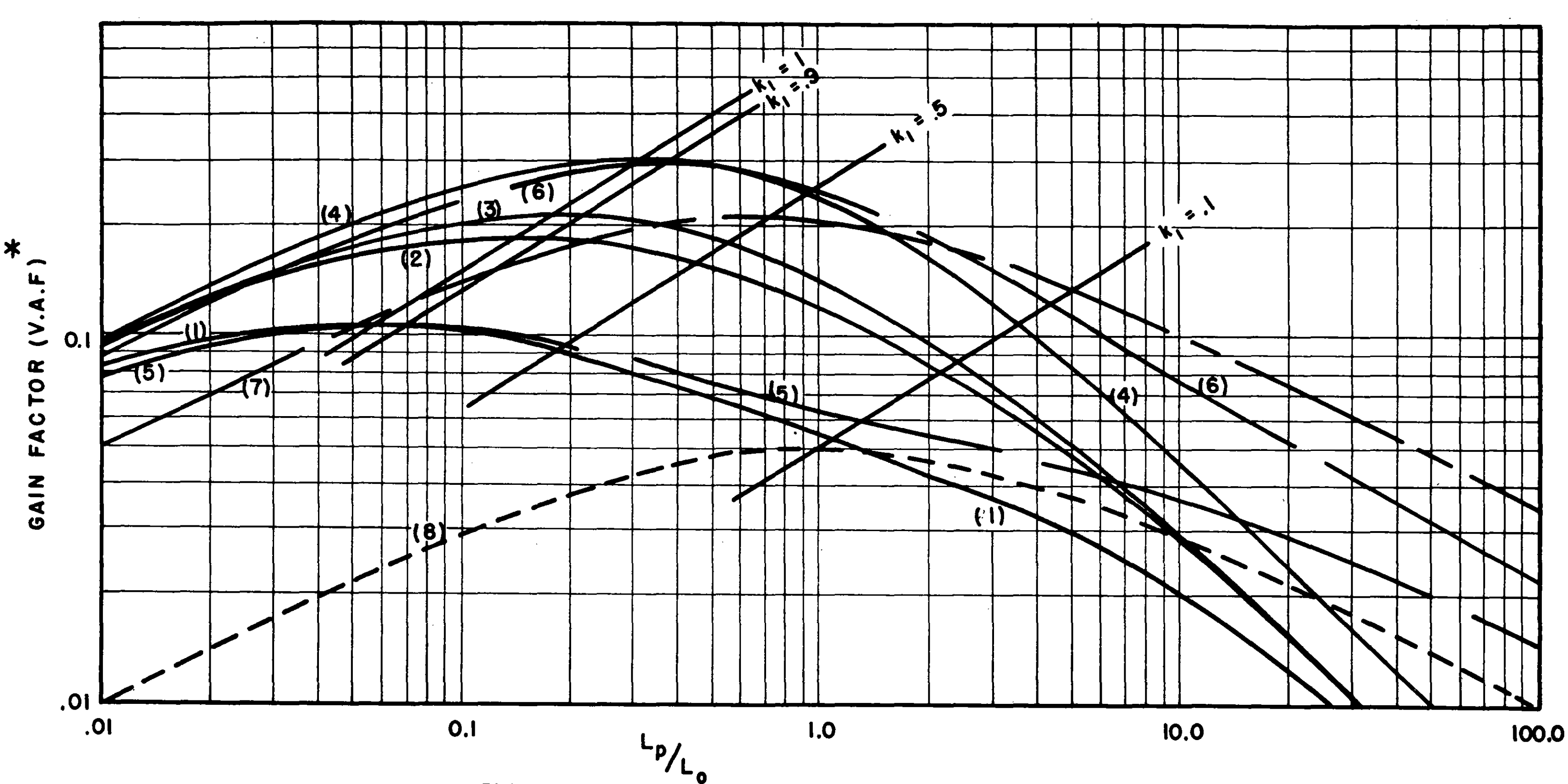
It is interesting to note that in the family of curves shown in Figure 3, for a particular coefficient of coupling, for example unity, a straight line can be drawn joining the maximum point of each curve to the maxima of the others. The maxima of other design cases (different Q's) for the same transformer coefficient of coupling will also lie on this straight line (See curves 1, 2, 3, and 4 of Figure 3). The same result can be obtained for other coefficients of coupling and values of Q. All of the straight lines thus obtained will be parallel. These curves further simplify the choice of the approximate value of the ratio L_p/L_o for the various values of Q and k_1. To complete the design of the transformer, the value of the secondary inductance, L_2, is determined from Equation (54) of Appendix IV.

The curves for Gain Factor (V.A.F.) shown in Figure 3 are directly useful for design purposes, but do not immediately indicate the relative effect of the various transformer design parameters on absolute gain. Figure 4 has been prepared to show this relationship for any set of constant values of tuning-capacity reactance (X_c), loop-inductance reactance (X_{Lo}), and loop Q (Q_o). The Gain Factor plotted therein is derived from Equation (36) and takes the form:

$$\text{Gain Factor} \qquad (\delta) = \frac{G}{\left(-X_c/X_{Lo}\right)^{1/2}\, Q_o} \quad \text{, or}$$

$$\delta = \frac{Q_2}{Q_o}\;\frac{k_1\left[\dfrac{L_p/L_o}{L_p/L_o+1}\left(1-\dfrac{L_p/L_o}{L_p/L_o+1}\right)\left(1-k_1^2\dfrac{L_p/L_o}{L_p/L_o+1}\right)\right]^{1/2}}{1+\dfrac{k_1^2\,L_p/L_o}{L_p/L_o+1}\left(1-\dfrac{L_p/L_o}{L_p/L_o+1}\right)\left(\dfrac{L_p}{L_o}\,\dfrac{Q_2}{Q_p}+\dfrac{Q_2}{Q_o}\right)}$$

It is evident from these curves that, with given values of loop inductance, loop Q, and secondary tuning capacity, the voltage amplification is highest when the transformer Q's are high. The curves also show that although loose coupling in the transformer is undesirable, the circuit coupling as indicated by the L_p/L_o ratio should be quite low for high transformer Q values. This illustrates the fact that the loop inductor (excepting cable inductance) has useful pickup or "effective height" regardless of the absolute value of its inductance, whereas the transformer leakage inductance

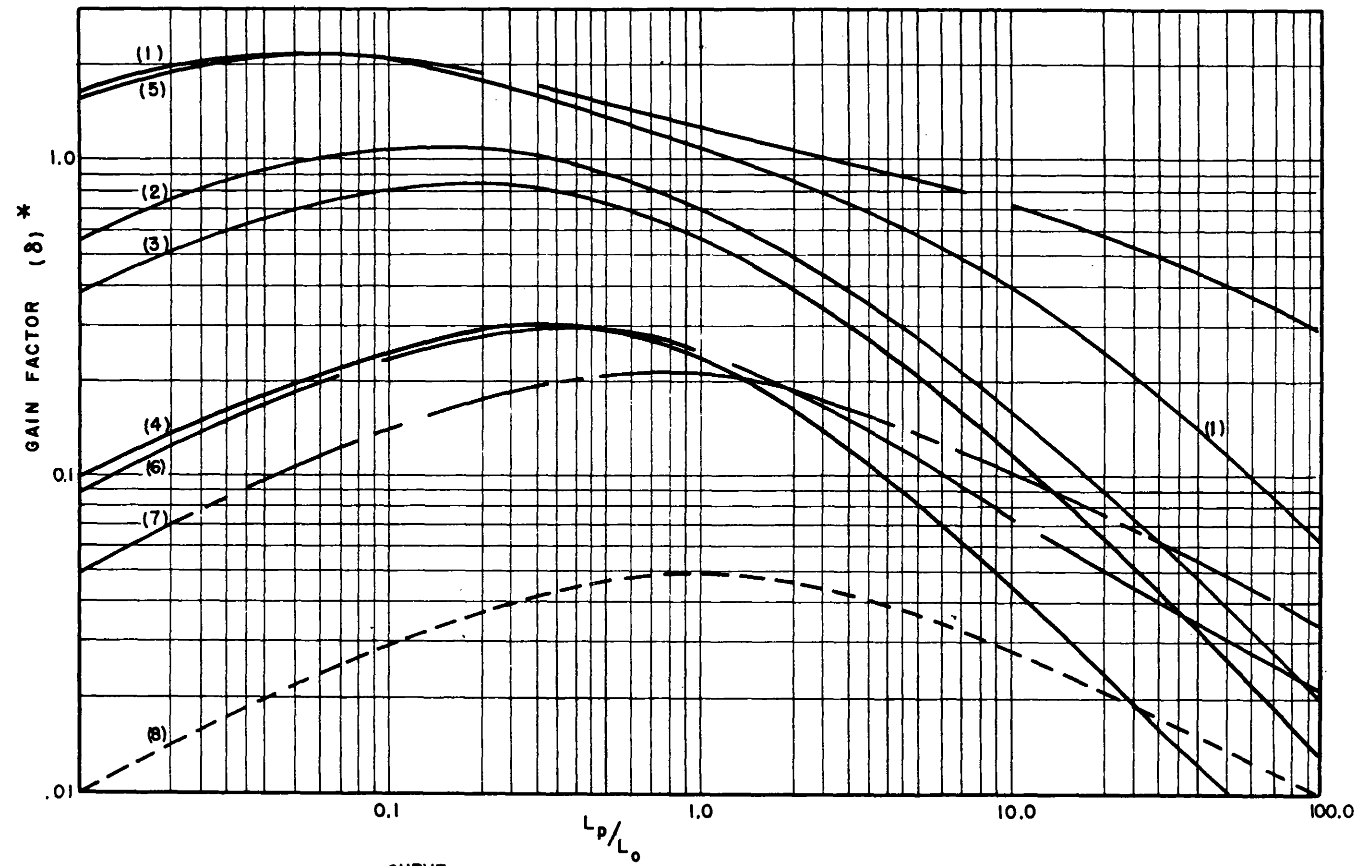


CURVE NUMBER	CONDITIONS		
(1)	k_1 = 1	Q_p = 10 Q_o	Q_2 = 20 Q_o
(2)	k_1 = 1	Q_p = 3 Q_o	Q_2 = 6 Q_o
(3)	k_1 = 1	Q_p = 2 Q_o	Q_2 = 4 Q_o
(4)	k_1 = 1	Q_p = Q_o	Q_2 = Q_o
(5)	k_1 = .9	Q_p = 10 Q_o	Q_2 = 20 Q_o
(6)	k_1 = .9	Q_p = Q_o	Q_2 = Q_o
(7)	k_1 = .5	Q_p = Q_o	Q_2 = Q_o
(8)	k = .1	Q_p = Q_o	Q_2 = Q_o

L_p = XFMR. UNLOADED PRIMARY INDUCTANCE
L_o = LOOP ANTENNA INDUCTANCE
k_1 = XFMR. COEFFICIENT OF COUPLING
Q_o = LOOP ANTENNA UNLOADED FIGURE OF MERIT.
Q_p = XFMR UNLOADED PRIMARY FIGURE OF MERIT.
Q_2 = XFMR UNLOADED SECONDARY FIGURE OF MERIT.

*FOR THE DEFINITION OF (V.A.F) REFER TO TEXT.

Figure 3- Variation of Gain Factor (V.A.F.) with Changes in the Ratio L_p/L_o.



*FOR THE DEFINITION OF (δ) REFER TO TEXT

CURVE NUMBER	CONDITIONS		
(1)	$k_1 = 1$	$Q_p = 10 Q_o$	$Q_2 = 20 Q_o$
(2)	$k_1 = 1$	$Q_p = 3 Q_o$	$Q_2 = 6 Q_o$
(3)	$k_1 = 1$	$Q_p = 2 Q_o$	$Q_2 = 4 Q_o$
(4)	$k_1 = 1$	$Q_p = Q_o$	$Q_2 = Q_o$
(5)	$k_1 = .9$	$Q_p = 10 Q_o$	$Q_2 = 20 Q_o$
(6)	$k_1 = .9$	$Q_p = Q_o$	$Q_2 = Q_o$
(7)	$K_1 = .5$	$Q_p = Q_o$	$Q_2 = Q_o$
(8)	$k_1 = .1$	$Q_p = Q_o$	$Q_2 = Q_o$

L_p = XFMR. UNLOADED PRIMARY INDUCTANCE
L_o = LOOP ANTENNA INDUCTANCE
k_1 = XFMR. COEFFICIENT OF COUPLING
Q_o = LOOP ANTENNA UNLOADED FIGURE OF MERIT.
Q_p = XFMR UNLOADED PRIMARY FIGURE OF MERIT.
Q_2 = XFMR UNLOADED SECONDARY FIGURE OF MERIT

Figure 4- Variation of Gain Factor (δ) with Changes in the Ratio of L_p/L_o.

being of the nature of a "loading" inductance has no useful pickup or collector action.

This method of determining transformer design constants is useful for loop-coupling circuits in which the main problem is optimum voltage amplification with no regard to optimum signal-to-noise ratio. Examples are receiving systems where the other main source of noise is the mixer circuit or other elements following the input circuits. Design criteria for optimum signal-to-noise ratio at the input to the first amplifier in the system are treated in the following section of this report.

DESIGN CRITERIA FOR OPTIMUM SIGNAL-TO-NOISE RATIO IN A TRANSFORMER-COUPLED LOOP SYSTEM

For a given signal input, the output signal-to-noise ratio of a receiver, designed for the lower radio frequencies, is usually determined by the circuits preceding the first tube. The field strength which determines the amount of loop-induced voltage is fixed by considerations involving the transmitter and propagation effects. If the size and acceptable inductance of the loop collector are predetermined, the "effective height," or effective signal voltage at the loop terminals is also fixed. Proper choice of circuit parameters to obtain the optimum signal-to-noise ratio before amplification by the first tube of a receiver affords the major possibility of good sensitivity in most cases.

When design is for the purpose of optimum overall signal-to-noise ratio with an input circuit such as that of Figure 5, all factors contributing to the total noise output from the receiver can be referred to that circuit for convenience in analysis. For instance, as shown in Figure 5, all noise sources in the first tube and circuits following can be represented by a single noise-voltage generator, E_{eq}, in series with the grid circuit of that tube.[6] For applications which require evaluation of the increase in noise due to E_{eq}, the equations of Appendix III indicate that the term $(1 + R_{eq}/R_{res})^{1/2}$

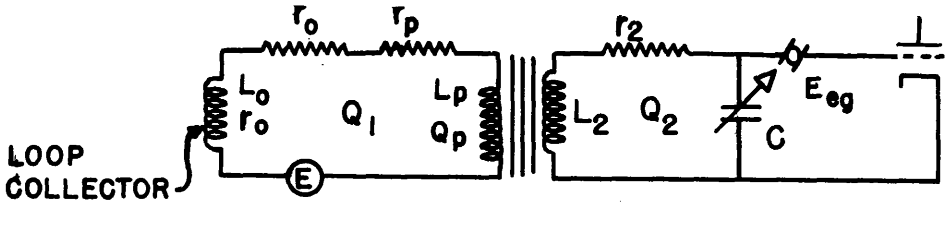


Fig. 5 - *Transformer-Coupled Loop Collector with Equivalent Receiver Noise Voltages in the Grid Circuit*

[6] "Fluctuations in Vacuum Tube Amplifiers and Input Systems" by W. A. Harris, RCA Review, Vol. V, No. 4, pp. 505-524, April 1941 and Vol. VI, No. 1, pp. 114-124, July 1941.

will provide a direct measure of the noise increase. In receivers designed for VLF operation, however, the equivalent resonant resistance (R_{res}) of the input circuits is usually very high compared to the noise resistance (R_{eg}) represented by the equivalent noise generator (see Equation 49 of Appendix III).[7] Because the equivalent noise resistance is usually relatively small and negligible in this application, the noise voltage generator E_{eq} can be omitted in the following analysis.

Considering these assumptions and the circuit of Figure 5, the signal-to-noise ratio, as measured at the input to the first tube, takes the form indicated by Equation (50). This expression takes into account the total thermal noise generated in the input circuits.

$$\frac{\text{Signal Voltage}}{\text{Noise Voltage}} = \psi = \frac{7.94 \times 10^9}{(\Delta f)^{1/2}} E\left(\frac{Q_2}{X_{Lo}}\right)^{1/2} \frac{k_1}{\left[\left(\frac{L_o}{L_p} + \frac{L_p}{L_o} + 2\right) + k_1^2\left(\frac{Q_2}{Q_o} + \frac{Q_2}{Q_p} + \frac{L_p}{L_o}\right)\right]^{1/2}} \quad (50)$$

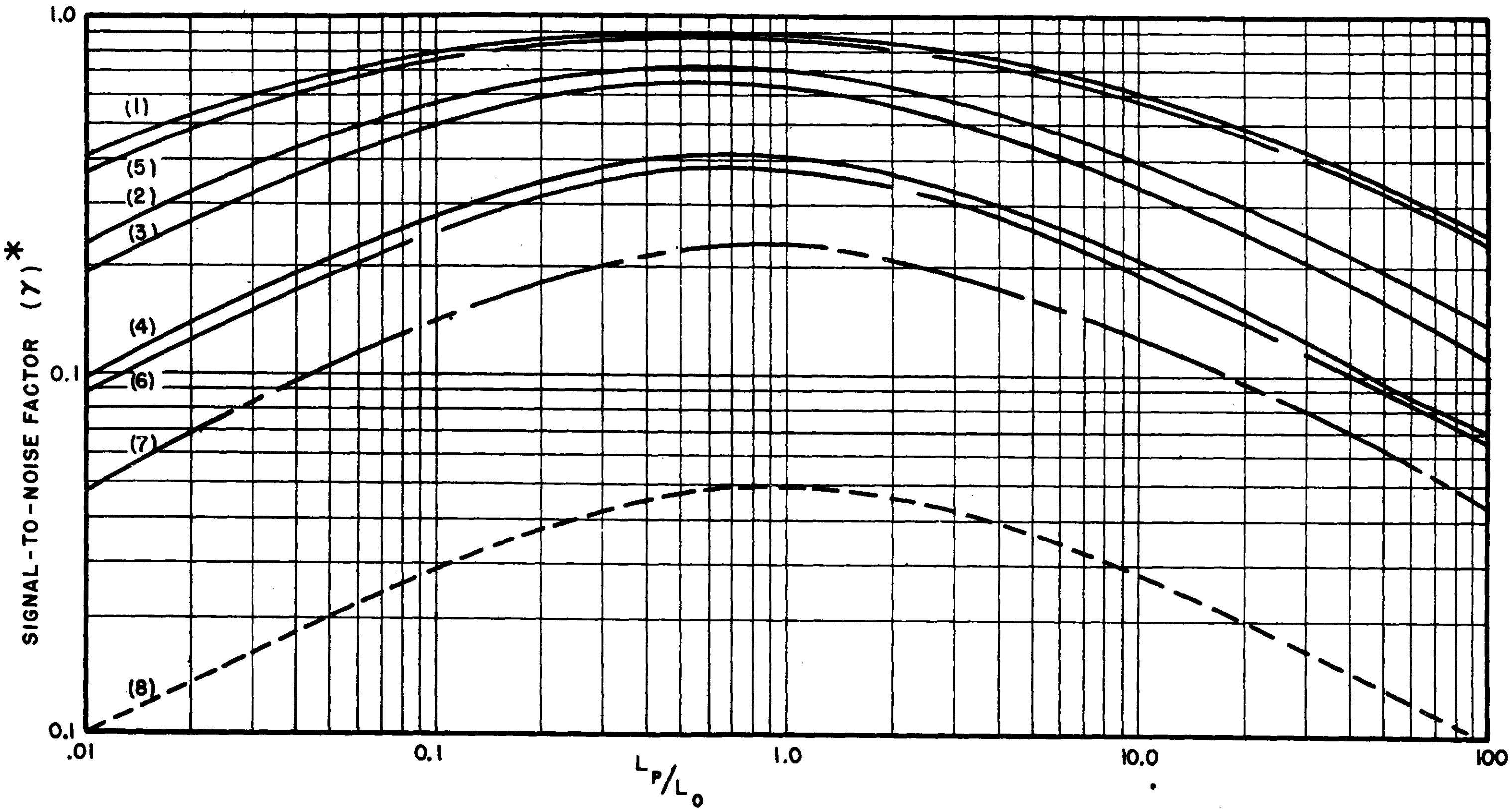
The derivation for Equation (50) is given in the Appendix III. If both sides of equation (50) are divided by the factor $7.94 \times 10^9/(\Delta f)^{1/2}\ E\ (Q_o/X_{Lo})^{1/2}$, an expression is obtained which can be called the "signal-to-noise factor" (Equation 51). This factor, when plotted as shown in Figure 6 for the various values of transformer-primary to loop inductance ratios, Q values, and coefficients of coupling will yield a family of curves which indicate that the L_p/L_o ratio is not critical with optimum signal-to-noise ratio occuring in the region of $L_p/L_o = .7$. With the loop collector inductance known, and a practicable value of transformer coefficient of coupling determined, the primary inductance L_p can be derived from the optimum value of L_p/L_o for the particular Q values involved. The preliminary design of the transformer is completed by determination of the secondary inductance L_2 from Equation (54) of Appendix IV.

In summary, for designs in which optimum signal-to-noise ratio for a given input is of prime importance and optimum voltage amplification in the input circuits is a secondary consideration, preliminary design is best based on Equation (50).

COMPROMISE IN DESIGN BETWEEN OPTIMUM VOLTAGE AMPLIFICATION AND OPTIMUM SIGNAL-TO-NOISE RATIO

The curves shown in Figure 3 have been plotted for cases in which the transformer Q's are equal to or better than the loop Q value. This is representative of most loop systems, since it is generally possible to provide higher Q in the transformers than in the loops. Systems with transformer Q's substantially higher than the loop Q require looser circuit coupling

[7] "Design Values for Loop-Antenna Input Circuits" by J. E. Browder and V. J. Young, IRE Proc., 35, pp 519-525, May 1947



* FOR THE DEFINITION OF (γ) REFER TO TEXT.

CURVE NUMBER	CONDITIONS		
(1)	$k_1 = 1$	$Q_p = 10\,Q_o$	$Q_2 = 20\,Q_o$
(2)	$k_1 = 1$	$Q_p = 3\,Q_o$	$Q_2 = 6\,Q_o$
(3)	$k_1 = 1$	$Q_p = 2\,Q_o$	$Q_2 = 4\,Q_o$
(4)	$k_1 = 1$	$Q_p = Q_o$	$Q_2 = Q_o$
(5)	$k_1 = .9$	$Q_p = 10\,Q_o$	$Q_2 = 20\,Q_o$
(6)	$k_1 = .9$	$Q_p = Q_o$	$Q_2 = Q_o$
(7)	$k_1 = .5$	$Q_p = Q_o$	$Q_2 = Q_o$
(8)	$k_1 = .1$	$Q_p = Q_o$	$Q_2 = Q_o$

L_p = XFMR. UNLOADED PRIMARY INDUCTANCE
L_o = LOOP ANTENNA INDUCTANCE
k_1 = XFMR. COEFFICIENT OF COUPLING
Q_o = LOOP ANTENNA UNLOADED FIGURE OF MERIT.
Q_p = XFMR UNLOADED PRIMARY FIGURE OF MERIT.
Q_2 = XFMR UNLOADED SECONDARY FIGURE OF MERIT

Figure 6- Variation of Signal-to-Noise Factor with Changes in the Ratio of L_p/L_o.

(lower values of k) between L_o and L_2, a condition readily fulfilled in most designs by reduction of the L_p/L_o ratio. As indicated in Figure 3, for example, with the transformer primary and secondary Q two and four times as high, respectfully, as the loop Q, and with 100-percent transformer coupling (k_1 = 1) optimum voltage amplification occurs with an L_p/L_o ratio of about .17. On the other hand, the same system, as shown in Figure 6, would require an L_p/L_o ratio of about .5 for optimum signal-to-noise ratio. Designing for optimum signal-to-noise ratio would result in a decrease from optimum voltage amplification of about 1.8 db, while designing for optimum voltage amplification would decrease the signal-to-noise ratio by about 1.1 db below optimum. With the compromise value of L_p/L_o = .33, there would be a decrease of 0.4 db from optimum voltage amplification and 0.3 db from optimum signal-to-noise value. Such a compromise appears to be the best course when amplification and signal-to-noise ratio are of equal importance.

It is interesting to note that as the various Q's in the loop and transformer system become more nearly equal, the L_p/L_o ratio approaches values for both optimum voltage amplification and optimum signal-to-noise ratio which involve practically no loss in a compromise design.

PRACTICAL CONSIDERATIONS IN TRANSFORMER-COUPLED AND DIRECT-CONNECTED LOOP SYSTEMS

The theoretical solutions given above have been based on the assumption that the loop is closely adjacent to its tuning-condenser or transformer-condenser network. In practical applications in naval service, however, the loop is often located at a considerable distance from these networks, necessitating incorporation of substantial lengths of interconnecting cable. Such cable adds series-inductance and shunt-capacitance to the terminating network, the added capacitance being of particular concern in VLF applications. The cable can also add appreciable series resistance if its wire size is too small, and considerable shunt resistance if leakage due to moisture, etc., is present. The following example illustrates the effect of the cable capacity in a direct-coupled and a transformer-coupled loop system designed for VLF reception. With a frequency range of 15 to 25 kc and 50 feet of coupling cable having a capacitance of 20 µµf per foot between the loop and receiver, the following minimum capacities will appear in the input circuit prior to the first amplifier tube:

(a) Total distributed capacities of loop structure	= 100 µµf
(b) Total capacitance of 50-foot cable	= 1000 µµf
(c) Wiring and stray capacities (including junction boxes, fittings, etc.)	= 40 µµf
(d) Loop circuit trimmer (average setting)	= 50 µµf
(e) Tuning capacitor minimum	= 150 µµf
Total	1340 µµf

In order to cover a 15 to 25 kc range with such a high value of minimum circuit capacity and a direct-coupled loop, it would be necessary to employ a tuning capacitor with a maximum capacity of 2390 μμf, providing a total circuit maximum capacitance of 3730 μμf, an undesirably high figure. The inductance of the loop would have to be 30 millihenries to resonate with 3730 μμf at 15 kc. A typical transformer-coupled loop system covering the same frequency range and presently used in the naval service has the following approximate values for the various circuit parameters:

(a) Loop collector inductance (L_o)	= 8 millihenries
(b) Transformer coefficient of coupling (k_1)	= .94
(c) Total tuning capacitance (minimum)	= 150 μμf
(d) Total tuning capacitance (maximum)	= 525 μμf
(e) Inductance necessary to resonate the total tuning capacitance over the required frequency range	= 220 millihenries

Assuming that the Q's of the various circuits of both systems are all equal to 35, the voltage gain or amplification for the transformer case at 15 kc is determined as follows:

With the required tuning inductance of 220 millihenries, the necessary capacitive reactance for resonance is 22,000 ohms. The loop reactance (L_p = 8 millihenries) is 755 ohms. From Figure 3, a value of .295 is obtained for optimum gain factor. From Equation (36), the voltage amplification is computed to be approximately 55.7. This system, with 50 feet of two-wire cable totaling 1000 μμf, covers the specified frequency range. It should be noted that the cable inductance, which will have a value between 20 to 50 microhenries, can be safely ignored as being negligible in comparison to the loop and transformer inductance values. Adequate wire size in the cable and a negligible value of leakage resistance has also been assumed.

For the purpose of comparing both systems as to the voltage actually developed across the tuning capacitor, a constant field-strength, which induces 1000 μv in the transformer-coupled loop collector, can be assumed. The inductance values of the two loops being considered indicates that there are approximately 1.935 times as many turns in the loop of the direct connected system (L_o = 30 millihenries) as are required in the transformer-coupled case (L_o = 8 millihenries). The induced voltage in the direct-connected case will then be 1.935 (0.001) volts, and the voltage appearing across the variable capacitor, will be 1.935 (0.001) Q or 0.0678 volts, with Q = 35 (see Equation 57 of Appendix V). The voltage developed across the variable capacitor for the transformer-coupled case, as derived from Equation (36), is 55.7 (0.001) or 0.0557 volts.

These data indicate that, on the basis of designing for optimum voltage developed across the tuning capacitor, the direct-connected loop system in this example provides approximately 1.7 db more voltage gain or amplification than the transformer-coupled loop system. This difference in performance is so small that other factors will usually have to be considered in making a choice.

For the purpose of comparing the two systems on the basis of the better signal-to-noise ratio at the input to the first tube, design of the transformer in accordance with Equation (50) should also be undertaken. A signal-to-noise factor of approximately 0.4 is obtained from Figure 6 for the case of $Q_o = Q_p = Q_2$ and $k_1 = .94$. Utilizing Equations (50) and (61) of Appendices III and V, respectively, it will be found that the direct-connected loop has approximately 7.8 db better signal-to-noise ratio for a given field-strength than the transformer-coupled loop case. To illustrate the effects of various parameters on circuit performance with regard to signal-to-noise ratio eight sets of conditions are shown in Table I. This tabulation compares systems designed for optimum voltage and optimum signal-to-noise ratio, in both the direct-connected and the transformer-coupled form. The examples show the relative improvement possible for optimum voltage and signal-to-noise ratio design of coupling transformers for various minimum circuit capacitances and Q values. Examples 3 and 4 are typical of present conditions for underwater reception systems in regard to cable capacitance and loop inductance. For these cases, Table I indicates that the transformer-coupled system is better than the direct-coupled system on the basis of voltage amplification. The improvement is greater if the transformer is designed for optimum voltage amplification rather than optimum signal-to-noise ratio. Comparing these examples on a signal-to-noise ratio basis, however, the tabulation indicates that for none of the conditions chosen is the transformer-coupled superior to the direct-coupled case. Obviously, use of a transformer results in a loss of signal energy expended as heat in the transformer windings and core, which depresses the signal-to-noise ratio.

PRACTICAL DESIGN OF LOOP INPUT SYSTEMS

The foregoing discussion has shown that the direct-connected loop always provides a better signal-to-noise ratio than a transformer-coupled system, but that transformer-coupling can provide the greater voltage amplification or overall "effective height" when relatively long loop cables must be used in an installation. Transformer-coupling also permits the use of loop tuning condensers of a size and capacitance normally feasible in a receiving system. This allows ganging and tracking of circuits in the receiver not possible with the much larger capacitors required in a direct-connected system, as, for instance, Example 3 of Table I. The low-inductance loops feasible with transformer-coupling are simpler in structure and easier to manufacture, and can be used for more than one tuning band without relative loss of efficiency, by provision of appropriate transformers. The transformer

TABLE I COMPARISON EXAMPLES OF
TRANSFORMER-COUPLED AND DIRECT-CONNECTED
LOOP COLLECTOR SYSTEMS AT 15 KC

Cable				Direct-Connected Loop				Transformer-Coupled Loop					Tuning Capacity
Example	Length (ft)	Cap./ft (μμf)	Total cap. (μμf)	Minimum fixed cap (μμf)	Maximum tuning cap (μμf)	Loop Ind. (Mh)	Loop Q	Loop Ind. (Mh)	Loop Q_o	Xfmr. primary Q_p	Xfmr. secondary Q_2	Xfmr. coefficient of coupling k_1	Min. 25 μμf Max. 500 μμf X_c Ohms (At 15 kc)
1	---	None	----	*None	500	220	35	8	35	35	35	1	22,000
2	---	None	----	*None	500	220	35	8	35	35	350	1	22,000
3	50	20	1000	340	3730	30	35	8	35	35	90	.9	22,000
4	100	20	2000	400	6670	16.8	35	8	35	35	90	.9	22,000
5	50	10	500	260	2111	53.3	35	8	35	35	35	.9	22,000
6	50	10	500	260	2111	53.3	100	.15	100	100	100	.9	22,000
7	50	10	500	260	2111	53.3	35	8	35	35	350	1	22,000
8	50	10	500	260	2111	53.3	35	8	35	35	70	1	22,000

	Transformer Designed for Optimum Signal-to-Noise Ratio				Transformer Designed for Optimum Voltage Amplification			
	Voltage Comparison at the Grid of the First R F Tube		S/N Ratio Comparison at the Grid of the First R F Tube		.Voltage Comparison at the Grid of the First R F Tube		S/N Ratio Comparison at the Grid of the First R F Tube	
Example	Superiority of direct-connected system (in db)	Superiority of transformer-coupled system (in db)	Superiority of direct-connected system (in db)	Superiority of transformer-coupled system (in db)	Superiority of direct-connected system (in db)	Superiority of transformer-coupled system (in db)	Superiority of transformer-coupled system (in db)	Superiority of direct-connected system (in db)
1	11.3	---	7.5	-----	10.2	---	8.2	-----
2	1.7	---	2.5	-----	----	3.4	3.4	-----
3	----	2.8	5.6	-----	----	3.7	5.8	-----
4	----	5.4	5.6	-----	----	6.2	5.8	-----
5	5.2	---	8.5	-----	4.5	---	8.7	-----
6	5.1	---	8.3	-----	4.5	---	8.8	-----
7	----	4.5	3.1	-----	----	9.5	3.6	-----
8	----	1.0	5.7	-----	----	.8	6.4	-----

* For these examples, the stray capacity of the loop is assumed to be included in the tuning capacitance.

coupled system will also, in general, be less sensitive to change of cable length and to the use of cable with high losses.

The coupling transformers should be designed for the highest possible Q value within practical limitations, to provide the best possible signal-to-noise ratio performance. Fortunately, as shown in Table I, the designs for optimum voltage and optimum signal-to-noise ratio provide performance figures not widely different, and, in some cases, a compromise design will be desirable.

It should be noted that this report is primarily concerned with VLF operation and the final equations for signal-to-noise ratio performance were derived on the basis of no noise voltages appearing in the receiver from sources other than the loop or loop and transformer proper. For operation above the VLF range, Equation (51) should include the expression $(1 + r_{eq}/r_{res})^{1/2}$ shown in Equation (49).

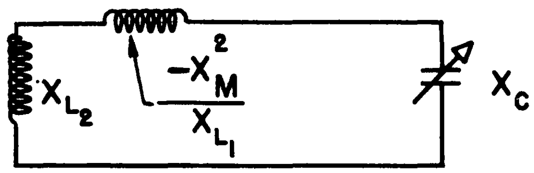
CONCLUSIONS

Direct-coupled loop circuits provide the maximum possible signal-to-noise ratio for a given Q value and should be used where no appreciable length of cable is needed between loop and tuning condenser, and where tuning coverage is limited to one band. Practical systems for shipboard use, however, generally require considerable cable length and ganged tuning—conditions favoring use of transformer-coupled circuits, in spite of loss in signal-to-noise ratio. Provision of small transformer structures with Q values high relative to loop Q is often feasible and reduces the signal-to-noise ratio loss. "Lossy" cable has less shunting effect on the relatively low-inductance loops of the transformer-coupled type of system, and changes of cable length which vary cable capacitance also are less deleterious in such circuits.

APPENDIX I

DETERMINATION OF THE APPROXIMATE SECONDARY INDUCTANCE OF A TRANSFORMER-COUPLED LOOP SYSTEM

A circuit equivalent to Figure 2 can be drawn showing just the secondary and the effects of the primary parameters on the secondary circuit. The equivalent circuit is shown below.



The above figure is approximately correct if it is assumed that:

(a) The Q's of both the primary and secondary circuits are greater than 10.

(b) The inductive reactance reflected into the secondary circuit is $-X_M^2/X_{L1}$ [8].

(c) The term X_{L1} is the total inductive reactance of the primary circuit. Then, at resonance the parameters can be equated as follows:

$$X_{L2} - \frac{X_M^2}{X_{L1}} = X_c \quad . \qquad (1)$$

Equation (1) states that at resonance the inductive and capacitive reactances are equal. The inductive reactance required to tune the condenser which is called X_T is

$$X_T = X_{L2} - \frac{X_M^2}{X_{L1}} \quad . \qquad (2)$$

If it is assumed that the loop inductance L_o is equal to the transformer primary inductance L_p (see Figure 2) then

$$X_{L1} = 2\ X_{Lp} \quad . \qquad (3)$$

For any transformer it is known that

$$X_M^2 = k_1^2\ X_{Lp}\ X_{L2} \ . \qquad (4)$$

[8] Radio Engineering by F. E. Terman, p. 153, McGraw Hill, (1943)

Substituting Equations (3) and (4) in Equation (2) and solving for the transformer secondary inductance X_{L2} , Equation (5) is obtained:

$$X_{L2} = X_T \left(\frac{2}{2 - k_1^2} \right) \quad \text{, or} \qquad (5)$$

$$L_2 = L_T \left(\frac{2}{2 - k_1^2} \right) \quad . \qquad (6)$$

APPENDIX II

DERIVATION OF OPTIMUM VOLTAGE-AMPLIFICATION EQUATIONS FOR A TRANSFORMER-COUPLED LOOP SYSTEM

The following relations are derived from Figure 2:

$$L_1 = L_o + L_p \tag{7}$$

$$R_1 = r_o + r_p \tag{8}$$

$$k_1 = M/(L_p\ L_2)^{1/2} \tag{9}$$

$$k = M/(L_1\ L_2)^{1/2} \tag{10}$$

$$Z = R_1 + jX_{L1} \tag{11}$$

$$Z_2 = r_2 + j(X_{L2} + X_c)\ . \tag{12}$$

Using these relations and applying Kirchoff's Laws to the primary and secondary circuits of Figure 2, the following equations are obtained:

$$-i_1\ Z_1 + i_2\ jX_M + E = 0 \quad ; \tag{13}$$

$$i_1\ jX_M - i_2\ Z_2 = 0\ . \tag{14}$$

Solving Equations (13) and (14) for i_1 and i_2, the following will be obtained:

$$i_1 = \frac{E}{Z_1 + \dfrac{X_M^2}{Z_2}} \tag{15}$$

$$i_2 = j\ \frac{X_M\ i_1}{Z_2}\ . \tag{16}$$

The voltage across the variable capacitor is

$$E_g = -j\ X_c\ i_2\ . \tag{17}$$

Substituting Equation (16) in (17), E_g becomes

$$E_g = \frac{X_c\ X_M\ i_1}{Z_2}\ . \tag{18}$$

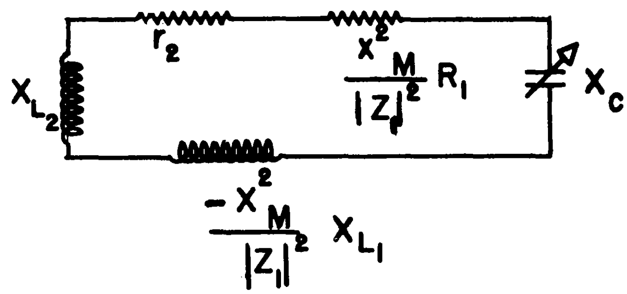
From Equation (15), the loop-induced voltage is

$$E = i_1 \left(Z_1 + \frac{X_M^2}{Z_2}\right). \tag{19}$$

The ratio of equation (18) to equation (19) gives the voltage amplification, or

$$G = \frac{E_g}{E} = \frac{X_c\ X_M}{Z_1\ Z_2 + X_M^2} \qquad (20)$$

An equivalent circuit for Figure 2 can be drawn, showing the parameters due to the transformer primary circuit reflected into the secondary circuit.[9] This circuit is shown below.



Considering the above circuit at resonance, the following equation can be written:

$$X_{L2} - \frac{X_M^2}{|Z_1|^2} X_{L1} + X_c = 0 \qquad (21)$$

Assuming that the Q of the primary circuit is 10 or greater, Equation (21) becomes:

$$X_M^2 \approx X_{L1}\ (X_{L2} + X_c) \qquad (22)$$

Substituting Equations (9), (10), (11), (12), and (22) in Equation (20) and assuming that the Q's of the circuits are 10 or greater, Equation (20) becomes

$$G = \frac{X_c\ X_M}{R_1\ (X_{L2} + X_c) + (r_2\ X_{L1})} \qquad (23)$$

[9] Terman, loc. cit.

From Equations (9) and (10), the following is obtained:

$$k^2 = k_1^2 \frac{L_p}{L_1} \tag{24}$$

From Figure 2, the following equations are obtained:

$$R_1 = \frac{X_{L1}}{Q_1} \tag{25}$$

$$r_2 = \frac{X_{L2}}{Q_2} \tag{26}$$

$$r_o = \frac{X_{Lo}}{Q_o} \tag{27}$$

$$r_p = \frac{X_{Lp}}{Q_p} \tag{28}$$

From Equation (22), the value of $(X_{L2} + X_c)$ is obtained:

$$(X_{L2} + X_c) \approx \frac{X_M^2}{X_{L1}} \tag{29}$$

Equation (10) may be rewritten to give

$$X_M = k\ (X_{L1}\ X_{L2})^{1/2} \tag{30}$$

Upon substitution of Equations (29), (25), (26), and (30) in Equation (23), the following expression for G is obtained:

$$G = \frac{X_c\ k}{(X_{L1}\ X_{L2})^{1/2}} \left(\frac{Q_1\ Q_2}{k^2\ Q_2 + Q_1} \right). \tag{31}$$

Putting Equations (24) and (25) in (31), gives the following:

$$G = \frac{X_c\ k_1 \left(\frac{L_p}{L_1}\right)^{1/2}}{(X_{L1}\ X_{L2})^{1/2}} \left(\frac{\frac{X_{L1}}{R_1}\ Q_2}{\frac{X_{L1}}{R_1} + k_1^2 \frac{L_p}{L_1}\ Q_2} \right). \tag{32}$$

Equation (9) can be expressed as follows:

$$X_M = k_1 \left(X_{Lp}\, X_{L2}\right)^{1/2} \quad . \tag{33}$$

Equating expressions (30) and (33), and remembering that $X_{Lp} = X_{L1} - X_{Lo}$, the following is obtained:

$$X_{L1} = \frac{X_{Lo}}{1 - \dfrac{k^2}{k_1^2}} \quad . \tag{34}$$

Equating expressions (22) and (30), it follows that

$$X_{L2} = \frac{-X_c}{1 - k^2} \quad . \tag{35}$$

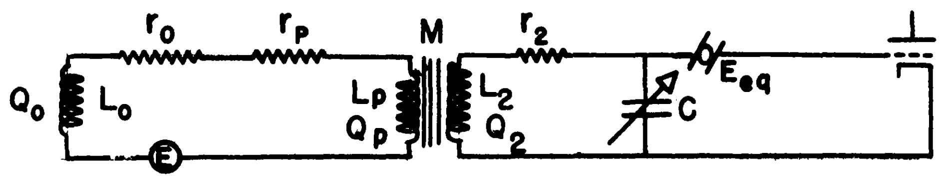
Therefore, substituting Equations (34), (35) and (24) in Equation (32) and solving in terms of the ratio of transformer primary to loop inductance and Q's, the final expression of voltage amplification is obtained. This result takes the form

$$G = \left(\frac{-X_c}{X_{Lo}}\right)^{1/2} Q_2 \frac{k_1 \left[\dfrac{\frac{L_p}{L_o}}{\frac{L_p}{L_o}+1}\left(1 - \dfrac{\frac{L_p}{L_o}}{\frac{L_p}{L_o}+1}\right)\left(1 - k_1^2 \dfrac{\frac{L_p}{L_o}}{\frac{L_p}{L_o}+1}\right)\right]^{1/2}}{1 + \dfrac{k_1^2 \frac{L_p}{L_o}}{\frac{L_p}{L_o}+1}\left(1 - \dfrac{\frac{L_p}{L_o}}{\frac{L_p}{L_o}+1}\right)\left(\dfrac{L_p}{L_o}\dfrac{Q_2}{Q_p} + \dfrac{Q_2}{Q_o}\right)} \quad . \tag{36}$$

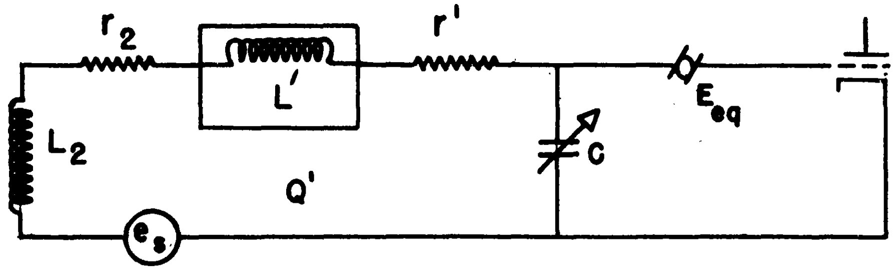
APPENDIX III

DERIVATION OF THE OPTIMUM SIGNAL-TO-NOISE RATIO EQUATIONS FOR A TRANSFORMER-COUPLED LOOP SYSTEM

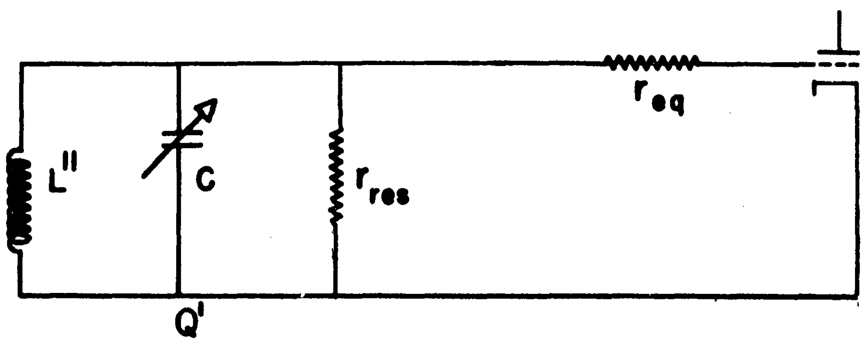
The total of noise voltages (E_{eq}) generated in those parts of a receiver following the input circuit can be represented by an equivalent noise generator in a circuit as shown in (A) below.



(A)



(B)



(C)

Assuming Q values greater than ten, Circuit (B) is equivalent to Circuit (A) when

$$L' = -\frac{M^2}{L_o + L_p} \tag{37}$$

$$r' = \frac{M^2}{(L_o + L_p)^2}\left(\frac{\omega L_o}{Q_o} + \frac{\omega L_p}{Q_p}\right) \tag{38}$$

$$L'' = (L_2 + L') \tag{39}$$

$$E_s = \frac{M}{(L_o + L_p)} E \quad . \tag{40}$$

From Circuit (B)

$$Q' = \frac{\omega\ (L_2 + L')}{r_2 + r'} \tag{41}$$

Circuit (C) is equivalent to Circuit (B) when E_{eq} is replaced by its equivalent noise resistance r_{eq} and

$$Q' = \frac{r_{res}}{\omega\ (L_2 + L')} \quad . \tag{42}$$

Equating expressions (41) and (42), it will be found that

$$r_{res} = \frac{\omega^2\ (L_2 + L')^2}{r_2 + r'} \tag{43}$$

The noise voltages in the grid circuit of (C) will be developed across the resistors r_{eq} and r_{res}. Applying the Nyquist equation[10] to this circuit, the rms noise voltage due to these resistances becomes

$$E_M = 1.26 \times 10^{-10}\ (\Delta f)^{1/2}\ (r_{res} + r_{eq})^{1/2} \tag{44}$$

where Δf is the width in cps of the equivalent rectangle having the same area as the squared transmission characteristic of the input circuit; the number (1.26×10^{-10}) is a factor derived from Boltzmann's Constant at a

[10] Reference Data for Radio Engineers, Federal Telephone and Radio Corporation, Second Edition, p. 248

temperature of 293°K (20°C).

In Circuit (B), the voltage appearing at the grid is

$$E_g = E_s Q' \quad . \tag{45}$$

Substituting Equations (40) and (42) in equation (45), the following is obtained:

$$E_g = \frac{ME}{L_o + L_p} \; \frac{r_{res}}{\omega (L_2 + L')} \tag{46}$$

The signal to noise ratio, ψ, is the ratio of E_g/E_n. Therefore, from Equations (46) and (44):

$$\psi = \frac{E_g}{E_n} = \frac{\dfrac{ME}{L_o + L_p} \; \dfrac{r_{res}}{\omega (L_2 + L')}}{1.26 \times 10^{-10} \; (\Delta f)^{1/2} (r_{res} + r_{eq})^{1/2}} \tag{47}$$

Substituting Equation (9) in (47) and rearranging the terms, the equation becomes

$$\psi = \frac{7.94 \times 10^{9}}{(\Delta f)^{1/2}} \; E \; \frac{k_1 \; (L_p \; L_2)^{1/2}}{\omega \; (L_o + L_p)(L_2 + L')} \left(\frac{r_{res}}{1 + \dfrac{r_{eq}}{r_{res}}} \right)^{1/2} \tag{48}$$

Substituting Equations (43), (26), and (38) in Equation (48) and simplifying to obtain Q and inductance ratios, the equation becomes:

$$\psi = \frac{\dfrac{7.94 \times 10^{9}}{(\Delta f)^{1/2}} \; E \left(\dfrac{Q_2}{X_{L'o}} \right)^{1/2} \dfrac{k_1}{\left[\left(\dfrac{L_o}{L_p} + \dfrac{L_p}{L_o} + 2 \right) + k_1^2 \left(\dfrac{Q_2}{Q_o} + \dfrac{Q_2}{Q_p} \dfrac{L_p}{L_o} \right) \right]^{1/2}}}{\left(1 + \dfrac{r_{eq}}{r_{res}} \right)^{1/2}} \tag{49}$$

From Equation (49), it is evident that the factor $(1 + r_{eq}/r_{res})^{1/2}$ is very close to unity in VLF receiver applications, since the ratio of r_{eq}/r_{res} is usually so small as to be negligible in such cases. (See discussion in section of text entitled "Design Criteria for Optimum Signal-To-Noise Ratio in a Transformer-Coupled Loop System.")

Therefore, it can be stated that

$$\psi = \frac{7.94 \times 10^{9}}{(\Delta f)^{1/2}} \; E\left(\frac{Q_2}{X_{Lo}}\right)^{1/2} \frac{k_1}{\left[\left(\frac{L_o}{L_p} + \frac{L_p}{L_o} + 2\right) + k_1^2\left(\frac{Q_2}{Q_o} + \frac{Q_2}{Q_p}\frac{L_p}{L_o}\right)\right]^{1/2}} \quad . \qquad (50)$$

Dividing both sides of this expression by $(Q_o)^{1/2}$, Equation (50) can also be expressed as the "signal-to-noise factor":

$$\gamma = \frac{\psi}{\frac{7.94 \times 10^{9}}{(\Delta f)^{1/2}} \; E\left(\frac{Q_o}{X_{Lo}}\right)^{1/2}} = \frac{\left(\frac{Q_2}{Q_o}\right)^{1/2} k_1}{\left[\left(\frac{L_o}{L_p} + \frac{L_p}{L_o} + 2\right) + k_1^2\left(\frac{Q_2}{Q_o} + \frac{Q_2}{Q_p}\frac{L_p}{L_o}\right)\right]^{1/2}} \quad . \qquad (51)$$

APPENDIX IV

DETERMINATION OF THE TRANSFORMER SECONDARY INDUCTANCE REQUIRED TO PROVIDE A DESIRED VALUE OF SECONDARY-CIRCUIT TANK INDUCTANCE

Substitution of Equations (7) and (9) in Equation (2), results in the following expression for the required tank inductance:

$$L_T = L_2 - \frac{k_1^2 L_p L_2}{L_p + L_o} \tag{52}$$

Solving for L_2,

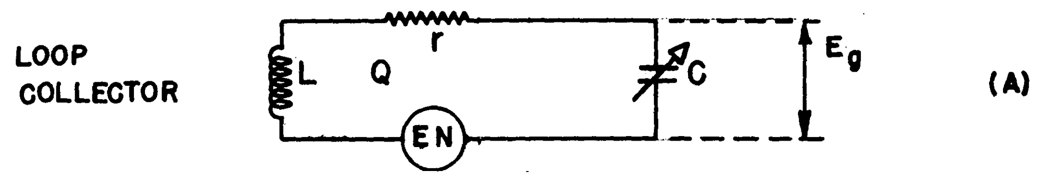
$$L_2 = \frac{L_T}{1 - k_1^2 \dfrac{L_p}{L_p + L_o}}\text{, or} \tag{53}$$

$$L_2 = \frac{L_T}{1 - k_1^2 \dfrac{L_p/L_o}{L_p/L_o + 1}} \tag{54}$$

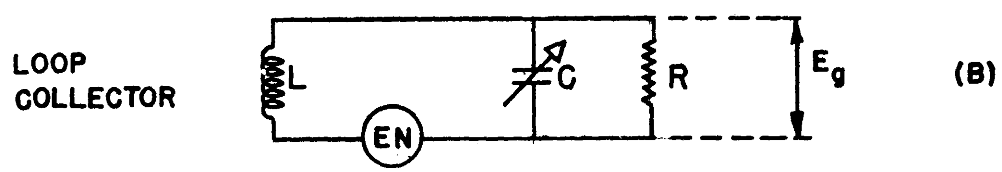
APPENDIX V

DERIVATION OF THE SIGNAL-TO-NOISE RATIO EQUATION FOR A DIRECT-CONNECTED LOOP SYSTEM

With both systems of coupling (direct-connected and transformer-coupled) using the same value of tuning capacitor, and with the direct-connected loop system having N times the number of turns in its loop as a loop of the same diameter in the transformer-coupled collector system, the induced voltage for the directly connected loop will be EN, when the voltage induced in the transformer-coupled loop is E.



DIRECT-CONNECTED LOOP COLLECTOR



EQUIVALENT CIRCUIT

In circuit (A):

$$Q = \frac{X_L}{r} \quad , \text{ and} \tag{55}$$

$$Q = \frac{E_g}{EN} \quad . \tag{56}$$

Therefore,

$$E_g = E\,N\,Q \quad . \tag{57}$$

In circuit (B):

$$R = Q\,X_L \quad . \tag{58}$$

Applying the Nyquist equation to Ciıcuit (B) for the noise voltage developed by the resistor R, it will be found that

$$E_n = 1.26 \times 10^{-10} (\Delta f)^{1/2} (R)^{1/2} \quad . \tag{59}$$

The rms signal-to-noise voltage ratio is the ratio of Equations (57) and (59):

$$\psi = \frac{E_g}{E_n} = \frac{E\,N\,Q}{1.26 \times 10^{-10} (\Delta f)^{1/2} (R)^{1/2}} \quad . \tag{60}$$

Substituting Equation (58) in equation (60) and simplifying, the following expression is obtained:

$$\psi = \frac{7.94 \times 10^{9}}{(\Delta f)^{1/2}} EN \left(\frac{Q}{X_L}\right)^{1/2} \quad . \tag{61}$$

ACKNOWLEDGMENT

The author gratefully acknowledges the use of much material derived from papers referred to in the text of this report, particularly those written by Mr. W. S. Bachman of General Electric Company, Bridgeport, Connecticut, and Messrs. J. E. Browder and V. J. Young of Sperry Gyroscope Company, Inc., New York.

PRNC-6871-8-16-48-100